Journal of Mathematical Sciences, Vol. 110, No. 2, 2002

MAPPINGS OF GENERALIZED VARIATION AND COMPOSITION OPERATORS

V. V. Chistyakov UDC 517.983.3

1. Introduction

Let $\mathbb{R}^{[a,b]}$ be the algebra of all real functions $f : [a,b] \to \mathbb{R}$ on the closed interval $[a,b]$, $BV[a,b]$ be the subalgebra of functions of bounded (in the sense of Jordan) variation in $\mathbb{R}^{[a,b]}$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a given function of two variables, and $H : \mathbb{R}^{[a,b]} \to \mathbb{R}^{[a,b]}$ be the Nemytskii composition operator generated by the function h according to the following rule:

$$(Hf)(t) = h(t, f(t)), \qquad t \in [a,b], \quad f \in \mathbb{R}^{[a,b]}.$$

In [18], Matkowski and Miś have shown that if the operator H acts from $BV[a,b]$ onto itself and satisfies the Lipschitz condition, then we have the following for the function h that generates it: if $h^*(t,x) = \lim\limits_{s\to t-0} h(s,x)$ is the left regularization of h with respect to the first argument, then $h^*(t,x) = h_0(t) + h_1(t)x$ for all $t \in (a,b]$ and $x \in \mathbb{R}$, where the functions h_0, $h_1 \in BV[a,b]$ are left continuous. This result is of interest since, as is shown by examining the function $h(t,x) = \sin x$, the space $BV[a,b]$ cannot be replaced by the space $C[a,b]$ of continuous functions and by the space $L^p(a,b)$ of p-power ($p \geq 1$) Lebesgue-integrable functions. The representation of the generating function h (often without applying a regularization) in the above form has been established by many authors in different spaces, namely, in the space of Hölder functions [13] and in the space of Lipschitzian functions [14] (where such a representation was found for the first time) and also in the space of Lipschitzian mappings [16], in the space of differentiable functions [15], in the space of functions of bounded second p-variation [17], in the space of functions of Riesz-bounded p-variation [19, 20], in the space of mappings of bounded generalized variation of the Riesz–Orlicz type [3, 4], and in some others (see also the references in [3]).

The goal of the present paper is to describe generating Lipschitzian functions of Nemytskii composition operators mapping between spaces of mappings having bounded generalized variation in the sense of Wiener–Young–Orlicz. The generalized variation of this type was studied mainly for real-valued functions [8, 12, 21, 25, 26]. Much less is known about the properties of mappings of bounded generalized variation taking values in normed spaces (see, e.g., [1, 2, 7]), because of the fundamental difficulties. In Sec. 2, we develop the theory of mappings of bounded generalized variation, and then, in Sec. 3, we apply this theory for the characterization of composition operators satisfying the Lipschitz condition (Theorems 7 and 8 and Corollary 9).

The results of the present paper were presented at the workshop "Theory of functions, its applications and related issues," dedicated to the 130th anniversary of the birth of D. F. Egorov, held on September 13–18, 1999 in Kazan' [5].

2. Mappings of Bounded Variation

Notation. Let $I = [a,b]$ be a closed interval of the real line $\mathbb{R}$ (a, $b \in \mathbb{R}$, $a < b$). For the set X, let X^I denote the set of all mappings $f : I \to X$ acting from I into X. Let Φ be the set of all convex continuous functions $\varphi : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}^+$ such that $\varphi(\rho) = 0$ only for $\rho = 0$, and let $\Phi_0 = \left\{\varphi \in \Phi : \lim\limits_{\rho\to 0} \dfrac{\varphi(\rho)}{\rho} = 0\right\}$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 79, Dynamical Systems–10, 2000.

Any function $\varphi \in \Phi$ is strictly increasing; its inverse is denoted by φ^{-1}. For φ, $\psi \in \Phi$, we write $\psi \underset{0}{\preccurlyeq} \varphi$ if $\limsup\limits_{\rho \to 0} \dfrac{\psi(\rho)}{\varphi(C\rho)} < \infty$ for a certain constant $C > 0$ (or, equivalently, if there exist constants $C_0 > 0$ and $\rho_0 > 0$ such that $\psi(\rho) \le C_0 \varphi(C\rho)$ for all $\rho \in [0, \rho_0]$). If $\psi \underset{0}{\preccurlyeq} \varphi$ and $\varphi \underset{0}{\preccurlyeq} \psi$, then we say that φ and ψ *are equivalent in a neighborhood of zero* and write $\varphi \underset{0}{\sim} \psi$. We use the following abbreviations: LNS = linear normed space, BS = Banach space, NA = normalized algebra, BA = Banach algebra.

Throughout this paper, the letters X, Y, and Z (possibly with subscripts) denote an LNS; the norms in these spaces (which are different in general) are denoted by $\|\cdot\|$. Unless otherwise stated, we assume that φ and ψ (with subscripts) are the elements of the set Φ.

Definition. The (*generalized*) *φ-variation* of the mapping $f \in X^I$ is

$$\mathsf{V}_\varphi(f) = \mathsf{V}_\varphi(f, I) = \sup_T \sum_{i=1}^m \varphi\big(\|f(t_i) - f(t_{i-1})\|\big), \tag{1}$$

where the supremum is taken over all positive integers m and over all tuples of points $T = \{t_i\}_{i=0}^m$ such that $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ (i.e., over all *partitions* of the closed interval I).

For $\varphi_q(\rho) = \rho^q$ ($\rho \ge 0$, $q \ge 1$), (1) gives the classical concept of the variation of the mapping f in the sense of C. Jordan [23, Chapter 8] whenever $q = 1$, and in the sense of N. Wiener [25] whenever $q > 1$. The general definition (1) involving the (nondecreasing continuous) function φ is due to L. Young [26].

It is known that the functional V_φ is *nondecreasing*, that is, $\mathsf{V}_\varphi(f, J) \le \mathsf{V}_\varphi(f, I)$ if I contains the closed interval J; it is *semiadditive*, that is, $\mathsf{V}_\varphi(f, [a, c]) + \mathsf{V}_\varphi(f, [c, b]) \le \mathsf{V}_\varphi(f, [a, b])$ for $a \le c \le b$, and it is (*sequentially*) *lower semicontinuous*, that is, $\mathsf{V}_\varphi(f, I) \le \liminf\limits_{n\to\infty} \mathsf{V}_\varphi(f_n, I)$ if the sequence $f_n \in X^I$ pointwise converges to $f \in X^I$ on I as $n \to \infty$.

