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Solution for One-Stage Bidding Game with Incomplete Information*

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Abstract We investigate a model of one-stage bidding between two differently informed stockmarket agents for a risky asset (share). The random liquidation price of a share may take two values: the integer positive m with probability p and 0 with probability $1-p$. Player 1 (insider) is informed about the price, Player 2 is not. Both players know the probability p. Player 2 knows that Player 1 is an insider. Both players propose simultaneously their bids. The player who posts the larger bid buys one share from his opponent for this price. Any integer bids are admissible. The model is reduced to a zero-sum game with lack of information on one side. We construct the solution of this game for any p and m: we find the optimal strategies of both players and describe recurrent mechanism for calculating the game value. The results are illustrated by means of computer simulation.

Keywords: insider trading, asymmetric information, equalizing strategies, optimal strategies.

1. Introduction

The model of bidding for risky asset (a share) with different agent's information about liquidation value of a share was introduced by De Meyer and Saley, 2002.

A liquidation price of a share, which can take two values – high and low share price, depends on a random "state of nature". Before bidding starts a chance move determines the "state of nature" and therefore the liquidation value of a share. The probability p of choice of high share price is known to both players. Besides, Player 1 (insider) is informed about the "state of nature", Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent step both players simultaneously propose their bids for one share. The maximal bid wins and one share is transacted at this price. After this bids are reported to both players. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

De Meyer and Saley reduce the model to a zero-sum repeated game with lack of information on the side of Player 2, they solve the game for any number of steps, find optimal behavior of both players and expected profit of insider.

* This work was supported by the grant 10-06-00348-a of Russian Foundation of Basic Research

In De Meyer and Saley model any real bids are admissible.

It is more realistic to assume that players may assign only discrete bids proportional to a minimal currency unit. The models with admissible discrete bids are investigated by Domansky, 2007. These models are the models of n-stage bidding game $G_n^m(p)$ with incomplete information with high share price equal to integer positive m, low share price equal to zero, and admissible discrete bids. Solving of n-stage bidding game is combinatorially difficult. The solution was only obtained for the game $G_\infty^m(p)$ with unlimited duration.

The solution of finitely-stage games is an open problem. The solution has been found only for the difference between high and low share prices equal or less than 3 ($m \le 3$) (Kreps, 2009b).

In this paper we give the complete solution for the one-stage bidding game $G_1^m(p)$ with arbitrary integer m and with any probability $p \in (0,1)$ of the high share price.

We develop recurrent approach to computing optimal strategies of uninformed player for any probability p based on analysis of structure of bids used in optimal strategies of both players. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in optimal mixed strategy.

The optimal strategy of insider equalizes spectrum of obtained optimal strategy of Player 2, one can obtain distribution of it's weights solving the system of difference equations arizing from equalizing conditions. As a base for calculating solution elements we use solution on the ends of interval $[0,1]$: for probabilities p sufficiently close to 0 and 1 the game has solutions in pure strategies. When p runs from 0 (or 1) till some limit these pure strategies holds as optimal ones, and so on.

For the special finite set of p depending on m another approach to finding the solution of one-stage bidding game was suggested in the paper of Kreps, 2009a.

2. Model of one-stage bidding with arbitrary bids

Here we give the explicit solution for one-stage bidding game $G_1(p)$ with arbitrary admissible bids. We put $m = 1$ to make formulae more clear. It follows from the work of De Meyer and Saley, 2002 that the game value $V_1(p) = p(1-p)$.

Note that the optimal strategy of Player 1 is to post the bid zero at the state L and randomization of bids at the state H for any prior probability. This observation allows to reduce solving one-stage game $G_1(p)$ with incomplete information to solving the game on unit square with payoff function

$$K_p(x,y) = \begin{cases} (1-p)y + p(1-x), & \text{for } x > y; \\ (1-p)y, & \text{for } x = y; \\ (1-p)y - p(1-y), & \text{for } x < y. \end{cases}$$

Applying well-known heuristic methods of solving games on unit square with payoff function that has a break on the principal diagonal (Karlin, 1964) we find strategies of informed and uninformed players. Further in Theorem 1 it is shown that these strategies are optimal.

Denote by the same letter a mixed strategy of a player and corresponding cumulative density function: F_p is a mixed strategy of Player 1, G_p is a mixed strategy of Player 2. Write down the gain of Player 1 when he applies mixed strategy F_p and

Player 2 applies pure strategy y:

$$K_p(F_p, y) = (1-p)y - \int_0^y p(1-y)dF_p(x) + \int_y^1 p(1-x)dF_p(x) -$$

$$-\int_y^1 x dF_p(x) = (1-p)y - p(1-y)F_p(y) + p(1-F_p(y)) - \int_y^1 x dF_p(x).$$

The optimal strategy of Player 1 equalizes the payoffs for pure strategies in spectrums of optimal strategies of Player 2 (see Karlin, 1964).

