

Cubic Algebra and Averaged Hamiltonian for the Resonance $3 : (-1)$ Penning–Ioffe Trap

O. Blagodyreva, M. Karasev, and E. Novikova

*Moscow Institute for Electronics and Mathematics at HSE,
E-mail: mkarasev@hse.ru, emnovikova@hse.ru*

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Abstract. For the $3 : (-1)$ resonance Penning trap, we describe the algebra of symmetries which turns out to be a non-Lie algebra with cubic commutation relations. The irreducible representations and coherent states of this algebra are constructed explicitly. The perturbing inhomogeneous magnetic field of Ioffe type, after double quantum averaging, generates an effective Hamiltonian of the trap. In the irreducible representation, this Hamiltonian becomes a second-order ordinary differential operator of the Heun type.

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1. INTRODUCTION

Quantum devices which can localize single electrons and manipulate with the properties of their states are called nanotraps. For example, the Penning and Paul traps are well known [1–5]. In the leading approximation, such systems are simply harmonic oscillators in a homogeneous magnetic field. This leading term is supplemented with the perturbation introduced by anharmonicity of the electric field or by inhomogeneity of the magnetic field. Usually, the trap anharmonicity is regarded as a disturbance that must be constructively avoided or compensated in a controlled way, for example, when creating an ideal “geonium” [6, 7, 8]. However, on the other hand, one can try to use the anharmonicity to obtain nontrivial excited states.

True enough, as long as there are no resonances in the system, the anharmonicity only slightly perturbs the frequencies of the leading (harmonic) part practically without any change in its phase structure. In this case, the quantum states can be constructed by using the long-established and well-developed oscillatory or semiclassical approximation [9, 10]. But in the cases where the frequencies of the harmonic part Hamiltonian are in resonance, there arises a somewhat more mathematically meaningful and physically interesting picture. Note that the technical realization of the situation where the frequencies are in resonance can easily be achieved by varying the trap controlling fields.

In the resonance case, the spectrum of the harmonic part of the trap is strongly degenerate. For the Penning traps, the resonance degeneracy is infinite. Degeneracy leads to freedom that is absent in the nonresonance case. In the “life” evolving inside this freedom, the perturbation (anharmonicity) of the trap begins to play a significant role. The quantum states are sharply reconstructed, and new quantum numbers appear in the system.

The Penning trap without anharmonicity taken into account is an axially symmetric three-dimensional oscillator in a homogeneous magnetic field directed along the axial axis. The signature $(+, -)$ of normal modes of this system in the directions transverse to the axis is not elliptic but hyperbolic. Although the spectrum of the leading part Hamiltonian is discrete, it is unbounded either from above or from below, which may cause instability in the resonance mode. This is precisely the situation arising in the Penning trap at the lowest resonance $1 : (-1)$ between transverse modes.

The higher resonance $2 : (-1)$ between transverse modes in the Penning trap does not suffer from such instability. But it turns out that, in this case, the third (longitudinal) mode is automatically in resonance with two transverse modes so that the complete resonance frequency proportion has the form $2 : (-1) : 2$. For such three- (and more) frequency resonances, the algebra of symmetries of the trap is more complicated in principle than the two-frequency case; see in [11]. Leaving this interesting mode aside for a while, we consider a higher resonance of transverse modes, namely

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3 : (-1). This is a good case, because it is stable and the third (longitudinal) frequency is incommensurate with the two transverse frequencies. Thus, to describe this case, it suffices to consider only the two-frequency resonance algebras, following [12, 13].

It turns out that in the case of the 3 : (-1) resonance under study, the algebra of symmetries of the harmonic part of the Penning trap is of non-Lie type and is determined by cubic commutation relations. The generators of this algebra are polynomials of degree four in the initial Heisenberg coordinates and momenta. For this non-Lie algebra, the irreducible representations and generalized coherent states can be constructed¹, following the general scheme [16]. The irreducible representations are realized by second-order differential Heun-type operators in the Hilbert space of antiholomorphic distributions on the complex plane.

The anharmonicity of the Penning nanotrap is determined by the simplest (linear) inhomogeneity of the magnetic field, which is called the Ioffe inhomogeneity [17,18], and by the fourth-order terms of inhomogeneity in the electric potential. It is shown that, after (double) quantum averaging, the Ioffe inhomogeneity generates a Hamiltonian linearly depending on the generators of the non-Lie algebra of symmetries. In the irreducible representation, this effective Hamiltonian of the Penning-Ioffe trap is given by an ordinary second-order differential operator of Heun type.

2. AVERAGING TRANSFORMATION OF THE HAMILTONIAN

The Penning trap is generated by electric and magnetic fields. The electric potential in the Cartesian coordinates x, y, z is given by

$$\frac{1}{2}\omega^2 U_0 + \frac{\varepsilon^2}{2}(aU_1 + bU_2),$$

where U_j are polynomials, obeying the Laplace equation $\Delta U_j = 0$:

$$U_0 = z^2 - (x^2 + y^2)/2, \quad U_1 = 8z^4 - 24z^2(x^2 + y^2) + 3(x^2 + y^2)^2, \quad U_2 = x^4 - 6x^2y^2 + y^4,$$

where a, b are some constants, ε is a small parameter, and ω is the axial frequency. The magnetic field consists of the homogeneous part directed along z -axis and of an inhomogeneous perturbation. We choose this perturbation to be linear in x, y -coordinates and directed along the x, y -plane (the so-called "Ioffe field"). Thus the magnetic field of the trap is given by

$$\mathbf{B} = \omega_0 \mathbf{e}_z + \varepsilon((n_2x + 2n_3y)\mathbf{e}_x - (2n_1x + n_2y)\mathbf{e}_y),$$

where e_x, e_y, e_z are unit vectors along the Cartesian axes, and ω_0 is the Larmor frequency.

