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Families of Lagrangian fibrations on hyperkähler manifolds

Ljudmila Kamenova [a,*], Misha Verbitsky [b,*,1]

[a] *Department of Mathematics, Room 3-115, Stony Brook University, Stony Brook, NY 11794-3651, USA*

[b] *Laboratory of Algebraic Geometry, Faculty of Mathematics, National Research University HSE, 7 Vavilova Str., Moscow, Russia*

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A holomorphic Lagrangian fibration on a holomorphically symplectic manifold is a holomorphic map with Lagrangian fibers. It is known (due to Huybrechts) that a given compact manifold admits only finitely many holomorphic symplectic structures, up to deformation. We prove that a given compact, simple hyperkähler manifold with $b_2 \geqslant 7$ admits only finitely many deformation types of holomorphic Lagrangian fibrations. We also prove that all known hyperkähler manifolds are never Kobayashi hyperbolic.

Contents

* Corresponding authors.

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1. Introduction

Irreducible compact hyperkähler manifolds, or irreducible holomorphic symplectic manifolds, are a natural generalization of K3 surfaces in higher dimensions. The geometry of K3 surfaces is well studied. In particular, it is known that any two K3 surfaces are deformation equivalent to each other, i.e., there is only one deformation type of K3 surfaces.

A natural question to ask is whether the same is true in higher dimensions. The answer is negative due to Beauville's examples. In every possible complex dimension $2n$ there are at least the Hilbert scheme of n points on a K3 surface S, $\mathrm{Hilb}^n(S)$, and the generalized Kummer varieties $K^{n+1}(A)$, where A is an Abelian surface. These two examples are not deformation equivalent since they have different Betti numbers. There are two more exceptional examples due to K. O'Grady in dimensions 6 and 10.

It is conjectured that in every fixed dimension there are finitely many deformation types of irreducible compact hyperkähler manifolds. It is also conjectured that every hyperkähler manifold can be deformed to one that admits a holomorphic Lagrangian fibration. It would be interesting to classify the deformation types of the pairs (M, L) of a hyperkähler manifold together with a Lagrangian fibration on it. In the present paper, we show that the number of deformational classes of such pairs is finite, if one fixes the smooth manifold underlying M.

In [18] Huybrechts proved that for a fixed compact manifold there are at most finitely many deformation types of hyperkähler structures on it. Therefore, to prove that the number of deformation classes of pairs (M, L) is finite, it would suffice to prove it when a deformational class of M is fixed.

Let $M \xrightarrow{\pi} X$ be a Lagrangian fibration, where X is a normal projective variety. Then $H^2(X) = \mathbb{C}$, hence $\operatorname{rk}\operatorname{Pic}(X) = 1$. Therefore, the primitive ample bundle L_X on X is unique (up to torsion). Denote by L_M the semiample bundle $\pi^*(L_X)$ on M. Clearly, $c_1(L_M)^{\operatorname{rk} M} = 0$; a (1,1)-class satisfying this equation is called **parabolic**. The Lagrangian fibration $M \xrightarrow{\pi} X$ is uniquely determined by a class $[c_1(L_M)] \in \operatorname{Pic}(M)$ which is parabolic and semiample (this is due to D. Matsushita, [23]; see [29] for a detailed exposition of an early work on Lagrangian fibrations). Therefore, to classify the Lagrangian fibrations it would suffice to classify pairs (M, L_M), where L_M is a parabolic semiample line bundle.

We prove that in the Teichmüller space of hyperkähler manifolds with a fixed parabolic class the pairs admitting a Lagrangian fibration form a dense and open subset. The other main result is that the action of the monodromy group has finitely many orbits. As a corollary of these results we obtain that for a fixed compact manifold, there are only finitely many deformation types of hyperkähler structures equipped with a Lagrangian fibration.

