REMARKS ON CURVATURE IN THE TRANSPORTATION METRIC

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Abstract. According to a classical result of E. Calabi any hyperbolic affine hypersphere endowed with its natural Hessian metric has a non-positive Ricci tensor. The affine hyperspheres can be described as the level sets of solutions to the "hyperbolic" toric Kähler-Einstein equation $e^{\Phi} = \det D^2\Phi$ on proper convex cones. We prove a generalization of this theorem showing that for every Φ solving this equation on a proper convex domain Ω the corresponding metric measure space $(D^2\Phi, e^{\Phi}dx)$ has a non-positive Bakry-Émery tensor. Modifying the Calabi's computations we obtain this result by applying tensorial maximum principle to the weighted Laplacian of the Bakry-Émery tensor. All of the computations are carried out in the generalized framework adapted to the optimal transportation problem for arbitrary target and source measures. For the optimal transportation of probability measures we prove a third-order uniform dimension-free a priori estimate in spirit of the second-order Caffarelli's contraction theorem.

1. Introduction

The toric Kähler-Einstein equation

$$e^{-\alpha\Phi} = \det D^2\Phi \tag{1.1}$$

for a convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a real analog of the complex Monge-Ampère equation, which is instrumental in the theory of Kählerian manifolds. Here, $D^2\Phi$ is the Hessian of the function Φ. The equation (1.1) is also connected to convex geometry. According to the standard classification, we distinguish between the parabolic case $\alpha = 0$, elliptic case $\alpha > 0$, and hyperbolic case $\alpha < 0$. The reader should be not confused by the fact that according to the standard PDE terminology, equation (1.1) is *always* (non-uniformly) elliptic.

Having in mind potential applications in analysis and probability we deal with the Monge-Ampère equation of a more general type

$$e^{-V} = e^{-W(\nabla\Phi)} \det D^2\Phi. \tag{1.2}$$

When the functions e^{-V}, e^{-W} are densities of finite (probability) measures, the equation (1.2) is related to the optimal transportation problem ([3], [24]) and can be analyzed using various functional-analytical and measure-theoretical methods. The measures $\mu = e^{-V}\,dx$, $\nu = e^{-W}dx$ are called source and target measures respectively. However, here we are interested in situations when the associated measures are not always finite. An example is given by the hyperbolic case of (1.1), i.e. $\alpha < 0$, where

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$\mu = e^{-\alpha\Phi}dx$ and ν is the Lebesgue measure on the set $\nabla\Phi(\mathbb{R}^n)$ which is typically unbounded.

In this paper we extend the celebrated Calabi's approach for affine spheres and compute applications of a second-order differential operator to various geometric quantities. Some on them can be estimated with the help of the maximum principle. These computations, yet elementary, are tedious and admit non-trivial geometrical interpretation. We try to present them in the most simple but general form, sometimes using associated diagrams. In our opinion the most natural object to study is the so-called metric measure space

$$(h = D^2\Phi, \mu),$$

i.e. the space $\mathbb{R}^n$ (or a subset of it) equipped with the Hessian metric

$$h = \sum_{i,j=1}^{n} \Phi_{ij} dx^i dx^j$$

and the mesure $\mu = e^{-V}dx$. The reader can find explanations justifying this viewpoint and motivating applications in [12], [13], [14], [15],[19]. The corresponding second-order differential operator L is the weighted Laplacian

$$L = \Delta_h - \nabla_h P \cdot \nabla_h,$$

where ∇_h is the Riemannian gradient, Δ_h is the Riemannian (Laplace-Beltrami) Laplacian, and P is the potential of μ with respect to the Riemannian volume:

$$e^{-V} = e^{-P}\sqrt{\det D^2\Phi}.$$

Given a tensor T of any type one can always compute the corresponding weighted Laplacian

$$LT = \Phi^{pq}\Big(\nabla_p\nabla_q T - \nabla_p P \nabla_q T\Big).$$

Our central technical result is the exact expression of LT for several important tensors T and arbitrary measures μ and ν. Our computations imply, in particular, that for Φ solving (1.1) the following differential inequality holds:

$$L\big(\mathrm{Ric}_\mu\big) \geq 4\mathrm{Ric}_\mu \odot g, \tag{1.3}$$

where $\odot$ is symmetric product (see below), where

$$\mathrm{Ric}_\mu = \mathrm{Ric} + \nabla_h^2 P$$

is the Bakry-Émery tensor of (h, μ), Ric is the standard Ricci tensor of h, and

$$g_{ij} = \Phi_{iab}\Phi_j^{ab}.$$

The latter tensor was introduced by Calabi in [5]. It consitutes the most substantial (non-negative) part of the Bakry-Émery tensor. Applying the maximum principle we obtain from (1.3) that the largest eigenvalue of Ric_μ is non-positive at its maximum point. Therefore:

Theorem 1.1. *Consider a proper, open convex domain $\Omega \subset \mathbb{R}^n$. Let α be a negative number and Φ is the (unique) solution to (1.1) on Ω satisfying $\lim_{x\to\partial\Omega}\Phi(x) = +\infty$. Then*

$$\mathrm{Ric}_\mu \leq 0.$$

In the case when Ω is a proper convex cone the unique solution Φ to (1.1) must be logarithmically-homogenous, i.e., $\Phi(\lambda x) = \Phi(x) + 2(n/\alpha) \log \lambda$ for any $\lambda > 0$. It can be easily verified (see Lemma 5.2) that every logarithmically-homogeneous $\Phi = 2P$ satisfies

$$\nabla_h^2 \Phi = 0.$$

In particular, the Ricci and Bakry-Émery tensors coincide in this case and Theorem 1.1 immediately implies

Corollary 1.2. *(Calabi [5], see also Fox [12], Loftin [21] and Sasaki [23]). Consider a proper open convex cone $\Omega \subset \mathbb{R}^n$. Let α be a negative number and Φ is the (unique) solution to (1.1) on Ω satisfying $\lim_{x \to \partial\Omega} \Phi(x) = +\infty$. Then*

$$\mathrm{Ric} \le 0$$

and moreover the Ricci tensors of the level sets of Φ (hyperbolic affine spheres) endowed with the metric h are non-positive.

Using the same approach we prove third-derivatives analog of the so-called contraction theorem (L. Caffarelli). According to this theorem every (see more general statement in Section 5) optimal transportation mapping $\nabla \Phi$ pushing forward the standard Gaussian measure $\mu = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{x^2}{2}} dx$ onto probability measure $\nu = e^{-W} dx$ is a contraction provided $D^2 W \ge \mathrm{Id}$. This result is important in probability theory because it implies that the isoperimetric properties of μ are comparable (not worse) with the isoperimetric properties of γ. Contractivity of the optimal mappings corresponds to the following uniform estimate

$$D^2 \Phi(x) \le \mathrm{Id}, \ \forall x \in \mathbb{R}^n.$$

We prove that a similar estimate (global, dimension free, uniform) for 3^{rd}-order derivatives of Φ holds under natural assumptions for the second and third-order derivatives of V and W.

2. Notations, definitions, and previously known results

It will be assumed throughout that we are given a smooth (at least C^5) convex function Φ on $\mathbb{R}^n$ such that its gradient $\nabla \Phi$ pushes forward the measure

$$\mu = e^{-V} dx$$

onto the measure

$$\nu = e^{-W} dx.$$

The potentials V, W are assumed to be sufficiently regular (at least C^3). Equivalently, Φ solves the corresponding Monge-Ampére equation (1.2).

Sometimes we assume that μ is supported on a convex domain Ω. The domain Ω is called proper if it does not contain a complete affine line.

The function Φ can arise as a solution to any of the following problems:

(1) Optimal transportation problem: given probability measures μ, ν find the (unique up to a set of μ-measure zero) function Φ solving (1.2). We refer to [3], [24], where the reader can find comprehensive information about the solvability, uniqueness, and regularity issues.

