Functional Analysis and Its Applications, Vol. 44, No. 1, pp. 36–47, 2010
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 44, No. 1, pp. 44–58, 2010

Simple Hurwitz Numbers of a Disk*

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Received March 31, 2008

To the memory of Israel Moiseevich Gelfand

Abstract. Let D be the closed unit disk. We study the Hurwitz numbers corresponding to the coverings of D whose only multiple critical value lies on the boundary of D and find differential equations describing the generating function of these numbers.

Key words: Hurwitz numbers, topological field theory, cut-and-join equation.

1. Introduction

A classical Hurwitz number is a weighted number of coverings of a compact genus g surface S with critical values of given type [6]. The precise definition is as follows. Consider a ramified covering $\varphi\colon \Omega \to S$ of S by a compact surface Ω. Two coverings φ and $\varphi'\colon \Omega' \to S$ are said to be equivalent if there exists a homeomorphism $f\colon \Omega \to \Omega'$ such that $\varphi' f = \varphi$. Let $\mathrm{Aut}(\varphi)$ be the group of self-equivalences of φ, and let $|\mathrm{Aut}(\varphi)|$ be its order.

In a neighborhood of each point $y \in \Omega$, the covering φ is equivalent to the quotient map by the group of rotations of order $n(y)$ about y. The *type* of a value $x \in S$ is defined as the monomial $a_{n(y_1)} \cdots a_{n(y_k)}$, where $\varphi^{-1}(x) = \{y_1, \dots, y_k\}$. The values of all types except for a_1^k are said to be *critical*. The critical values of type $a_1^{k-1}a_2$ are said to be *simple*.

Take finitely many points $x_1, \dots, x_v \in S$ and fix some monomials $a^1, \dots, a^v$ in the variables a_i. A *classical Hurwitz number* is defined as the number $\langle a^1, \dots, a^v\rangle_g = \sum |\mathrm{Aut}(\varphi)|^{-1}$, where the sum is taken over all equivalence classes of coverings with values $x_1, \dots, x_v$ of the respective types $a^1, \dots, a^v$ and with no other critical values. A Hurwitz number of the form $\langle a\rangle^m = \langle a, a_1^{k-1}a_2, \dots, a_1^{k-1}a_2\rangle_0$, where a is an arbitrary monomial in the variables a_i and m is the number of monomials $a_1^{k-1}a_2$ in the brackets, is called a *classical simple Hurwitz number*. The classical simple Hurwitz numbers are closely related to the intersection theory on the moduli space of compact Riemann surfaces with marked points [7].

We assign variables p_i to the variables a_i, and the monomials $p_a = p_{i_1} \cdots p_{i_r}$ to the monomials $a = a_{i_1} \cdots a_{i_r}$. The generating function of simple Hurwitz numbers is given by the formula $\Phi(\lambda, p_1, p_2, \dots) = \sum_{m\geqslant 0} \frac{\lambda^m}{m!} \sum_a \langle a\rangle^m p_a$, where the second sum is taken over all monomials. According to [5], the generating function $\Phi(\lambda, p_1, p_2, \dots)$ satisfies the cut-and-join differential equation

$$\frac{\partial \Phi}{\partial \lambda} = L_\lambda \Phi, \qquad L_\lambda = \frac{1}{2}\sum_{ij}(i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + \sum_{ij} ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}\,.$$

The proof is essentially based on the fact that the classical Hurwitz numbers are correlators of a closed 2D topological field theory [3]. This equation has some remarkable properties. For example, it is satisfied by the generating function of Hodge integrals [12].

The definition of Hurwitz numbers can be extended to coverings of arbitrary surfaces with boundary by arbitrary surfaces with boundary [1]. In the present paper, we define and study the *simple Hurwitz numbers of a disk*. They correspond to the coverings of the disk that have only one multiple critical value and satisfy the additional condition that this value lies on the boundary

*Supported in part by grants NSh-709.2008.1 and RFBR 08-01-00110-a.

 0016–2663/10/4401–0036

of the disk. In particular, we find recursion relations that permit computing all simple Hurwitz numbers of the disk.

The generating functions of simple Hurwitz numbers of a disk depend on complex parameters α, β, and γ (which are analogs of the parameter λ in the classical situation) and four infinite sets of variables $\acute{p}_i$, $\grave{p}_i$, $\bar{p}_i$, and $\dot{p}_i$, which describe the topological type of the multiple critical value (analogs of the variables p_i in the classical situation). The equation $\partial\Phi/\partial\lambda = L_\lambda\Phi$ has a counterpart in the form of three differential equations.

The Hurwitz numbers of surfaces with boundary are correlators of an open-closed (noncommutative) 2D topological field theory [1]. That is precisely why the generating function of simple Hurwitz numbers of a disk satisfies some differential equations.

The author is grateful to V. M. Buchstaber for useful discussion of topological properties of coverings.

2. Hurwitz Numbers

Let D be the closed disk $\{z \in \mathbb{C} : |z| \leqslant 1\}$ with oriented boundary, and let Ω be a compact surface with boundary. In this paper, we use the term "covering" for degree k Smith–Dold coverings such that the function taking each point of the disk to the number of its preimages has the following property: its restriction to the interior $D^\circ = D \setminus \partial D$ of the disk is discontinuous at at most finitely many points, and so is its restriction to the boundary ∂D. One can readily show that this class of coverings is the same as the one considered in [1]. Recall that a continuous function $\varphi\colon \Omega \to D$ is called a Smith–Dold covering ([11], [4]) if there exists a continuous function $t\colon D \to \operatorname{Sym}^k(\Omega)$ such that (1) $x \in t\varphi(x)$ for all $x \in \Omega$ and (2) $\operatorname{Sym}^k(\varphi)(ty) = ky$ for all $y \in D$.

In particular, dianalytic morphisms of Klein surfaces ([2], [8]) are coverings in our sense. Moreover, for every topological covering $\varphi\colon \Omega \to D$ there exists a dianalytic structure on Ω making φ a morphism of Klein surfaces [9]. Morphisms of Klein surfaces are in a one-to-one correspondence with real meromorphic functions in the sense of [10]. Namely, each dianalytic morphism can be obtained from a real meromorphic function f on a real algebraic curve (P, τ) as the composition $h_2 f h_1^{-1}$, where $h_1\colon P \to P/\langle\tau\rangle$ is the natural projection and $h_2(z) = \operatorname{Re} z + i|\operatorname{Im} z|$.

The preimage $\varphi^{-1}(x)$ of an interior point $x \in D \setminus \partial D$ consists of $n = n(x) \leqslant k$ points. Consider a simple contour $r \in D^\circ$ bounding a small neighborhood of x. Its preimage $\varphi^{-1}(r)$ splits into simple contours $C_1, \dots, C_n \in \Omega^\circ = \Omega \setminus \partial\Omega$. The unordered set $(\deg(\varphi|_{C_1}), \dots, \deg(\varphi|_{C_n}))$ of degrees of the restrictions of φ to these contours is called the (*topological*) *type of the interior value* $x \in D^\circ$ and will be denoted by a monomial $a_1^{t_1} \cdots a_k^{t_k}$ in commuting formal variables a_i, where t_i is the number of indices j such that $\deg(\varphi|_{C_j}) = i$. (Here $a_i^0 = 1$.)

A value $x \in D^\circ$ is said to be *regular* if the restriction of φ to each connected component of the preimage of a small neighborhood of x is a homeomorphism. All other $x \in D^\circ$ are called *interior critical values*. In other words, the interior critical values are all interior values of topological type not equal to a_1^k. Any covering has at most finitely many interior critical values. The interior critical values of type $a_1^{k-1}a_2$ are said to be *simple*.