In order to work with the case of 3 : (-1) resonance, we choose $\omega_0^2 = 4, \omega^2 = 3/2$.

The Hamiltonian of such a resonance trap is given by $H = H_0 + H_1$, where the leading part H_0 represents a harmonic oscillator in a homogeneous magnetic field:

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - (xp_y - yp_x) + \frac{1}{8}(x^2 + y^2) + \frac{3}{4}z^2,$$

and the perturbing part H_1 is generated by cubic and quarternary inhomogeneities:

$$H_1 = -\varepsilon(n_1x^2 + n_2xy + n_3y^2)p_z + \frac{\varepsilon^2}{2}(n_1x^2 + n_2xy + n_3y^2)^2 + \frac{\varepsilon^2}{2}(aU_1 + bU_2).$$

Let us transform the Hamiltonian H_0 to the normal form by means the following symplectic change of coordinates:

$$q_+ = x/2 - p_y, \quad q_- = x/2 + p_y, \quad q_3 = \sqrt[4]{3/2}z, \quad p_+ = y/2 + p_x, \quad p_- = -y/2 + p_x, \quad p_3 = \sqrt[4]{2/3}p_z.$$

¹The usual coherent states (of a harmonic oscillator) for Penning traps were considered, for example in [14, 15].

In the new coordinates, the quantum Penning–Ioffe trap is described by the Hamiltonians

$$\widehat{H}_0 = \frac{1}{4}\widehat{V}_0, \quad \widehat{H}_1 = -\varepsilon\widehat{V}_1 + \frac{\varepsilon^2}{2}(\widehat{V}_2 + a\widehat{V}_3 + b\widehat{V}_4), \quad (2.1)$$

where V_0 represents the harmonic $3 : (-1)$ resonance term:

$$V_0 = 3A_+ - A_- + \sqrt{6}A_3, \quad A_j \triangleq p_j^2 + q_j^2 \quad (j = +, -, 3),$$

and the perturbing cubic and quaternary terms are given by

$$\begin{aligned} V_1 &= \sqrt[4]{3/2}(n_1(q_+ + q_-)^2 + n_2(q_+ + q_-)(p_+ - p_-) + n_3(p_+ - p_-)^2)p_3, \\ V_2 &= (n_1(q_+ + q_-)^2 + n_2(q_+ + q_-)(p_+ - p_-) + n_3(p_+ - p_-)^2)^2, \\ V_3 &= (16/3)q_3^4 - 24\sqrt{2/3}q_3^2((q_+ + q_-)^2 + (p_+ - p_-)^2) + 3((q_+ + q_-)^2 + (p_+ - p_-)^2)^2, \\ V_4 &= (q_+ + q_-)^4 - 6(q_+ + q_-)^2(p_+ - p_-)^2 + (p_+ - p_-)^4. \end{aligned}$$

The upper checks in the Hamiltonians (2.1) stand for the quantization substitution

$$p_x \rightarrow \widehat{p}_x = -i\hbar\partial/\partial x, \quad p_y \rightarrow \widehat{p}_y = -i\hbar\partial/\partial y, \quad p_z \rightarrow \widehat{p}_z = -i\hbar\partial/\partial z.$$

The first perturbing Hamiltonian \widehat{V}_1 in (2.1) does not commute with the leading term \widehat{H}_0 . After a unitary transformation, it does commute up to $O(\varepsilon^2)$ (following the algebraic averaging method [19, 13]). The unitary transformation is of the form

$$(\widehat{H}_0 - \varepsilon\widehat{V}_1)e^{i\varepsilon\widehat{D}_1/\hbar} = e^{i\varepsilon\widehat{D}_1/\hbar}(\widehat{H}_0 - \varepsilon\widehat{V}_1 + O(\varepsilon^2)),$$

where \widehat{D}_1 and \widehat{V}_1 are the solution of the “homological” equations

$$\begin{aligned} (i/\hbar)[\widehat{H}_0, \widehat{D}_1] &= \widehat{V}_1 - \widehat{V}_1, \\ [\widehat{H}_0, \widehat{V}_1] &= 0. \end{aligned}$$