The set of all mappings $f \in X^I$ for which the quantity defined by (1) is finite is convex (but it is not necessarily a linear space), and $\mathsf{V}_\varphi(\cdot) = \mathsf{V}_\varphi(\,\cdot\,, I)$ is a convex functional on this set. We denote by $W_\varphi(I; X)$ the linear space of all $f \in X^I$ such that $\mathsf{V}_\varphi(f/\lambda) < \infty$ for a certain constant $\lambda > 0$ (depending on f). For $f \in X^I$ the following criterion holds: $f \in W_\varphi(I; X)$ if and only if $f(t) = g(\chi(t))$ for all $t \in I$, where $\chi : I \to \mathbb{R}$ is a nondecreasing bounded function and the mapping $g : J \to X$, acting from the image $J = \chi(I)$ of the function χ in the space X, has the following property for a certain constant $\lambda > 0$:

$$\|g(t) - g(s)\| \le \lambda \varphi^{-1}(|t - s|)$$

for all $t, s \in J$. This statement can be obtained from [7, Theorem 3.2] (if one sets $\Phi(\rho) = \varphi(\rho/\lambda)$). In particular, a similar criterion (with $\varphi(\rho) = \rho$ and $\lambda = 1$) holds for mappings of bounded variation in the sense of Jordan [1,2], where it was just found.

In the linear space $W_\varphi(I; X)$, we introduce the norm $\|f\|_\varphi = \|f(a)\| + p_\varphi(f)$, where

$$p_\varphi(f) = p_\varphi(f, I) = \inf\left\{\lambda > 0 : \mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}, \quad f \in W_\varphi(I; X). \tag{2}$$

For $X = \mathbb{R}$, the LNS $W_\varphi(I; \mathbb{R})$ was studied by Musielak and Orlicz [21], Ciemnoczołowski and Orlicz [8], and Maligranda and Orlicz [12]. In particular, it is shown in [12] that the space $W_\varphi(I; \mathbb{R})$ is a BA. The seminorm defined by (2) is called the *Luxemburg–Nakano–Orlicz seminorm* [10, 22, 24].

Certain properties of p_φ are reflected in the following lemma.

Lemma 1. *For the mapping $f \in W_\varphi(I; X)$, we have*:

(a) *if t, $s \in I$, then $\|f(t) - f(s)\| \le \varphi^{-1}(1) p_\varphi(f)$;*
(b) *if $p_\varphi(f) > 0$, then $\mathsf{V}_\varphi\big(f/p_\varphi(f)\big) \le 1$;*
(c) *for $\lambda > 0$, we have that $p_\varphi(f) \le \lambda$ if and only if $\mathsf{V}_\varphi(f/\lambda) \le 1$; if $\mathsf{V}_\varphi(f/\lambda) = 1$, then $p_\varphi(f) = \lambda$ (the converse is not true in general);*
(d) *if a sequence $f_n \in W_\varphi(I; X)$ pointwise converges to f as $n \to \infty$ on I, then $p_\varphi(f) \le \liminf\limits_{n\to\infty} p_\varphi(f_n)$.*

Proof. (a) For any t, $s \in I$, by virtue of (1) and (2), we have

$$\varphi\left(\frac{\|f(t)-f(s)\|}{\lambda}\right) \le \mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) \le 1 \quad \forall\, \lambda > p_\varphi(f),$$

whence, taking the inverse function φ^{-1}, we obtain (a).

(b) Suppose that a sequence of numbers $\lambda_n > \lambda = p_\varphi(f)$ converges to λ as $n \to \infty$. It follows from the definition of the number λ that $\mathsf{V}_\varphi(f/\lambda_n) \le 1$ for all positive integers n. Since f/λ_n pointwise converges to f/λ on I as $n \to \infty$, by the lower semicontinuity of the functional $\mathsf{V}_\varphi(\cdot)$, we obtain that $\mathsf{V}_\varphi(f/\lambda) \le \liminf_{n\to\infty} \mathsf{V}_\varphi(f/\lambda_n) \le 1$.

(c) To prove the first assertion, it suffices to show that if $0 < p_\varphi(f) < \lambda$, then $\mathsf{V}_\varphi(f/\lambda) < 1$, and this is directly implied by the convexity of $\mathsf{V}_\varphi(\cdot)$ and assertion (b), that is,

$$\mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) \le \frac{p_\varphi(f)}{\lambda}\mathsf{V}_\varphi\left(\frac{f}{p_\varphi(f)}\right) \le \frac{p_\varphi(f)}{\lambda} < 1.$$

To prove the second assertion, it suffices to observe that the cases where $p_\varphi(f) > \lambda$ and $p_\varphi(f) < \lambda$ are impossible.

(d) We set $\alpha = \liminf_{n\to\infty} p_\varphi(f_n)$ and assume that $\alpha < \infty$. Then $\alpha = \lim_{k\to\infty} p_\varphi(f_{n_k})$ for a certain subsequence $\{f_{n_k}\}_{k=1}^\infty$ in $\{f_n\}_{n=1}^\infty$; therefore, for any $\varepsilon > 0$, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that $p_\varphi(f_{n_k}) < \alpha+\varepsilon$ for all $k \ge k_0(\varepsilon)$. It follows from the definition of $p_\varphi(f_{n_k})$ that $\mathsf{V}_\varphi\big(f_{n_k}/(\alpha+\varepsilon)\big) \le 1$ for $k \ge k_0(\varepsilon)$, and since $f_{n_k}/(\alpha+\varepsilon) \to f/(\alpha+\varepsilon)$ pointwise on I as $k \to \infty$, the lower semicontinuity of $\mathsf{V}_\varphi(\,\cdot\,)$ yields

$$\mathsf{V}_\varphi\left(\frac{f}{\alpha+\varepsilon}\right) \le \liminf_{k\to\infty} \mathsf{V}_\varphi\left(\frac{f_{n_k}}{\alpha+\varepsilon}\right) \le 1.$$

Therefore, $p_\varphi(f) \le \alpha+\varepsilon$ for all $\varepsilon > 0$. □

Property (a) in Lemma 1 means that any mapping $f \in W_\varphi(I;X)$ is bounded. This property can be substantially refined: the image $f(I)$ of a mapping $f \in W_\varphi(I;X)$ is a *completely bounded* and *separable* subset of X, and if, in addition, X is a BS, then $f(I)$ is *precompact* (i.e., the closure of $f(I)$ in X is compact). Indeed, assume the contrary: for some $\varepsilon > 0$, the image $f(I)$ cannot be covered by finitely many balls from X of radius ε centered at $f(I)$. Proceeding by induction, we construct a sequence $\{x_n\}_{n=0}^\infty \subset f(I)$ as follows: setting $x_0 = f(a)$ and determining the elements $x_1, \dots, x_{n-1} \in f(I)$, we choose $x_n \in f(I)$ in such a way that $\|x_n - x_j\| \ge \varepsilon$ for all $j = 0, 1, \dots, n-1$. If $x_n = f(t_n)$, where $t_n \in I$, $n \in \mathbb{N}$, then it is clear that $t_n \ne t_k$ for $n \ne k$; therefore, we can assume without loss of generality that $t_{n-1} < t_n$ for all $n \in \mathbb{N}$. Then for the partition $T_n = \{a\} \cup \{t_i\}_{i=1}^n \cup \{b\}$ of the closed interval I we have the following:

$$\mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) \ge \sum_{i=1}^n \varphi\left(\frac{\|f(t_i)-f(t_{i-1})\|}{\lambda}\right) \ge n\varphi\left(\frac{\varepsilon}{\lambda}\right)$$

for any $\lambda > 0$. Therefore, $\mathsf{V}_\varphi(f/\lambda) = \infty$ for all $\lambda > 0$; this contradicts the condition $f \in W_\varphi(I;X)$. It remains to note that a bounded set is separable in an LNS and precompact in a BS.