The preimage $\varphi^{-1}(y)$ of a boundary point $y \in \partial D$ also consists of $n = n(y) \leqslant k$ points. Consider a simple arc $l \subset D$ with endpoints on ∂D bounding a small neighborhood of y. Its preimage $\varphi^{-1}(l) \subset \Omega$ is a graph with k edges. The vertices of the graph split into two groups corresponding to the preimages of the two endpoints of l. The endpoints of l and the corresponding groups of vertices are assumed to be ordered in the sense of the orientation of the boundary ∂D. For convenience, we say that one endpoint is left and the other is right, so that the positive orientation of the disk boundary corresponds to moving in the neighborhood from left to right between the endpoints. Accordingly, we distinguish between *right* and *left* vertices of the graph $\varphi^{-1}(l)$.

Thus, the graph $\varphi^{-1}(l)$ is bipartite. Its topological type is called the (*topological*) *type of the boundary value* $y \in \partial D$ [1]. The degree of every vertex of the graph does not exceed 2. Thus, the connected components of the graph $\varphi^{-1}(l)$ can be of the following types:

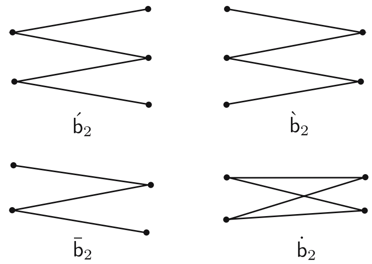

Fig. 1

- $\acute{\mathsf{b}}_i$ (a graph with i left and $i+1$ right vertices).
- $\grave{\mathsf{b}}_i$ (a graph with $i+1$ left and i right vertices).
- $\bar{\mathsf{b}}_i$ (an open graph with i left and i right vertices).
- $\dot{\mathsf{b}}_i$ (a closed graph with i left and i right vertices).

See Fig. 1 for the corresponding graphs with $i = 2$.

Here is an example explaining the geometrical meaning of these invariants. The Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ bears an action of the dihedral group G generated by the reflection $z \mapsto \bar{z}$ and the rotation $z \mapsto e^{2\pi i/k} z$. The action defines a covering $\varphi\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}/G$; its critical points are 0 and ∞. The boundary values corresponding to these points have the type $\dot{\mathsf{b}}_k$. Consider the restriction $\tilde{\varphi}$ of the mapping φ to the disk $\{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\} \cup \infty$. For k odd, the topological types of both values $\tilde{\varphi}(0)$ and $\tilde{\varphi}(\infty)$ are the same and equal to $\bar{\mathsf{b}}_{(k+1)/2}$. For k even, one of the values has the topological type $\acute{\mathsf{b}}_{k/2}$ and the other, $\grave{\mathsf{b}}_{k/2}$.

Let us assign commuting formal variables $\acute{b}_i$, $\grave{b}_i$, $\bar{b}_i$, $\dot{b}_i$ to the graphs $\acute{\mathsf{b}}_i$, $\grave{\mathsf{b}}_i$, $\bar{\mathsf{b}}_i$, $\dot{\mathsf{b}}_i$. In what follows, we almost always denote graphs by the same symbols as the corresponding monomials. To a union of graphs, we assign the product of the corresponding monomials. Thus, the topological type of any boundary value is encoded by a monomial $b = \acute{b}_1^{\acute{s}_1} \cdots \acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{b}_1^{\grave{s}_1} \cdots \grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{b}_1^{\bar{s}_1} \cdots \bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{b}_1^{\dot{s}_1} \cdots \dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$ where $k = \sum_{i=1}^{\acute{n}} 2\acute{s} + \sum_{i=1}^{\bar{n}} (2\bar{s}_i - 1) + \sum_{i=1}^{\grave{n}} 2\grave{s}_i + \sum_{i=1}^{\dot{n}} 2\dot{s}_i$. Let $\operatorname{Aut}(b)$ be the automorphism group of the graph corresponding to b, and let $|\operatorname{Aut}(b)|$ be the order of this group.

Reversing the order of the groups of vertices of the graph gives rise to an involution $b \mapsto b^*$ on the set of monomials. In particular, $\acute{b}_i^{\,*} = \grave{b}_i$, $\grave{b}_i^{\,*} = \acute{b}_i$, $\bar{b}_i^{\,*} = \bar{b}_i$, and $\dot{b}_i^{\,*} = \dot{b}_i$.

A value $x \in \partial D$ is said to be *regular* if the restriction of φ to each connected component of the preimage of a small boundary segment containing x is a homeomorphism. All other boundary values are called *boundary critical values.* In other words, boundary critical values are all critical values of type other than $\bar{b}_1^{\bar{s}} \dot{b}_1^{\dot{s}}$. Any covering has at most finitely many boundary critical values. Boundary critical values of types $\acute{b}_1 \bar{b}_1^{\bar{s}} \dot{b}_1^{\dot{s}}$ and $\grave{b}_1 \bar{b}_1^{\bar{s}} \dot{b}_1^{\dot{s}}$ are called *simple boundary critical values* and will be referred to as *aigu values* and *grave values*, respectively.

We say that two coverings $\varphi_1\colon \Omega_1 \to D$ and $\varphi_2\colon \Omega_2 \to D$ are equivalent if there exists a homeomorphism $\phi\colon \Omega_1 \to \Omega_2$ such that $\varphi_1 = \varphi_2 \phi$. The automorphism groups $\operatorname{Aut}(\varphi_i)$ of equivalent coverings φ_i are isomorphic. We denote their order by $|\operatorname{Aut}(\varphi_i)|$.

Example 2.1. A covering $\varphi\colon \Omega \to D$ of the disk $D = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\} \cup \infty$ has no critical values if and only if its restriction to each connected component is either a homeomorphism or is equivalent to a two-sheeted covering $h\colon \mathbb{C} \cup \infty \to \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\} \cup \infty$, where $h(z) = \operatorname{Re} z + i|\operatorname{Im} z|$. Its automorphism group $\operatorname{Aut}(\varphi)$ is isomorphic to $S_n \times S_m \times (\mathbb{Z}/2\mathbb{Z})^m$, where n and m stand for the numbers of connected components of Ω on which φ is a homeomorphism or a two-sheeted covering, respectively.

On the disk D, let us fix finitely many interior points $x_1, \dots, x_v \in D^\circ$ and finitely many boundary points $y_1, \dots, y_w \in \partial D$ numbered in accordance with the orientation of the boundary. We also fix monomials $a^1, \dots, a^v$ in the variables a_i and monomials $b^1, \dots, b^w$ in the variables $\acute{b}_i$, $\grave{b}_i$, $\bar{b}_i$, $\dot{b}_i$.

The *Hurwitz number* is defined as $\langle a^1,\dots,a^v,(b^1,\dots,b^w)\rangle = \sum |\mathrm{Aut}(\varphi)|^{-1}$, where the sum is over the equivalence classes of coverings having the values $x_1,\dots,x_v$, $y_1,\dots,y_w$ of the respective types $a^1,\dots,a^v$, $b^1,\dots,b^w$ and no other critical values.

Example 2.2. $\langle(\bar{b}_1^{\bar{s}}\dot{b}_1^{\dot{s}})\rangle = (\bar{s}_1!)^{-1}(2^{\dot{s}}\dot{s}_1!)^{-1}$.

One can readily see that the Hurwitz number is independent of the position of the critical points. It does not change under any permutation of the monomials a^i and under a cyclic permutation of the monomials b^i. The above-described correspondence between coverings and real meromorphic functions permits one to interpret these Hurwitz numbers as a counterpart of the classical Hurwitz numbers for real algebraic curves [1].

Lemma 2.1. *The Hurwitz number $\langle(c,b)\rangle$ is nonzero only if $c=b^*$, and then $\langle(b^*,b)\rangle = |\mathrm{Aut}(b)|^{-1}$.*

The Hurwitz number $\langle(c,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},b)\rangle$ is nonzero only in the following cases:

- $\langle(\dot{b}_i d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\grave{b}_i d)\rangle = \frac{1}{2}|\mathrm{Aut}(d)|^{-1}$.
- $\langle(\bar{b}_{i+j} d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\bar{b}_i\grave{b}_j d)\rangle = |\mathrm{Aut}(d)|^{-1}$.
- $\langle(\acute{b}_{i+j} d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\grave{b}_i\grave{b}_j d)\rangle = (1-\frac{1}{2}\delta_{ij}|\mathrm{Aut}(d)|^{-1}$.
- $\langle(\grave{b}_{i+j-1} d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\bar{b}_i\bar{b}_j d)\rangle = (1-\frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$.