1.1. Hyperkähler manifolds

Definition 1.1. A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

Definition 1.2. A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Theorem 1.3 *(Bogomolov's Decomposition Theorem). (See [4,3].) Any hyperkähler manifold admits a finite covering, which is a product of a torus and several simple hyperkähler manifolds.* $\square$

Remark 1.4. Further on, all hyperkähler manifolds are assumed to be simple.

A note on terminology. Speaking of hyperkähler manifolds, people usually mean one of two different notions. One either speaks of holomorphically symplectic Kähler manifold, or of a manifold with a *hyperkähler structure*, that is, a triple of complex structures satisfying quaternionic relations and parallel with respect to the Levi-Civita connection. The equivalence (in compact case) between these two notions is provided by the Yau's solution of Calabi's conjecture [3]. Throughout this paper, we use the complex algebraic geometry point of view, where "hyperkähler" is synonymous with "Kähler holomorphically symplectic", in lieu of the differential-geometric approach. The reader may check [3] for an introduction to hyperkähler geometry from the differential-geometric point of view.

Notice also that we included compactness in our definition of a hyperkähler manifold. In the differential-geometric setting, one does not usually assume that the manifold is compact.

1.2. The Bogomolov–Beauville–Fujiki form

Theorem 1.5. *(See [13].) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some integer quadratic form q on $H^2(M)$ and a constant $c > 0$.* $\square$

Definition 1.6. This form is called **Bogomolov–Beauville–Fujiki form**. It is defined by this relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville; [2], [17, 23.5])

$$\lambda q(\eta,\eta) = (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}$$

$$+ (1-n)\frac{(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n)(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1})}{\int_M \Omega^n \wedge \overline{\Omega}^n}$$

where Ω is the holomorphic symplectic form, and λ a positive constant.

Remark 1.7. The form q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on the space $\langle \Omega, \overline{\Omega}, \omega \rangle$ where ω is a Kähler form, as seen from the following formula

$$\mu q(\eta_1, \eta_2) = \int_X \omega^{2n-2} \wedge \eta_1 \wedge \eta_2 - \frac{2n-2}{(2n-1)^2} \frac{\int_X \omega^{2n-1} \wedge \eta_1 \cdot \int_X \omega^{2n-1} \wedge \eta_2}{\int_M \omega^{2n}}, \quad \mu > 0 \quad (1.1)$$

(see e.g. [33, Theorem 6.1], or [17, Corollary 23.9]).

Definition 1.8. Let $[\eta] \in H^{1,1}(M)$ be a real (1,1)-class on a hyperkähler manifold M. We say that $[\eta]$ is **parabolic** if $q([\eta], [\eta]) = 0$. A line bundle L is called **parabolic** if $c_1(L)$ is parabolic.

1.3. The hyperkähler SYZ conjecture

Theorem 1.9. *(See D. Matsushita, [23].) Let $\pi : M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, with $0 < \dim X < \dim M$. Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian tori (this means that the symplectic form vanishes on the fibers).*[2]

Definition 1.10. Such a map π to a normal projective variety X is called **a holomorphic Lagrangian fibration**.

Remark 1.11. The base of π is conjectured to be rational. J.-M. Hwang [19] proved that $X \cong \mathbb{C}P^n$, if it is smooth. D. Matsushita [24] proved that it has the same rational cohomology as $\mathbb{C}P^n$.

Remark 1.12. The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be used to determine the topology of M [21,15].

Definition 1.13. Let (M, ω) be a Calabi–Yau manifold, Ω the holomorphic volume form, and $Z \subset M$ a real analytic subvariety, Lagrangian with respect to ω. If $\Omega|_Z$ is proportional to the Riemannian volume form, Z is called **special Lagrangian** (SpLag).

[2] Here, as elsewhere, we assume that the hyperkähler manifold M is simple.

The special Lagrangian varieties were defined in [16] by Harvey and Lawson, who proved that they minimize the Riemannian volume in their cohomology class. This implies, in particular, that their moduli are finite-dimensional. In [26], McLean studied deformations of non-singular special Lagrangian subvarieties and showed that they are unobstructed.