(2) Given a probability measure ν satisfying $\int x d\nu = 0$ find a solution to the elliptic Kähler-Einstein equation
$$e^{-\Phi} = e^{-W(\nabla\Phi)} \det D^2\Phi.$$
The existence and uniqueness results are presented in [2, 7, 25].

(3) Given a proper open convex domain $\Omega \subset \mathbb{R}^n$ find a solution to the hyperbolic Kähler-Einstein equation
$$e^{\Phi} = \det D^2\Phi. \tag{2.1}$$
on Ω which satisfies $\lim_{x\to\partial\Omega}\Phi(x) = +\infty$.

The existence and uniqueness of a solution to (2.1) was obtained by Cheng and Yau, who continued the investigations of Calabi and Nirenberg. The formulation below is taken from [12].

Theorem 2.1. *(S.Y. Cheng, S.T. Yau,* [8], [9], [10]*) For every proper open convex domain $\Omega \subset \mathbb{R}^n$ there exists a unique convex function Φ solving (2.1) and satisfying* $\lim_{x\to\partial\Omega}\Phi(x) = +\infty$. *The Riemannian metric $h = D^2\Phi$ is complete on Ω.*

We assume throughout that we are given the standard Euclidean coordinate system $\{x^i\}$.The interior of $\Omega = \mathrm{supp}(\mu)$ is equipped with metric
$$h = h_{ij}dx^i dx^j = \Phi_{ij}dx^i dx^j = (\partial^2_{x_i x_j}\Phi)dx^i dx^j$$
and with the measure μ. The Legendre transform
$$\Psi(y) = \sup_{x\in\Omega}\big(\langle x, y\rangle - \Phi(x)\big)$$
defines the dual convex potential Ψ, satisfying $\nabla\Phi\circ\nabla\Psi(y) = y$ and pushing forward ν onto μ.

We give below a list of useful computational formulas, the reader can find the proof in [19]. It is convenient to use the following notation:
$$V_i = \partial_{x_i}V, \ V_{ij} = \partial^2_{x_i x_j}V, \ V_{ijk} = \partial^3_{x_i x_j x_k}V$$
$$W^i = (\partial_{x_i}W)\circ\nabla\Phi, \ W^{ij} = (\partial^2_{x_i x_j}W)\circ\nabla\Phi, \ W^{ijk} = (\partial^3_{x_i x_j x_k}W)\circ\nabla\Phi.$$
We follow the standard conventions of Riemannian geometry (i.e., Φ^{ij} is inverse to Φ_{ij}, Einstein summation, raising indices etc.).

The measure μ has the following density with respect to the Riemannian volume
$$\mu = e^{-P}dvol_g, \quad P = \frac{1}{2}\big(V + W(\nabla\Phi)\big).$$

The associated diffusion generator (weighted Laplacian) L has the form
$$Lf = \Phi^{ij}f_{ij} - W^i f_i = \Delta_h f - \frac{1}{2}(V^i + W^i)f_i,$$
where Δ_h is the Riemannian Laplacian.

The following non-negative symmetric tensor g plays prominent role in our analysis
$$g_{ij} = \Phi_{iab}\Phi^{ab}_j.$$

In order to distinguish between the (weighted) Laplacians of a tensor T and the Laplacian of its component in the fixed Euclidean coordinate system we use for the latter the square brackets. For instance $(LT)_{ij}$ will denote the Laplacian of the $(0,2)$-tensor T and
$$L[T_{ij}]$$

denotes the Laplacian of the scalar function T_{ij} with fixed indices i, j. The proof of the following lemma can be found in [19].

Lemma 2.2. *The weighted Laplacians of the partial derivatives of Φ for fixed i, j, k satisfy the following relations:*

$$L[\Phi_i] = -V_i = -W_i + \Phi^{kl}\Phi_{ikl}. \tag{2.2}$$

$$L[\Phi_{ij}] = -V_{ij} + W_{ij} + g_{ij} \tag{2.3}$$

$$\begin{aligned} L[\Phi_{ijk}] = & - V_{ijk} + W_{ijk} + \big(W_i^s\Phi_{sjk} + W_j^s\Phi_{sik} + W_k^s\Phi_{sij}\big) \\ & + \big(\Phi_{abi}\Phi^{ab}_{jk} + \Phi_{abj}\Phi^{ab}_{ik} + \Phi_{abk}\Phi^{ab}_{ij}\big) - 2\Phi^a_{bi}\Phi^b_{cj}\Phi^c_{ak}. \end{aligned} \tag{2.4}$$

Recall that for two tensors T_{ij}, S_{ij}, their symmetric product is defined as follows:

$$(T \odot S)_{ij} = \frac{1}{2}\big(T_{ik}S^k_j + T_{jk}S^k_i\big).$$

Finally, we give a list of formulas for the most important quantities.

(1) Connection

$$\Gamma^k_{ij} = \frac{1}{2}\Phi^k_{ij}.$$

(2) Hessian of f

$$\nabla^2_h f_{ij} = f_{ij} - \frac{1}{2}\Phi^k_{ij} f_k.$$

(3) Riemann tensor

$$\mathrm{R}_{ikjl} = \frac{1}{4}(\Phi_{ila}\Phi^a_{kj} - \Phi_{ija}\Phi^a_{kl}).$$

(4) Ricci tensor

$$\mathrm{Ric}_{ij} = \frac{1}{4}\Big(\Phi_{iab}\Phi^{ab}_j + \Phi_{ijk}(V^k - W^k)\Big) = \frac{1}{4}(g_{ij} + \Phi_{ijk}(V^k - W^k)).$$

(5) Bakry-Emery tensor

$$(\mathrm{Ric}_\mu)_{i,j} = \mathrm{Ric}_{ij} + \frac{1}{2}\nabla^2_h(V + W(\nabla\Phi))_{ij} = \frac{1}{4}g_{ij} + \frac{1}{2}V_{ij} + \frac{1}{2}W_{ij}.$$

3. Laplacians for tensors

This section is devoted to computations of the weighted Laplacian of several important tensors. We stress that in this section **we omit the subscript** h for the sake of simplicity, i.e. the symbols ∇, ∇^2 etc. are always related to the Hessian metric h, but not to Eudlidean metric.

Given a tensor T we define its Laplacian as follows:

$$\Delta T = \Phi^{pq}\nabla_p\nabla_q T.$$

Here $\nabla_p T$ is the covariant derivative, which means, in particular, that

$$\nabla_p\Phi_{ij} = 0, \ \Delta\Phi_{ij} = 0.$$

Similarly

$$LT = \Delta T - \frac{1}{2}(V^k + W^k)\nabla_k T.$$

Lemma 3.1. *Let*

$$f_i = \partial_{x_i} f$$

for some function f. Then

$$\nabla_p f_i = f_{ip} - \frac{1}{2}\Phi^k_{ip} f_k$$

$$\nabla_q \nabla_p f_i = \big(f_{ip} - \frac{1}{2}\Phi^k_{ip} f_k\big)_q - \frac{1}{2}\Phi^m_{qi}\big(f_{mp} - \frac{1}{2}\Phi^k_{mp} f_k\big) - \frac{1}{2}\Phi^m_{qp}\big(f_{mi} - \frac{1}{2}\Phi^k_{mi} f_k\big)$$

Taking the trace we get

$$\Delta f_i = \Phi^{pq}\big(f_{ip} - \frac{1}{2}\Phi^k_{ip} f_k\big)_q - \frac{1}{2}\Phi^{mp}_i\big(f_{mp} - \frac{1}{2}\Phi^k_{mp} f_k\big) + \frac{1}{2}(V^m - W^m)\big(f_{mi} - \frac{1}{2}\Phi^k_{mi} f_k\big)$$