Note that *Helly's selection principle*, which is known for functions/mappings of Jordan bounded variation (i.e., when $\varphi(\rho) = \rho$; see [9], [23, Chapter 8, Sec. 4], and [2, Sec. 5]) and bounded φ-variation (see [21, 1.3] and [7, Sec. 6]), is extended to the space $W_\varphi(I;X)$ as well. In proving the following theorem, we use certain ideas of [21, 1.3].

Theorem 2 (generalized Helly's selection principle)**.** *Let X be an LNS* (*over $\mathbb{R}$ or $\mathbb{C}$*) *and $\mathcal{F} \subset W_\varphi(I;X)$ be an infinite family of mappings such that the set $\mathcal{F}(t) = \{f(t) \mid f \in \mathcal{F}\}$ is precompact in X for any $t \in I$ and $\sup_{f\in\mathcal{F}} p_\varphi(f) < \infty$. Then the family $\mathcal{F}$ contains a sequence of mappings that converges pointwise on I to a certain mapping from $W_\varphi(I;X)$.*

Proof. Choosing a number $\lambda > 0$ in such a way that $\sup_{f\in\mathcal{F}} p_\varphi(f) \le \lambda$, we have by Lemma 1(c) that $\mathsf{V}_\varphi(f/\lambda) \le 1$ for all $f \in \mathcal{F}$. Determining the nonnegative function $v_f(t) = \mathsf{V}_\varphi(f/\lambda, [a,t])$, $t \in I = [a,b]$, for $f \in \mathcal{F}$, we have that the infinite family of nondecreasing functions $\{v_f \mid f \in \mathcal{F}\}$ is uniformly bounded (from above by the constant 1); therefore, by Helly's selection principle for monotonic functions (see [9] and [23, Chapter 8, Sec. 4, Lemma 2]), there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ such that the functions v_{f_n} converge pointwise on I to a nondecreasing bounded function $v \in \mathbb{R}^I$ as $n \to \infty$. Taking into account that for any $t \in I$, the set $\{f_n(t)\}_{n=1}^\infty$ is precompact in X, we can assume without loss of generality (passing, if necessary, to a subsequence in $\{f_n\}_{n=1}^\infty$ on the basis of the standard diagonal process) that $f_n(t)$ is convergent in X as $n \to \infty$ at all rational points $t \in I$ and at points $t = a$ and $t = b$.

If we show that the sequence $f_n(t)$ converges in X at all irrational points $t \in (a,b)$ that are points of continuity of the function v, then, taking into account that the set of points of discontinuity of the function v is no more than countable and the sequences $\{f_n(s)\}_{n=1}^\infty$ are precompact in X for all $s \in I$, we apply the diagonal process to choose a subsequence from the sequence $\{f_n\}_{n=1}^\infty$; this subsequence converges to a certain mapping $f \in X^I$ at each point $t \in I$. By virtue of the lower semicontinuity of the functional $\mathsf{V}_\varphi(\,\cdot\,)$, we obtain $\mathsf{V}_\varphi(f/\lambda) \le 1$ or $p_\varphi(f) \le \lambda$.

Thus, let us prove that if $t \in (a,b)$ is an irrational point of continuity of the function v, then $f_n(t)$ converges in X as $n \to \infty$. We take $\varepsilon > 0$ and a rational number $s \in (a,t)$ such that $0 \le v(t) - v(s) < \frac{1}{3}\varphi\big(\varepsilon/(3\lambda)\big)$. By virtue of the pointwise convergence of v_{f_n} to v, we choose a number $n_0 \in \mathbb{N}$ in such a way that

$$\max\big\{|v_{f_n}(t) - v(t)|, |v_{f_n}(s) - v(s)|\big\} < \frac{1}{3}\varphi\left(\frac{\varepsilon}{3\lambda}\right), \quad n \ge n_0.$$

By virtue of definition (1) and the semiadditivity of $\mathsf{V}_\varphi(\,\cdot\,)$ we have

$$\begin{aligned}\varphi\left(\frac{\|f_n(t) - f_n(s)\|}{\lambda}\right) &\le \mathsf{V}_\varphi\left(\frac{f_n}{\lambda}, [s,t]\right) \le v_{f_n}(t) - v_{f_n}(s)\\ &\le \big|v_{f_n}(t) - v(t)\big| + \big(v(t) - v(s)\big) + \big|v(s) - v_{f_n}(s)\big|,\end{aligned}$$

which implies $\|f_n(t) - f_n(s)\| \le \varepsilon/3$ for all $n \ge n_0$. We choose a number $m_0 \in \mathbb{N}$ such that $\|f_n(s) - f_m(s)\| \le \varepsilon/3$ for all $n, m \ge m_0$ and set $N = \max\{n_0, m_0\}$. Then for $n, m \ge N$, we obtain

$$\|f_n(t) - f_m(t)\| \le \|f_n(t) - f_n(s)\| + \|f_n(s) - f_m(s)\| + \|f_m(s) - f_m(t)\| \le \varepsilon,$$

i.e., the sequence $\{f_n(t)\}_{n=1}^\infty$ is fundamental in X. Now, the fact that this sequence is precompact in X implies that it converges to a certain element from X as $n \to \infty$. □

Remark 1. Theorem 2 holds under weaker constraints, namely, one can assume that X is an arbitrary metric space, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing unbounded continuous function, and $\varphi(\rho) = 0$ for $\rho = 0$ only, while the condition $\sup_{f\in\mathcal{F}} p_\varphi(f) < \infty$ can be replaced by the following one: there exist constants $\lambda > 0$ and $C > 0$ such that $\mathsf{V}_\varphi(f/\lambda) \le C$ for all $f \in \mathcal{F}$.

Lemma 3. *If $f : I = [a,b] \to \mathbb{R}$ is a bounded monotonic function, then $f \in W_\varphi(I;\mathbb{R})$ and*

$$p_\varphi(f) = |f(b) - f(a)|/\varphi^{-1}(1)$$

for any function $\varphi \in \Phi$.