The Hurwitz number $\langle(c,\grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},b)\rangle$ is nonzero only in the following cases:

- $\langle(\acute{b}_i d,\grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\dot{b}_i d^*)\rangle = \frac{1}{2}|\mathrm{Aut}(d)|^{-1}$.
- $\langle(\bar{b}_i\acute{b}_j d,\grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\bar{b}_{i+j} d^*)\rangle = |\mathrm{Aut}(d)|^{-1}$.
- $\langle(\acute{b}_i\acute{b}_j d,\grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\grave{b}_{i+j} d^*)\rangle = (1-\frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$.
- $\langle(\bar{b}_i\bar{b}_j d,\grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\acute{b}_{i+j-1} d^*)\rangle = (1-\frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$.

Proof. The first statement is obvious. Let $\langle(c,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},b)\rangle \neq 0$. By definition, this means that there exists a covering $\varphi\colon \Omega\to D$ with boundary critical values $y_1,y_2,y_3\in\partial D$ of the respective types c, $\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}$, b and with no other critical values. Consider points $z_1\in(y_1,y_2)$ and $z_2\in(y_2,y_3)$. The preimage of z_1 consists of $\bar{m}$ points in a neighborhood of which φ is one-to-one (simple points) and $\dot{m}+1$ points in a neighborhood of which φ is two-to-one (double points). The preimage of z_2 consists of $\bar{m}+2$ simple points and $\dot{m}$ double points.

Let $p_1,p_2\in\varphi^{-1}(z_2)$ be the simple points corresponding to the graph $\acute{b}_1$. Let q_1 and q_2 be the corresponding points of the graph b. Then one of the following cases takes place:

- q_1 and q_2 belong to the same connected component of type $\grave{b}_i$.
- q_1 and q_2 belong to connected components of types $\bar{b}_i$ and $\grave{b}_j$.
- q_1 and q_2 belong to distinct connected components of types $\bar{b}_i$ and $\bar{b}_j$.
- q_1 and q_2 belong to distinct connected components of types $\grave{b}_i$ and $\grave{b}_j$.

In the first case, $b=\grave{b}_i d$. Consider the restriction $\varphi'\colon\Omega'\to D$ of the covering φ to the connected component containing the points q_1 and q_2 and the restriction $\varphi''\colon\Omega''\to D$ of φ to the complement $\Omega''=\Omega\setminus\Omega'$. The covering φ'' has critical values $y_1,y_3\in\partial D$ of types d^* and d. Therefore, the covering φ' has critical values $y_1,y_2,y_3\in\partial D$ of types $\dot{b}_i$, $\acute{b}_1 e$, and $\grave{b}_i$. Thus, the covering φ has critical values $y_1,y_2,y_3\in\partial D$ of types $\dot{b}_i d^*$, $\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}$, and $\grave{b}_i d$.

The equivalence class of the covering φ' (respectively, φ'') contains all coverings having critical values of the same types as φ' (respectively, φ'') itself. Furthermore, $\mathrm{Aut}(\varphi)=\mathrm{Aut}(\varphi')\times\mathrm{Aut}(\varphi'')$. Thus,

$$\langle(\dot{b}_i d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\grave{b}_i d)\rangle = \frac{1}{|\mathrm{Aut}(\varphi)|} = \frac{1}{|\mathrm{Aut}(\varphi')|}\,\frac{1}{|\mathrm{Aut}(\varphi'')|} = \frac{1}{|\mathrm{Aut}(\varphi')|}\,\frac{1}{|\mathrm{Aut}(d)|}\,.$$

The group $\mathrm{Aut}(\varphi')$ is generated by the involution transposing the points p_1 and p_2. Consequently, $\langle(\dot{b}_i d^*,\acute{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}},\grave{b}_i d)\rangle = \frac{1}{2}|\mathrm{Aut}(d)|^{-1}$.

The other cases can be treated in a similar way. The main difference is in the properties of the group $\mathrm{Aut}(\varphi')$. It is nontrivial only for $i=j$ in the last two cases.

By changing the orientation of the disk boundary, we find that $\langle (a,b,c)\rangle = \langle (c^*, b^*, a^*)\rangle$. Thus, the second assertion of the lemma implies the third one. $\square$

Whenever the difference between the graphs $\acute{b}_i$ and $\grave{b}_i$ is not important, we denote both graphs also by $\hat{b}_i$. We define the length $|b|$ of a connected graph b as the number of its edges. In particular, $|\bar{b}_i| = 2i-1$ and $|\hat{b}_i| = |\acute{b}_i| = |\grave{b}_i| = 2i$. For an arbitrary graph b, let $|b|$ be the maximum length of its connected components.

Lemma 2.2. *The Hurwitz number* $\langle a_1^m a_2, (c,b)\rangle$ *is not zero only in the following cases*:

- $\langle a_1^m a_2, (\dot{b}_i \dot{b}_j d^*, \dot{b}_{i+j} d)\rangle = (1 - \frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\dot{b}_i \bar{b}_j d^*, \bar{b}_{i+j} d)\rangle = |\bar{b}_j|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\dot{b}_i \grave{b}_j d^*, \acute{b}_{i+j} d)\rangle = |\hat{b}_j|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\dot{b}_i \acute{b}_j d^*, \grave{b}_{i+j} d)\rangle = |\hat{b}_j|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\bar{b}_i \grave{b}_j d^*, \bar{b}_k \acute{b}_l d)\rangle = \delta_{(i+j)(k+l)}|\bar{b}_i \hat{b}_j \bar{b}_k \hat{b}_l|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\acute{b}_i \grave{b}_j d^*, \bar{b}_k \bar{b}_l d)\rangle = \delta_{(i+j+1)(k+l)}|\hat{b}_i \hat{b}_j \bar{b}_k \bar{b}_l|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\grave{b}_i \grave{b}_j d^*, \acute{b}_k \acute{b}_l d)\rangle = (1 - \frac{1}{2}\delta_{ij}\delta_{kl})\delta_{(i+j)(k+l)}|\hat{b}_i \hat{b}_j \hat{b}_k \hat{b}_l|\,|\mathrm{Aut}(d)|^{-1}$.
- $\langle a_1^m a_2, (\bar{b}_i \bar{b}_j d^*, \bar{b}_k \bar{b}_l d)\rangle = (1 - \frac{1}{2}\delta_{ij}\delta_{kl})\delta_{(i+j)(k+l)}|\bar{b}_i \bar{b}_j \bar{b}_k \bar{b}_l|\,|\mathrm{Aut}(d)|^{-1}$.

Proof. Let $\langle a_1^m a_2, (c,b)\rangle \neq 0$. This means by definition that there exists a covering $\varphi\colon \Omega \to D$ with critical values x, y_1, and y_2 of types $a_1^m a_2$, c, and b, respectively, and with no other critical values. Consider a simple curve $l_i \subset D$ having endpoints on ∂D and bounding a small neighborhood of the point y_i. Consider the graphs $c = \varphi^{-1}(l_1)$ and $b = \varphi^{-1}(l_2)$ describing the topological types of the points y_i. Consider a segment $l \subset D$ joining the l_1 and l_2 and passing through x.

The preimage $\varphi^{-1}(l)$ consists of $\deg\varphi$ segments; exactly two of them meet. The intersection is a critical point of φ. The segments join the edges of the graphs b and c. Thus, the graph c^* is obtained from b by the following perestroika: one cuts two edges of the graph b and joins the resulting half-edges differently.