Rearranging the terms we finally obtain using Lemma 2.2

$$\begin{aligned} L f_i &= \Phi^{pq}\big(f_{ipq} - \frac{1}{2}\Phi^k_{ipq} f_k + \frac{1}{2}\Phi^{km}_q \Phi_{imp} f_k - \frac{1}{2}\Phi^k_{ip} f_{kq}\big) - \frac{1}{2}\Phi^{mp}_i f_{mp} + \frac{1}{4} g^k_i f_k - W^m(f_{mi} - \frac{1}{2}\Phi^k_{mi} f_k) \\ &= L[f_i] + W^k f_{ik} - \frac{1}{2}(L[\Phi_{ik}] + W^p \Phi_{ikp}) f^k - \Phi^{mk}_i f_{mk} + \frac{3}{4} g^k_i f_k - W^m f_{mi} + \frac{1}{2} W^m \Phi^k_{mi} f_k \\ &= L[f_i] - \frac{1}{2} L[\Phi_{ik}] f^k + \frac{3}{4} g^k_i f_k - \Phi^{mk}_i f_{mk} = L[f_i] - \Phi^{mk}_i f_{mk} + \frac{1}{2}(V_{ik} - W_{ik}) f^k + \frac{1}{4} g^k_i f_k. \\ &= \Phi^{mk} f_{imk} - \Phi^{mk}_i f_{mk} - W_k f^k_i + \frac{1}{2}(V_{ik} - W_{ik}) f^k + \frac{1}{4} g^k_i f_k \end{aligned}$$

Corollary 3.2.

$$L\Phi_i = \frac{1}{2}(V_{ik} - W_{ik})\Phi^k + \frac{1}{4} g^k_i \Phi_k - W_i.$$

All the following calculations are essentially based on the next Lemma, which is obtained by direct computations with the help of Lemma 2.2. The computation is long and quite standard. See Section 4 for a graphical method for performing this computation relatively quickly.

Lemma 3.3.

$$\begin{aligned} L\Phi_{iab} &= -V_{iab} + W_{iab} + \frac{1}{2}\big((V^m_i + W^m_i)\Phi_{mab} + (V^m_a + W^m_a)\Phi_{mib} + (V^m_b + W^m_b)\Phi_{mia}\big) \\ &\quad - \frac{1}{2}\Phi^l_{ik}\Phi^m_{al}\Phi^k_{bm} + \frac{1}{4}\Big(g^k_i \Phi_{kab} + g^k_a \Phi_{kib} + g^k_b \Phi_{kia}\Big). \end{aligned}$$

Proposition 3.4.

$$\begin{aligned} L g_{ij} &= (-V_{iab} + W_{iab})\Phi^{ab}_j + (-V_{jab} + W_{jab})\Phi^{ab}_i \\ &\quad + \frac{1}{2}\Big((V_{is} + W_{is}) g^s_j + (V_{js} + W_{js}) g^s_i\Big) + 2(V_{am} + W_{am})\Phi^m_{ib}\Phi^{ab}_j \\ &\quad + \frac{1}{2} g_{ki} g^k_j + 2\nabla_p \Phi_{iab} \nabla^p \Phi^{ab}_j + 8\mathrm{R}_{iabc}\mathrm{R}^{abc}_j. \end{aligned}$$

In particular, if Φ satisfies (1.1) then

$$L g_{ij} = \big(g_{ki}(\mathrm{Ric}_\mu)^k_j + g_{kj}(\mathrm{Ric}_\mu)^k_i\big) + 2\nabla_p \Phi_{iab} \nabla^p \Phi^{ab}_j + 8\mathrm{R}_{iabc}\mathrm{R}^{abc}_j,$$

which implies

$$L g \geq 2 g \odot \mathrm{Ric}_\mu.$$

Proof. Applying L to $g_{ij} = \Phi_{iab}\Phi_j^{ab}$, one gets

$$L(g_{ij}) = (L\Phi_{iab})\Phi_j^{ab} + 2\nabla_p\Phi_{iab}\nabla^p\Phi_j^{ab} + \Phi_{iab}(L\Phi_j^{ab}).$$

Lemma 3.4 implies

$$(L\Phi_{iab})\Phi_j^{ab} = \Big[-V_{iab} + W_{iab} + \frac{1}{2}\big((V_i^m + W_i^m)\Phi_{mab} + (V_a^m + W_a^m)\Phi_{mib} + (V_b^m + W_b^m)\Phi_{mia}\big)\Big]\Phi_j^{ab}$$
$$- \frac{1}{2}\Phi_{ik}^l\Phi_{al}^m\Phi_{bm}^k\Phi_j^{ab} + \frac{1}{4}\Big(g_i^k\Phi_{kab} + g_a^k\Phi_{kib} + g_b^k\Phi_{kia}\Big)\Phi_j^{ab}.$$

The similar formula for $(L\Phi_{jab})\Phi_i^{ab}$ is obtained by interchanging i and j. Using the relations $g_i^k\Phi_{kab}\Phi_j^{ab} = g_i^k g_{kj}$ and $g_a^k\Phi_{kib}\Phi_j^{ab} - \Phi_{ik}^l\Phi_{al}^m\Phi_{bm}^k\Phi_j^{ab} = 8\mathrm{R}_{iabc}\mathrm{R}_j^{abc}$ one gets

$$(L\Phi_{iab})\Phi_j^{ab} = \Big[-V_{iab} + W_{iab} + \frac{1}{2}\big((V_i^m + W_i^m)\Phi_{mab} + (V_a^m + W_a^m)\Phi_{mib} + (V_b^m + W_b^m)\Phi_{mia}\big)\Big]\Phi_j^{ab}$$
$$+ \frac{1}{4}g_i^k g_{kj} + 4\mathrm{R}_{iabc}\mathrm{R}_j^{abc}$$

and the claim follows. □

4. Computations with diagrams

Some of the computations of the previous sections are rather tedious, and some of the formulas are not very pleasant to the eye. Consider, for example, the following expression from formula (2.4):

$$-V_{ijk}+W_{ijk}+\big(W_i^s\Phi_{sjk}+W_j^s\Phi_{sik}+W_k^s\Phi_{sij}\big)+\big(\Phi_{abi}\Phi_{jk}^{ab}+\Phi_{abj}\Phi_{ik}^{ab}+\Phi_{abk}\Phi_{ij}^{ab}\big)-2\Phi_{bi}^a\Phi_{cj}^b\Phi_{ak}^c.$$

We propose to replace it by the diagram in Figure 1.

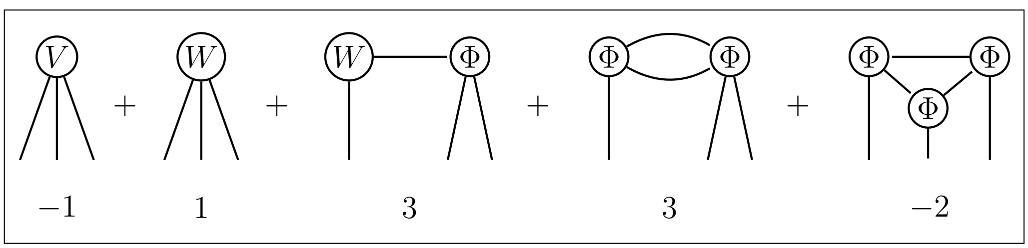

Figure 1

The diagram in Figure 1 is the weighted sum of five basic diagrams, the number below each basic diagram is its coefficient. A basic diagram D consists of a set of vertices $V = V(D)$ and two collections of edges $E_{int} = E_{int}(D)$ and $E_{ext} = E_{ext}(D)$. Each vertex is marked with a letter, which is usually either Φ, V or W. An internal edge $e \in E_{int}$ connects two vertices $x, y \in V$. An external edge is connected only to a single vertex. To each basic diagram D with $L = \#(E_{ext})$ there corresponds a symmetric $(0, L)$-tensor constructed via the following mechanism:

(1) Associate a new index with any edge. Assume that the indices associated with the external edges are $i_1, \dots, i_L$ and that those associated with internal edges are $i_{L+1}, \dots, i_{L+L'}$.
(2) Orient all of the internal edges in an arbitrary manner. For each vertex v, let v^{up} (respectively, v_{down}) be the set of all indices of edges arriving at v (respectively, emanating from v). Let v_{ext} be the set of all indices of external edges connected to v.