We assume that optimal strategies of both players have the same spectrums. Putting the derivative by y of the function $K_p(F_p, y)$ equal to zero we obtain the differential equation for points of the common spectrum:

$$\frac{dF_p(x)}{dx} 2p(1-x) = pF_p(x) + (1-p). \tag{1}$$

It is easy to see that $F_p(0) = 0$. The equation (1) has the solution

$$F_p^*(x) = \frac{1-p}{p}((1-x)^{-1/2} - 1) \quad x \in [0, 1-(1-p)^2].$$

Applying similar reasoning for mixed strategy G_p of Player 2 we get:

$$K_p(x, G_p) = (1-p)\int_0^1 y dG_p(y) + p(1-x)G_p(x)\int_x^1 p(1-y)dG_p(y);$$

$$(1-y)\frac{dG_p(y)}{dy} = G_p(y). \tag{2}$$

Solving equation (2) and choosing the solution with the same spectrum as one's of solution of equation (1) we obtain

$$G_p^*(y) = \frac{1-p}{\sqrt{1-y}}, \text{ for } y \in [0, 1-(1-p)^2].$$

Observe that using this strategy Player 2 proposes a bid 0 with the positive probability $G_p^*(0) = 1-p$.

Theorem 1. *For the one-stage bidding game $G_1(p)$ the unique optimal strategy of Player 1 is to post 0 at the state L. At the state H this strategy is given by the cumulative depending on p density function on $[0, 1]$*

$$F_p^*(x) = \begin{cases} \frac{(1-p)(1-\sqrt{1-x})}{p\sqrt{1-x}}, & \textit{for } x \le (1-(1-p)^2); \\ 1, & \textit{for } x > (1-(1-p)^2). \end{cases}$$

The unique optimal strategy of Player 2 is given by the cumulative density function

$$G_p^*(y) = \begin{cases} \frac{1-p}{\sqrt{1-y}}, & \textit{for } y \le (1-(1-p)^2); \\ 1, & \textit{for } y > (1-(1-p)^2). \end{cases}$$

Proof. Check that the strategy F_p^* of Player 1 equalizes points in spectrum of strategy G_p^* of Player 2, i.e. for all $y \in [0, 1-(1-p)^2]$ value $K_p(F_p^*, y)$ is constant and guarantees to Player 1 the gain $p(1-p)$. Really,

$$\int_0^{1-(1-p)^2} K_p(x,y)dF_p^*(x)$$

$$= (1-p)y - p(1-y)F_p^*(y) + 1/2\int_y^{1-(1-p)^2} (1-p)(1-x)^{-1/2}dx$$

$$= (1-p)y - (1-p)[(1-y)^{1/2} - (1-y)] + (1-p)(1-y)^{1/2} - (1-p)^2 = p(1-p).$$

It's obvious that for $y > 1-(1-p)^2$ the payoff function decreases. Thus the strategy F_p^* guarantees to Player 1 the gain $p(1-p)$.

Proof for the strategy G_p^* is the similar.

$$\int_0^1 K_p(x,y)dG_p^*(y) = \int_0^{1-(1-p)^2} (1-p)\frac{dG_p^*(y)}{dy}dy + p(1-x)G_p^*(x) -$$

$$\int_x^{1-(1-p)^2} p(1-y)\frac{dG_p^*(y)}{dy}dy = p(1-p).$$

□

Remark 1. Continuous distribution corresponding to the optimal strategy of Player 1 and continuous component of distribution corresponding to the optimal strategy of Player 2 have the same spectrum $(0, 1-(1-p)^2)$ and the similar density proportional to

$$(1-x)^{-3/2}.$$

Remark 2. Changing a scale in results obtained to m we get a formulae for game value

$$V_1(p) = m \cdot p(1-p),$$

following expressions for cumulative density functions: for optimal strategy of Player 1 at state H

$$F_p^*(x) = \begin{cases} \frac{(1-p)(\sqrt{m}-\sqrt{m-x})}{p\sqrt{m-x}}, & \text{for } x \le m(1-(1-p)^2); \\ 1, & \text{for } x > m(1-(1-p)^2). \end{cases}$$

for optimal strategy of Player 2

$$G_p^*(y) = \begin{cases} \frac{(1-p)\sqrt{m}}{\sqrt{m-y}}, & \text{for } y \le m(1-(1-p)^2); \\ 1, & \text{for } y > m(1-(1-p)^2). \end{cases}$$

Densities (similar within a coefficient) of these distributions with the same spectrum $(0, m(1-(1-p)^2))$ are proportional to

$$(m-x)^{-3/2}.$$

3. Model of one-stage bidding with integer bids

Against the model of De Meyer we change a scale: at state H the share price is equal to integer positive number m, at state L the share price is zero.

We consider a model with admissible integer bids proportional to a minimal currency unit. The reasonable bids are only $0, 1, \ldots, m-1$.

The model is reduced to a zero-sum game with lack of information on the side of Player 2. The state space is $S = \{L, H\}$, the action sets of both players are $I = J = \{0, 1, \ldots, m-1\}$.

At state L payoffs that insider receives are given by matrix $A^{L,m}$

$$A^{L,m} = \begin{pmatrix} 0 & 1 & 2 & \ldots\, m-1 \\ -1 & 0 & 2 & \ldots\, m-1 \\ -2 & -2 & 0 & \ldots\, m-1 \\ \vdots & \vdots & \vdots & \ddots \quad \ldots \\ -m+1 & -m+1 & -m+1 & \ldots \quad 0 \end{pmatrix}.$$

At state H the payoff matrix $A^{H,m}$ is

$$A^{H,m} = \begin{pmatrix} 0 & -m+1 & -m+2 & \ldots -1 \\ m-1 & 0 & -m+2 & \ldots -1 \\ m-2 & m-2 & 0 & \ldots -1 \\ \vdots & \vdots & \vdots & \ddots \ldots \\ 1 & 1 & 1 & \ldots \ 0 \end{pmatrix}$$

Rows of matrix are the bids of Player 1 with numeration starts from zero, column are the bids of Player 2.

