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A restricted shift completeness problem ☆

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Abstract

We solve a problem about the orthogonal complement of the space spanned by restricted shifts of functions in $L^2[0,1]$ posed by M. Carlsson and C. Sundberg.

Keywords: Exponentials systems; Completeness; Fourier transform; Polya sets

Recently, Marcus Carlsson and Carl Sundberg posed the following problem. Let $f \in L^2[0,1]$. Consider the Fourier transform

$$\hat{f}(\lambda) = \int_0^1 f(x)e^{i\lambda x}\,dx$$

of f and assume that the zeros of the entire function $\hat{f}$ are simple. Put $\Lambda = \{\lambda\colon\ \hat{f}(-\bar{\lambda}) = 0\}$. Suppose that $\operatorname{conv}(\operatorname{supp} f) = [0, 1/2]$, and put

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$$\mathfrak{A}_f = \mathrm{Clos}_{L^2[0,1]} \mathrm{Lin}\{\tau_t f\colon\ 0 \leqslant t \leqslant 1/2\},$$

where $\tau_t f(x) = f(x-t)$. It is clear that $\{e^{i\lambda x}\}_{\lambda\in\Lambda} \perp \mathfrak{A}_f$ in $L^2[0,1]$. The problem by Carlsson and Sundberg is whether the family

$$\{e^{i\lambda x}\}_{\lambda\in\Lambda} \cup \{\tau_t f\}_{0\leqslant t\leqslant 1/2}$$

is complete in $L^2[0,1]$. In this article we solve (a slightly more general form of) this problem. Our solution involves two components: a non-harmonic Fourier analysis in the Paley–Wiener space developed recently in [1], and sharp density results of Beurling–Malliavin type from [4,5].

Theorem 1. *Let $0 < a < 1$, $f \in L^2[0,1]$, and let $\mathrm{conv}(\mathrm{supp}\, f) = [0,a]$. Denote $\Lambda = \{(\lambda_k, n_k)\colon\ \hat{f}^{(s)}(-\bar{\lambda}_k) = 0,\ 0 \leqslant s < n_k\}$ (i.e. Λ is the zero divisor of $\hat{f}(-\bar{z})$). Then the family*

$$\{x^s e^{i\lambda_k x}\}_{(\lambda_k,n_k)\in\Lambda,\, 0\leqslant s<n_k} \cup \{\tau_t f\}_{0\leqslant t\leqslant 1-a}$$

is complete in $L^2[0,1]$.

Proof. We apply the Fourier transform and a simple rescaling to reduce our problem to the following one. Let F belong to the Paley–Wiener space $\mathcal{P}W_{\pi a}$ (the Fourier image of $L^2[-\pi a, \pi a]$), and let $\Lambda = \{(\lambda_k, n_k)\}$ be the zero divisor of F. Then the family

$$\{F(z)e^{itz}\}_{|t|\leqslant\pi(1-a)} \cup \{K_\lambda^s\}_{(\lambda_k,n_k)\in\Lambda,\, 0\leqslant s<n_k} \tag{1}$$

is complete in $\mathcal{P}W_\pi$. Here, $K_\lambda^0(z) = K_\lambda(z) = \frac{\sin[\pi(z-\bar{\lambda})]}{\pi(z-\bar{\lambda})}$ is the reproducing kernel of the space $\mathcal{P}W_\pi$, and

$$K_\lambda^s = \left(\frac{d}{d\bar{\lambda}}\right)^s K_\lambda$$

reproduce the s-th derivatives:

$$\langle f, K_\lambda^s\rangle_{\mathcal{P}W_\pi} = f^{(s)}(\lambda), \quad f \in \mathcal{P}W_\pi,\ \lambda \in \mathbb{C},\ s \geqslant 0.$$

It is easy to show that for every $\beta \in \mathbb{R}$, the functions

$$F(z)\frac{\sin[\pi(1-a)(z-\beta)]}{z-\beta-n(1-a)^{-1}}, \quad n \in \mathbb{Z},$$

belong to the closed linear span of $\{F(z)e^{itz}\}_{|t|\leqslant\pi(1-a)}$ in $\mathcal{P}W_\pi$. We set $G(z) = F(z)\sin[\pi(1-a)(z-\beta)]$, and fix β in such a way that G has only simple zeros. Denote $\Lambda' = \{\beta + \frac{n}{1-a}\}_{n\in\mathbb{Z}}$. It remains to verify that the family

$$\left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda\in\Lambda'} \cup \{K_{\lambda_k}^s\}_{(\lambda_k,n_k)\in\Lambda,\, 0\leqslant s<n_k}$$

is complete in $\mathcal{P}W_\pi$.