The solution is given by the integrals

$$\begin{aligned} \widehat{V}_1 &= \frac{1}{(2\pi)^2} \int_0^{2\pi} dt \int_0^{2\pi} d\tau e^{-\frac{i}{\hbar}t(3\widehat{A}_+ - \widehat{A}_-)} e^{-\frac{i}{\hbar}\tau\widehat{A}_3} \widehat{V}_1 e^{\frac{i}{\hbar}\tau\widehat{A}_3} e^{\frac{i}{\hbar}t(3\widehat{A}_+ - \widehat{A}_-)}, \\ \widehat{D}_1 &= \frac{1}{(2\pi)^2} \int_0^{2\pi} dt \int_0^{2\pi} d\tau \phi(t, \tau) e^{-\frac{i}{\hbar}t(3\widehat{A}_+ - \widehat{A}_-)} e^{-\frac{i}{\hbar}\tau\widehat{A}_3} \widehat{V}_1 e^{\frac{i}{\hbar}\tau\widehat{A}_3} e^{\frac{i}{\hbar}t(3\widehat{A}_+ - \widehat{A}_-)}, \quad \phi(t, \tau) = -4i \sum_{k \neq 0} \frac{e^{i(k_1 t + k_3 \tau)}}{k_1 + \sqrt{6}k_3}. \end{aligned}$$

It is easy to see that after the averaging procedure we obtain the zero operator $\widehat{V}_1 = 0$. Thus the term of order ε disappears in the Hamiltonian.

But, at the same time, after this averaging the term of order ε^2 is changed, namely, the additional summand

$$\frac{\varepsilon^2}{2} \frac{i}{\hbar} [\widehat{D}_1, \widehat{V}_1]$$

occurs. The operator \widehat{D}_1 in this summand can be computed explicitly:

$$\begin{aligned} \widehat{D}_1 &= (\sqrt[4]{3/2}/2) \left((n_1 - n_3 - in_2) \left(\frac{1}{5}\widehat{C}_+^2 + \frac{4}{5}\widehat{B}_- \widehat{C}_+ - \widehat{B}_-^2 \right) + (n_1 - n_3 + in_2) \left(\widehat{C}_-^2 - \frac{4}{5}\widehat{B}_+ \widehat{C}_- - \frac{1}{5}\widehat{B}_+^2 \right) \right. \\ &\quad \left. + 2(n_1 + n_3) (\widehat{B}_+ \widehat{B}_- - \widehat{C}_+ \widehat{C}_-) \right) (\widehat{C}_3 - \widehat{B}_3) + \frac{\sqrt[4]{3/2}}{2} \sqrt{3/2} ((n_1 - n_3 - in_2) \left(-\frac{1}{15}\widehat{C}_+^2 - \frac{2}{5}\widehat{B}_- \widehat{C}_+ + \widehat{B}_-^2 \right) \\ &\quad + (n_1 - n_3 + in_2) \left(\widehat{C}_-^2 - \frac{2}{5}\widehat{B}_+ \widehat{C}_- - \frac{1}{15}\widehat{B}_+^2 \right) + 2(n_1 + n_3) (\widehat{B}_+ \widehat{B}_- + \frac{1}{3}(\widehat{B}_+ \widehat{C}_+ + \widehat{B}_- \widehat{C}_- + 2\hbar) + \widehat{C}_+ \widehat{C}_-)) (\widehat{C}_3 + \widehat{B}_3). \end{aligned}$$

Here we have introduced the creation-annihilation operators $\widehat{B}_j = \widehat{q}_j - i\widehat{p}_j$ and $\widehat{C}_j = \widehat{q}_j + i\widehat{p}_j$ ($j = +, -, 3$).

Thus, the second application of the algebraic averaging method (in the order ε^2) produces several operators commuting with \widehat{H}_0 ,

$$\frac{\varepsilon^2}{2}(\widehat{V}_2 + a\widehat{V}_3 + b\widehat{V}_4 + \widehat{V}_5), \quad \widehat{V}_5 \equiv \frac{i}{\hbar}[\widehat{D}_1, \widehat{V}_1]. \tag{2.2}$$

After computations, we obtain the following explicit expressions:

$$\begin{aligned} \widehat{V}_2 = & (3n_1^2 + 2n_1n_3 + 3n_3^2 + n_2^2) \left(\frac{1}{8}\widehat{B}_+^2\widehat{C}_+^2 + \frac{1}{2}\widehat{B}_+\widehat{C}_+\widehat{B}_-\widehat{C}_- + \frac{1}{8}\widehat{B}_-^2\widehat{C}_-^2 + \hbar\widehat{B}_+\widehat{C}_+ + \hbar\widehat{B}_-\widehat{C}_- + \hbar^2 \right) \\ & + \frac{1}{4}(n_1^2 - n_3^2)(\widehat{B}_+\widehat{B}_-^3 + \widehat{C}_+\widehat{C}_-^3) - \frac{1}{4}i(n_1n_2 + n_2n_3)(\widehat{B}_+\widehat{B}_-^3 - \widehat{C}_+\widehat{C}_-^3), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \widehat{V}_3 = & (2\widehat{B}_3^2\widehat{C}_3^2 + 8\hbar\widehat{B}_3\widehat{C}_3 + 4\hbar^2) - 12\sqrt{2/3}(\widehat{B}_3\widehat{C}_3 + \hbar)(\widehat{B}_+\widehat{C}_+ + \widehat{B}_-\widehat{C}_- + 2\hbar) \\ & + 3(\widehat{B}_+^2\widehat{C}_+^2 + 4\widehat{B}_+\widehat{C}_+\widehat{B}_-\widehat{C}_- + \widehat{B}_-^2\widehat{C}_-^2 + 8\hbar\widehat{B}_+\widehat{C}_+ + 8\hbar\widehat{B}_-\widehat{C}_- + 8\hbar^2), \quad \widehat{V}_4 = 0, \end{aligned} \tag{2.4}$$