It is obvious that at state L insider proposes 0 for any probability p, but at state H insider doesn't use 0. Therefore in the sequel we will be interested in the strategy of Player 1 at state H.

The strategy of Player 2 doesn't depend on the state of nature.

These observations allows to reduce solving of the game $G_1^m(p)$ with incomplete information to solving the game with complete information with payoff matrix

$$A^m(i,j) = \begin{cases} (1-p)j + p(m-i), & \text{for } i > j; \\ (1-p)j, & \text{for } i = j; \\ (1-p)j + p(-m+j), & \text{for } i < j, \end{cases}$$

here $i \in I$ is the bid of insider at state H, $j \in J$ is the bid of uninformed player.

The matrix of gains of insider at state L is reduced to easier view:

$$A^{L,m} = \begin{pmatrix} 0\ 1\ 2 \ldots m-1 \end{pmatrix}.$$

The matrix A^m is written down in compact view:

$$A^m(p) = A^{L,m} \cdot (1-p) + A^{H,m} \cdot p.$$

By $V_1^m(p)$ denote the value of the game $G_1^m(p)$ which is expected gain of insider, denote by $v^{H,m}$ and $v^{L,m}$ his gains at states H and L respectively:

$$V_1^m(p) = v^{L,m} \cdot (1-p) + v^{H,m} \cdot p.$$

Logic of constructing optimal strategies of both players according to a probability of choosing high share price is following: when p increases the bids of both players also grow. So for little (close to zero) p bids of both players are minimal. When p increases growth of bids leads to extension of spectrum of selectable strategies.

For p close to 1/2 Player 2 has maximal uncertainty and hence maximal bids spectrum. Uncertainty is minimal for p close to zero or unity.

It follows from the general theory that the value $V_1^m(p)$ of the game $G_1^m(p)$ is a continuous concave piecewise linear function over $[0, 1]$ with a finite numbers of domains of linearity. Moreover, the optimal strategy of the uninformed Player 2 is constant over linearity domains.

Lets start from p close to zero. Then (due to domination) there is an equilibrium situation in pure strategies, the optimal strategy of Player 2 is proposing zero, ones of Player 1 is proposing 1.

When p grows up starting with certain p_1 a bid 1 of Player 1 becomes to be dominated. Then Player 2 needs to include the bid 1 in his spectrum and similarly Player 1 needs to randomize between bids 1 and 2. Probability p_1 of this strategy changing is the first peak point of function $V_1^m(p)$ and so on.

At each stage uninformed player equalizes insider gains for pure strategies from insider's optimal strategy spectrum. Observing the changes of sets of bids used is optimal strategy by insider one can obtain a peak points of piecewise linear function $V_1^m(p)$.

Similarly for p near 1 one starts from bid $m - 1$ and then add lower bids sequentially. Combination of this approaches (to start from bottom or to start from top) allows to find al peak points of function $V_1^m(p)$ over $[0, 1]$.

4. Analysis of bids used

Consider probabilities from the left part of the interval [0;1]. Let's analyze optimal behavior of players at these probabilities.

Fix p. Denote by x any strategy of insider, by y strategy of Player 2.

Denote the probability (the weight) of action i in this mixed strategy by $x(i)$ and $y(i)$ for Players 1 and 2 respectively.

Definition 1. Set of bids is called *spectrum of strategy* of player if the he use this bids in this mixed strategy with positive probability.

The notation is $\operatorname{Spec} x$, $\operatorname{Spec} y$.

Definition 2. We say that *the strategy y of Player 2 equalizes the subset B of bids* of Player 1 if for all pure strategies of Player 1 from B he receive the same payoff. It means that the following equality holds

$$\sum_j A^m(i, j)y(j) = v \quad \text{for all } i \in B,$$

here v is a common equalization value for all $i \in B$.

In the sequel if we don't mention any set B it means that $B = \operatorname{Spec} x$. The similar definition holds for the case when the strategy x of Player 1 is equalizing for bids of Player 2.

Consider a mixed strategy y, let k_2 is maximal bid in $\operatorname{Spec} y$. Let insider use the maximal bid k_1. Let the pair of strategies (x, y) be optimal. Then spectrums of these strategies are connected by the following way.

Proposition 1. *Let p is not a peak point of function $V_1^m(p)$. Then maximal bid in spectrum of insider's optimal strategy is equal to uninformed player maximal bid or coincides with bid following the maximal uninformed player one, i.e.*

$$k_1 = k_2 \ \ or \ \ k_1 = k_2 + 1.$$

Proof. Show first that $k_2 \leq k_1$.

It follows from the structure of matrix $A^m(p)$ that

$$\forall j \in \operatorname{Spec} y \quad j \leq k_1 + 1.$$

Let $j = k_1 + 1 \in \operatorname{Spec} y$. Then the pure strategy $k_1 + 1$ equalizes the strategy of Player 2 and because of uniqueness of extreme optimal strategy $k_1 + 1$ is optimal strategy of Player 2.

If insider uses bids less than or equal to k_1 his gain is equal to

$$A^m(x, k_1 + 1) = -mp + k_1 + 1.$$

But if insider use bid $k_1 + 1$ he will earn

$$A^m(k_1 + 1, k_1 + 1) = -(k_1 + 1)p + k_1 + 1,$$

that is more than $-mp+k+1$. So it is profitable to insider to deviate from strategy x. It contradicts the optimality of insider's strategy.