(3) For a vertex v write $S(v)$ for the letter with which it is marked. Write S_L for the collection of all permutations of $\{1,\ldots,L\}$. For a permutation $\sigma \in \sigma_L$ set $\sigma(v_{ext}) = \{i_{\sigma(j)}\,;\, i_j \in v_{ext}\}$.

(4) The resulting symmetric tensor is

$$\frac{1}{L!}\sum_{\sigma\in S_L}\prod_{v\in V} S(v)^{v^{up}}_{v_{down},\sigma(v_{ext})} dx^{i_1}\ldots dx^{i_L}. \tag{4.1}$$

We may also accommodate non-symmetric tensors, by marking the external edges of the diagram with the indices $i_1,\ldots,i_L$. In this case, the tensor corresponding to the diagram is constructed in the same way, except that in (4.1), we always take σ to be the identity permutation, i.e., there is no need for a sum and for the normalizing $1/L!$ factor. From our experience, after a bit of training it is easier to compute the symmetric contraction product, the covariant derivative and the weighted Laplacian of a tensor in terms of these diagrams.

We proceed to describe the contraction product of two basic diagrams, an example is presented in Figure 2.

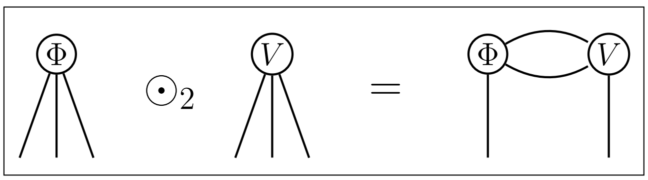

FIGURE 2. The tensor $\Phi_{ik\ell}V_j^{k\ell}$, which is the symmetric contraction product of Φ_{ijk} and V_{ijk}, where we contract two indices.

Formally, assume that we are given two basic diagrams D_1 and D_2. Set $L_i = \#(E_{ext}(D_i))$ for $i=1,2$, and assume that $L_1 \geq k$ and $L_2 \geq k$ where $k \geq 1$ is an integer. The symmetric contraction product $D_1 \odot_k D_2$ is described as follows:

(1) For each subset $A \subseteq E_{ext}(D_1)$ and $B \subseteq E_{ext}(D_2)$, both of size exactly k, and for each invertible map $f : A \to B$ we construct a basic diagram. The weight of this basic diagram is $1/n$, where $n = (L_1!L_2!)/(k!(L_1-k)!(L_2-k)!)$ is the number of all possible choices of A, B and f.

(2) The basic diagram corresponding to A, B and f, is the disjoint union of D_1 and D_2, except that for any $e \in A$, we replace the pair of edges $e \in A$ and $f(e) \in B$ by a single edge connecting the vertex of e and the vertex of $f(e)$.

The integer k in the notation $\odot_k$ is referred to as the order of the symmetric contraction product. Recall that the contraction of two indices in a given tensor corresponds to a symmetric contraction product with the tensor Φ_{ij}. We move on to the description of covariant differentiation. The covariant derivative of a symmetric tensor is not necessarily symmetric.

Here are the general rules for depicting the covariant derivative ∇_p of the basic diagram D:

(1) For each vertex v, we add a basic diagram D_v whose weight is $+1$. The basic diagram D_v is constructed from D by adding an external edge emanating from v and marked by p.

(2) For each internal edge e, we add a basic diagram D_e whose weight is -1. The basic diagram D_e is constructed from D by adding an external edge,

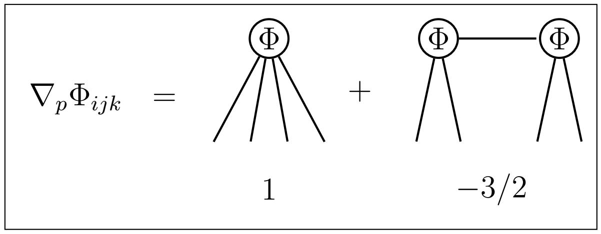

FIGURE 3. The covariant derivative $\nabla_p(\Phi_{ijk})$ has turned out to be a symmetric tensor.

marked by p, which is emanating from a new vertex, marked by Φ, in the middle of the edge e.

(3) For each external edge e we add a basic diagram whose weight is $-1/2$, which is constructed exactly as in the case of an internal edge.

An internal edge $e \in E_{int}$ is called a loop if it connects a vertex to itself. The Monge-Ampère equation (1.2) allows us to eliminate any loop which connects a vertex marked by Φ to itself. For example, by differentiating (1.2) we obtain

$$\Phi^j_{ji} = -V_i + W_i, \qquad \Phi^k_{ijk} = -V_{ij} + W_{ij} + \Phi^\ell_{ik}\Phi^k_{j\ell} + \Phi^k_{ij}W_k.$$

Thus, we may replace a Φ-vertex having a loop and additional k edges by a certain sum of basic diagrams. The rules for loop elimination in the case where $k = 1, 2, 3$ are depicted in Figure 4.

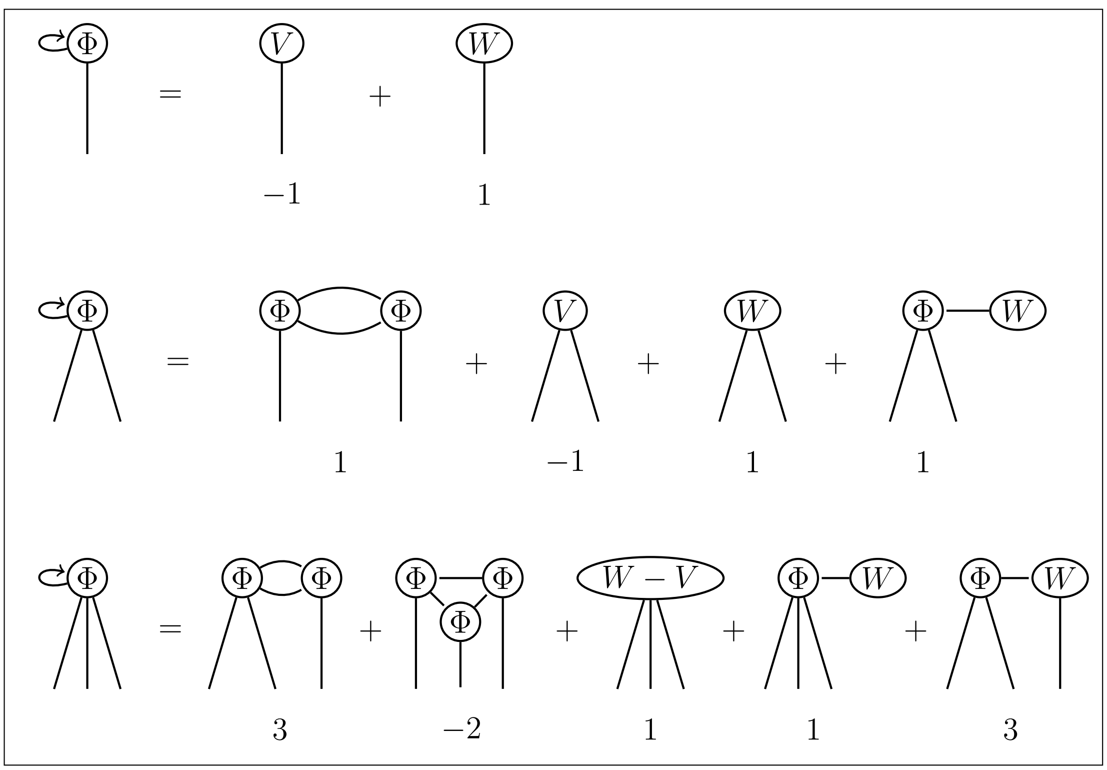

FIGURE 4. Loop elimination

Let us emphasize that when applying the loop elimination rules, we count all the edges related to the loop. For example, if a Φ-vertex has a loop, plus one external and one internal edge, then the second picture in Figure 4 is applicable. The combinatorics of loop elimination for an arbitrary $k \geq 4$ is not too complicated, but it will not be needed here. We move on to the weighted Laplacian L. Here are the rules for depicting the weighted Laplacian L of a basic diagram D:

(1) For $a \in V \cup E_{ext} \cup E_{int}$ we denote

$$w(a) = \begin{cases} 1 & a \in V \\ -1 & a \in E_{int} \\ -1/2 & a \in E_{ext} \end{cases}$$

(2) For any $a, b \in V \cup E_{ext} \cup E_{int}$ with $a \neq b$ we add a basic diagram $D_{a,b}$ whose weight is $w(a)w(b)$. The basic diagram $D_{a,b}$ is constructed from D by adding an internal edge connecting a and b. Note that when a is an edge, adding an internal edge introduces a new vertex, marked by Φ, in the middle of the edge a.

(3) For any $a \in V \cup E_{ext} \cup E_{int}$ we add a basic diagram whose weight is $w(a)$ which is obtained from D by adding a loop around a. Note that when a is an edge, we add a vertex marked by Φ in the middle of the edge a and a loop around this vertex.

(4) For any edge $e \in E_{ext} \cup E_{int}$ we add a basic diagram that is obtained from D by adding two vertices on the edge e and connecting one to the other via an internal edge. The weight of this basic diagram is 2 if $a \in E_{int}$ and is $3/4$ if $a \in E_{ext}$.

Note that the basic diagram $D_{a,b}$ in the first rule is the same as $D_{b,a}$, hence it appears twice in the resulting diagram. We urge the reader to verify the computation of $L\Phi_{ijk}$ in Figure 5 by applying the above rules. This establishes Lemma 3.3 in a relatively painless manner.

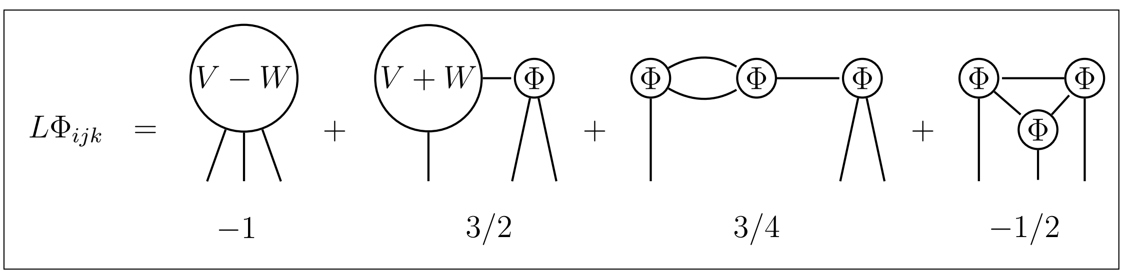

FIGURE 5

In order to verify Proposition 3.4, one needs to compute the symmetric contraction product of order two of the two tensors $L\Phi_{ijk}$ and Φ_{ijk}. Again, this appears easier to perform by using the diagrams, see Figure 6.

5. NEGATIVITY OF RIC_μ

We are ready to prove Theorem 1.1. The existence of the unique solution to (1.1) is established in Theorem 2.1. We will also apply the following lemma which is a generalization of some classical facts proved already in the works of Calabi [5] and Osserman [22]. Extension to the metric-measure space is based on the estimates for weighted Laplacians of the distance (see, for instance, [1]). A more general statement with the proof the reader can find in [12], Theorem 3.3.

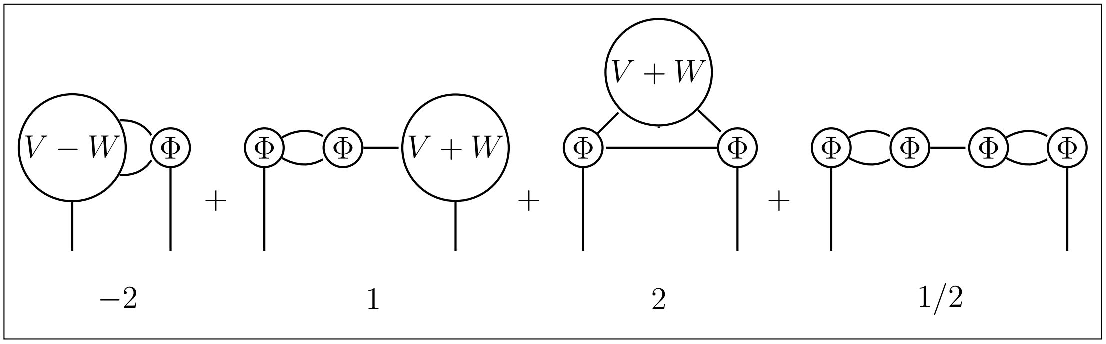

FIGURE 6. The tensor $Lg_{ij} - 2\nabla_p \Phi_{iab} \nabla^p \Phi_j^{ab} - 8R_{iabc}R_j^{abc}$

Lemma 5.1. *Let (M, g) be a complete Riemannian manifold equipped with a measure $\mu = e^{-V} dvol_g$ with twice continuously differentiable density. Assume that its generalized Bakry-Émery tensor $\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_\mu - \frac{1}{N-n}\nabla V \otimes \nabla V$ satisfies*

$$\mathrm{Ric}_{\mu,N} \geq K$$

for some $K \in \mathbb{R}$ and $N > n$. Let $u \in C^2(M)$ be a non-negative function satisfying

$$Lu \geq Bu^2 - Au$$

for some $A, B > 0$ at every point x with $u(x) \neq 0$, where $L = \Delta - \nabla V \cdot \nabla$ is the weighted Laplacian. Then

$$\sup_M u \leq \frac{A}{B}.$$

Proof of Theorem 1.1 According to Proposition 3.4

$$Lg \geq 2g \odot \mathrm{Ric}_\mu.$$

Taking into acount that $g = 4\mathrm{Ric}_\mu - 2\alpha h$, we can rewrite it as

$$L\mathrm{Ric}_\mu \geq 2\mathrm{Ric}_\mu^2 - \alpha \mathrm{Ric}_\mu.$$

Let us estimate $L\lambda$ for arbitrary point x_0, where λ is the largest eigenvalue of Ric_μ. One has $\lambda(x_0) = (\mathrm{Ric}_\mu)_{ij}(x_0)\eta^i\eta^j$ for some tangent unit vector η. Extend η in such a way that

$$\nabla\eta = 0, \Delta\eta = 0$$

at x_0 (see, for instance, Theorem 4.6 of [11]). Next we note that the function $\lambda - (\mathrm{Ric}_\mu)_{ij}\eta^i\eta^j$ is non-negative and equals to zero at x_0. Hence $L\lambda - L((\mathrm{Ric}_\mu)_{ij}\eta^i\eta^j) \geq 0$ at x_0. One obtains the following relation at x_0

$$L\lambda \geq L((\mathrm{Ric}_\mu)_{ij}\eta^i\eta^j) = L(\mathrm{Ric}_\mu)\eta^i\eta^j \geq 2\lambda^2 - \alpha\lambda.$$

We will apply Lemma 5.1. Note that the completeness of the space follows from Theorem 2.1. In addition, it is easy to check (see [19]) that tensor $\mathrm{Ric}_{\mu,2n}$ is non-negative, thus Lemma 5.1 is applicable. We note that $\tilde{\lambda} = \lambda - \frac{1}{2}\alpha$ is the largest eigemvalue of $\frac{1}{4}g$, hence nonnegative. The above inequality implies

$$L\tilde{\lambda} \geq 2\tilde{\lambda}^2 + \alpha\tilde{\lambda}.$$

By Lemma 5.1 $\tilde{\lambda} \leq -\frac{\alpha}{2}$, hence $\lambda \leq 0$.

