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SHORT COMMUNICATIONS

On Limit Cycles of Systems with a Partial Integral

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Abstract—For a certain class of two-dimensional autonomous systems of differential equations with an invariant curve that contains ovals, we indicate necessary and sufficient conditions for these ovals to be limit cycles of phase trajectories.

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Two-dimensional autonomous systems of differential equations with given programmed motions were considered by numerous authors. The statement of the problem goes back to G. Darboux and N.P. Erugin.

The paper [1] deals with the systems

$$\begin{aligned} \frac{dx}{dt} &= F(x,y)H_y(x,y) + a(x,y)H(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= -F(x,y)H_x(x,y) + b(x,y)H(x,y) \equiv Q(x,y), \end{aligned} \tag{1}$$

where the functions a, b, F, and H are single-valued and analytic in the domain G, and conditions for system (1) to be conservative near the ovals (nonsingular compact connected components) of the curve $H = 0$, as well as conditions for the existence or absence of limit cycles other than ovals of $H = 0$, were obtained. In particular, the following assertion holds.

Theorem 1. *Let the function $F(x,y)$ be nonzero on an oval $L \subset \{(x,y): \ H = 0\}$ and inside the domain $B \subset G$, $\partial B = L$; in addition, let*

$$(a/F)_x + (b/F)_y \equiv 0 \tag{2}$$

in B. Then system (1) has an integrating factor $\mu = (FH)^{-1}$, and all trajectories of system (1) are closed in some neighborhood $S(L,\varepsilon)$ of the oval L.

In the present paper, we indicate classes of systems satisfying condition (2); in addition, we do not exclude the case in which $F(x,y) = 0$ in the domain B.

1. Let us show that the condition $F(x,y) \neq 0$ inside B, $\partial B = L$, is important for the conservativeness of system (1) near L.

Indeed, if

$$a(x,y) \equiv x, \qquad b(x,y) \equiv y, \qquad F \equiv x^2 + y^2, \qquad H \equiv x^2 + y^2 - 1, \tag{3}$$

then system (1) has the unique equilibrium $(0,0)$, the limit cycle is described by the equation $x^2 + y^2 = 1$, has the integrating factor $\mu = (x^2 + y^2)^{-1}(x^2 + y^2 - 1)^{-1}$, and has the first Darboux integral $(x^2+y^2-1)(x+iy)^{-i/2}(x-iy)^{i/2} = C$. In addition, system (1) with right-hand sides defined in accordance with (3) satisfies condition (2) everywhere except for the point $(0,0)$ at which $F = 0$.

The set of systems (1) satisfying relation (2) contains real systems of the form

$$\begin{aligned}\frac{dx}{dt} &= \beta_1\Phi_{1y}\Phi_2\cdots\Phi_k + \Phi_1\sum_{j=2}^{k}\beta_j\Phi_2\cdots\Phi_{j-1}\Phi_{jy}\Phi_{j+1}\cdots\Phi_k \equiv P,\\ \frac{dy}{dt} &= -\beta_1\Phi_{1x}\Phi_2\cdots\Phi_k - \Phi_1\sum_{j=2}^{k}\beta_j\Phi_2\cdots\Phi_{j-1}\Phi_{jx}\Phi_{j+1}\cdots\Phi_k \equiv Q,\end{aligned} \tag{4}$$

where the Φ_j are coprime polynomials in real variables x and y and are irreducible over the field of complex numbers, and the quantities $\beta_j \neq 0$, $j = 1,\ldots,k$, and the coefficients of $\Phi_j(x,y)$ are complex in the general case and such that P and Q are coprime polynomials.

The properties of systems (4) were considered in [2–10] for various values of the number k and degree of the polynomials Φ_j and under changes of the quantities β_j and the coefficients multiplying Φ_j.

Note that if $k = 1$ or $k = 2$, then the real system (4) has no limit cycles [2]. If $k \geq 3$, then system (4) can have only algebraic limit cycles; in this case, the cycles are hyperbolic, the polynomials specifying the limit cycles occur in an analytic expression for the first Darboux integral $\Gamma(x,y) = \Phi_1^{\beta_1}\cdots\Phi_k^{\beta_k} = C$ and the integrating factor $\mu = (\Phi_1\cdots\Phi_k)^{-1}$, and the function $\Gamma(x,y)$ is multivalued in a neighborhood of an arbitrary limit cycle [2]. By [11], for algebraically nonintegrable systems (4) (which have finitely many algebraic invariant curves), we have $k \leq n(n+1)/2 + 1$, where $n = \max(\deg P, \deg Q)$.