Show now that $k_2 > k_1 - 2$. If maximal bid of Player 2 is less than or equal to $k_1 - 2$ then it's not profitable for Player 1 to use the bid k_1 (it is dominated by the bid $k_1 - 1$), but it is impossible because of conditions of the theorem (k_1 is used by insider the in optimal strategy). □

To make notation more clear let's write k instead of k_1; so the maximal bid in spectrum of Player 2 is k or $k - 1$.

Denote the game value $V_1^m(p)$ by $v_k(p)$ if the maximal bid in optimal strategy of insider is k. Also attach index k to the gains of insider at states L and H

$$v_k(p) = v_k^H \cdot p + v_k^L \cdot (1 - p).$$

Consider probabilities p for which maximal bids in spectrums of optimal strategies of both players coincide. Denote the distribution function of weights in strategy y of Player 2 by G_k:

$$G_k(i) = \sum_{j=1}^{i} y(j), \quad G_k(k) = 1,$$

$$y(k) > 0, \quad y(k+1) = \ldots = y(m-1) = 0.$$

Suppose spectrum of insider's strategy is maximal that means it contains all bid from 1 till k without lacunas. As the number of bids should be the same there should exist unusable bid among the bids of Player 2 from 0 till k.

Proposition 2. *Suppose that both players use maximal spectrums in optimal strategies with maximal bid k. Then Player 2 misses either bid 1 or bid 2 in his optimal strategy.*

Proof. It follows from the Shapley-Snow theorem (see Karlin, 1964, Chapter 2) that optimal strategy of Player 2 equalizes the bids of insider. The condition of equalization gives the system of difference equations on G_k:

$$(m-j)G_k(j-1) = (m-j-1)G_k(j+1), \quad j = \overline{1, k-1}. \tag{3}$$

So we obtain $k-1$ equations with k unknown quantities: $G_k(0), G_k(1), \ldots, G_k(k-1)$. ($G_k(k) = 1$ is known.)

Hence *there is one-parametric family of solutions.*

Use a natural monotonicity property of cumulative distribution function: $G_k(j) \leq G_k(j+1)$. Suppose $j > 2$. Let's find a relation between $y(1)$ and $y(2)$.

It follows from (3) that

$$(m-1)y(0) = (m-2)(y(0) + y(1) + y(2))$$

$$y(0) = (m-2)(y(1) + y(2)) \tag{4}$$

Let us suppose, for the sake of definiteness, that k is even. Then using conditions $G_k(k) = 1$ and (3) we move to step 2 and calculate $G_k(0) = y(0)$.

Let's show that for any $y(1)$, $y(2)$ satisfying (4) there exists the equalizing strategy with such $y(1)$, $y(2)$. It is enough to prove that all $y(j) \geq 0$.

Choose $y(1)$. Specify $G_k(1) = y(0) + y(1)$. From the difference equation we find $G_k(3), G_k(5), \ldots, G_k(k-1)$. One can show monotonicity

$$G_k(2j) \leq G_k(2j+1) \leq G_k(2j+2)$$

by induction.

So one can parameterize the family of equalizing strategies of Player 2 by $y(1)$ or $y(2)$ with condition

$$y(1) + y(2) = c \geq 0.$$

The optimal strategies corresponds to the extreme points: $y(1) = 0$ and $y(2) = 0$. Therefore the uninformed player misses either bid 1 or bid 2 in his optimal strategy. □

The example of optimal strategies with maximal spectrums is presented on Figure 1.

The spectrum "without lacunas" is not the only possible structure. For some probabilities p spectrums of optimal strategies of both players have lacunas. The possible structures of lacunas are described in the following theorem.

Proposition 3. *The maximal length of lacuna (number of missed bids) in spectrum of optimal strategy of both players is not greater than 1 if players use bids less than $m-1$. The greater lacuna can be only before the bid $m-1$.*

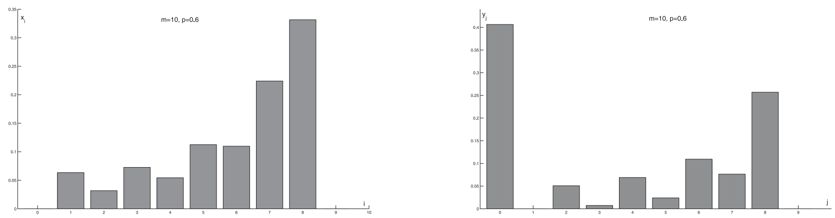

Figure1: Weights of bids of Player 1 and 2 respectively for $m = 10$, $p = 0.6$

Proof. I. Let's verify the statement of the theorem for uninformed player. Proof is by reductio ad absurdum. Suppose Player 2 misses two or more bids. So there exists the bid λ such that

$$y(\lambda - 1) = 0, \quad y(\lambda) = 0, \quad y(\lambda + 1) > 0.$$

Assume that insider uses λ in optimal strategy with positive probability. Then it is profitable for insider to move the weight from bid λ to bid $\lambda - 1$, his additional profit will be

$$x(\lambda)(y(0) + \ldots + y(\lambda - 2)) > 0.$$

It means that the strategy of insider in not optimal and *insider does't post λ in optimal strategy*:

$$x(\lambda) = 0.$$

Assume that insider doesn't use bid $\lambda+1$, i.e. $x(\lambda+1) = 0$. Then it is profitable for Player 2 to move the weight from bid $\lambda + 1$ to bid λ, his profit will be

$$y(\lambda + 1)(x(0) + \ldots + x(\lambda - 1)) > 0.$$

It means that the strategy of Player 2 in not optimal and *insider proposes bid* $\lambda + 1$ *with positive probability*:

$$x(\lambda + 1) > 0.$$

It is not profitable for insider to use $\lambda - 1$ instead $\lambda + 1$, that implies

$$A^m(\lambda + 1, y) \geq A^m(\lambda - 1, y).$$

From this inequality we obtain the estimate

$$y(\lambda + 1) \geq \frac{2}{m - (\lambda - 1)} - \frac{(y(\lambda + 2) + \ldots + y(k)) \cdot 2}{m - (\lambda - 1)}. \tag{5}$$