and finally

$$\begin{aligned} \widehat{V}_5 = & \frac{\sqrt{6}}{5} \left((6n_1^2 + n_2^2 + 8n_1n_3 + 6n_3^2)\widehat{B}_+\widehat{C}_+ - (2n_1^2 + 7n_2^2 - 24n_1n_3 + 2n_3^2)\widehat{B}_-\widehat{C}_- \right. \\ & + 2(2n_1^2 - 3n_2^2 + 16n_1n_3 + 2n_3^2)\hbar \left. \right) (\widehat{B}_3\widehat{C}_3 + \hbar) - \frac{3}{4} \left(\frac{1}{15}(9n_1^2 - n_2^2 + 22n_1n_3 + 9n_3^2)\widehat{B}_+^2\widehat{C}_+^2 \right. \\ & + \frac{1}{3}(5n_1^2 + 3n_2^2 - 2n_1n_3 + 5n_3^2)\widehat{B}_-^2\widehat{C}_-^2 + \frac{4}{15}(17n_1^2 - 3n_2^2 + 46n_1n_3 + 17n_3^2)\widehat{B}_+\widehat{C}_+\widehat{B}_-\widehat{C}_- \\ & + \frac{8}{15}(13n_1^2 - 2n_2^2 + 34n_1n_3 + 13n_3^2)\hbar\widehat{B}_+\widehat{C}_+ + \frac{8}{5}(7n_1^2 + 2n_2^2 + 6n_1n_3 + 7n_3^2)\hbar\widehat{B}_-\widehat{C}_- \\ & + \frac{8}{15}(27n_1^2 + 7n_2^2 + 26n_1n_3 + 27n_3^2)\hbar^2 + 2(n_1^2 - n_3^2)(\widehat{B}_+\widehat{B}_-^3 + \widehat{C}_+\widehat{C}_-^3) \\ & \left. - 2i(n_1n_2 + n_2n_3)(\widehat{B}_+\widehat{B}_-^3 - \widehat{C}_+\widehat{C}_-^3) \right). \end{aligned} \tag{2.5}$$

3. SYMMETRY ALGEBRA AND ITS IRREDUCIBLE REPRESENTATIONS

The main property of the averaged operators in (2.2) (computed in (2.3), (2.4), (2.5)) is that they commute with the leading part $\widehat{H}_0 = \frac{1}{4}\widehat{V}_0$ of the Penning trap Hamiltonian. Let us now describe the algebra of all operators, commuting with the leading part \widehat{H}_0 . This symmetry algebra is generated by five self-adjoint operators:

$$\widehat{A}_1 = \widehat{B}_+\widehat{C}_+ + \hbar, \quad \widehat{A}_2 = \widehat{B}_-\widehat{C}_- + \hbar, \quad \widehat{A}_3 = \widehat{B}_3\widehat{C}_3 + \hbar, \quad \widehat{A}_4 = -\frac{i}{2}(\widehat{B}_-^3\widehat{B}_+ - \widehat{C}_-^3\widehat{C}_+), \quad \widehat{A}_5 = \frac{1}{2}(\widehat{B}_-^3\widehat{B}_+ + \widehat{C}_-^3\widehat{C}_+). \tag{3.1}$$

The commutation relations between these generators are nonlinear, namely, they are cubic:

$$\begin{aligned} [\widehat{A}_1, \widehat{A}_2] = 0, \quad [\widehat{A}_k, \widehat{A}_3] = 0 \quad (k = 1, 2, 4, 5), \quad [\widehat{A}_1, \widehat{A}_4] = -2i\hbar\widehat{A}_5, \quad [\widehat{A}_2, \widehat{A}_4] = -6i\hbar\widehat{A}_5, \\ [\widehat{A}_1, \widehat{A}_5] = 2i\hbar\widehat{A}_4, \quad [\widehat{A}_2, \widehat{A}_5] = 6i\hbar\widehat{A}_4, \quad [\widehat{A}_4, \widehat{A}_5] = i\hbar(15\hbar^2\widehat{A}_1 + 23\hbar^2\widehat{A}_2 + 9\widehat{A}_1\widehat{A}_2^2 + \widehat{A}_2^3). \end{aligned} \tag{3.2}$$

The Casimir operators have the form

$$\widehat{V}_0 = 3\widehat{A}_1 - \widehat{A}_2 + \sqrt{6}\widehat{A}_3, \tag{3.3}$$

$$\widehat{K} = \widehat{A}_4^2 + \widehat{A}_5^2 - \widehat{A}_1\widehat{A}_2^3 - 23\hbar^2\widehat{A}_1\widehat{A}_2 - 9\hbar^2\widehat{A}_2^2 - 15\hbar^4. \tag{3.4}$$

They commute with all generators of the algebra of symmetries, i.e., $[\widehat{V}_0, \widehat{A}_j] = 0$ and $[\widehat{K}, \widehat{A}_j] = 0$. For the representation under study, the Casimir operator (3.4) vanishes, $\widehat{K}|_{\text{in our representation}} = 0$.