Such a perestroika of bipartite graphs can be realized by a covering $\varphi\colon \Omega \to D$ if and only if the orientation of the edges induced by the ordering of the groups of graph vertices is consistent with one of the orientations of the boundary $\partial\Omega$. All pairs (b,c) satisfying these conditions are listed in Lemma 2.2.

The number of coverings corresponding to a pair (b,c) coincides with the number of pairs of edges of the graph b such that cutting and joining them gives the graph c^*. This number, in turn, depends on the number of edges in the connected components of b and c to be cut and joined.

For $b = \bar{b}_{i+j} d$ and $c = \dot{b}_i \bar{b}_j d^*$, the number of coverings coincides with the number of sequences of i successive edges of the graph $\bar{b}_{i+j}$. There are exactly $|\bar{b}_j|$ such sequences. Thus, $\langle a_1^m a_2, (\dot{b}_i \bar{b}_j d^*, \bar{b}_{i+j} d)\rangle = |\bar{b}_j|\,|\mathrm{Aut}(d)|^{-1}$.

For $b = \bar{b}_k \acute{b}_l d$ and $c = \bar{b}_i \grave{b}_j d^*$, the number of coverings coincides with the number of pairs of edges such that one edge belongs to $\bar{b}_k$, the other to $\acute{b}_l$, and after cutting and joining by these edges the graph $\bar{b}_i \acute{b}_j$ arises. Thus, $\langle a_1^m a_2, (\bar{b}_i \grave{b}_j d^*, \bar{b}_k \acute{b}_l d)\rangle = \delta_{(i+j)(k+l)}|\bar{b}_i \hat{b}_j \bar{b}_k \hat{b}_l|\,|\mathrm{Aut}(d)|^{-1}$.

The other Hurwitz numbers can be computed in a similar way. $\square$

3. Simple Hurwitz Numbers of a Disk

We say that a boundary critical value p' of a covering φ *precedes* a boundary critical value p'' of the same covering if the orientation of the disk boundary is from p' to p'' on the curve (p', p'') and if (p', p'') contains no critical values.

Consider the set $\mathscr{H}(m, \hat{m}, b)$ of equivalence classes of coverings with m simple interior critical values, $\hat{m}$ simple boundary critical values, and a single critical value, not necessarily simple, of type b. (This value is said to be *special.*) The set $\mathscr{H}(m, \hat{m}, b)$ splits into the subsets $\mathscr{H}(m, \acute{m}, \grave{m}, b)$ containing coverings with $\acute{m}$ aigu values and $\grave{m}$ grave values. The set $\mathscr{H}(m, \acute{m}, \grave{m}, b)$ splits into two subsets $\acute{\mathscr{H}}(m, \acute{m}, \grave{m}, b)$ and $\grave{\mathscr{H}}(m, \acute{m}, \grave{m}, b)$. The first (respectively, second) of them contains coverings such that the special critical value is preceded by an aigu value (respectively, a grave value).

Let $\mathsf{H}(m, \hat{m}, b)$ (respectively, $\mathsf{H}(m, \acute{m}, \grave{m}, b)$, $\acute{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$, or $\grave{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$) be the sum of Hurwitz numbers of coverings in the set $\mathscr{H}(m, \hat{m}, b)$ (respectively, $\mathscr{H}(m, \acute{m}, \grave{m}, b)$, $\acute{\mathscr{H}}(m, \acute{m}, \grave{m}, b)$, or $\grave{\mathscr{H}}(m, \acute{m}, \grave{m}, b)$). We extend the definitions of $\mathsf{H}(m, \hat{m}, b)$, $\mathsf{H}(m, \acute{m}, \grave{m}, b)$, $\acute{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$, and $\grave{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$ to the ratios $b = b^1/b^2$ of monomials assuming all these numbers to be 0 unless b is a monomial. We refer to all these numbers as the *simple Hurwitz numbers of a disk*.

Example 3.1. One can readily see that a covering cannot have exactly one critical value [1]. By comparing this with the previous examples, one finds that if $\mathsf{H}(0, 0, b) > 0$, then $b = \bar{b}_1^{\bar{s}_1} \dot{b}_1^{\dot{s}_1}$ and $\mathsf{H}(0, 0, \bar{b}_1^{\bar{s}_1} \dot{b}_1^{\dot{s}_1}) = (\bar{s}_1!)^{-1}(2^{\dot{s}} \dot{s}_1!)^{-1}$.

Lemma 3.1. *Let* $b = \acute{b}_1^{\acute{s}_1} \cdots \acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{b}_1^{\grave{s}_1} \cdots \grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{b}_1^{\bar{s}_1} \cdots \bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{b}_1^{\dot{s}_1} \cdots \dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$. *Then*

$$\begin{aligned}\acute{\mathsf{H}}(m, \acute{m}+1, \grave{m}, b) &= \sum_i i(\dot{s}_i + 1)\mathsf{H}\biggl(m, \acute{m}, \grave{m}, b\frac{\dot{b}_i}{\grave{b}_i}\biggr) \\ &\quad + \sum_{ij} \biggl((\bar{s}_{i+j} + 1)\mathsf{H}\biggl(m, \acute{m}, \grave{m}, b\frac{\bar{b}_{i+j}}{\bar{b}_i \grave{b}_j}\biggr) + (\grave{s}_{i+j} + 1)\mathsf{H}\biggl(m, \acute{m}, \grave{m}, b\frac{\grave{b}_{i+j}}{\grave{b}_i \grave{b}_j}\biggr) \\ &\qquad + (\acute{s}_{i+j-1} + 1)\mathsf{H}\biggl(m, \acute{m}, \grave{m}, b\frac{\acute{b}_{i+j-1}}{\bar{b}_i \bar{b}_j}\biggr)\biggr).\end{aligned}$$

Proof. Consider a covering $\varphi\colon \Omega \to D$ in the set $\acute{\mathscr{H}}(m, \acute{m}+1, \grave{m}, b)$. Let y and y' be the special critical value of the covering and the preceding boundary critical value, respectively. Consider a segment $l \subset D$ with endpoints in ∂D separating y and y' from the other critical values. Let us shrink l into a point y_l; then the disk splits into two disks, D' and D'', and the covering φ splits into two coverings, $\varphi'\colon \Omega' \to D'$ and $\varphi''\colon \Omega'' \to D''$. The critical values of φ' are y_l, y', and y. By identifying D'' with D, we see that $\varphi'' \in \mathscr{H}(m, \acute{m}, \grave{m}, c)$ for some monomial c. By [1], this implies that $\acute{H}(m, \acute{m}+1, \grave{m}, b) = \sum_{pq} H(m, \acute{m}, \grave{m}, \beta_p)\, F^{pq} \langle (\beta_q, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b) \rangle$, where $\{\beta_p\}$ is the set of all monomials, $\{F^{pq}\}$ is the inverse matrix of $\{\langle (\beta_p, \beta_q) \rangle\}$, and the sum is taken over all pairs of monomials. By Lemma 2.1, $F^{pq} = \delta_{\beta_p, \beta_q^*} |\mathrm{Aut}(\beta_p)|$. Further, again by Lemma 2.1, the sum on the right-hand side of the equation consists of four subsums, each of which is determined by the respective form of the monomials b and $\beta_q = \beta_p^*$.

Let $b = \grave{b}_i d$ and $\beta_q = \dot{b}_i d^*$. Then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\dot{b}_i d)| = \frac{|\mathrm{Aut}(d)|}{(2i)^{\dot{s}_i} \dot{s}_i!} (2i)^{\dot{s}_i + 1} (\dot{s}_i + 1)! = 2i |\mathrm{Aut}(d)| (\dot{s}_i + 1).$$

Moreover, $\langle (\beta_q, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b) \rangle = \langle (\dot{b}_i d^*, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, \grave{b}_i d) \rangle = \frac{1}{2} |\mathrm{Aut}(d)|^{-1}$ by Lemma 2.1. Thus, the first subsum is $\sum\limits_i i(\dot{s}_i + 1)\mathsf{H}(m, \acute{m}, \grave{m}, b\frac{\dot{b}_i}{\grave{b}_i})$.