And it is not profitable to deviate from $\lambda+1$ to $\lambda+2$ (here we use the fact that $\lambda + 1 < m - 1$, i.e. the lacuna is not before the bid $m - 1$), so

$$A^m(\lambda + 1, y) \geq A^m(\lambda + 2, y).$$

And we obtain an estimate on $y(\lambda + 1)$:

$$y(\lambda + 1) \leq \frac{1}{m - (\lambda + 1)} - \frac{(y(\lambda + 2) + \ldots + y(k))}{m - (\lambda + 1)} - \frac{y(\lambda + 2)(m - (\lambda + 2))}{m - (\lambda + 1)}. \tag{6}$$

Comparing (5) and (6) we get

$$\frac{1}{m-(\lambda+1)} \geq (y(\lambda+2)+\ldots+y(k))\left(\frac{1}{m-(\lambda+1)}-\frac{2}{m-(\lambda-1)}\right)+$$
$$+\frac{2}{m-(\lambda-1)}+\frac{y(\lambda+2)(m-(\lambda+2))}{m-(\lambda+1)}.$$

Using natural conditions one have $y(\lambda+2)+\ldots+y(k) < 1$, $\frac{1}{m-(\lambda+1)} - \frac{2}{m-(\lambda-1)} < 0$:

$$\frac{1}{m-(\lambda+1)} > \frac{1}{m-(\lambda+1)} + \frac{y(\lambda+2)(m-(\lambda+2))}{m-(\lambda+1)},$$

$$0 > y(\lambda+2).$$

Contradiction.
The theorem is proved for Player 2.

II. The proof for insider is similar. □

The example of optimal strategies with lacunas in spectrums is presented on Figure 2.

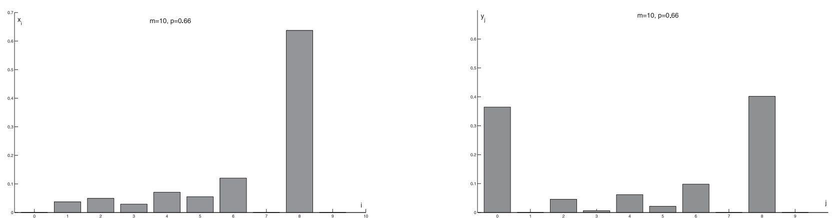

Figure2: Weights of bids of Player 1 and 2 respectively for $m = 10$, $p = 0.66$

5. Procedure of computing of game value

Consider the function $V_1^m(p) = v_k(p)$ of game value. If probability p is not a peak point of function $V_1^m(p)$ then the game value (it coincides with the equalizing value) can be determined the unique way by spectrums of optimal strategies of both players at this probability p. So the following notation takes place

$$v_k(p) = v_k(\operatorname{Spec} x, \operatorname{Spec} y),$$

k is a maximal bid in $\operatorname{Spec} x$.

Consider strategies of players for small probabilities of high share price.

1) Consider p near 0. The game with reduced matrix $A^m(p)$ has an equilibrium situation in pure strategies. The optimal strategy of Player 2 is proposing 0, ones of insider is proposing 1. The vector payoff of insider is

$$v_1^H = m - 1,$$

$$v_1^L = 0.$$

The game value is

$$v_1 = (m-1)p.$$

2) Let p increase. Then players pass from pure strategies to mixed strategies. Insider adds the bid 2 to the set of his bids used. Uninformed player adds the bid 1 with such weight that equalize the gain of insider's bids 1 and 2. Therefore the optimal strategy of Player 2 is mixed strategy with weights

$$y = \left(\frac{m-2}{m-1}, \frac{1}{m-1}, 0, 0, \ldots, 0\right),$$

that means $\operatorname{Spec} y = \{0,\, 1\}$.
The vector payoff of insider and the game value are the following

$$v_2^H = m-2,$$

$$v_2^L = \frac{1}{m-1},$$

$$v_2(p) = (m-2)p + \frac{1}{m-1}(1-p).$$

Hence we find the first peak point p_1 of function $v_k = v(p)$. It's enough to equate the game values from cases 1) and 2):

$$(m-1)p = (m-2)p + \frac{1}{m-1}(1-p),$$

from here

$$1 - p_1 = \frac{m-1}{m}.$$

3) Let p increases a little more. Then Player 2 adds the bid 2 in the spectrum of bids. By property 3 he can't use the bids 1 and 2 in optimal strategy at the same time, so he misses the bid 1. In fact it's the other way to equalize the spectrum of insider $\operatorname{Spec} x = \{1,\, 2\}$.
The optimal strategy of Player 2 is

$$y = \left(\frac{m-2}{m-1}, 0, \frac{1}{m-1}, 0, \ldots, 0\right).$$

The vector payoff of insider and the game value are the following

$$v_2^H = \frac{(m-2)^2}{m-1},$$

$$v_2^L = \frac{2}{m-1},$$

$$v_2 = \frac{(m-2)^2}{m-1}p + \frac{2}{m-1}(1-p).$$

The second peak point p_2 (when strategy changes from 2) to 3)) admits

$$1 - p_2 = \frac{m-2}{m-1}.$$

Suppose that we know equalizing strategies of Player 2 with all possible spectrums with the maximal bid less than k. Describe in this terms an optimal strategies when insider adds the bid k.