In [30], Strominger, Yau and Zaslow tried to explain the mirror symmetry phenomenon using the special Lagrangian fibrations. They conjectured that any Calabi–Yau manifold admits a Lagrangian fibration with special Lagrangian fibers. Taking its dual fibration, one obtains “the mirror dual” Calabi–Yau manifold.

Definition 1.14. A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

Remark 1.15. From semiampleness it obviously follows that L is nef. Indeed, let $\pi : M \longrightarrow \mathbb{P}H^0(L^N)^*$ be the standard map. Since the sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of L is a pullback of the Kähler form on $\mathbb{C}P^n$. However, the converse is false: a nef bundle is not necessarily semiample (see e.g. [11, Example 1.7]).

Remark 1.16. Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X an ample class on X. Then $\eta := \pi^*\omega_X$ is semiample and parabolic. The converse is also true, by Matsushita's theorem: if L is semiample and parabolic, L induces a Lagrangian fibration. This is the only known source of non-trivial special Lagrangian fibrations.

Conjecture 1.17 *(Hyperkähler SYZ conjecture). Let L be a parabolic nef line bundle on a hyperkähler manifold. Then L is semiample.*

Remark 1.18. This conjecture was stated by many people (Tyurin, Bogomolov, Hassett, Tschinkel, Huybrechts, Sawon); please see [29] for an interesting and historically important discussion, and [31] for details and references.

Remark 1.19. The SYZ conjecture can be seen as a hyperkähler version of the “abundance conjecture” (see e.g. [12, 2.7.2]).

Claim 1.20. *Let M be an irreducible hyperkähler manifold in one of 4 known classes known, that is, a deformation of a Hilbert scheme of points on K3, a deformation of generalized Kummer variety, or a deformation of one of two examples by O'Grady. Then M admits a deformation equipped with a holomorphic Lagrangian fibration.*

Proof. When S is an elliptic K3 surface, the Hilbert scheme of points $\mathrm{Hilb}^n(S)$ has an induced Lagrangian fibration with smooth fibers that are products of n elliptic curves: $\mathrm{Hilb}^n(S) \to Sym^n(\mathbb{P}^1) \simeq \mathbb{P}^n$. Similarly, when A is an elliptic Abelian surface, the generalized Kummer variety $K^n(A)$ admits a Lagrangian fibration. Another construction gives Lagrangian fibrations on $\mathrm{Hilb}^n(S)$ and on $K^n(A)$ if S contains a smooth genus n curve and if A contains a smooth genus $n+2$ curve (see Examples 3.6 and 3.8 in [29]).

O'Grady's examples are deformation equivalent to Lagrangian fibrations, as follows from Corollary 1.1.10 in [28]. $\square$

2. Hyperkähler geometry: preliminary results

2.1. Teichmüller space and the moduli space

Here we cite the relevant result from the deformation theory of hyperkähler manifolds. We follow [32].

Let M be a hyperkähler manifold (compact and simple, as usual), and Comp_0 be the Frèchet manifold of all complex structures of hyperkähler type on M. The quotient $\mathrm{Teich} := \mathrm{Comp}_0 / \mathrm{Diff}^0$ of Comp_0 by isotopies is a finite-dimensional complex analytic space [9]. This quotient is called the **Teichmüller space** of M. When M is a complex curve, the quotient $\mathrm{Comp}_0 / \mathrm{Diff}^0$ is the Teichmüller space of this curve.

The mapping class group $\Gamma = \mathrm{Diff}^+ / \mathrm{Diff}^0$ acts on Teich in the usual way, and its quotient Mod is the **moduli space** of M.

As shown in [18], Teich has a finite number of connected components. Take a connected component Teich^I containing a given complex structure I, and let $\Gamma^I \subset \Gamma$ be the set of elements of Γ fixing this component. Since Teich has only a finite number of connected components, Γ^I has finite index in Γ. On the other hand, as shown in [32], the image of the group Γ is commensurable to $O(H^2(M,\mathbb{Z}), q)$.