Let $b = \bar{b}_i \grave{b}_j d$ and $\beta_q = \bar{b}_{i+j} d^*$. Then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\bar{b}_{i+j} d)| = \frac{|\mathrm{Aut}(d)|}{\bar{s}_{i+j}!} (\bar{s}_{i+j} + 1)! = |\mathrm{Aut}(d)| (\bar{s}_{i+j} + 1).$$

Moreover, $\langle (\beta_q, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b) \rangle = \langle (\bar{b}_{i+j} d^*, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, \bar{b}_i \grave{b}_j d) \rangle = |\mathrm{Aut}(d)|^{-1}$. Thus, the second subsum is $\sum_{ij} (\bar{s}_{i+j} + 1)\mathsf{H}(m, \acute{m}, \grave{m}, b\frac{\bar{b}_{i+j}}{\bar{b}_i \grave{b}_j})$.

Let $b = \grave{b}_i \grave{b}_j d$ and $\beta_q = \acute{b}_{i+j} d^*$. Then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\grave{b}_{i+j} d)| = \frac{|\mathrm{Aut}(d)|}{2^{\grave{s}_{i+j}} \grave{s}_{i+j}!} 2^{(\grave{s}_{i+j} + 1)} (\grave{s}_{i+j} + 1)! = 2 |\mathrm{Aut}(d)| (\grave{s}_{i+j} + 1).$$

Moreover, $\langle(\beta_q, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b)\rangle = \langle(\acute{b}_{i+j}d^*, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, \grave{b}_i\grave{b}_j d)\rangle = (1-\frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$. Thus, the third subsum is $\sum_{ij}(\grave{s}_{i+j}+1)\mathsf{H}(m, \acute{m}, \grave{m}, b\frac{\grave{b}_{i+j}}{\grave{b}_i\grave{b}_j})$.

Let $b = \bar{b}_i\bar{b}_j d$ and $\beta_q = \grave{b}_{i+j-1}d^*$. Then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\acute{b}_{i+j-1}d)| = \frac{|\mathrm{Aut}(d)|}{2^{(\acute{s}_{i+j-1})}\acute{s}_{i+j-1}!}\,2^{(\acute{s}_{i+j-1}+1)}(\acute{s}_{i+j-1}+1)! = 2|\mathrm{Aut}(d)|(\acute{s}_{i+j-1}+1).$$

Moreover, $\langle(\beta_q, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b)\rangle = \langle(\grave{b}_{i+j-1}d^*, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, \bar{b}_i\bar{b}_j d)\rangle = (1-\frac{1}{2}\delta_{ij})|\mathrm{Aut}(d)|^{-1}$. Thus, the fourth subsum is

$$\sum_{ij}(\acute{s}_{i+j-1}+1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\acute{b}_{i+j-1}}{\bar{b}_i\bar{b}_j}\right). \qquad \square$$

Lemma 3.2. *Let* $b = \acute{b}_1^{\acute{s}_1}\cdots\acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}}\grave{b}_1^{\grave{s}_1}\cdots\grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}}\bar{b}_1^{\bar{s}_1}\cdots\bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}}\dot{b}_1^{\dot{s}_1}\cdots\dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$*. Then*

$$\begin{aligned}\grave{\mathsf{H}}(m, \acute{m}, \grave{m}+1, b) = &\sum_i(\grave{s}_i+1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i}{\dot{b}_i}\right) + \sum_{ij}\left(2(\bar{s}_i+1)(\grave{s}_j+1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\bar{b}_i\grave{b}_j}{\bar{b}_{i+j}}\right)\right.\\ &+ 2((\grave{s}_i+1)(\grave{s}_j+1)+\delta_{ij}(\grave{s}_i+1))\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i\grave{b}_j}{\grave{b}_{i+j}}\right)\\ &\left.+ \frac{1}{2}((\bar{s}_i+1)(\bar{s}_j+1)+\delta_{ij}(\bar{s}_i+1))\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\bar{b}_i\bar{b}_j}{\acute{b}_{i+j-1}}\right)\right).\end{aligned}$$

Proof. The lemma can be proved along the same lines as the preceding one; the proof uses the second part of Lemma 2.1.

If $(\beta_q, \grave{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b) = (\acute{b}_i d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \dot{b}_i d)$, then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\grave{b}_i d)| = 2|\mathrm{Aut}(d)|(\grave{s}_i+1)$$

and

$$\langle(\beta_q, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b)\rangle = \langle(\acute{b}_i d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \dot{b}_i d)\rangle = \frac{1}{2}\frac{1}{|\mathrm{Aut}(d)|}.$$

Thus, the corresponding subsum is $\sum_i(\grave{s}_i+1)\mathsf{H}(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i}{\dot{b}_i})$.

If $(\beta_q, \grave{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b) = (\bar{b}_i\acute{b}_j d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \bar{b}_{i+j}d)$, then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\bar{b}_i\grave{b}_j d)| = 2|\mathrm{Aut}(d)|(\bar{s}_i+1)(\grave{s}_j+1)$$

and

$$\langle(\beta_q, \acute{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b)\rangle = \langle(\bar{b}_i\acute{b}_j d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \bar{b}_{i+j}d)\rangle = |\mathrm{Aut}(d)|^{-1}.$$

Thus, the corresponding subsum is

$$2\sum_{ij}(\bar{s}_i+1)(\grave{s}_j+1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\bar{b}_i\grave{b}_j}{\bar{b}_{i+j}}\right).$$

If $(\beta_q, \grave{b}_1\bar{b}_1^{\bar{n}}\dot{b}_1^{\dot{n}}, b) = (\acute{b}_i\acute{b}_j d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \grave{b}_{i+j}d)$, then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\grave{b}_i\grave{b}_j d)| = 4|\mathrm{Aut}(d)|((\grave{s}_i+1)(\grave{s}_j+1)+\delta_{ij}(\grave{s}_i+1))$$

and

$$\langle(\acute{b}_i\acute{b}_j d^*, \grave{b}_1\bar{b}_1^{\bar{m}}\dot{b}_1^{\dot{m}}, \grave{b}_{i+j}d)\rangle = \left(1-\frac{1}{2}\delta_{ij}\right)\frac{1}{|\mathrm{Aut}(d)|}.$$

Thus, the corresponding subsum is

$$2\sum_{ij}((\grave{s}_i+1)(\grave{s}_j+1)+\delta_{ij}(\grave{s}_i+1))\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i\grave{b}_j}{\grave{b}_{i+j}}\right).$$

If $(\beta_q, \grave{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b) = (\bar{b}_i \bar{b}_j d^*, \grave{b}_1 \bar{b}_1^{\bar{m}} \dot{b}_1^{\dot{m}}, \acute{b}_{i+j-1} d)$, then

$$|\mathrm{Aut}(\beta_p)| = |\mathrm{Aut}(\bar{b}_i \bar{b}_j d)| = |\mathrm{Aut}(d)|((\bar{s}_i + 1)(\bar{s}_j + 1) + \delta_{ij}(\bar{s}_i + 1))$$

and

$$\langle(\beta_q, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b)\rangle = \langle(\bar{b}_i \bar{b}_j d, \grave{b}_1 \bar{b}_1^{\bar{m}} \dot{b}_1^{\dot{m}}, \acute{b}_{i+j-1} d^*)\rangle = \left(1 - \frac{\delta_{ij}}{2}\right) \frac{1}{|\mathrm{Aut}(d)|}.$$

Thus, the corresponding subsum is

$$\frac{1}{2} \sum_{ij} ((\bar{s}_i + 1)(\bar{s}_j + 1) + \delta_{ij}(\bar{s}_i + 1)) \mathsf{H}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_i \bar{b}_j}{\acute{b}_{i+j-1}}\right). \qquad \square$$

The corresponding result for the interior simple critical points is as follows:

Lemma 3.3. *Let* $b = \acute{b}_1^{\acute{s}_1} \cdots \acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{b}_1^{\grave{s}_1} \cdots \grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{b}_1^{\bar{s}_1} \cdots \bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{b}_1^{\dot{s}_1} \cdots \dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$. *Then the following equation holds for* $\tilde{\mathsf{H}} = \acute{\mathsf{H}}$ *and* $\tilde{\mathsf{H}} = \grave{\mathsf{H}}$:

$$\begin{aligned}
&\tilde{\mathsf{H}}(m+1, \acute{m}, \grave{m}, b) \\
&\quad = \sum_{ij} (i+j)(\dot{s}_{i+j} + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\dot{b}_{i+j}}{\dot{b}_i \dot{b}_j}\right) + 2ij((\dot{s}_i + 1)(\dot{s}_j + 1) \\
&\qquad + \delta_{ij}(\dot{s}_i + 1)) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\dot{b}_i \dot{b}_j}{\dot{b}_{i+j}}\right) + |\bar{b}_j|(\bar{s}_{i+j} + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_{i+j}}{\dot{b}_i \bar{b}_j}\right) \\
&\qquad + 2i|\bar{b}_j|(\dot{s}_i + 1)(\bar{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\dot{b}_i \bar{b}_j}{\bar{b}_{i+j}}\right) + 2|\hat{b}_j|(\acute{s}_{i+j} + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \acute{m}, b \frac{\acute{b}_{i+j}}{\dot{b}_i \acute{b}_j}\right) \\
&\qquad + 4i|\hat{b}_j|(\dot{s}_i + 1)(\acute{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \acute{m}, b \frac{\dot{b}_i \acute{b}_j}{\acute{b}_{i+j}}\right) + 2|\hat{b}_j|(\grave{s}_{i+j} + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\grave{b}_{i+j}}{\dot{b}_i \grave{b}_j}\right) \\
&\qquad + 4i|\hat{b}_j|(\dot{s}_i + 1)(\grave{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\dot{b}_i \grave{b}_j}{\grave{b}_{i+j}}\right) \\
&\qquad + \sum_{i+j=k+l} \left(2|\bar{b}_i \hat{b}_j \bar{b}_k \hat{b}_l|(\bar{s}_i + 1)(\acute{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_i \acute{b}_j}{\bar{b}_k \acute{b}_l}\right)\right. \\
&\qquad\qquad \left. + 2|\bar{b}_i \hat{b}_j \bar{b}_k \hat{b}_l|(\bar{s}_i + 1)(\grave{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_i \grave{b}_j}{\bar{b}_k \grave{b}_l}\right)\right) \\
&\qquad + \sum_{i+j+1=k+l} 2(1 + \delta_{kl})|\hat{b}_i \hat{b}_j \bar{b}_k \bar{b}_l|(\acute{s}_i + 1)(\grave{s}_j + 1) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\acute{b}_i \grave{b}_j}{\bar{b}_k \bar{b}_l}\right) \\
&\qquad + \sum_{i+j=k+l+1} \frac{1 + \delta_{ij}}{2} |\bar{b}_i \bar{b}_j \hat{b}_k \hat{b}_l|((\bar{s}_i + 1)(\bar{s}_j + 1) + \delta_{ij}(\bar{s}_i + 1)) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_i \bar{b}_j}{\acute{b}_k \grave{b}_l}\right) \\
&\qquad + \sum_{i+j=k+l} (1 + \delta_{ij})(1 + \delta_{kl}) \left(\frac{1}{4} |\bar{b}_i \bar{b}_j \bar{b}_k \bar{b}_l|((\bar{s}_i + 1)(\bar{s}_j + 1) + \delta_{ij}(\bar{s}_i + 1)) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_i \bar{b}_j}{\bar{b}_k \bar{b}_l}\right)\right. \\
&\qquad + |\hat{b}_i \hat{b}_j \hat{b}_k \hat{b}_l|((\acute{s}_i + 1)(\acute{s}_j + 1) + \delta_{ij}(\acute{s}_i + 1)) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\acute{b}_i \acute{b}_j}{\acute{b}_k \acute{b}_l}\right) \\
&\qquad \left. + |\hat{b}_i \hat{b}_j \hat{b}_k \hat{b}_l|((\grave{s}_i + 1)(\grave{s}_j + 1) + \delta_{ij}(\grave{s}_i + 1)) \tilde{\mathsf{H}}\left(m, \acute{m}, \grave{m}, b \frac{\grave{b}_i \grave{b}_j}{\grave{b}_k \grave{b}_l}\right)\right).
\end{aligned}$$

Proof. Consider a covering $\varphi\colon \Omega \to D$ in the set $\acute{\mathscr{H}}(m+1, \acute{m}, \grave{m}, b)$. Let y be the special critical value of φ, and let x be an interior critical value of φ. Consider a segment $l \subset D$ with endpoints on the boundary separating y and x from the other critical values of the covering and shrink it into a point y_l; then the disk D splits into two disks, D' and D'', and the covering φ splits into two coverings, $\varphi'\colon \Omega' \to D'$ and $\varphi''\colon \Omega'' \to D''$. The critical values of φ' are y_l, y, and x. By identifying D'' with D, we see that $\varphi'' \in \acute{\mathscr{H}}(m, \acute{m}, \grave{m}, c)$ for some monomial c. By [1], this implies that $\acute{H}(m+1, \acute{m}, \grave{m}, b) = \sum_{pq} \acute{H}(m, \acute{m}, \grave{m}, \beta_p) F^{pq}\langle(\beta_q, \acute{b}_1 \bar{b}_1^{\bar{n}} \dot{b}_1^{\dot{n}}, b)\rangle$. By Lemma 2.2, the right-hand side contains fifteen subsums. (To every case in the lemma except for the last, there correspond two subsums.) Each subsum can be computed by the scheme used in the proofs of Lemmas 3.1 and 3.2. □

Lemmas 3.1, 3.2, and 3.3 permit one to find all numbers $\acute{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$ and $\grave{\mathsf{H}}(m, \acute{m}, \grave{m}, b)$ starting from $\mathsf{H}(0, 0, b)$.

4. Differential Equations for the Generating Functions

Consider the algebra of formal power series in the commuting variables $\acute{p}_i$, $\grave{p}_i$, $\bar{p}_i$, $\dot{p}_i$. To each monomial $p_b = \acute{p}_1^{\acute{s}_1} \cdots \acute{p}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{p}_1^{\grave{s}_1} \cdots \grave{p}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{p}_1^{\bar{s}_1} \cdots \bar{p}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{p}_1^{\dot{s}_1} \cdots \dot{p}_{\dot{n}}^{\dot{s}_{\dot{n}}}$ in this algebra, we assign the monomial $b = \acute{b}_1^{\acute{s}_1} \cdots \acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{b}_1^{\grave{s}_1} \cdots \grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{b}_1^{\bar{s}_1} \cdots \bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{b}_1^{\dot{s}_1} \cdots \dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$.

Consider the generating functions

$$\acute{H}(\alpha, \beta, \gamma \mid \acute{p}_1, \grave{p}_1, \bar{p}_1, \dot{p}_1, \acute{p}_2, \dots) = \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \acute{\mathsf{H}}(m, \acute{m}, \grave{m}, b) p_b,$$
$$\grave{H}(\alpha, \beta, \gamma | \acute{p}_1, \grave{p}_1, \bar{p}_1, \dot{p}_1, \acute{p}_2, \dots) = \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \grave{\mathsf{H}}(m, \acute{m}, \grave{m}, b) p_b,$$

where the inner sums are over all monomials b. In particular, $\acute{H}(0, 0 \mid \dots) = \grave{H}(0, 0 \mid \dots) = \exp(\bar{p}_1 + \dot{p}_1/2)$ according to Example 3.1.