In section 4. we described the properties of spectrums of optimal strategies of both players. As optimal strategies are equalizing we are interested in equalizing strategies with spectrums satisfying properties 1, 2, 3.

Define the operation of *discarding the last* (maximal) *bid* in the strategy of uninformed player by the following way: if the strategy y has a spectrum $\operatorname{Spec} y$ with maximal bid k (for the sake of definiteness we add index k to strategy: y_k) then after discarding the last bid the new strategy (denote it by $\bar{y}_{k-1}$) has a spectrum $\operatorname{Spec} \bar{y}_{k-1} = \operatorname{Spec} y_k \setminus \{k\}$ and weights $\bar{y}_{k-1}(j)$ in strategy $\bar{y}_{k-1}$ are proportional to corresponding weights in initial strategy. Let's introduce the proportionality factor α_k by formulae

$$y_k(j) = \bar{y}_{k-1}(j) \cdot \alpha_k, \quad j = 0, \dots, k-1.$$

Hence

$$y_k(k) = 1 - \alpha_k.$$

5.1. The situation when maximal bid is less than $m-1$

1) Consider the case when maximal bids in spectrums of optimal strategies of both players coincides and equal to k ($k < m-1$) in detail. Assume that insider uses the bid $k-1$.

Denote by i the pure strategy of player proposing the bid i.

Player 2 equalizes the spectrum of bids of Player 1. The condition of equalization in matrix terms is the following

$$(iA^m(p), (y_k)^t) = v_k, \quad i = 1, \dots, k.$$

(we equalize only the rows according to bids from insider's spectrum) Here v_k is, technically speaking, not a game value, it's an equalization value.

As all rows in matrix $A^{L,m}(p)$ are the same let's rewrite the system above the following way

$$(iA^{H,m}(p), (y_k)^t) = v_k^H, \tag{7}$$

in $A^{H,m}(p)$ only rows and columns corresponding to spectrums elements are used.

Note that it is enough for Player 2 to use bids less than k to equalize insider's spectrum with maximal bid less then k. In other words the k's column in matrix doesn't influence the equalizing a "small" spectrum of insider $\operatorname{Spec} x_k \setminus \{k\}$. So we can discard the last bid in insider's strategy and proceed to consider the strategy $\bar{y}_{k-1}$.

The strategy $\bar{y}_{k-1}$ equalizes the spectrum of insider

$$\operatorname{Spec} x_{k-1} = \operatorname{Spec} x_k \setminus \{k\},$$

with equalization value $v_{k-1} = v_{k-1}^H p + v_{k-1}^L (1-p)$.

The system (7) can be reduced to

$$\begin{cases} v_{k-1}^H \alpha_k + (-m+k) y_k(k) = v_k^H, \\ (m-k)\alpha_k = v_k^H, \\ y_k(k) = 1 - \alpha_k. \end{cases} \tag{8}$$

If one knows v^H_{k-1} for all available combinations of spectrums than he can easily solve the system to obtain the recurrent relation for v^H_k:

$$\alpha_k = \frac{m-k}{v^H_{k-1}},$$

$$y_k(k) = 1 - \frac{m-k}{v^H_{k-1}},$$

$$v^H_k = \frac{(m-k)^2}{v^H_{k-1}},$$

$$v^L_k = (v^L_{k-1} - k)\frac{m-k}{v^H_{k-1}} + k.$$

Obtaining the proportionality factor α_k and the maximal bid weight $y_k(k)$, one constructs the equalizing strategy of Player 2.

The equalization value is

$$v_k(p) = \frac{(m-k)^2}{v^H_{k-1}} \cdot p + \left((v^L_{k-1} - k)\frac{m-k}{v^H_{k-1}} + k\right) \cdot (1-p).$$

2) Here we consider the case when maximal bids in spectrums of optimal strategies of both players equal k, but both players don't use the bid $k-1$.

Forbidding insider to post k and applying the similar reasoning we obtain recurrent relation of the second order:

$$y_k(j) = \bar{y}_{k-2}(j) \cdot \alpha_k, \quad i = 1, \ldots, k-2,$$

$$y_k(k-1) = 0,$$

$$\alpha_k = \frac{m-k}{v^H_{k-2}},$$

$$y_k(k) = 1 - \frac{m-k}{v^H_{k-2}},$$

$$v^H_k = \frac{(m-k)^2}{v^H_{k-2}},$$

$$v^L_k = (v^L_{k-2} - k)\frac{m-k}{v^H_{k-2}} + k,$$

$$v_k(p) = \frac{(m-k)^2}{v^H_{k-2}} \cdot p + \left((v^L_{k-2} - k)\frac{m-k}{v^H_{k-2}} + k\right) \cdot (1-p).$$

3) One more possible case of spectrum structure is the following: maximal bid of insider is k and maximal bid of uninformed player is $k-1$. Moreover there can be lacunas before the maximal bids (their length is not greater than 1). Analysis needed is almost the same as in previous cases.

The mechanism of recursion. *Define the number d by the structure of spectrums of strategies of both players so that the last d bids of uninformed player don't influence the equalizing a "shortened" spectrum of bids of insider*

$$\operatorname{Spec} x \setminus \{k, k-1, \ldots, k-(d-1)\}.$$

Let d is minimal ones of all possible.