5.1. **Cone case.** Let us analyse the case when Ω is a cone. Then Φ is logarithmically homogeneous. This implies, in particular, the following relation:

$$\Phi_i x^i = 2(n/\alpha)$$

Differentiating this relation twice we get

$$\Phi_{ij} x^j + \Phi_i = 0$$

$$2\Phi_{ij} + \Phi_{ijk} x^k = 0.$$

We obtain from the first identity $x^j = -\Phi^{ij}\Phi_i$. Substituting this into the second identity, we finally get

Lemma 5.2. *If Φ is logarithmically homogeneus (in particular, if Φ solves (1.1) in a cone for some negative α), then*

$$\nabla^2_h \Phi_{ij} = \Phi_{ij} - \frac{1}{2}\Phi^k \Phi_{ijk} = 0.$$

This implies

$$\nabla^2_h P = \frac{1}{2}\nabla^2_h \Phi_{ij} = 0$$

and

$$\mathrm{Ric} = \mathrm{Ric}_{\mu}.$$

provided W is constant.

Proof of Corollary 1.2. The statement $\mathrm{Ric} \le 0$ follows immediately from Theorem 1.1 and Lemma 5.2. To prove the statement for a level set $M = \{\Phi = c\}$ we use the following formula (see, for instance Lemma 7.1 in [20])

$$\mathrm{Ric}_M = \big(\mathrm{Ric} - \mathrm{R}(\cdot, \eta, \cdot, \eta)\big)|_{TM} + \big(Hh|_{TM} - II_M\big) II_M.$$

Here η is the unit normal to M, II_M is the second fundamental form of M, and H is mean curvature of M. Since $\nabla^2_h \Phi = 0$ and $M = \{\Phi = c\}$, necessarily $II_M = 0$. Note that $\eta = x/|x|_h$ and

$$4\mathrm{R}(i, x, j, x) = \Phi_{iax}\Phi^a_{jx} - \Phi_{axx}\Phi^a_{ij}.$$

Taking into account that $\Phi_{iax} = -2\Phi_{ia}$ we easily get $\mathrm{R}(i, x, j, x) = \nabla^2_h \Phi_{ij} = 0$. Hence $\mathrm{Ric}_M = \mathrm{Ric}|_{TM} \le 0$.

6. Application to log-concave measures

In this section we deal with probability measures

$$\mu = e^{-V} dx, \ \nu = e^{-W} dx$$

and the solution $T = \nabla \Phi$ of the corresponding optimal transportation problem. The functions V, W, and Φ are assumed to be sufficiently smooth (by the regularity theory for the Monge-Ampère equation the smoothness of Φ follows from the smoothness of the potentials under additional assumptions).

The contraction theorem of L. Caffarelli has numerous applications in probability and analysis. It can be stated in the following form (see [17], [18]): under the assumption

$$D^2 V \le C, \ D^2 W \ge c,$$

where c, C are positive constants, the potential Φ satisfies

$$D^2 \Phi \le \sqrt{\frac{C}{c}}.$$

In this section we prove a kind of extension of this result to the third-order derivatives.

Given a quadratic form Q we denote by $\|Q\|_h$ its Riemannian norm:

$$\|Q\|_h = \sup_{v:h(v,v)=1} Q(v,v).$$

Let us recall that

$$V_{ij} = \partial^2_{x_i x_j} V,\ V_{ijk} = \partial^3_{x_i x_j x_k} V$$

but

$$W_{ij} = \sum_{a,b} \Phi_{ai}\Phi_{bj} \cdot \partial^2_{x_b x_a} W,\ W_{ijk} = \sum_{a,b,c} \Phi_{ai}\Phi_{bj}\Phi_{ck} \cdot \partial^3_{x_a x_b x_c} W.$$

Proposition 6.1. *Assume that the matrix whose elements are $V_{ij} + W_{ij}$ is positive semi-definite (this holds, in particular, when both measures are log-concave). Then the following inequality holds*

$$\|H\|_h + 2L\|g\|_h \geq \|g\|_h^2 \tag{6.1}$$

where

$$H_{ij} = \mathrm{Tr}\Bigl[\bigl[(V+W)^{(2)}\bigr]^{-1}(V-W)^{(3)}_i (D^2\Phi)^{-1}(V-W)^{(3)}_j\Bigr],$$

$(V-W)^{(3)}_i$, $(V+W)^{(2)}$ are the matrices with the entries $V_{iab} - W_{iab}$, $V_{ab} + W_{ab}$ respectively.

Proof. With some abuse of notation, when we write $V_{ij} \geq 0$ we mean that the symmetric matrix $(V_{ij})^n_{i,j=1}$ is positive semi-definite. According to Proposition 3.4

$$\begin{aligned} Lg_{ij} &\geq (-V_{iab} + W_{iab})\Phi^{ab}_j + (-V_{jab} + W_{jab})\Phi^{ab}_i \\ &+ \frac{1}{2}\Bigl((V_{is} + W_{is})g^s_j + (V_{js} + W_{js})g^s_i\Bigr) + 2(V_{am} + W_{am})\Phi^m_{ib}\Phi^{ab}_j + \frac{1}{2}g_{ki}g^k_j. \end{aligned}$$

First we note that

$$2(V_{iab} - W_{iab})\Phi^{ab}_i = 2\mathrm{Tr}\Bigl[(D^2\Phi)^{-1}(V-W)^{(3)}_i (D^2\Phi)^{-1} D^2\Phi_{e_i}\Bigr], \tag{6.2}$$

$$2(V_{am} + W_{am})\Phi^m_{ib}\Phi^{ab}_i = 2\mathrm{Tr}\Bigl[(D^2\Phi)^{-1}(V+W)^{(2)}(D^2\Phi)^{-1}A_i\Bigr], \tag{6.3}$$

where A_i is the matrix with the entries $\Phi_{iac}\Phi^c_{ib}$. We apply the Cauchy inequality

$$4\mathrm{Tr}\bigl(XY\bigr) \leq 4\mathrm{Tr}\bigl(QX^2\bigr) + \mathrm{Tr}\bigl(Q^{-1}Y^2\bigr)$$

which is valid for non-negative symmetric matrices Q, X, Y. Setting

$$X = (D^2\Phi)^{-1/2}D^2\Phi_{e_i}(D^2\Phi)^{-1/2},\ Y = (D^2\Phi)^{-1/2}(V-W)^{(3)}_i(D^2\Phi)^{-1/2},$$
$$Q = (D^2\Phi)^{-1/2}(V+W)^{(2)}(D^2\Phi)^{-1/2}$$

one gets

$$\begin{aligned} &2\mathrm{Tr}\Bigl[(D^2\Phi)^{-1}(V-W)^{(3)}_i(D^2\Phi)^{-1}D^2\Phi_{e_i}\Bigr] \\ &\leq \frac{1}{2}\mathrm{Tr}\Bigl[\bigl[(D^2\Phi)^{-1/2}(V+W)^{(2)}(D^2\Phi)^{-1/2}\bigr]^{-1}\bigl((D^2\Phi)^{-1/2}(V-W)^{(3)}_i(D^2\Phi)^{-1/2}\bigr)^2\Bigr] \\ &+ 2\mathrm{Tr}\Bigl[\bigl[(D^2\Phi)^{-1/2}(V+W)^{(2)}(D^2\Phi)^{-1/2}\bigr]\bigl((D^2\Phi)^{-1/2}D^2\Phi_{e_i}(D^2\Phi)^{-1/2}\bigr)^2\Bigr]. \end{aligned}$$

Taking into account (6.2), (6.3) we rewrite this relation as follows:

$$2(V_{iab} - W_{iab})\Phi^{ab}_i \leq 2(V_{am} + W_{am})\Phi^m_{ib}\Phi^{ab}_i + \frac{1}{2}H_{ab}.$$

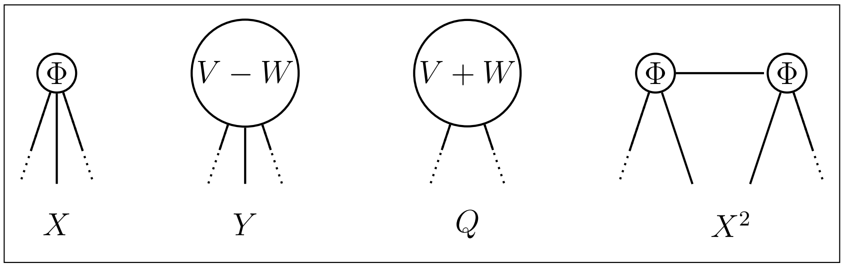

FIGURE 7. We let the reader guess the meaning of these operator-valued tensors.