Assume the converse. Then there exists $h \in \mathcal{P}W_\pi \setminus \{0\}$ such that

$$\left(\frac{G(z)}{z-\lambda}, h\right) = 0, \quad \lambda \in \Lambda', \tag{2}$$

$$\left(h, K_\lambda^s\right) = 0, \qquad (\lambda_k, n_k) \in \Lambda, \quad 0 \leqslant s < n_k. \tag{3}$$

For $0 \leqslant \gamma < 1$, we expand h with respect to the orthogonal basis $K_{n+\gamma}$:

$$h = \sum_{n\in\mathbb{Z}} \bar{a}_{n,\gamma} K_{n+\gamma}, \quad \{a_{n,\gamma}\} \in \ell^2.$$

Then (2)–(3) can be rewritten as

$$\sum_{n\in\mathbb{Z}} \frac{a_{n,\gamma} G(n+\gamma)}{n+\gamma-\lambda} = 0, \quad \lambda \in \Lambda',$$

$$\sum_{n\in\mathbb{Z}} \frac{\bar{a}_{n,\gamma}(-1)^n}{(n+\gamma-\lambda_k)^s} = 0, \quad (\lambda_k, n_k) \in \Lambda,\ 0 < s \leqslant n_k.$$

Changing γ if necessary we can assume that $a_{n,\gamma} \neq 0$, $G(n+\gamma) \neq 0$, $n \in \mathbb{Z}$. Therefore there exist entire functions S_γ and T_γ such that

$$\sum_{n\in\mathbb{Z}} \frac{a_{n,\gamma} G(n+\gamma)}{n+\gamma-z} = \frac{T_\gamma(z)\sin[\pi(1-a)(z-\beta)]}{\sin[\pi(z-\gamma)]}, \tag{4}$$

$$\sum_{n\in\mathbb{Z}} \frac{\bar{a}_{n,\gamma}(-1)^n}{n+\gamma-z} = \frac{S_\gamma(z)F(z)}{\sin[\pi(z-\gamma)]} = \frac{h(z)}{\sin[\pi(z-\gamma)]}. \tag{5}$$

Since $h = FS_\gamma$ does not depend on γ, we write in what follows $S = S_\gamma$.

Put $V_\gamma = ST_\gamma$. Comparing the residues in Eqs. (4)–(5) at the points $n+\gamma$, $n \in \mathbb{Z}$, we conclude that

$$V_\gamma(n+\gamma) = (-1)^n |a_{n,\gamma}|^2, \quad n \in \mathbb{Z}. \tag{6}$$

By construction, V_γ is of at most exponential type π. Therefore, we have the representation

$$V_\gamma(z) = Q_\gamma(z) + \sin\bigl[\pi(z-\gamma)\bigr] R_\gamma(z), \tag{7}$$

where

$$Q_\gamma(z) = \sin\pi(z-\gamma) \sum_{n\in\mathbb{Z}} \frac{|a_{n,\gamma}|^2}{z-n-\gamma},$$

and R_γ is a function of zero exponential type. Thus, the conjugate indicator diagram of V_γ is $[-\pi, \pi]$, and hence, the conjugate indicator diagrams of T_γ and S are $[-\pi a, \pi a]$ and

$[-\pi(1-a), \pi(1-a)]$ correspondingly. Therefore, each of the functions V_γ^*/V_γ, T_γ^*/T_γ, and S^*/S is a ratio of two Blaschke products. Here we use the notation $H^*(z) = \overline{H(\bar{z})}$.

It follows from (5) that

$$\frac{S(z)F(z)}{\sin[\pi(z-\gamma)]} \cdot \frac{S^*(z)}{S(z)} = \sum_{n\in\mathbb{Z}} \frac{\bar{a}_{n,\gamma}(-1)^n}{n+\gamma-z} \cdot \frac{S^*(n+\gamma)}{S(n+\gamma)} + H(z)$$

for some entire function H. Since $FS^* \in \mathcal{P}W_\pi$, we conclude that H is of zero exponential type and tends to 0 along the imaginary axis. Thus, $H = 0$.