The equalization value on "short" spectrums of players is v^H_{k-d}. Weights of the last d bids of uninformed player are determined the unique way by conditions of equalizing rows $k, k-1, \ldots, k-(d-1)$.

Because of properties 1 - 3 of spectrums the maximal depth of recursion d is not more than 3.

5.2. The situation when players use the bid $m-1$

It follows from property 3 that strategies containing bid $m-1$ have the following structure: they can contain a large lacuna (with length more than 1) before the bid $m-1$, for example from the bid $\tilde{k}$, but there are no lacunas with length more than 1 in interval of bids from 0 to $\tilde{k}$.

Using the notation for weighs of strategies we conclude (for insider) that there exist $\tilde{k} < m-1$ such that

$$x_{m-1}(m-1) > 0, \quad x_{m-1}(m-2) = x_{m-1}(m-3) = \ldots = x_{m-1}(\tilde{k}+1) = 0,$$

$$x_{m-1}(\tilde{k}) > 0, \quad x_{m-1}(j) \geq 0 \quad j = 1, 2, \ldots, \tilde{k}-1.$$

Let the maximal bid less than $m-1$ of uninformed player be $\tilde{l}$. Then applying the similar reasoning as in property 1 one can infer the following relation:

$$\tilde{l} = \tilde{k}, \text{ or } \tilde{l} = \tilde{k} - 1.$$

Denote the strategy x_{m-1} of insider (with maximal bid $m-1$) by $\tilde{x}_k$. Denote the spectrum of strategy $\tilde{x}_k$ by

$$\operatorname{Spec} \tilde{x}_k = \operatorname{Spec} x_{\tilde{k}} \cup \{m-1\}.$$

We use the same notation for Player 2.

Denote the game value at situation when insider uses the optimal strategy with spectrum $\operatorname{Spec} \tilde{x}_k$ by $\tilde{v}_k$.

Write down the condition of Player 1 optimal strategy spectrum equalization by Player 2:

$$(iA^m(p), (\tilde{y}_l)^t) = \tilde{v}_k, \quad i \in \operatorname{Spec} \tilde{x}_k.$$

It follows from the structure of matrix that the bid $m-1$ of Player 2 does'n influence the equalization of insider's strategy without the last bid $\operatorname{Spec} x_{\tilde{k}}$.

Similarly to the case $k < m-1$ above we introduce the proportionality factor and use the equalization value for strategies with maximal bid equals to $\tilde{k}$. This way we obtain the recurrent formulae on the game value:

$$\tilde{v}^H_k = \frac{1}{v^H_{\tilde{k}}},$$

$$\tilde{v}^L_k = (v^L_{\tilde{k}} - m + 1) \cdot \frac{1}{v^H_{\tilde{k}}} + m - 1,$$

$$\tilde{v}_k = \frac{1}{v^H_{\tilde{k}}} \cdot p + \left((v^L_{\tilde{k}} - m + 1) \cdot \frac{1}{v^H_{\tilde{k}}} + m - 1 \right) \cdot (1-p).$$

Consider a special case when $\tilde{k}$ doesn't exist that means that the optimal strategies of both players are pure strategies – the bid $m-1$. It occurs when probability p is near 1 namely $p \in [\frac{m-1}{m}, 1]$.

When recursion goes to step 2 we use the initial conditions (were found above). As a result we obtain the weights for case when players use strategies with maximal spectrums. As for $k = 2$ where one has two families of equalizing strategies of Player 2 at each step of recursion we get two families of equalizing strategies with maximal spectrums.

The number of possible combinations of spectrums of equalizing strategies is limited by properties 1, 2, 3. Therefore the number of equalizing strategies for each probability p (and so for domains of linearity of function $V_1^m(p)$) is limited. Choose as an optimal strategy of Player 2 the one that gives to insider minimal guaranteed benefit. The corresponding equalizing value is a game value for this p.

We have in mind in this case the following. As the same spectrum of insider can be equalized not uniquely by Player 2, he must choose the strategy that gives the minimal benefit to insider. In this case insider chooses the strategy with such spectrum to make this minimum the biggest from all possible minimums. Therefore the choice is realized on base of min-max theorem applying to the "small" set of equalizing strategies.

6. Peak points of function $V_1^m(p)$

It the property 2 it was established that if players use maximal possible bids then Player 2 can equalize the spectrum of Player 1 in two ways: using all bids excluding the bid 1 or 2. Denote these equalizing strategies of Player 2 by y_k^1 (if the bid 1 is used) y_k^2 (if the bid 2 is used), k is a maximal bid in spectrums of both players.

We obtain two families of equalizing strategies. Denote the probabilities when the strategies y_k^1 and y_k^2 interchange (for $k < m-1$) by p_k, $\{p_k\} = P^m$. Denote the probabilities when the strategies $\tilde{y}_k^1$ and $\tilde{y}_k^2$ interchange by q_k, $\{q_k\} = Q^m$.

One can explicitly compute this probabilities. In section 5. were compute p_1 and p_2. For $k > 2$ the recurrent formula holds:

$$1 - p_1 = \frac{m-1}{m}, \quad 1 - p_2 = \frac{m-2}{m-1}, \quad 1 - p_k = (1 - p_{k-2}) \frac{m-k}{m-k+1}.$$

The similar formulae for the set Q_m:

$$1 - q_1 = \frac{1}{m}, \quad 1 - q_2 = \frac{1}{m-1}, \quad 1 - q_k = (1 - p_{k-2}) \frac{1}{m-k+1}.$$

These formulae determine the same families of probabilities which were obtained in the work of Kreps, 2009a. These families have properties established in the mentioned paper. Here we adduce these properties.