This readily implies the following inequality:

$$H_{ij}+2Lg_{ij}\geq (V_{is}+W_{is})g^s_j+(V_{js}+W_{js})g^s_i+g_{ki}g^k_j.$$

The differential inequality for the corresponding norm $\|g\|$ can be obtained in the same way as in the proof of Theorem 1.1. It remains to note that given the eigenvector v of g which corresponds to the largest eigenvalue λ one has $H_{vv}\leq \|H\|_h$, $\big((V_{is}+W_{is})g^s_j\big)(v,v)=\lambda(V_{vv}+W_{vv})\geq 0$. □

Lemma 6.2. *Assume that measure μ has full support and $V_{ij}+W_{ij}\geq 0$. Assume, in addition, that there exists $p>2$ such that the Euclidean operator norm $\|(D^2\Phi)^{-1}\|$ of $(D^2\Phi)^{-1}$ belongs to $L^{p'}(\mu)$ with $p'>p$. Then*

$$\int \|g\|^p_h \, d\mu \leq \int \|H\|_h^{\frac{p}{2}} d\mu.$$

Proof. Set for brevity $\Lambda=\|g\|_h$. Take a compactly supported smooth nonnegative function ξ. Multiply (6.1) by $\xi\Lambda^{p-2}$ and integrate over μ.

$$\int \|H\|_h\Lambda^{p-2}\xi d\mu+2\int \xi L(\Lambda)\Lambda^{p-2}\geq \int \xi\Lambda^p d\mu.$$

Integrating by parts we get

$$\int \|H\|_h\Lambda^{p-2}\xi d\mu-2\int \langle\nabla_h\xi,\nabla_h\Lambda\rangle_h\Lambda^{p-2}d\mu\geq \int \xi\Lambda^p d\mu+2(p-2)\int \xi\Lambda^{p-3}\|\nabla_h\Lambda\|^2 d\mu.$$

Next

$$-2\int \langle\nabla_h\xi,\nabla_h\Lambda\rangle_h\Lambda^{p-2}d\mu\leq 2(p-2)\int \xi\|\nabla_h\Lambda\|^2\Lambda^{p-3}d\mu+\frac{1}{2(p-2)}\int \frac{\|\nabla_h\xi\|_h^2}{\xi}\Lambda^{p-1}d\mu.$$

This estimate yields

$$\int \|H\|_h\Lambda^{p-2}\xi d\mu+\frac{1}{2(p-2)}\int \frac{\|\nabla_h\xi\|_h^2}{\xi}\Lambda^{p-1}d\mu\geq \int \xi\Lambda^p d\mu.$$

For every $\varepsilon>0$ one gets by the Hölder inequality

$$\frac{1}{2(p-2)}\int \frac{\|\nabla_h\xi\|_h^2}{\xi}\Lambda^{p-1}d\mu\leq \int \varepsilon\Lambda^p\xi d\mu+c(\varepsilon,p)\int \Big|\frac{\nabla_h\xi}{\xi}\Big|_h^{2p}\xi d\mu.$$

Finally, one has for every $\varepsilon>0$

$$\begin{aligned}(1-\varepsilon)\int \xi\Lambda^p d\mu &\leq c(\varepsilon,p)\int \Big|\frac{\nabla_h\xi}{\xi}\Big|_h^{2p}\xi d\mu+\int \|H\|_h\Lambda^{p-2}\xi d\mu\\ &\leq c(\varepsilon,p)\int \Big|\frac{\nabla_h\xi}{\xi}\Big|_h^{2p}\xi d\mu+\Big(\int \|H\|_h^{\frac{p}{2}}\xi d\mu\Big)^{\frac{2}{p}}\Big(\int \Lambda^p\xi d\mu\Big)^{\frac{p-2}{p}}.\end{aligned}$$

It remains to show that there exists a sequence of non-negative smooth compactly supported functions $\{\xi_n\}$ with the properties

$$\xi_n \nearrow 1 \text{ pointwise}, \quad \lim_n \int \Bigl|\frac{\nabla_h \xi_n}{\xi_n}\Bigr|_h^{2p} \xi_n d\mu = 0.$$

Estimating

$$|\nabla_h \xi_n|_h^2 \le \|(D^2\Phi)^{-1}\| |\nabla \xi_n|^2$$

and applying Hölder inequality we get that it is enough to have

$$\xi_n \nearrow 1 \text{ pointwise}, \quad \lim_n \int \Bigl|\frac{\nabla \xi_n}{\xi_n}\Bigr|^{m} \xi_n d\mu = 0$$

for any (or sufficiently big) $m > 2$. To solve this problem in dimension one we find a non-negative compactly supported function η with the properties

$$\eta|_{\{x:|x|\le 1\}} = 1, \ \eta(-x) = \eta(x), \ \sup_x \eta(x) \Bigl|\frac{\eta'(x)}{\eta(x)}\Bigr|^m < C_m < \infty, \ \forall m > 0.$$

The construction of such a function is standard. To obtain the desired sequence set $\xi_n^{(1)}(x) = 1$ provided $|x| \le n$ and $\xi_n^{(1)}(x) = \eta(|x| - n + 1)$ provided $|x| \ge n$. In the multidimensional case set $\xi_n(x) = \prod_{i=1}^n \xi_n^{(1)}(x_i)$. □

Letting p to ∞ we get

Corollary 6.3. *Assume that μ and ν are log-concave, μ has full support, and the Euclidean operator norm $\|(D^2\Phi)^{-1}\|$ of $(D^2\Phi)^{-1}$ is integrable in any power. Moreover, assume that*

$$\|H\|_h \le K$$

for some constant $K > 0$. Then

$$\|g\|_h \le \sqrt{K}.$$

Corollary 6.4. *Assume that there exist positive constants c, C, B such that*

$$c \le D^2 V \le C, \ c \le D^2 W \le C,$$

$$\sup_{e:|e|=1} \operatorname{Tr}\bigl[D^2 V_e\bigr]^2 \le B, \ \sup_{e:|e|=1} \operatorname{Tr}\bigl[D^2 W_e\bigr]^2 \le B$$

(here $|\cdot|$ is the Euclidean norm).