Proof. We first show that $\mathsf{V}_\varphi(f/\lambda) = \varphi\big(|f(b)-f(a)|/\lambda\big)$, $\lambda > 0$. This relation is well known (see [21, 1.03]), but we give an alternative proof of it. Since $\{a,b\}$ is a partition of I, we have $\varphi\big(|f(b)-f(a)|/\lambda\big) \le \mathsf{V}_\varphi(f/\lambda)$. To prove the converse inequality, assume that $T = \{t_i\}_{i=0}^m$ is a partition of I and $t \in I$, so that $t_{k-1} \le t \le t_k$ for some $1 \le k \le m$. Let $V_\varphi[f,T]$ denote the sum under the supremum sign in (1) and corresponding to the partition T. Since f is monotonic, we have

$$|f(t) - f(t_{k-1})| + |f(t_k) - f(t)| = |f(t_k) - f(t_{k-1})|;$$

therefore, applying the inequality $\varphi(\rho_1)+\varphi(\rho_2)\le\varphi(\rho_1+\rho_2)$, $\rho_1\ge 0$, $\rho_2\ge 0$, we obtain

$$V_\varphi\left[\frac{f}{\lambda},T\cup\{t\}\right]=V_\varphi\left[\frac{f}{\lambda},T\right]-\varphi\left(\frac{|f(t_k)-f(t_{k-1})|}{\lambda}\right)\\ +\varphi\left(\frac{|f(t)-f(t_{k-1})|}{\lambda}\right)+\varphi\left(\frac{|f(t_k)-f(t)|}{\lambda}\right)\le V_\varphi\left[\frac{f}{\lambda},T\right].$$

Thus, the more points T contains, the less is $V_\varphi[f/\lambda,T]$. Therefore, $V_\varphi[f/\lambda,T]\le\varphi\big(|f(b)-f(a)|/\lambda\big)$ for any partition T of the closed interval I, and the converse inequality is proved.

Let $f(a)\ne f(b)$. By virtue of the formula obtained above, $\mathsf{V}_\varphi(f/\lambda)=1$ if and only if $\lambda=|f(b)-f(a)|/\varphi^{-1}(1)$; therefore, it remains to apply Lemma 1(c). The case where $f(a)=f(b)$ is obvious. □

Theorem 4. (a) *Let (X,Y,Z) be a triple of LNSs for which there exists a bilinear mapping $M: X\times Y\to Z$ such that $\|xy\|\le\|x\|\cdot\|y\|$ for all $x\in X$ and $y\in Y$, where $xy=M(x,y)$. If $f\in W_\varphi(I;X)$ and $g\in W_\varphi(I;Y)$, then for the product $fg\in Z^I$ defined by the rule $(fg)(t)=f(t)g(t)$, $t\in I$, we have*

$$fg\in W_\varphi(I;Z),\qquad \|fg\|_\varphi\le\gamma\|f\|_\varphi\|g\|_\varphi,$$

where $\gamma=\gamma(\varphi)=\max\{1,2\varphi^{-1}(1)\}$. If, in addition, X is a BS, then $W_\varphi(I;X)$ is a BS as well.

(b) *If $\psi\underset{0}{\preccurlyeq}\varphi$, then $W_\varphi(I;X)\subset W_\psi(I;X)$, and if, in addition, X is a BS, then there exists a constant $\kappa=\kappa(\varphi,\psi)>0$ such that $\|f\|_\psi\le\kappa\|f\|_\varphi$ for all $f\in W_\varphi(I;X)$.*

Proof. (a) 1. By Lemma 1(a), for any $f\in W_\varphi(I;X)$, we have the estimate

$$\|f\|_u=\sup_{t\in I}\|f(t)\|\le\|f(a)\|+\varphi^{-1}(1)p_\varphi(f). \tag{3}$$

If $g\in W_\varphi(I;Y)$, let us show that the following inequality holds:

$$p_\varphi(fg)\le p_\varphi(f)\|g\|_u+\|f\|_u p_\varphi(g)\equiv\lambda. \tag{4}$$

One can assume without loss of generality that $p_\varphi(f)$, $p_\varphi(g)$, $\|f\|_u$, and $\|g\|_u$ are different from zero. Let $T=\{t_i\}_{i=0}^m$ be an arbitrary partition of the closed interval I. Setting $\Delta f_i=f(t_i)-f(t_{i-1})$ and similarly for Δg_i and $\Delta(fg)_i$, using the monotonicity and the convexity of the function φ, and applying Lemma 1(b), we obtain

$$\begin{aligned}\sum_{i=1}^m\varphi\left(\frac{\|\Delta(fg)_i\|}{\lambda}\right)&=\sum_{i=1}^m\varphi\left(\frac{\|(\Delta f_i)g(t_i)+f(t_{i-1})(\Delta g_i)\|}{\lambda}\right)\\&\le\sum_{i=1}^m\varphi\left(\frac{\|\Delta f_i\|\cdot\|g\|_u+\|f\|_u\cdot\|\Delta g_i\|}{\lambda}\right)\\&\le\frac{p_\varphi(f)\|g\|_u}{\lambda}\sum_{i=1}^m\varphi\left(\frac{\|\Delta f_i\|}{p_\varphi(f)}\right)+\frac{\|f\|_u p_\varphi(g)}{\lambda}\sum_{i=1}^m\varphi\left(\frac{\|\Delta g_i\|}{p_\varphi(g)}\right)\\&\le\frac{p_\varphi(f)\|g\|_u}{\lambda}\mathsf{V}_\varphi\left(\frac{f}{p_\varphi(f)}\right)+\frac{\|f\|_u p_\varphi(g)}{\lambda}\mathsf{V}_\varphi\left(\frac{g}{p_\varphi(g)}\right)\\&\le\frac{p_\varphi(f)\|g\|_u}{\lambda}+\frac{\|f\|_u p_\varphi(g)}{\lambda}=1.\end{aligned}$$

Since T is arbitrary, we obtain $\mathsf{V}_\varphi(fg/\lambda)\le 1$; this implies $p_\varphi(fg)\le\lambda$.

The inequality $\|fg\|_\varphi\le\gamma\|f\|_\varphi\|g\|_\varphi$ now follows from (3), (4), and the definition of the norm $\|\cdot\|_\varphi$.

2. If X is complete, we show that $W_\varphi(I;X)$ is a complete space as well. Let $\{f_n\}_{n=1}^\infty\subset W_\varphi(I;X)$ be a fundamental sequence. Then the sequence $\{f_n(t)\}_{n=1}^\infty$ is fundamental in X for any $t\in I$ by Lemma 1(a).

The completeness of X implies that the sequence f_n converges pointwise on I to a certain mapping $f \in X^I$ as $n \to \infty$. By Lemma 1(d), we have

$$\|f_n - f\|_\varphi \le \liminf_{m\to\infty} \|f_n - f_m\|_\varphi = \lim_{m\to\infty} \|f_n - f_m\|_\varphi \in \mathbb{R}^+, \quad n \in \mathbb{N},$$

and since $\{f_n\}_{n=1}^\infty$ is fundamental in $W_\varphi(I;X)$, we have

$$\limsup_{n\to\infty} \|f_n - f\|_\varphi \le \lim_{n\to\infty} \lim_{m\to\infty} \|f_n - f_m\|_\varphi = 0,$$

which implies $\lim\limits_{n\to\infty} \|f_n - f\|_\varphi = 0$. To prove that $f \in W_\varphi(I;X)$, we note that the sequence $\{p_\varphi(f_n)\}_{n=1}^\infty$ is fundamental; therefore, it is bounded, thus it remains to apply Lemma 1(d).