Theorem 2. *For $p \in P^m \bigcup Q^m$ the value of the game with admissible integer bids coincides with the value of the game with arbitrary bids (De Meyer's model):*

$$V_1^m(p) = m \cdot p \cdot (1-p) \quad \textit{for all } p \in P^m \bigcup Q^m.$$

Theorem 3. *The set $P^m \bigcup Q^m$ becomes everywhere dense over $[0, 1]$ as $m \to \infty$. One has as corollary:*

$$\lim_{m\to\infty} V_1^m(p)/m = p(1-p).$$

So the probabilities from the set $P^m \bigcup Q^m$ have a marvelous property that at these points there are no difference for players to play the game with real or integer bids. At other points p insider prefers to play the game with arbitrary bids because in this game he has the greater freedom of action that guarantees him the greater benefit.

7. On the optimal strategy of insider

Consider the strategy x_k of insider (with maximal bid k in it's spectrum). Denote the distribution function of weights in strategy x_k by F_k:

$$F_k(i) = \sum_{j=1}^{i} x_k(j), \quad F_k(k) = 1,$$

$$y(k) > 0, \quad y(k+1) = \ldots = y(m-1) = 0.$$

In section 5. for fixed p we found the optimal strategy y of uninformed player (so $\operatorname{Spec} y$ was found too) and the game value $v_k(p)$. Withal the spectrum $\operatorname{Spec} x$ of optimal strategy x of insider at state H was found.

Strategy of insider equalizes active bids of Player 2. If $j-1$ and j are active bids of Player 2 then the condition of equalization is in the form of second-order difference equation:

$$F_k(j-2) - \frac{m-j}{m-j+1} F_k(j) + \frac{1}{m-j+1} \cdot \frac{1-p}{p} = 0.$$

If the bids $j-2$ and j are active, but Player 2 misses bid $j-1$, then one should use the condition of "gluing"

$$(x_k \cdot A^m(p), (j)^t) = (x_k \cdot A^m(p), (j-2)^t) \quad (= v_k(p)).$$

From this it follows the third-order difference equation on the cumulative distribution function:

$$F_k(j-3)(m-j) + (F_k(j-2) - F_k(j+1))(m-j+1) - F_k(j)(m-j) + 2\frac{1-p}{p} = 0.$$

In the case of $k = m-1$ the similar condition is given for the bids $m-1$ and $\tilde{k}$.

Another conditions on F_k follows from the analysis of $\operatorname{Spec} x$. If the bid i is not in $\operatorname{Spec} x$. then this gives the following simple condition on F_k:

$$F_k(i) = F_k(i-1).$$

So we obtain a sufficient number of independent linear equations to uniquely determine the distributions of weights in optimal strategy of insider. Solution of the system of equations gives the optimal strategy of insider.

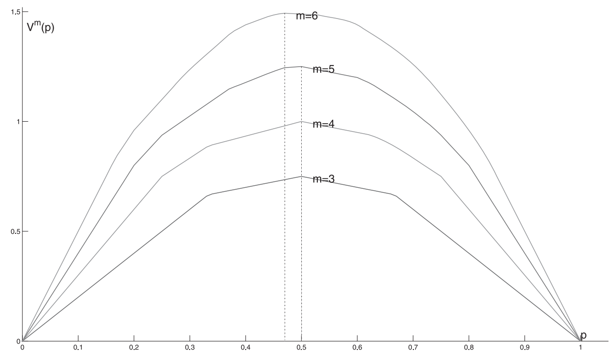

Figure3: The gains of insider in one-stage game for m from 3 till 6

8. The results of computer simulation

Here our aim is to investigate the game value which was described in section 5. by means of computer modeling. The function $V_1^m(p)$ for $m = 3, 4, 5$ and 6 is presented on a Figure 3.

By heuristic analysis of the game it seems to be natural that Player 2 has maximal uncertainty for $p = 1/2$. It means that insider gets the maximal gain at $p = 1/2$. Moreover, the result of continuous model of De Meyer and Saley confirms this idea. As it was shown in section **??** the value of continuous game is a quadratic function with it's maximum at the point $p = 1/2$:

$$V^m(p) = m \cdot p\,(1-p).$$

In discrete model for small m ($m \leq 5$) the maximum of the game value function is observed at the point $p = 1/2$. But starting with $m = 6$ the maximum of $V_1^m(p)$ shifts a little (it is shown dashed on Figure 3). It means that *the maximum of uncertainty is not obliged to be at* $p = 1/2$. It's the first counterintuitive property of the discrete model.

The second counterintuitive property is *observed dissymmetry of the dame value with respect to the probability* 1/2. It means that in the general case the game values at points p and $1-p$ aren't equal.

One more nontrivial effect of the model is *possibility of lacunas in spectrums of optimal strategies*. In the continuous model there are no any lacunas in the interval between minimal and maximal bids used.

9. Conclusion

The obtained results demonstrate that the discrete model possesses a number of specific characteristics that distinguish it from the continuous models. Hence the methods of analysis of discrete models are fundamentally different. The developed recurrent approach demonstrates complexity and originality of solving this class of models.

Acknowledgments The authors express their gratitude to V. Kreps and F. Sandomirskii for useful discussions on the subjects.

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