We set $\bar{b}_{n,\gamma} = \bar{a}_{n,\gamma} \frac{S^*(n+\gamma)}{S(n+\gamma)}$, and obtain

$$\sum_{n\in\mathbb{Z}} \frac{\bar{b}_{n,\gamma}(-1)^n}{n+\gamma-z} = \frac{S^*(z)F(z)}{\sin\pi(z-\gamma)}.$$

Analogously, using the fact that the function $z \mapsto T_\gamma(z)\sin[\pi(1-a)(z-\beta)]$ belongs to $\mathcal{P}W_\pi$ and the fact that ST_γ is real on $\mathbb{Z}+\gamma$, we deduce from (4) that

$$\sum_{n\in\mathbb{Z}} \frac{b_{n,\gamma}G(n+\gamma)}{n+\gamma-z} = \frac{T_\gamma^*(z)\sin[\pi(1-a)(z-\beta)]}{\sin[\pi(z-\gamma)]}.$$

Thus, the function

$$g = \sum_{n\in\mathbb{Z}} \bar{b}_n K_{n+\gamma}$$

is orthogonal to the system (1), whence the elements $h+g$, $ih-ig$ are also orthogonal to (1), and correspond to the pairs $(S+S^*, T_\gamma+T_\gamma^*)$, $(iS-iS^*, -iT_\gamma+iT_\gamma^*)$. Therefore, from now on we assume that S, T_γ, and hence, V_γ are real on the real line.

Now it follows from (6) that the function V_γ has at least one zero in every interval $(n+\gamma, n+1+\gamma)$, $n\in\mathbb{Z}$. By (7), the zeros of V_γ coincide with the zeros of the function

$$R_\gamma(\lambda) + \sum_{n\in\mathbb{Z}} \frac{|a_{n,\gamma}|^2}{\lambda-n-\gamma}. \tag{8}$$

Next we fix $\gamma\in[0,1)$ and a sufficiently small $\delta>0$ for which there exist two subsets Σ, Σ_1 of the zero set $\mathcal{Z}(S)$ of the function S with the following properties:

- Σ has exactly one point in those intervals $[n+\gamma, n+1+\gamma]$ where $\mathcal{Z}(S)\cap[n+\gamma, n+1+\gamma)\neq\emptyset$, and

$$\operatorname{dist}(x, \mathbb{Z}+\gamma) > \frac{\delta}{1+x^2}, \quad x\in\Sigma;$$

- Σ_1 has positive upper density, and $\operatorname{dist}(x,\mathbb{Z}+\gamma)>\delta$, $x\in\Sigma_1$.

From now on, we use the notations $R = R_\gamma$, $a_n = a_{n,\gamma}$, $V = V_\gamma$, $T = T_\gamma$. We need to consider three cases. If R is a nonzero polynomial, then the zeros of the function (8) approach $\mathbb{Z} + \gamma$ and we obtain a contradiction to the existence of Σ_1. If $R = 0$, then [1, Proposition 3.1] implies that the density of Σ_1 is zero. Finally, if R is not a polynomial, we can divide it by $(z - z_1)(z - z_2)$, where z_1 and z_2 are two arbitrary zeros of R, $z_1, z_2 \notin \Sigma$, to get a function R_1 of zero exponential type which is bounded on Σ.

Next, we obtain some information on Σ. For a discrete set $X = \{x_n\} \subset \mathbb{R}$ we consider its counting function $n_X(t) = \operatorname{card}\{n\colon\ x_n \in [0, t)\}$, $t \geqslant 0$, and $n_X(t) = -\operatorname{card}\{n\colon\ x_n \in (-t, 0)\}$, $t < 0$. If f is an entire function and X is the set of its real zeros (counted according to multiplicities), then there exists a branch of the argument of f on the real axis, which is of the form $\arg f(t) = \pi n_X(t) + \psi(t)$, where ψ is a smooth function. Such choice of the argument is unique up to an additive constant and in what follows we always assume that the argument is chosen to be of this form.

We use the (easy to show) fact that for every function $f \in \mathcal{P}W_\pi$ with the conjugate indicator diagram $[-\pi, \pi]$ and all zeros in $\overline{\mathbb{C}_+}$, one has

$$\arg f = \pi x + \tilde{u} + c, \tag{9}$$

where $u \in L^1((1 + x^2)^{-1}\,dx)$, $c \in \mathbb{R}$. Here $\tilde{u}$ denotes the conjugate function (the Hilbert transform) of u,

$$\tilde{u}(x) = \frac{1}{\pi} v.p. \int_{\mathbb{R}} \left(\frac{1}{x - t} + \frac{t}{t^2 + 1} \right) u(t)\,dt.$$