We now demonstrate how to construct irreducible representations of the non-Lie symmetry algebra (3.2) following the general method [16]. Introduce the creation-annihilation generators $\widehat{A}_+ = \widehat{A}_5 + i\widehat{A}_4$ and $\widehat{A}_- = \widehat{A}_5 - i\widehat{A}_4$. Then the commutation relations of the symmetry algebra can be written in the form

$$[\widehat{A}_-, \widehat{A}_+] = f(\widehat{A}), \quad \widehat{A}_- \widehat{A} = \varphi(\widehat{A}) \widehat{A}_-, \quad \widehat{A} \widehat{A}_+ = \widehat{A}_+ \varphi(\widehat{A}), \tag{3.5}$$

where $\widehat{A} = (\widehat{A}_1, \widehat{A}_2, \widehat{A}_3)$ is the set of commuting generators, $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a vector-valued function, and $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a scalar function:

$$\varphi(a) = (a_1 + 2\hbar, a_2 + 6\hbar, a_3), \quad f(a) = 2\hbar(15\hbar^2 a_1 + 23\hbar^2 a_2 + 9a_1 a_2^2 + a_3^3). \tag{3.6}$$

Introduce a base function $\mathcal{F}_a: \mathbb{Z}_+ \rightarrow \mathbb{R}$ given by

$$\mathcal{F}_a(n) = \frac{1}{n+1} \sum_{j=0}^n f(\varphi^j(a)),$$

i.e.,

$$\mathcal{F}_a(n) = 432\hbar^4 \left(n^3 + \frac{2\hbar + a_1 + a_2}{2\hbar} n^2 + \frac{14\hbar^2 + 9\hbar a_1 + 9\hbar a_2 + 9a_1 a_2 + 3a_2^2}{36\hbar^2} n + \frac{15\hbar a_1 + 23\hbar^2 a_2 + 9a_1 a_2^2 + a_3^3}{216\hbar^3} \right).$$

Define the subset $\mathbf{R} \subset \mathbb{R}^3$ as follows: the vector $a = (a_1, a_2, a_3)$ with components $a_j = \hbar(2N_j + 1)$ (where $N_j \in \mathbb{Z}_+$) belongs to \mathbf{R} if $\mathcal{F}_a(n) > 0$ for all $n \in \mathbb{Z}_+$.

For each $a \in \mathbf{R}$, decompose the function $\mathcal{F}_a(n)$ into two real multipliers,

$$\mathcal{F}_a(n) = \mathcal{F}_a^+(n) \mathcal{F}_a^-(n), \tag{3.7}$$

where

$$\mathcal{F}_a^+(n) = 24\hbar^2 (n - q_2 - iq_3)(n - q_2 + iq_3)$$

and

$$\mathcal{F}_a^-(n) = 18\hbar^2 (n - q_1).$$

Here

$$\begin{aligned} q_1 &= -\xi - 2\eta + 2\zeta, \quad q_2 = -\xi + \eta - \zeta, \quad q_3 = \sqrt{3}(\eta + \zeta), \quad \xi = \frac{1}{3}(N_1 + N_2 + 2), \quad \eta = \frac{\nu}{6\hbar r}, \quad \zeta = \frac{r}{72\hbar}, \\ r &= (108s + 12\sqrt{12\nu^3 + 81s^2})^{1/3}, \quad s = -8\hbar^3 N_1(2N_1^2 + 3N_1 - 3N_1 N_2 - 6N_2 + 3N_2^2 + 1), \\ \nu &= -4\hbar^2(3N_1^2 + 3N_1 - 3N_1 N_2 + 1). \end{aligned}$$

Introduce the positive numbers

$$s_0(a) = 1, \quad s_n(a) = \frac{n! \mathcal{F}_a(n-1) \dots \mathcal{F}_a(0)}{|\mathcal{F}_a^+(n-1)|^2 \dots |\mathcal{F}_a^+(0)|^2}, \quad 0 \leq n < \infty,$$

i.e.,

$$s_n(a) = (3/4)^n n! \prod_{j=1}^n \frac{(n-j-q_1)}{(n-j-q_2-iq_3)(n-j-q_2+iq_3)}.$$

We use the numbers $s_n(a)$ to introduce the Hilbert space of distributions on \mathbb{R}^2 . These distributions are antiholomorphic, i.e., they are annihilated by the operator $\partial/\partial w$, where w is the complex coordinate on the plane \mathbb{R}^2 . More precisely, by $P_{s(a)}$ we denote the space of all distributions

$$g(\bar{w}) = \sum_{n=0}^{\infty} g_n \bar{w}^n$$

on \mathbb{R}^2 , which satisfy the relation

$$\|g\|_{P_{s(a)}} \stackrel{\text{def}}{=} \left(\sum_{n=0}^{\infty} s_n(a) |g_n|^2 \right)^{1/2} < \infty.$$

The inner product in $P_{s(a)}$ is defined as

$$(g', g)_{P_{s(a)}} = \sum_{n=0}^{\infty} s_n(a) g'_n \bar{g}_n.$$

In the general case, if H is an abstract Hilbert space, P is the space of antiholomorphic distributions, and there exists a monomorphism $\mathcal{J}: P \rightarrow H$, then the “integral kernel” \mathcal{P} of this monomorphism is an H -valued holomorphic distribution such that

$$(\psi, \mathcal{J}(g))_H = ((\mathcal{P}, \psi)_H, g)_P$$

for any $g \in P$ and $\psi \in H$.