Theorem 4.1. *The equation*

$$\frac{\partial \acute{H}}{\partial \beta} = L_\beta H$$

holds, where

$$L_\beta = \sum_i i \dot{p}_i \frac{\partial}{\partial \dot{p}_i} + \sum_{ij} \left(\bar{p}_i \grave{p}_j \frac{\partial}{\partial \bar{p}_{i+j}} + \grave{p}_i \grave{p}_j \frac{\partial}{\partial \grave{p}_{i+j}} + \bar{p}_i \bar{p}_j \frac{\partial}{\partial \acute{p}_{i+j-1}} \right)$$

Proof. Assume that $b = \acute{b}_1^{\acute{s}_1} \cdots \acute{b}_{\acute{n}}^{\acute{s}_{\acute{n}}} \grave{b}_1^{\grave{s}_1} \cdots \grave{b}_{\grave{n}}^{\grave{s}_{\grave{n}}} \bar{b}_1^{\bar{s}_1} \cdots \bar{b}_{\bar{n}}^{\bar{s}_{\bar{n}}} \dot{b}_1^{\dot{s}_1} \cdots \dot{b}_{\dot{n}}^{\dot{s}_{\dot{n}}}$, where $\acute{n} = \acute{n}(b)$, $\acute{s}_i = \acute{s}_i(b)$, $\grave{n} = \grave{n}(b)$, $\grave{s}_i = \grave{s}_i(b)$, $\bar{n} = \bar{n}(b)$, $\bar{s}_i = \bar{s}_i(b)$, and $\dot{n} = \dot{n}(b)$, $\dot{s}_i = \dot{s}_i(b)$.

By Lemma 3.1,

$$\begin{aligned}
\frac{\partial \acute{H}}{\partial \beta} &= \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \acute{\mathsf{H}}(m, \acute{m}+1, \grave{m}, b) p_b \\
&= \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \sum_{i=1}^\infty i(\dot{s}_i + 1) \mathsf{H}\left(m, \acute{m}, \grave{m}, b \frac{\dot{b}_i}{\dot{b}_i}\right) p_b \\
&\quad + \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \sum_{i=1}^\infty \sum_{j=1}^\infty (\bar{s}_{i+j} + 1) \mathsf{H}\left(m, \acute{m}, \grave{m}, b \frac{\bar{b}_{i+j}}{\bar{b}_i \grave{b}_j}\right) p_b \\
&\quad + \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \sum_{i=1}^\infty \sum_{j=1}^\infty (\grave{s}_{i+j} + 1) \mathsf{H}\left(m, \acute{m}, \grave{m}, b \frac{\grave{b}_{i+j}}{\grave{b}_i \grave{b}_j}\right) p_b \\
&\quad + \sum_{m, \acute{m}, \grave{m} \geqslant 0} \frac{\alpha^m}{m!} \frac{\beta^{\acute{m}}}{\acute{m}!} \frac{\gamma^{\grave{m}}}{\grave{m}!} \sum_b \sum_{i=1}^\infty \sum_{j=1}^\infty (\acute{s}_{i+j-1} + 1) \mathsf{H}\left(m, \acute{m}, \grave{m}, b \frac{\acute{b}_{i+j-1}}{\bar{b}_i \bar{b}_j}\right) p_b
\end{aligned}$$

Let us make the change of variables $c = b\frac{\dot{b}_i}{\grave{b}_i}$ in the first summand. Then $p_c = p_b\frac{\dot{p}_i}{\grave{p}_i}$ and $p_b = p_c\frac{\grave{p}_i}{\dot{p}_i} = \grave{p}_i\frac{\partial p_c}{\partial \dot{p}_i}(\dot{s}_i+1)^{-1}$. Thus,

$$\sum_b\sum_{i=1}^{\infty} i(\dot{s}_i+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\dot{b}_i}{\grave{b}_i}\Big)p_b = \sum_c\sum_{i=1}^{\infty} i\mathsf{H}(m,\acute{m},\grave{m},a)\grave{p}_i\frac{\partial p_c}{\partial \dot{p}_i},$$

where the sum is over all monomials c.

Let us make the change of variables $c = b\frac{\bar{b}_{i+j}}{\bar{b}_i\grave{b}_j}$ in the second summand. Then $p_c = p_b\frac{\bar{p}_{i+j}}{\bar{p}_i\grave{p}_j}$ and $p_b = p_c\frac{\bar{p}_i\grave{p}_j}{\bar{p}_{i+j}} = \bar{p}_i\grave{p}_j\frac{\partial p_c}{\partial \bar{p}_{i+j}}(\bar{s}_{i+j}+1)^{-1}$. Thus,

$$\sum_b\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\bar{s}_{i+j}+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\bar{b}_{i+j}}{\bar{b}_i\grave{b}_j}\Big)p_b = \sum_c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathsf{H}(m,\acute{m},\grave{m},c)\bar{p}_i\grave{p}_j\frac{\partial p_c}{\partial \bar{p}_{i+j}}\,.$$

Likewise,

$$\sum_b\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\grave{s}_{i+j}+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\grave{b}_{i+j}}{\grave{b}_i\grave{b}_j}\Big)p_b = \sum_c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathsf{H}(m,\acute{m},\grave{m},c)\grave{p}_i\grave{p}_j\frac{\partial p_c}{\partial \grave{p}_{i+j}}$$

and

$$\sum_b\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\acute{s}_{i+j-1}+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\acute{b}_{i+j-1}}{\bar{b}_i\bar{b}_j}\Big)p_b = \sum_c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathsf{H}(m,\acute{m},\grave{m},c)\bar{p}_i\bar{p}_j\frac{\partial p_c}{\partial \acute{p}_{i+j-1}}\,.$$

Hence $\frac{\partial \acute{H}}{\partial \beta} = L_\beta H$. □

Theorem 4.2. *The equation*

$$\frac{\partial \grave{H}}{\partial \gamma} = L_\gamma H$$

holds, where

$$L_\gamma = \sum_{i=1}^{\infty}\dot{p}_i\frac{\partial}{\partial \grave{p}_i} + \sum_{ij}\Big(2\bar{p}_{i+j}\frac{\partial^2}{\partial \bar{p}_i\partial \grave{p}_j} + 2\grave{p}_{i+j}\frac{\partial^2}{\partial \grave{p}_i\partial \grave{p}_j} + \frac{1}{2}\acute{p}_{i+j-1}\frac{\partial^2}{\partial \bar{p}_i\partial \bar{p}_j}\Big).$$

Proof. By Lemma 3.2,

$$\begin{aligned}
\frac{\partial \grave{H}}{\partial \gamma} &= \sum_{\acute{m},\grave{m}\geqslant 0}\frac{\alpha^m}{m!}\frac{\beta^{\acute{m}}}{\acute{m}!}\frac{\gamma^{\grave{m}}}{\grave{m}!}\sum_b \grave{\mathsf{H}}(m,\acute{m},\grave{m}+1,b)p_b \\
&= \sum_{m,\acute{m},\grave{m}\geqslant 0}\frac{\alpha^m}{m!}\frac{\beta^{\acute{m}}}{\acute{m}!}\frac{\gamma^{\grave{m}}}{\grave{m}!}\sum_b\sum_{i=1}^{\infty}(\grave{s}_i+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\grave{b}_i}{\dot{b}_i}\Big)p_b \\
&= 2\sum_{m,\acute{m},\grave{m}\geqslant 0}\frac{\alpha^m}{m!}\frac{\beta^{\acute{m}}}{\acute{m}!}\frac{\gamma^{\grave{m}}}{\grave{m}!}\sum_b\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(\bar{s}_i+1)(\grave{s}_j+1)\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\bar{b}_i\grave{b}_j}{\bar{b}_{i+j}}\Big)p_b \\
&\quad + 2\sum_{m,\acute{m},\grave{m}\geqslant 0}\frac{\alpha^m}{m!}\frac{\beta^{\acute{m}}}{\acute{m}!}\frac{\gamma^{\grave{m}}}{\grave{m}!}\sum_b\sum_{i=1}^{\infty}((\grave{s}_i+1)(\grave{s}_j+1)+\delta_{ij}(\grave{s}_i+1))\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\grave{b}_i\grave{b}_j}{\grave{b}_{i+j}}\Big)p_b \\
&\quad + \frac{1}{2}\sum_{m,\acute{m},\grave{m}\geqslant 0}\frac{\alpha^m}{m!}\frac{\beta^{\acute{m}}\frac{\gamma^{\grave{m}}}{\grave{m}!}}{\acute{m}!}\sum_b\sum_{i=1}^{\infty}((\bar{s}_i+1)(\bar{s}_j+1)+\delta_{ij}(\bar{s}_i+1))\mathsf{H}\Big(m,\acute{m},\grave{m},b\frac{\bar{b}_i\bar{b}_j}{\acute{b}_{i+j-1}}\Big)p_b.
\end{aligned}$$