It follows from (4)–(5) that $FV \in \mathcal{P}W_{\pi a + \pi}$. Now let us replace all zeros λ of the functions h, F, S, T, and V in $\mathbb{C}_-$ by $\bar{\lambda}$. Since the Paley–Wiener space is closed under division by Blaschke products, we still have for the new functions h, F, S, T, and V (which we denote by the same letters) that $h \in \mathcal{P}W_\pi$ and $FV \in \mathcal{P}W_{\pi a + \pi}$. Recall that the function V has at least one zero in each of the intervals $(n + \gamma, n + 1 + \gamma)$, $n \in \mathbb{Z}$. Let us consider its representation $V = V_0 H$, where the zeros of V_0 are simple, interlacing with $\mathbb{Z} + \gamma$ and $V_0|_\Sigma = 0$. It is clear that $\arg V_0 = \pi x + O(1)$. Since, by (9),

$$\arg(FV) = \pi a x + \pi x + \tilde{u} + c,$$

we conclude that

$$\arg(FH) = \pi a x + \tilde{u} + O(1).$$

Consider the equality $h = FHS/H$ and note that

$$\arg\left(\frac{S}{H}\right) = \pi n_\Sigma - \alpha,$$

where α is some nondecreasing function on $\mathbb{R}$. This follows from the fact that S/H vanishes only on a subset of the real axis which contains Σ and $\frac{S^* H}{S H^*}$ is a Blaschke product. Applying the representation (9) to h, we conclude that

$$\pi n_\Sigma(x) = \pi(1 - a)x + \tilde{u} + v + \alpha, \tag{10}$$

where $u \in L^1((1 + x^2)^{-1}\,dx)$, $v \in L^\infty(\mathbb{R})$, and α is nondecreasing.

Summing up, we have an entire function R_1 of zero exponential type which is not a polynomial, and which is bounded on a set $\Sigma \subset \mathbb{R}$ satisfying (10).

To deduce a contradiction from this, we use some information on the classical Polya problem and on the second Beurling–Malliavin theorem. We say that a sequence $X = \{x_n\} \subset \mathbb{R}$ is a *Polya sequence* if any entire function of zero exponential type which is bounded on X is a constant. We say that a disjoint sequence of intervals $\{I_n\}$ on the real line is *a long sequence of intervals* if

$$\sum_n \frac{|I_n|^2}{1 + \operatorname{dist}^2(0, I_n)} = +\infty.$$

A complete solution of the Polya problem was obtained by Mishko Mitkovski and Alexei Poltoratski [5]. In particular,[1] a separated sequence $X \subset \mathbb{R}$ is not a Polya sequence if and only if there exists a long sequence of intervals $\{I_n\}$ such that

$$\frac{\operatorname{card}(X \cap I_n)}{|I_n|} \to 0.$$

Applying this result to our R and Σ (formally speaking, Σ is not a separated sequence but by construction it is a union of two separated sequences which are interlacing), we find a long system of intervals $\{I_n\}$ such that

$$\frac{\operatorname{card}(\Sigma \cap I_n)}{|I_n|} \to 0.$$

Given $I = [a, b]$, denote $I^- = [a, (2a + b)/3]$, $I^+ = [(a + 2b)/3, b]$,

$$\Delta_I^* = \inf_{I^+}\bigl[\pi(1-a)x - \pi n_\Sigma(x) + v\bigr] - \sup_{I^-}\bigl[\pi(1-a)x - \pi n_\Sigma(x) + v\bigr].$$

Now, for a long system of intervals $\{I_n\}$ and for some $c > 0$ we have

$$\Delta_{I_n}^* \geqslant c|I_n|.$$

Next we use a version of the second Beurling–Malliavin theorem given by Nikolai Makarov and Alexei Poltoratski in [4, Theorem 5.9]. This theorem (or rather its proof) gives that if the function $\pi(1-a)x - \pi n_\Sigma(x) + v$ may be represented as $-\alpha - \tilde{u}$ for α and u as above, then there is no such long family of intervals. This contradiction completes the proof. $\square$

Remark 2. It is easy to see that in the limit case $a = 1$ the statement analogous to Theorem 1 is not true: there exists $f \in L^2[0, 1]$ such that $\operatorname{conv}(\operatorname{supp} f) = [0, 1]$, $\hat{f}$ has only simple zeros which form a set $\Lambda \subset \mathbb{R}$, and the family

$$\{e^{-i\lambda x}\}_{\lambda \in \Lambda} \cup \{f\}$$

[1] The “only if” part of this statement is implicitly contained in the results of Louis de Branges in the 1960s: [2, Theorem XI], [3, Theorems 66, 67]; see also [5, Remark, p. 1068].

is not complete in $L^2[0,1]$. Rescaling the problem to the interval $[-\pi,\pi]$, it suffices to find a function G in $\mathcal{P}W_\pi$ which is of the form $G(z)=\frac{\sin\pi z}{S(z)}$, where G is some zero genus product with sufficiently sparse zeros, and define f by $\hat{f}=G$. E.g., one may take as S the canonical product with zeros 2^n, $n\geqslant 1$, or $S(z)=(z+1)\sqrt{z}\sin(\pi\sqrt{z})$. It is easy to show that in the latter case f does not have an L^2 derivative.

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