The monomorphism $\mathcal{J}: P \rightarrow H$ is referred to as a coherent transformation, and its integral kernel $\mathcal{P} \in H \otimes \bar{P}$ is referred to as a family of coherent states. The coherent transformation is a unitary mapping, and its inverse is defined as

$$\mathcal{J}^{-1}(\psi) = (\mathcal{P}, \psi)_H$$

for any $\psi \in H$.

If $P = P_{s(a)}$, then the family $\mathcal{P} = \{\mathcal{P}_w\}$ can be represented as a power series in w :

$$\mathcal{P}_w = \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{s_n}} e_n \tag{3.8}$$

with respect to any orthonormal system $e_n \in H$, $e_0 = \mathcal{J}(1)$.

A unit vector $p_0 \in L^2(\mathbb{R}^3)$ is called a vacuum vector if it is an eigenvector of the generators $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3$ and the annihilation operator \widehat{A}_- turns it into zero, i.e.,

$$\widehat{A}_k p_0 = a_k p_0 \quad (k = 1, 2, 3), \quad \widehat{A}_- p_0 = 0. \tag{3.9}$$

The first three equations in (3.9) imply that the eigenvalues a_k have the form $a_k = \hbar(2N_k + 1)$, where $N_k \in \mathbb{Z}_+$ ($k = 1, 2, 3$). The last equation (3.9) selects the following four opportunities:

- (1) $N_2 = 0, \quad N_1, N_3 \in \mathbb{Z}_+; p_0 = \text{const}(x - iy)^{N_1} \mathcal{H}_{N_3}(z/\sqrt{\hbar}) e^{-\frac{x^2+y^2+2z^2}{4\hbar}};$
- (2) $N_2 = 1, \quad N_1, N_3 \in \mathbb{Z}_+; p_0 = \text{const}(x + iy)(x - iy)^{N_1} \mathcal{H}_{N_3}(z/\sqrt{\hbar}) e^{-\frac{x^2+y^2+2z^2}{4\hbar}};$
- (3) $N_2 = 2, \quad N_1, N_3 \in \mathbb{Z}_+; p_0 = \text{const}(x + iy)^2(x - iy)^{N_1} \mathcal{H}_{N_3}(z/\sqrt{\hbar}) e^{-\frac{x^2+y^2+2z^2}{4\hbar}};$
- (4) $N_1 = 0, \quad N_2, N_3 \in \mathbb{Z}_+; p_0 = \text{const}(x + iy)^{N_2} \mathcal{H}_{N_3}(z/\sqrt{\hbar}) e^{-\frac{x^2+y^2+2z^2}{4\hbar}}.$

Here \mathcal{H}_N are Hermitian polynomials.

Note that the spectrum of the leading trap Hamiltonian $\widehat{H}_0 = \frac{1}{4}\widehat{V}_0$ is given by

$$\frac{1}{2}(3N_1 - N_2 + 1) + \sqrt{3/2}(N_3 + 1/2).$$

These numbers have the form $s + \sqrt{3/2}l$, where s are integers or half-integers, and l are half-integers, $l \geq 1/2$. For $s \leq 1/2$, we are in the case (4) of the above list, and, for $s \geq 1$, we are in one of the cases (1)–(3) of the above list. Thus, for each eigenvalue of the Hamiltonian \widehat{H}_0 (or of the Casimir element \widehat{V}_0) we have a unique choice of the quantum numbers N_1, N_2, N_3 obeying one of the conditions (1)–(4) and therefore a unique choice of the vacuum vector p_0 .

Each representation of algebra (3.2) with vacuum vector naturally generates a system $\{e_n\}$ in (3.8), namely, the system of eigenvectors of $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3$ with $e_0 = p_0$. This system can be used to construct a coherent transformation (i.e., an intertwining mapping) and to implement the representation of the algebra of symmetries in the space $P = P_{s(a)}$.

According to the results obtained in [16], there is a one-to-one correspondence between the set \mathbf{R} and the set of irreducible Hermitian representations of the algebra (3.2), which have the vacuum vector (3.9). For each $a \in \mathbf{R}$ and each decomposition into multipliers (3.7), the operators

$$\overset{\circ}{A}_+ = \bar{w}\mathcal{F}_a^+(\bar{w} d/d\bar{w}), \quad \overset{\circ}{A}_- = \mathcal{F}_a^-(\bar{w} d/d\bar{w}) d/d\bar{w}, \quad \overset{\circ}{A} = \mathcal{A}_a(\bar{w} d/d\bar{w}), \quad (3.11)$$

where $\mathcal{A}_a(n) = \varphi^n(a)$, represent the algebra (3.2). Moreover, this representation is irreducible Hermitian and has the vacuum vector 1 in the Hilbert space $P_{s(a)}$. Representations (3.11) corresponding to different vacuum vectors are not equivalent, but for each chosen $a \in \mathbf{R}$, the representations determined by different decompositions into multipliers (3.7) are equivalent.