(b) 1. Let $\psi \underset{0}{\preccurlyeq} \varphi$ and $f \in W_\varphi(I;X)$. Then there exist constants $C > 0$, $C_0 > 0$, and $\rho_0 > 0$ such that $\psi(\rho) \le C_0\varphi(C\rho)$ for all $0 \le \rho \le \rho_0$, and there exists a number $\lambda > 0$ such that $v = \mathsf{V}_\varphi(f/\lambda) < \infty$. Let $T = \{t_i\}_{i=0}^m$ be a partition of I and $\Delta f_i = f(t_i) - f(t_{i-1})$, $i \in A = \{1, \dots, m\}$. If B is the set of all subscripts $i \in A$ for which $\|\Delta f_i\|/(\lambda C) > \rho_0$, then, in view of $\sum\limits_{i\in A} \varphi\left(\dfrac{\|\Delta f_i\|}{\lambda}\right) \le v$, the number of elements in B does not exceed the number $v/\varphi(C\rho_0)$, and also the inequality $\|\Delta f_i\| \le \lambda\varphi^{-1}(v)$, $i \in A$, holds. Then

$$\begin{aligned}\sum_{i=1}^m \psi\left(\frac{\|\Delta f_i\|}{\lambda C}\right) &= \sum_{i\in A\setminus B} + \sum_{i\in B} \le \sum_{i\in A\setminus B} C_0\varphi\left(\frac{\|\Delta f_i\|}{\lambda}\right) + \sum_{i\in B} \psi\left(\frac{\varphi^{-1}(v)}{C}\right) \\ &\le C_0 v + \psi\left(\frac{\varphi^{-1}(v)}{C}\right) \cdot \frac{v}{\varphi(C\rho_0)}.\end{aligned}$$

Since the partition T is arbitrary, we conclude that $\mathsf{V}_\psi\big(f/(\lambda C)\big) < \infty$; therefore, $f \in W_\psi(I;X)$. Note that the inclusion $W_\varphi(I;X) \subset W_\psi(I;X)$ implies the relation $\psi \underset{0}{\preccurlyeq} \varphi$ (see [8, Theorem 4.1.1]).

2. Let X be a BS, and let $\psi \underset{0}{\preccurlyeq} \varphi$. By virtue of assertion (a), $W_\varphi(I;X)$ and $W_\psi(I;X)$ are BSs as well. The identity operator Id, which is defined by the rule $\mathrm{Id}(f) = f$, maps $W_\varphi(I;X)$ into $W_\psi(I;X)$ and is closed (by virtue of (3) and the definition of the norm $\|\cdot\|_\varphi$); therefore, by the closed-graph theorem, this operator is continuous. It remains to determine the desired number $\kappa = \kappa(\varphi,\psi)$ as an operator norm of the operator Id. □

Remark 2. (a) As a consequence of Theorem 4(a), we note that $W_\varphi(I;X)$ is an NA (respectively, a BA) if X is an NA (respectively, a BA). In this case, from the theory of Banach algebras it is known that on the space $W_\varphi(I;X)$, there exists a norm $|||\cdot|||$ such that $|||f||| \le \|f\|_\varphi \le \gamma|||f|||$ and $|||fg||| \le |||f|||\cdot|||g|||$ for all f, $g \in W_\varphi(I;X)$.

(b) Theorem 4(b) implies that for $\varphi \underset{0}{\sim} \psi$, the spaces $W_\varphi(I;X)$ and $W_\psi(I;X)$ consist of one and the same mappings and the norms in these spaces are equivalent. In particular, for $\varphi \in \Phi\setminus\Phi_0$, we have $\varphi \underset{0}{\sim} \varphi_1$ (see notation on p. 2456); therefore, $W_\varphi(I;X)$ coincides with the space of mappings of Jordan bounded variation. In what follows, we are interested primarily in the case where $\varphi \in \Phi_0$, which corresponds to the generalized φ-variation in the sense of Wiener–Young–Orlicz.

Lemma 5. *If X is a BS, then the mapping $f \in W_\varphi(I;X)$ has the left limit $f(t-0) \in X$ at each point $t \in (a,b]$ and the right limit $f(t+0) \in X$ at each point $t \in [a,b)$; moreover, the set of points of continuity of f on $I = [a,b]$ is no more than countable.*

Proof. It suffices to note that $\mathsf{V}_\varphi(f/\lambda) < \infty$ for a certain constant $\lambda > 0$ and apply Lemma 4.1 and Theorem 4.2 from [7]. □

Let $W_\varphi^*(I;X)$ denote the subset in $W_\varphi(I;X)$ that consists of those mappings that are left continuous on the half-interval $(a,b]$. If X is a BS and $f \in W_\varphi(I;X)$, define the *left regularization* $f^* : I \to X$ of the mapping f by the following rule:

$$f^*(t) = \lim_{s \to t-0} f(s) \quad \text{for} \quad a < t \le b,$$
$$f^*(a) = \lim_{t \to a+0} f^*(t).$$

Let us show that the limit of $f^*(a) \in X$ does exist, by using the Cauchy criterion for the existence of a limit in a complete space X. We fix $\varepsilon > 0$. The existence of the limit $f(a+0) = \lim_{t \to a+0} f(t)$ implies that there exists $\delta = \delta(\varepsilon) > 0$ such that if $a < t \le a + \delta$, then $\|f(t) - f(a+0)\| \le \varepsilon$. Assume that $t_j \in [a,b]$, $0 < t_j - a \le \delta$, $j = 1,2$. For small $\sigma > 0$ ($\sigma < t_j - a$, $j = 1,2$), we have

$$\|f^*(t_1) - f^*(t_2)\| \le \|f^*(t_1) - f(t_1 - \sigma)\| + \|f(t_1 - \sigma) - f(t_2 - \sigma)\| + \|f(t_2 - \sigma) - f^*(t_2)\|,$$

where the middle expression is estimated by

$$\|f(t_1 - \sigma) - f(t_2 - \sigma)\| \le \|f(t_1 - \sigma) - f(a+0)\| + \|f(a+0) - f(t_2 - \sigma)\| \le 2\varepsilon.$$

From the existence of $f^*(t_j) = \lim_{s \to t_j - 0} f(s)$, by choosing (by decreasing) a small $\sigma > 0$, we find that $\|f^*(t_j) - f(t_j - \sigma)\| \le \varepsilon$, $j = 1,2$. Finally, $\|f^*(t_1) - f^*(t_2)\| \le 4\varepsilon$ for $0 < t_j - a \le \delta(\varepsilon)$, $j = 1,2$, and it remains to apply the Cauchy criterion mentioned above.