Then there exists a constant $D(c, C)$ such that

$$\sup_{e:|e|=1} \operatorname{Tr}\bigl[D^2 \Phi_e\bigr]^2 \le DB.$$

Proof. First we note that by the contraction theorem

$$\sqrt{\frac{c}{C}} \le D^2\Phi \le \sqrt{\frac{C}{c}}. \tag{6.4}$$

Thus the metric h and the Euclidean metric are equivalent up to factors depending on c, C. In particular,

$$\sup_{e:|e|=1} \operatorname{Tr}\bigl[D^2\Phi_e\bigr]^2 \le \sup_{e:|e|=1} \frac{C}{c} \operatorname{Tr}\bigl[D^2\Phi^{-1} \cdot D^2\Phi_e\bigr]^2 = \sup_{e:|e|=1} \frac{C}{c} g(e,e)$$

$$\le \sup_{h:|u|_h=1} \Bigl(\frac{C}{c}\Bigr)^2 g(u,u) = \Bigl(\frac{C}{c}\Bigr)^2 \|g\|_h.$$

According to Corollary 6.3 one needs to estimate uniformly (in Euclidean or Riemannian norm) the matrix

$$H_{ij} = \mathrm{Tr}\Big[\big[(V+W)^{(2)}\big]^{-1}(V-W)^{(3)}_i (D^2\Phi)^{-1}(V-W)^{(3)}_j\Big] = \langle X_i, X_j\rangle,$$

here $\langle A, B\rangle = \mathrm{Tr}(A^t B)$ and

$$X_i = \big[(V+W)^{(2)}\big]^{-1/2}(V-W)^{(3)}_i (D^2\Phi)^{-1/2}.$$

It follows from (6.4) and assumptions of the Theorem that

$$c_1 \le (V+W)^{(2)} \le c_2,$$

where c_1, c_2 depends on c, C. Thus by the standard arguments

$$\begin{aligned} H_{ij} \le \langle X_i, X_j\rangle &\le \frac{C}{c}\big\langle [(V+W)^{(2)}]^{-1/2}(V-W)^{(3)}_i, [(V+W)^{(2)}]^{-1/2}(V-W)^{(3)}_j\big\rangle \\ &\le \frac{C}{cc_1}\langle (V-W)^{(3)}_i, (V-W)^{(3)}_j\rangle. \end{aligned}$$

Hence the Euclidean operator norm is controlled by the Euclidean operator norms of the following matrices

$$A_{ij} = \big\langle D^2 V_{x_i}, D^2 V_{x_j}\big\rangle,\ B_{ij} = \big\langle D^2\Phi\cdot D^2 W_{x_i}\cdot D^2\Phi, D^2\Phi\cdot D^2 W_{x_j}\cdot D^2\Phi\big\rangle.$$

The desired estimate follows immediately from the assumptions of the Theorem and (6.4). $\square$

Corollary 6.5. *Assume that $D^2 V > c > 0$ and $\nu = c_Q e^{-Q(x,x)}dx$ is Gaussian. Then*

$$\|g\|_h^2 \le \sup_{x\in\mathbb{R}^n} \|H(x)\|_h \le \frac{\|Q\|}{c} \sup_{x,e\in\mathbb{R}^n: |e|=1} \mathrm{Tr}\Big[(D^2V)^{-1}\big(D^2 V_e\big)^2\Big](x)$$

provided the right-hand side is finite.

Proof. According to the contraction theorem $D^2\Phi \ge \sqrt{\frac{c}{\|Q\|}}$. In particular, this implies

$$\|H\|_h = \sup_{v:\ \langle D^2\Phi v, v\rangle \le 1} H_{vv} \le \sup_{v:\ |v|^2 \le \sqrt{\frac{\|Q\|}{c}}} H_{vv} \le \sqrt{\frac{\|Q\|}{c}}\|H\|.$$

One has $W_{ijk} = 0, W_{ij} \ge 0$. This implies

$$H_{ij} \le \sqrt{\frac{\|Q\|}{c}}\mathrm{Tr}\Big[(D^2V)^{-1}\cdot D^2 V_{e_i}\cdot D^2 V_{e_j}\Big].$$

and the claim follows. $\square$

Remark 6.6. *The results of Corollaries 6.4, 6.5 are dimension-free and have natural analogues for infinite-dimensional measures. For instance, some natural estimates of this type holds for the potential φ of the optimal transporation $T(x) = x + \nabla\varphi(x)$ pushing forward $g\cdot\gamma$ onto a (infinite dimensional) Gaussian measure γ, where $\nabla\varphi$ is understood as a gradient along the Cameron-Martin space of γ (see* [16], [3], [4] *and the references therein).*

References

[1] Bakry D., Qian Z., Volume comparison theorems without Jacobi fields, Current trends in potential theory, Theta Ser. Adv. Math., vol. 4, Theta, Bucharest, 2005, pp. 115–122.

[2] Berman R. J., Berndtsson B., Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. Ann. Fac. Sci. Toulouse Math. (6) 22 (2013), no. 4, 649–711.

[3] Bogachev V.I., Kolesnikov A.V., The Monge–Kantorovich problem: achievements, connections, and perspectives, Russian Mathematical Surveys, 67(5), 2012, 785–890.

[4] Bogachev V.I., Kolesnikov A.V., Sobolev regularity for the Monge–Ampere equation in the Wiener space. Kyoto J. Math. Volume 53, Number 4 (2013), 713–738.

[5] Calabi E., Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. Michigan Math. J. 5 (1958), no. 2, 105–126.

[6] Calabi E., Complete affine hyperspheres. I, Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971), Academic Press, London, 1972, pp. 19–38.

[7] Cordero-Erausquin D., Klartag B., Moment measures. J. Functional Analysis, 268 (12), (2015), 3834–3866.

[8] Cheng S.-Y., Yau S.-T., On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Feffermans equation, Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544.

[9] Cheng S.-Y., Yau S.-T., The real Monge-Ampére equation and affine flat structures, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980) (Beijing), Science Press, 1982, pp. 339–370.

[10] Cheng S.-Y., Yau S.-T., Complete affine hypersurfaces. I. The completeness of affine metrics, Comm. Pure Appl. Math. 39 (1986), no. 6, 839–866.

[11] Chow B., Knopf D. The Ricci Flow: An Introduction Mathematical surveys and monographs, AMS, 2004.

[12] Fox D.J.F., A Schwarz lemma for Kähler affine metric and the canonical potential of a proper convex cone., Annali di Matematica Pura ed Applicata (2015), 194 (1), 1–42.

[13] Klartag B., Poincaré inequalities and moment maps. Ann. Fac. Sci. Toulouse Math., Vol. 22, No. 1, (2013), 1–41.

[14] Klartag B., Logarithmically-concave moment measures I. Geometric Aspects of Functional Analysis, Lecture Notes in Math. 2116 , Springer (2014), 231–260.

[15] Klartag B.B., Kolesnikov A.V., Eigenvalue distribution of optimal transportation. Anal. & PDE's. 2015, 8 (1), 33–55.

[16] Kolesnikov A.V. Convexity inequalities and optimal transport of infinite-dimensional measures. J. Math. Pures Appl., 2004, v. 83, n. 11, p 1373–1404.

[17] Kolesnikov A.V., On Sobolev regularity of mass transport and transportation inequalities, Theory Probab. Appl., (2013), 57(2), 243–264.

[18] Kolesnikov A.V., Mass transportation and contractions, MIPT Proc. (2010), 2(4), 90-99. arxiv: 1103.1479

[19] Kolesnikov A.V., Hessian metrics, CD(K,N)-spaces, and optimal transportation of log-concave measures. Discrete and Continuous Dynamical Systems - Series A., Vol. 34, No. 4, (2014), 1511–1532.

[20] Kolesnikov A.V., Milman E., Poincaré and Brunn-Minkowski inequalities on weighted Riemannian manifolds with boundary. arXiv:1310.2526.

[21] Loftin J., Affine spheres and Kähler-Einstein metrics. Math. Res. Lett., Vol. 9, no. 4, (2002), 425–432.

[22] Osserman R., On the inequality $\Delta u \geq f(u)$. Pacific J. Math., Vol. 7, (1957), 1641–1647.

[23] Sasaki T., A note on characteristic functions and projectively invariant metrics on a bounded convex domain. Tokyo J. Math., Vol. 8, no. 1, (1985), 49–79.

[24] Villani C., Topics in optimal transportation, Amer. Math. Soc. Providence, Rhode Island, 2003.

[25] Wang X.-J., Zhu X., Kähler–Ricci solitons on toric manifolds with positive first Chern class. Advances in Math., Vol. 188, (2004), 87–103.

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