In our case, the irreducible representation (3.11) is given by the formulas

$$\begin{aligned} \overset{\circ}{A}_+ &= 24\hbar^2\bar{w}(\bar{w} d/d\bar{w} - q_2 - iq_3)(\bar{w} d/d\bar{w} - q_2 + iq_3), \quad \overset{\circ}{A}_- = 18\hbar^2(\bar{w} d/d\bar{w} - q_1) d/d\bar{w}, \\ \overset{\circ}{A}_1 &= a_1 + 2\hbar\bar{w} d/d\bar{w}, \quad \overset{\circ}{A}_2 = a_2 + 6\hbar\bar{w} d/d\bar{w}, \quad \overset{\circ}{A}_3 = a_3. \end{aligned} \quad (3.12)$$

The abstract Hermitian representation of the algebra (3.2) in the Hilbert space H_a with vacuum vector p_0 can be intertwined with the representation (3.11) by using the following coherent states \mathcal{P}_w (3.8) which can be expressed in terms of the vacuum vector as follows:

$$p_w = p_0 + \sum_{n=1}^{\infty} \frac{1}{n!\mathcal{F}_a^-(n-1)\dots\mathcal{F}_a^-(0)} (w\widehat{A}_+)^n p_0.$$

In our case, we have

$$p_w = {}_0F_1\left(-q_1; \frac{w\widehat{A}_+}{18\hbar^2}\right)p_0, \quad (3.13)$$

where ${}_0F_1$ is a hypergeometric function [20].

We note that the “reproducing kernel” corresponding to (3.13), $\mathcal{K}_{s(a)}(\bar{w}, \nu) = (p_w, p_\nu)_{H_a}$, is the integral kernel of the identity operator in $P_{s(a)}$ and is determined by the distribution on $\mathbb{R}^2 \times \mathbb{R}^2$, namely,

$$\mathcal{K}_{s(a)}(\bar{w}, \nu) = \sum_{n=0}^{\infty} (\bar{w}\nu)^n / s_n(a).$$

In our case, we have

$$\mathcal{K}_{s(a)}(\bar{w}, \nu) = k(|w|^2), \quad \text{where } k(\xi) = {}_2F_1\left(-q_2 - iq_3, -q_2 + iq_3; -q_1; \frac{4}{3}\xi\right).$$

Let us consider the representations of the symmetry algebra (3.2) corresponding to points of the Casimir operator spectrum (3.3)

$$\widehat{H}_0 = \frac{1}{4} \widehat{V}_0$$

of the form $s + l\sqrt{3/2}$ with $s \geq 1$, i.e., the representations with vacuum vector of the form (1)–(3) in (3.10). We define the function

$$l(r) = \frac{4}{3} \frac{(1-\alpha)(1-\beta)}{(1-\gamma)} \frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta) - \Gamma(\gamma-1)\Gamma(\gamma-\alpha-\beta+1)} {}_2F_1\left(\alpha, \beta; \gamma; \frac{4}{3}r\right).$$

where $a = 1 - N_1$, $b = \frac{4}{3}$, and $c = \frac{5}{3}$ for $N_2 = 0$ in (3.10), $a = \frac{2}{3}$, $b = 1 - N_1$, and $c = \frac{4}{3}$ for $N_2 = 1$ in (3.10), $a = 1 - N_1$, $b = \frac{1}{3}$, $c = \frac{2}{3}$ for $N_2 = 2$ in (3.10), and ${}_2F_1(\alpha, \beta; \gamma; \cdot)$ is a hypergeometric function which is analytic in the unit disk [20]. Using the function l , one can write the inner product in the space $P_{s(a)}$ in integral form

$$(g', g)_{P_{s(a)}} = \frac{1}{2\pi} \iint_{|w| \leq \sqrt{3/2}} \overline{g'_n(\bar{w})} g_n(\bar{w}) l(|w|^2) dw d\bar{w}, \tag{3.14}$$

In this case, the key property of the reproducing kernel is satisfied:

$$k(|w|^2) = \frac{1}{2\pi} \int_{|\nu| \leq \sqrt{3/2}} k(\bar{w}\nu) k(\bar{\nu}w) l(|\nu|^2) d\nu d\bar{\nu}.$$

Then the coherent transformation \mathcal{J} intertwining the irreducible representation (3.12) with the initial representation (3.1) of the algebra of symmetries (3.2) in the Hilbert space $L^2(\mathbb{R}^3)$ can also be represented in integral form

$$\mathcal{J}(g) = \frac{1}{2\pi} \iint_{|w| \leq \sqrt{3/2}} g_n(\bar{w}) p_w l(|w|^2) dw d\bar{w}, \tag{3.15}$$

where p_w are the coherent states (3.13). The transformation (3.15) is unitary and takes the unit function 1 into the vacuum vector p_0 .