Note that the above example is obtained from system (4) for $k = 3$, $\beta_1 = 1$, $\beta_2 = -i/2$, $\beta_3 = i/2$, $H \equiv \Phi_1 \equiv x^2 + y^2 - 1$, and $F \equiv \Phi_2\Phi_3$, where $\Phi_2 = x + iy$ and $\Phi_3 = x - iy$.

2. Note that the problem on the limit cycles of system (1) for the case in which $F(x,y)$ is a linear function, $a(x,y) \equiv a$, $b(x,y) \equiv b$, and $a, b \in \mathbb{R}$ was considered in [12–14]. For the existence of a limit cycle specified by an oval ℓ of the curve $H = 0$, it is sufficient that the line $F = 0$ does not meet the oval ℓ and, in addition, $aF_x + bF_y \neq 0$. If these conditions are satisfied, then system (1) has no limit cycles other than ovals of $H = 0$; in addition, the limit cycles defined by the equation $H = 0$ are hyperbolic [1, 12, 13].

By a straightforward verification with the use of Theorem 1, one can justify the following assertion.

Lemma. *Let the following conditions be satisfied for system* (1):

(i) $F(x,y) = \alpha x + \beta y + \gamma$, $a(x,y) \equiv a$, $b(x,y) \equiv b$, *where* $a, b, \alpha, \beta, \gamma \in \mathbb{R}$, $|a| + |b| > 0$, *and* $|\alpha| + |\beta| > 0$,

(ii) *identity* (2) *holds*,

then there exists a number $\nu \neq 0$ *such that* $a = -\nu\beta$, $b = \nu\alpha$, *and system* (1) *has the first integral* $HF^{-\nu} = C$ *and the integrating factor* $\mu = (HF)^{-1}$ *and has no limit cycle.*

Theorem 2. *Let condition* (i) *of the lemma be satisfied, and let* $H(x,y)$ *be a polynomial of degree* n *such that* $\max(\deg P, \deg Q) < n$. *Then system* (1) *is algebraically integrable, has the first Darboux integral* $HF^{-n} = C$ *and integrating factor* $\mu = (HF)^{-1}$, *and has no limit cycles.*

Proof. Since $\max(\deg P, \deg Q) < n$, it follows that the homogeneous polynomial $H_n(x,y)$ of degree n contained in H satisfies the condition

$$(\alpha x + \beta y)H_{ny} + aH_n \equiv 0, \qquad (\alpha x + \beta y)H_{nx} - bH_n \equiv 0. \tag{5}$$

By (5), we have

$$(\alpha x + \beta y)(bH_{ny} + aH_{nx}) \equiv 0.$$

If $|\alpha|+|\beta| > 0$, then this implies that $H_n(x,y) = C(ay - bx)^n$, where $C \equiv \text{const} \neq 0$. By substituting this expression $H_n(x,y)$ into (5), for $|a| + |b| > 0$, we obtain the identity

$$(n\alpha - b)x + (n\beta + a)y \equiv 0,$$

whence we obtain $b = n\alpha$ and $a = -n\beta$. Consequently, relation (2) holds, and the lemma, where $\nu = n$, can be used. The proof of the theorem is complete.

Note that under the assumptions of Theorem 2, system (1) is obtained from (4) for $\beta_1 = 1$, $\beta_2 = -n$, $k = 2$, $\Phi_1 \equiv H$, and $\Phi_2 \equiv F$. If the polynomial H is irreducible, then the following assertion can be used.

Theorem 3 [8]. *Let* $k = 2$, $\max(\deg P, \deg Q) = n$, $\deg \Phi_1 = m_1$, $\deg \Phi_2 = m_2$, *and*, *in addition*, $m_1 + m_2 > n + 1$. *Then* $\beta_1 m_1 + \beta_2 m_2 = 0$, *and all trajectories of system* (4) *are algebraic.*

The lemma and Theorem 1 in [12] (Theorems 1 and 2 in [1]) imply the following assertion.

Theorem 4. *Suppose that assumptions* (i) *of the lemma are satisfied*; *then the limit cycles of phase trajectories of system* (1) *are given only by ovals of the curve* $H = 0$ *lying in the domain* G *if and only if the line* $F = 0$ *does not meet the ovals of* $H = 0$ *and* $aF_x + bF_y = \alpha a + \beta b \neq 0$; *in addition*, *the cycles are hyperbolic.*

Theorem 4 implies the assertions of Theorems 3.3 and 3.5 in [15] dealing with the limit cycle $x^2 + y^2 = 1$ of the quadratic system and its hyperbolicity.

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