Lemma 6. *If X is a BS and $f \in W_\varphi(I;X)$, then $f^* \in W^*_\varphi(I;X)$.*

Proof. 1. Let us show that the mapping f^* is left continuous on $(a,b]$. To this end, note that if f is left continuous at a point $a < t \le b$, then $f^*(t) = f(t)$. By Lemma 5, the set of points at which f is continuous is everywhere dense in $[a,b]$; therefore, if $t \in (a,b]$, then there exists a sequence $\{s_n\}_{n=1}^\infty$ of points of continuity of f lying strictly to the right of t and such that $s_n \to t$ as $n \to \infty$. Then we have

$$\lim_{s \to t-0} f^*(s) = \lim_{n \to \infty} f^*(s_n) = \lim_{n \to \infty} f(s_n) = \lim_{s \to t-0} f(s) = f^*(t) \quad \text{in } X.$$

2. Now, let us prove that $f^* \in W_\varphi(I;X)$, and, moreover, that $p_\varphi(f^*) \le p_\varphi(f)$. Without loss of generality, we assume that $\lambda = p_\varphi(f) > 0$. Let $Q = \{1,2,3,\dots\}$ be a finite or countable set, and let $\{\tau_n\}_{n \in Q} \subset (a,b]$ be the set of points of left discontinuity of the mapping $f \in W_\varphi(I;X)$. We define the mapping $f_1 : [a,b] \to X$, which is distinct from f only at the point τ_1, by the following rule: $f_1(t) = f(t)$ for $t \ne \tau_1$ and $f_1(\tau_1) = f(\tau_1 - 0)$, and show that $p_\varphi(f_1) \le \lambda$. Let $\tau_1 \ne b$. If the partition $T = \{t_i\}_{i=0}^m$ of the closed interval $I = [a,b]$ does not contain the point τ_1, then the following estimate is obvious:

$$\sum_{i=1}^m \varphi\left(\frac{\|\Delta(f_1)_i\|}{\lambda}\right) = \sum_{i=1}^m \varphi\left(\frac{\|\Delta f_i\|}{\lambda}\right) \le \mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) \le 1.$$

On the other hand, if the partition T contains the point τ_1, so that $\tau_1 = t_k$ for a certain number $k \in \{1,\dots,m-1\}$, then

$$\begin{aligned}\sum_{i=1}^m \varphi\left(\frac{\|\Delta(f_1)_i\|}{\lambda}\right) = \sum_{i=1}^{k-1} \varphi\left(\frac{\|\Delta f_i\|}{\lambda}\right) + \varphi\left(\frac{\|f(\tau_1 - 0) - f(t_{k-1})\|}{\lambda}\right)\\ + \varphi\left(\frac{\|f(t_{k+1}) - f(\tau_1 - 0)\|}{\lambda}\right) + \sum_{i=k+2}^m \varphi\left(\frac{\|\Delta f_i\|}{\lambda}\right).\end{aligned} \tag{5}$$

For any $\varepsilon > 0$, there exists s, $t_{k-1} < s < t_k = \tau_1$, such that the sum of the middle two summands on the right in the last relation does not exceed

$$\varphi\left(\frac{\|f(s) - f(t_{k-1})\|}{\lambda}\right) + \varphi\left(\frac{\|f(t_{k+1}) - f(s)\|}{\lambda}\right) + \varepsilon.$$

Since the set $\{t_i\}_{i=0}^{k-1} \cup \{s\} \cup \{t_i\}_{i=k+1}^{m}$ is a partition of I, the estimate

$$\sum_{i=1}^{m} \varphi\left(\frac{\|\Delta(f_1)_i\|}{\lambda}\right) \le \mathsf{V}_\varphi\left(\frac{f}{\lambda}\right) + \varepsilon \le 1 + \varepsilon, \quad \varepsilon > 0$$

follows. Therefore, $\mathsf{V}_\varphi(f_1/\lambda) \le 1$, and hence $p_\varphi(f_1) \le \lambda$. Now, if $\tau_1 = b$, then the arguments are similar (one should not take into account the summands in (5)).

If the mappings $f_1, \dots, f_{n-1}$ have already been defined, then for $t \in I$, we set $f_n(t) = f_{n-1}(t)$ for $t \ne \tau_n$ and $f_n(\tau_n) = f_{n-1}(\tau_n - 0) = f(\tau_n - 0)$, $n = 2, 3, \dots$. Then, by induction and by what was proved above, we have

$$p_\varphi(f_n) \le p_\varphi(f_{n-1}) \le \dots \le \lambda, \quad n \in Q.$$

For a finite Q, the lemma is proved; therefore, let Q be infinite. Defining the mapping $f_*(t) = f(t)$ for $t \in I$ for all $n \in Q$ if $t \ne \tau_n$, and $f_*(\tau_n) = f(\tau_n - 0)$ for $n \in Q$, and noting that f_n converges pointwise to f_* on $[a, b]$ as $n \to \infty$, by Lemma 1(d) we obtain

$$p_\varphi(f_*) \le \liminf_{n\to\infty} p_\varphi(f_n) \le \lambda = p_\varphi(f).$$

Finally, taking into account that $f^*(t) = f_*(t)$ for $t \ne a$ and $f^*(a) = f_*(a + 0)$, so that f^* differs from f_* only at the point a, by virtue of what was proved above we conclude that $p_\varphi(f^*) \le p_\varphi(f_*) \le p_\varphi(f)$, which is the result required. □

3. Composition Operators

Definition. The *Nemytskii composition operator* generated by the mapping $h : I \times X \to Y$ is the operator $H : X^I \to Y^I$ defined for $f \in X^I$ by the following rule:

$$(Hf)(t) \equiv H(f)(t) = h(t, f(t)), \qquad t \in I.$$

For each $x \in X$, we denote by $h^*(\,\cdot\,, x)$ the left regularization of the mapping $h(\,\cdot\,, x) : I \to Y$ if such a regularization exists, so that $h^* : I \times X \to Y$. For instance, if Y is a BS and $h(\,\cdot\,, x) \in W_\psi(I; Y)$ for all $x \in X$, then by Lemma 6 we have $h^*(\,\cdot\,, x) \in W_\psi^*(I; Y)$ for all $x \in X$.

Let $L(X; Y)$ be the LNS of all linear continuous operators mapping from X into Y that is equipped with the standard norm.

Theorem 7. *Let the Nemytskii composition operator $H : X^I \to Y^I$ be generated by the mapping $h : I \times X \to Y$, and let $\varphi, \psi \in \Phi$.*

(a) *If X is a real LNS, Y is a BS, and H acts from $W_\varphi(I; X)$ into $W_\psi(I; Y)$ and is a Lipschitzian operator* (*in the sense of norms of spaces indicated above*), *then there exists a constant $\mu_0 > 0$ such that*

$$\|h(t, x_1) - h(t, x_2)\| \le \mu_0 \|x_1 - x_2\|, \qquad t \in I, \quad x_1, x_2 \in X, \tag{6}$$

and there exists mappings $h_0 \in W_\psi^*(I; Y)$ and $h_1 \in L(X; Y)^I$ with the property that $h_1(\,\cdot\,)x \in W_\psi^*(I; Y)$ for all $x \in X$ such that

$$h^*(t, x) = h_0(t) + h_1(t)x, \qquad t \in I, \quad x \in X. \tag{7}$$

(b) *Conversely, if X is a BS, Y is an LNS, $\psi \underset{0}{\preccurlyeq} \varphi$, and $h(t, x) = h_0(t) + h_1(t)x$ for all $(t, x) \in I \times X$, where $h_0 \in W_\psi(I; Y)$ and $h_1 \in W_\psi(I; L(X; Y))$, then the operator H maps $W_\varphi(I; X)$ into $W_\psi(I; Y)$ and satisfies the global Lipschitz condition.*