Let us make the change of variables $c = b\frac{\grave{b}_i}{\dot{b}_i}$ in the first summand. Then

$$p_c = p_b\frac{\grave{p}_i}{\dot{p}_i} \quad \text{and} \quad p_b = p_c\frac{\dot{p}_i}{\grave{p}_i} = \dot{p}_i\frac{\partial p_c}{\partial \grave{p}_i}(\grave{s}_i+1)^{-1}.$$

Thus,

$$\sum_b \sum_{i=1}^{\infty} (\grave{s}_i + 1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i}{\dot{b}_i}\right)p_b = \sum_c \sum_{i=1}^{\infty} \mathsf{H}(m, \acute{m}, \grave{m}, c)\dot{p}_i \frac{\partial p_c}{\partial \grave{p}_i},$$

where the sum is over all monomials c. Likewise,

$$\sum_b \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\bar{s}_i + 1)(\grave{s}_j + 1)\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\bar{b}_i\grave{b}_j}{\bar{b}_{i+j}}\right)p_b = \sum_c \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathsf{H}(m, \acute{m}, \grave{m}, c)\bar{p}_{i+j} \frac{\partial^2 p_c}{\partial \bar{p}_i \partial \grave{p}_j}.$$

Let us make the change of variables $c = b\frac{\grave{b}_i\grave{b}_j}{\bar{b}_{i+j-1}}$ in the third summand. Then $p_c = p_b\frac{\grave{p}_i\grave{p}_j}{\bar{p}_{i+j-1}}$ and

$$p_b = p_c\frac{\grave{p}_{i+j-1}}{\grave{p}_i\grave{p}_j} = \grave{p}_{i+j-1}\frac{\partial^2 p_c}{\partial \bar{p}_i \partial \bar{p}_j}((\grave{s}_i + 1)(\grave{s}_j + 1) + \delta_{ij}(\grave{s}_i + 1))^{-1}.$$

Thus,

$$\sum_b \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ((\grave{s}_i + 1)(\grave{s}_j + 1) + \delta_{ij}(\grave{s}_i + 1))\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\grave{b}_i\grave{b}_j}{\acute{b}_{i+j-1}}\right)p_b$$
$$= \sum_c \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathsf{H}(m, \acute{m}, \grave{m}, c)\grave{p}_{i+j-1}\frac{\partial^2 p_c}{\partial \grave{p}_i \partial \grave{p}_j}.$$

Likewise,

$$\sum_b \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ((\bar{s}_i + 1)(\bar{s}_j + 1) + \delta_{ij}(\bar{s}_i + 1))\mathsf{H}\left(m, \acute{m}, \grave{m}, b\frac{\bar{b}_i\bar{b}_j}{\acute{b}_{i+j-1}}\right)p_b$$
$$= \sum_c \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathsf{H}(m, \acute{m}, \grave{m}, c)\acute{p}_{i+j-1}\frac{\partial^2 p_c}{\partial \bar{p}_i \partial \bar{p}_j}.$$

Hence $\frac{\partial \grave{H}}{\partial \gamma} = L_\gamma H$. ☐

Theorem 4.3. *The equations*

$$\frac{\partial \acute{H}}{\partial \alpha} = L_\alpha \acute{H}, \qquad \frac{\partial \grave{H}}{\partial \alpha} = L_\alpha \grave{H}$$

hold, where

$$L_\alpha = \sum_{ij}\Bigg((i+j)\dot{p}_i\dot{p}_j\frac{\partial}{\partial \dot{p}_{i+j}} + 2ij\dot{p}_{i+j}\frac{\partial^2}{\partial \dot{p}_i\partial \dot{p}_j} + |\bar{b}_j|\dot{p}_i\bar{p}_j\frac{\partial}{\partial \bar{p}_{i+j}} + i|\bar{b}_j|\bar{p}_{i+j}\frac{\partial^2}{\partial \dot{p}_i\partial \bar{p}_j}$$
$$+ 2|\hat{b}_j|\dot{p}_i\acute{p}_j\frac{\partial}{\partial \acute{p}_{i+j}} + 4i|\hat{b}_j|\acute{p}_{i+j}\frac{\partial^2}{\partial \dot{p}_i\partial \acute{p}_j} + 2|\hat{b}_j|\dot{p}_i\grave{p}_j\frac{\partial}{\partial \grave{p}_{i+j}} + 4i|\hat{b}_j|\grave{p}_{i+j}\frac{\partial^2}{\partial \dot{p}_i\partial \grave{p}_j}\Bigg)$$
$$+ \sum_{i+j=k+l} 2\left(|\bar{b}_i\hat{b}_j\bar{b}_k\hat{b}_l|\bar{p}_k\acute{p}_l\frac{\partial^2}{\partial \bar{p}_i\partial \acute{p}_j} + |\bar{b}_i\hat{b}_j\bar{b}_k\hat{b}_l|\bar{p}_k\grave{p}_l\frac{\partial^2}{\partial \bar{p}_i\partial \grave{p}_j}\right)$$
$$+ \sum_{i+j+1=k+l} 2(1+\delta_{kl})|\hat{b}_i\hat{b}_j\bar{b}_k\bar{b}_l|\bar{p}_k\bar{p}_l\frac{\partial^2}{\partial \acute{p}_i\partial \grave{p}_j} + \sum_{i+j=k+l+1}\frac{1}{2}(1+\delta_{ij})|\bar{b}_i\bar{b}_j\hat{b}_k\hat{b}_l|\acute{p}_k\grave{p}_l\frac{\partial^2}{\partial \bar{p}_i\partial \bar{p}_j}$$
$$+ \sum_{i+j=k+l}(1+\delta_{ij})(1+\delta_{kl})\Bigg(\frac{1}{4}|\bar{b}_i\bar{b}_j\bar{b}_k\bar{b}_l|\bar{p}_k\bar{p}_l\frac{\partial^2}{\partial \bar{p}_i\partial \bar{p}_j}$$
$$+ |\hat{b}_i\hat{b}_j\hat{b}_k\hat{b}_l|\grave{p}_k\grave{p}_l\frac{\partial^2}{\partial \grave{p}_i\partial \grave{p}_j} + |\hat{b}_i\hat{b}_j\hat{b}_k\hat{b}_l|\acute{p}_k\acute{p}_l\frac{\partial^2}{\partial \acute{p}_i\partial \acute{p}_j}\Bigg).$$

The proof uses Lemma 3.3 but otherwise follows the same scheme as the proofs of Theorems 4.1 and 4.2.

Let us conclude with the following observation due to the referee. Just as the classical cut-and-join equation, the equations in Theorems 4.1–4.3 are quasihomogeneous if the weight $|b|$ is assigned to the variable p_b. The initial conditions for these equations are given in Example 3.1. Therefore, the equations in Theorems 4.1–4.3 theoretically permit one to compute the generating function H by the scheme used in [7] for the classical simple Hurwitz numbers. To this end, one should consider the restriction of the operators to the finite-dimensional subspace of quasi-homogeneous polynomials of given degree and find the eigenvalues and eigenfunctions of this restriction. As was already noted, Lemmas 3.1–3.3 permit easily writing a computer algorithm for the straightforward computation of each Hurwitz number.

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Translated by Yu. M. Burman