4. AVERAGED HAMILTONIAN OVER THE ALGEBRA OF SYMMETRIES

Now we return to operators (2.2) commuting with \widehat{H}_0 . They can all be expressed in terms of generators of the symmetry algebra. As a result, the initial Hamiltonian \widehat{H} is unitarily equivalent up to $O(\varepsilon^4)$ to some Hamiltonian over the non-Lie algebra (3.2). Calculations lead to the result

$$\widehat{H} \sim \widehat{H}_0 + \frac{\varepsilon^2}{2} (\widehat{V}_2 + a\widehat{V}_3 + \widehat{V}_5) + O(\varepsilon^4), \tag{4.1}$$

where the leading part of the averaged Hamiltonian has the form

$$\widehat{H}_0 = \frac{1}{4} (3\widehat{A}_1 - \widehat{A}_2 + \sqrt{6}\widehat{A}_3),$$

and the perturbed part is quadratic in the generators of the algebra of symmetries:

$$\begin{aligned} \widehat{V}_2 &= (n_1^2 - n_2^2 + 6n_1n_3 + n_3^2) \left(\frac{1}{8}\widehat{A}_1^2 + \frac{1}{2}\widehat{A}_1\widehat{A}_2 + \frac{1}{8}\widehat{A}_2^2 + \frac{1}{4}\hbar^2 \right) + \frac{1}{2}((n_1^2 - n_3^2)\widehat{A}_4 + (n_1n_2 + n_2n_3)\widehat{A}_5), \\ \widehat{V}_3 &= 8\hbar^2 + 3\widehat{A}_1^2 + 12\widehat{A}_1\widehat{A}_2 + 3\widehat{A}_2^2 - 4\sqrt{6}(\widehat{A}_1 + \widehat{A}_2)\widehat{A}_3 + 2\widehat{A}_3^2, \end{aligned}$$

$$\begin{aligned}
 \widehat{V}_5 = & \frac{\sqrt{6}}{4}(3n_1^2 + 2n_1n_3 + 3n_3^2 + n_2^2)(\widehat{A}_1^2 + 4\widehat{A}_1\widehat{A}_2 + \widehat{A}_2 + 2\hbar^2) + \sqrt{6}((n_1n_2 + n_2n_3)\widehat{A}_4 \\
 & + (n_1^2 - n_3^2)\widehat{A}_5) + \frac{\sqrt{6}}{5}(- (6n_1^2 + 8n_1n_3 + 6n_3^2 + n_2^2)\widehat{A}_1 + (2n_1^2 - 24n_1n_3 + 2n_3^2 + 7n_2^2)\widehat{A}_2)\widehat{A}_3 \\
 & - \frac{3}{4}\left(\frac{1}{15}(9n_1^2 + 22n_1n_3 + 9n_3^2 - n_2^2)\widehat{A}_1^2 + \frac{1}{3}(5n_1^2 - 2n_1n_3 + 5n_3^2 + 3n_2^2)\widehat{A}_2^2 \right. \\
 & \left. + \frac{4}{15}(17n_1^2 + 46n_1n_3 + 17n_3^2 - 3n_2^2)\widehat{A}_1\widehat{A}_2 + \frac{4}{5}(9n_1^2 + 2n_1n_3 + 9n_3^2 + 4n_2^2)\hbar^2\right) \\
 & + 3((n_1^2 - n_3^2)\widehat{A}_4 + (n_1n_2 + n_2n_3)\widehat{A}_5).
 \end{aligned} \tag{4.2}$$

Now we use the coherent transformation to transfer the Hamiltonian (4.1), (4.2) into the irreducible representation (3.12) of the algebra of symmetries. In (4.2), all operators \widehat{A}_j will be replaced by the operators $\overset{\circ}{A}_j$ (3.2). As a result, we obtain an operator of the form

$$\begin{aligned}
 & \frac{\hbar}{4}(3(2N_1 + 1) - (2N_2 + 1) + \sqrt{6}(2N_3 + 1)) \\
 & + \frac{\varepsilon^2}{2}\left((\alpha_1\bar{w}^3 + \alpha_2\bar{w}^2 + \alpha_3\bar{w})\frac{d^2}{d\bar{w}^2} + (\beta_1\bar{w}^2 + \beta_2\bar{w} + \beta_3)\frac{d}{d\bar{w}} + \gamma_1\bar{w} + \gamma_2\right) + O(\varepsilon^4),
 \end{aligned} \tag{4.3}$$

whose coefficients $\alpha_j, \beta_j, \gamma_j$ can explicitly be calculated in terms of the coefficients in (4.2).

The ordinary second-order differential operator (4.3) acting with respect to the variable \bar{w} is a Heun-type operator from [20]. It is self-adjoint in the space with inner product (3.14). Its eigenvalues determine order ε^2 corrections to the leading part

$$\frac{\hbar}{4}(3(2N_1 + 1) - (2N_2 + 1) + \sqrt{6}(2N_3 + 1))$$

of the spectrum of the initial Hamiltonian \widehat{H} of the resonance Penning–Ioffe trap.

In the semiclassical approximation (where the quantum numbers $N_j \sim 1/\hbar$ are large as $\hbar \rightarrow 0$), the asymptotics of the spectrum of operator (3.4) is calculated as in [13, Part 3] in geometric terms by using a Bohr–Sommerfeld–Plank-type quantization condition for the energy levels of the symbol of the operator $\widehat{V}_2 + a\widehat{V}_3 + \widehat{V}_5$ on the quantum symplectic leaf corresponding to this irreducible representation of the algebra of symmetries.

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