Proof. (a) 1. Since the operator $H : W_\varphi(I;X) \to W_\psi(I;Y)$ is Lipschitzian, there exists a constant $\mu > 0$ such that $\|Hf_1 - Hf_2\|_\psi \le \mu\|f_1 - f_2\|_\varphi$ and, in particular, $p_\psi(Hf_1 - Hf_2) \le \mu\|f_1 - f_2\|_\varphi$ for all f_1, $f_2 \in W_\varphi(I;X)$. For $\|f_1 - f_2\|_\varphi > 0$, the last inequality (by Lemma 1(c)) is equivalent to the inequality

$$\mathsf{V}_\psi\left(\frac{Hf_1 - Hf_2}{\mu\|f_1 - f_2\|_\varphi}\right) \le 1;$$

therefore, from here and by the definitions of the operator H and the functional V_ψ, it follows that for any positive integer m and any $a \le \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m \le b$, we have

$$\sum_{i=1}^{m} \psi\left(\frac{\|h(\beta_i, f_1(\beta_i)) - h(\beta_i, f_2(\beta_i)) - h(\alpha_i, f_1(\alpha_i)) + h(\alpha_i, f_2(\alpha_i))\|}{\mu\|f_1 - f_2\|_\varphi}\right) \le 1. \tag{8}$$

For α, $\beta \in \mathbb{R}$, $\alpha < \beta$, we define the Lipschitzian function $\eta_{\alpha,\beta} : \mathbb{R} \to [0,1]$ by the following rule: $\eta_{\alpha,\beta}(s) = 0$ for $s \le \alpha$, $\eta_{\alpha,\beta}(s) = (s-\alpha)/(\beta-\alpha)$ for $\alpha \le s \le \beta$, and $\eta_{\alpha,\beta}(s) = 1$ for $s \ge \beta$.

2. Let us prove (6). Let x_1, $x_2 \in X$. If $a < t \le b$, then, setting $m = 1$, $\beta_1 = t$, and $\alpha_1 = a$ in (8), substituting the mappings $f_j(s) = \eta_{a,t}(s)x_j$, $s \in I$, $j = 1, 2$, in (8), and observing that $\|f_1 - f_2\|_\varphi = \|x_1 - x_2\|/\varphi^{-1}(1)$ by Lemma 3, we obtain inequality (6) with $\mu_0 = \mu\psi^{-1}(1)/\varphi^{-1}(1)$. If, on the other hand, $t = a$, then, setting $m = 1$, $\beta_1 = b$, and $\alpha_1 = a$ in (8) and substituting the mappings $f_j(s) = \big(1 - \eta_{a,b}(s)\big)x_j$, $s \in I$, $j = 1, 2$, into inequality (8), we obtain the inequality (6) with the desired constant $\mu_0 = \mu\psi^{-1}(1)\big(1 + 1/\varphi^{-1}(1)\big)$.

From the definition of h^*, we find that inequality (6) holds if one substitutes h^* for h in it. Thus, for any $t \in I$, the mapping $h^*(t, \cdot\,) : X \to Y$ is (Lipschitz) continuous.

3. Let us prove (7). Let $t \in (a, b]$, and let $a < \alpha_1 < \ldots < \beta_m < t$ in inequality (8), where m is arbitrary. We substitute the mappings $f_j(s) = \eta_m(s)x_1 + (2 - j)x_2$, $t \in I$, $x_j \in X$, $j = 1, 2$, in (8), where the function $\eta_m : I \to [0,1]$ is defined in the following way: $\eta_m(s) = 0$ for $a \le s \le \alpha_1$, $\eta_m(s) = \eta_{\alpha_i,\beta_i}(s)$ for $\alpha_i \le s \le \beta_i$, $i = 1, \ldots, m$, $\eta_m(s) = 1 - \eta_{\beta_i,\alpha_{i+1}}(s)$ for $\beta_i \le s \le \alpha_{i+1}$, $i = 1, \ldots, m-1$, and $\eta_m(s) = 1$ for $\beta_m \le s \le b$. Then we have

$$\sum_{i=1}^{m} \psi\left(\frac{\|h(\beta_i, x_1 + x_2) - h(\beta_i, x_1) - h(\alpha_i, x_2) + h(\alpha_i, 0)\|}{\mu\|x_2\|}\right) \le 1. \tag{9}$$

Since the constant mappings lie in $W_\varphi(I;X)$ and H takes its values in $W_\psi(I;Y)$, we have $h(\,\cdot\,, x) \in W_\psi(I;Y)$ for all $x \in X$. Taking into account the continuity of ψ and the definition of h^* and passing to the limit in (9) as $\alpha_1 \to t - 0$, we find that the following inequality holds for all $a < t \le b$:

$$\|h^*(t, x_1 + x_2) - h^*(t, x_1) - h^*(t, x_2) + h^*(t, 0)\| \le \mu\|x_2\|\psi^{-1}(1/m),$$

and, therefore, it holds also at the point $t = a$. In the limit, as $m \to \infty$, for all $t \in I$ and x_1, $x_2 \in X$, we obtain the relation

$$h^*(t, x_1 + x_2) - h^*(t, x_1) - h^*(t, x_2) + h^*(t, 0) = 0 \quad \text{in} \quad Y. \tag{10}$$

For each $t \in I$, we define the operator $S_t : X \to Y$ by the following rule: $S_t(x) = h^*(t, x) - h^*(t, 0)$, $x \in X$, and then we rewrite relation (10) in the form

$$S_t(x_1 + x_2) = S_t(x_1) + S_t(x_2), \quad x_1, x_2 \in X.$$

Thus, S_t is a continuous (by virtue of (6) for h^*) and additive operator; therefore, the assumption that X is real implies that $S_t \in L(X;Y)$ for all $t \in I$. Setting $h_0(t) = h^*(t, 0)$ and $h_1(t)x = S_t(x)$, $t \in I$, $x \in X$, we find that $h_0 \in Y^I$, $h_1 \in L(X;Y)^I$ and relation (7) holds. Noting that $h_0(\,\cdot\,) = h^*(\,\cdot\,, 0)$ and $h_1(\,\cdot\,)x = h^*(\,\cdot\,, x) - h^*(\,\cdot\,, 0)$, we conclude that h_0 and $h_1(\,\cdot\,)x$ belong to the space $W_\psi^*(I;Y)$ for all $x \in X$.

(b) To prove the converse statement, note that the operator H acts by the following rule: $(Hf)(t) = h_0(t) + h_1(t)f(t)$, $t \in I$, $f \in W_\varphi(I;X)$. Since $\psi \underset{0}{\preccurlyeq} \varphi$, $W_\varphi(I;X) \subset W_\psi(I;X)$ by Theorem 4(b); therefore, Theorem 4(a), if applied to the triple $(L(X;Y), X, Y)$ of LNSs, yields $h_1 f \in W_\psi(I;Y)$, so that H maps

$W_\varphi(I;X)$ into $W_\psi(I;Y)$. Once again applying Theorem 4(a, b), for all $f_1,\ f_2 \in W_\varphi(I;X)$ we obtain the inequality

$$\|Hf_1 - Hf_2\|_\psi = \|h_1(f_1 - f_2)\|_\psi \le \kappa(\varphi,\psi)\gamma(\psi)\|h_1\|_\psi\|f_1 - f_2\|_\varphi,$$

whence it follows that the operator H satisfies the Lipschitz condition. □

Remark 3. A statement similar to Theorem 7 also holds for the right regularization of the mapping $h(\,\cdot\,,x)$. However, generally speaking, one cannot substitute h for h^* in relation (7) (see the example in [18, p. 157], which is given for real-valued functions of Jordan bounded variation). For $X = Y = \mathbb{R}$ and $\varphi(\rho) = \rho$, Theorem 7 yields the results of [18].

Remark 4. For a BA X and for given $h_0,\ h_1 \in W_\varphi(I;X)$, it is not difficult to find the condition on the mapping h_1 under which the linear functional equation $x = h_0 + h_1 x$ is solvable for $x \in W_\varphi(I;X)$ using the contracting mapping principle, namely, $\|h_1\|_\varphi < 1/\gamma(\varphi)$. On the other hand, Theorem 7 means that one cannot directly apply the Banach principle mentioned above in the space $W_\varphi(I;X)$ for the solution of the equation $x = h(t,x)$ if the mapping $h(t,x)$ depends "nonlinearly" on the variable x.

The immediate consequence and generalization of Theorem 7 is the following theorem.

Theorem 8. *Let the Nemytskii operator* $H : (X_1 \times \cdots \times X_n)^I \to Y^I$ *be generated by a mapping* $h : I \times X_1 \times \cdots \times X_n \to Y$ *according to the following rule*:

$$H(f_1,\ldots,f_n)(t) = h\big(t, f_1(t),\ldots,f_n(t)\big),\quad t\in I,\quad f_i \in X_i^I,\quad i = 1,\ldots,n.$$

If $X_1,\ldots,X_n$ *are real LNSs,* Y *is a BS, and the operator* H *maps* $W_{\varphi_1}(I;X_1)\times\cdots\times W_{\varphi_n}(I;X_n)$ *into* $W_\psi(I;Y)$ *and is a Lipschitzian operator, then* $h^*(t,x_1,\ldots,x_n) = h_0(t) + \sum_{i=1}^n h_i(t)x_i$ *for all* $t \in I$ *and* $x_i \in X_i$, $i = 1,\ldots,n$, *where* $h_0 \in W_\psi^*(I;Y)$ *and* $h_i \in L(X_i;Y)^I$ *such that* $h_i(\,\cdot\,)x_i \in W_\psi^*(I;Y)$ *for all* $x_i \in X_i$, $i = 1,\ldots,n$.

Conversely, if X_i *is a BS,* $\psi \preccurlyeq_0 \varphi_i$, $h_i \in W_\psi(I;L(X_i;Y))$, $i = 1,\ldots,n$, Y *is an LNS,* $h_0 \in W_\psi(I;Y)$, *and* $h(t,x_1,\ldots,x_n) = h_0(t) + \sum_{i=1}^n h_i(t)x_i$ *for all* $t \in I$ *and* $x_i \in X_i$, $i = 1,\ldots,n$, *then* H *maps the direct product* $W_{\varphi_1}(I;X_1)\times\cdots\times W_{\varphi_n}(I;X_n)$ *into* $W_\psi(I;Y)$ *and is a globally Lipschitzian operator.*

Theorem 8, in turn, is extended to the case where $Y = Y_1 \times \cdots \times Y_k$.

We present a criterion for the Lipschitzness of a composition operator.

Corollary 9. *Let the Nemytskii operator* $H : (\mathbb{R}^n)^I \to \mathbb{R}^I$ *be generated by a function* $h : I \times \mathbb{R}^n \to \mathbb{R}$ *such that* $h^* = h$, *and let* $\psi \preccurlyeq_0 \varphi_i$, $i = 1,\ldots,n$. *The operator* H *maps* $W_{\varphi_1}(I;\mathbb{R})\times\cdots\times W_{\varphi_n}(I;\mathbb{R})$ *into* $W_\psi(I;\mathbb{R})$ *and is Lipschitzian if and only if* $h(t,x_1,\ldots,x_n) = h_0(t) + \sum_{i=1}^n h_i(t)x_i$ *for all* $t \in I$ *and* $x_i \in \mathbb{R}$, *where* $h_0,\ h_i \in W_\psi^*(I;\mathbb{R})$, $i = 1,\ldots,n$.

The results presented above are extended to multivalued mappings of bounded generalized φ-variation and multivalued composition operators ([6]); a detailed description of them is to appear.

Concluding this paper, we present a remark concerning composition operators on the space ℓ_φ, $\varphi \in \Phi$. We denote by ℓ_φ the space of all real sequences $x = \{x_n\}_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$ such that there exists a number $\lambda > 0$ (depending on x) for which $\sum_{n=1}^\infty \varphi\big(|x_n|/\lambda\big) < \infty$. The space ℓ_φ is an LNS with the norm

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : \sum_{n=1}^\infty \varphi\left(\frac{|x_n|}{\lambda}\right) \le 1\right\}$$

and with the BA, in which the following inequalities hold:

$$\sup_{n\in\mathbb{N}} |x_n| \le \varphi^{-1}(1)\|x\|_\varphi,$$
$$\|xy\|_\varphi \le \varphi^{-1}(1)\|x\|_\varphi\|y\|_\varphi \quad \text{for all } x,\, y \in \ell_\varphi,$$

where $xy = \{x_n y_n\}_{n=1}^\infty$ (see, e.g., [11, Sec. 3 and Theorem 10.3]).

The following example shows that an analog of Theorem 7 does not hold in the space ℓ_φ. For a function $h : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$, the composition operator $H : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is defined by the rule $(Hx)(n) = h(n, x_n)$, $n \in \mathbb{N}$, $x = \{x_n\}_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$. Let the function h be defined by the formula $h(n, x) = \sin x$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. Since

$$\sum_{n=1}^\infty \varphi\left(\frac{|\sin x_n|}{\|x\|_\varphi}\right) \le \sum_{n=1}^\infty \varphi\left(\frac{|x_n|}{\|x\|_\varphi}\right) \le 1,$$

we have $\|Hx\|_\varphi \le \|x\|_\varphi$ for all $x \in \ell_\varphi$, so that H maps ℓ_φ onto itself. Taking into account the inequalities

$$\sum_{n=1}^\infty \varphi\left(\frac{|\sin x_n - \sin y_n|}{\|x-y\|_\varphi}\right) \le \sum_{n=1}^\infty \varphi\left(\frac{|x_n - y_n|}{\|x-y\|_\varphi}\right) \le 1,$$

we find that $\|Hx - Hy\|_\varphi \le \|x - y\|_\varphi$ for all $x,\, y \in \ell_\varphi$.

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