In [32, Lemma 2.6] it was proved that any hyperkähler structure on a given simple hyperkähler manifold is also simple. Therefore, $H^{2,0}(M, I') = \mathbb{C}$ for all $I' \in \mathrm{Comp}$. This trivial observation is a key to the following well-known definition.

Definition 2.1. Let (M, I) be a hyperkähler manifold, and Teich its Teichmüller space. Consider a map $\mathsf{Per} : \mathrm{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$, sending J to the line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. It is easy to see that Per maps Teich into the open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{ l \in \mathbb{P}H^2(M,\mathbb{C}) \mid q(l,l) = 0,\ q(l,\bar{l}) > 0 \}.$$

The map $\mathsf{Per} : \mathrm{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ is called the **period map**, and the set $\mathbb{P}\mathsf{er}$ the **period space**.

The following fundamental theorem is due to F. Bogomolov [5].

Theorem 2.2. *Let M be a simple hyperkähler manifold, and* Teich *its Teichmüller space. Then the period map* $\mathsf{Per} : \mathrm{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ *is a local diffeomorphism (that is, an etale map). Moreover, it is holomorphic.* $\square$

Remark 2.3. Bogomolov's theorem implies that Teich is smooth. However, it is not necessarily Hausdorff (and it is non-Hausdorff even in the simplest examples). □

2.2. The polarized Teichmüller space

In [34, Corollary 2.6], the following proposition was deduced from [6] and [10].

Theorem 2.4. *Let M be a simple hyperkähler manifold, such that all integer $(1,1)$-classes satisfy $q(\nu,\nu) \geqslant 0$. Then its Kähler cone is one of the two connected components of the set $K := \{\nu \in H^{1,1}(M,\mathbb{R}) \mid q(\nu,\nu) > 0\}$.* □

Remark 2.5. From Theorem 2.4 it follows that on a hyperkähler manifold with $\operatorname{Pic}(M) = \mathbb{Z}$, for any rational class $\eta \in H^{1,1}(M)$ with $q(\eta,\eta) \geqslant 0$, either η or $-\eta$ is nef.

Remark 2.6. Consider an integer vector $\eta \in H^2(M)$ which is positive, that is, satisfies $q(\eta,\eta) > 0$. Denote by $\operatorname{Teich}^\eta$ the set of all $I \in \operatorname{Teich}$ such that η is of type $(1,1)$ on (M,I). The space $\operatorname{Teich}^\eta$ is a closed divisor in Teich. Indeed, by Bogomolov's theorem, the period map $\mathsf{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ is etale, but the image of $\operatorname{Teich}^\eta$ is the set of all $l \in \mathbb{P}\mathsf{er}$ which are orthogonal to η; this condition defines a closed divisor C_η in $\mathbb{P}\mathsf{er}$, hence $\operatorname{Teich}^\eta = \mathsf{Per}^{-1}(C_\eta)$ is also a closed divisor.

Remark 2.7. When $I \in \operatorname{Teich}^\eta$ is generic, Bogomolov's theorem implies that the space of rational $(1,1)$-classes $H^{1,1}(M,\mathbb{Q})$ is one-dimensional and generated by η. This is seen from the following argument. Locally around a given point I the period map $\operatorname{Teich}^\eta \longrightarrow \mathbb{P}\mathsf{er}$ is surjective on the set $\mathbb{P}\mathsf{er}^\eta$ of all $I \in \mathbb{P}\mathsf{er}$ for which $\eta \in H^{1,1}(M,I)$. However, the Hodge–Riemann relations give

$$\mathbb{P}\mathsf{er}^\eta = \bigl\{l \in \mathbb{P}\mathsf{er} \bigm| q(\eta,l) = 0\bigr\}. \tag{2.1}$$

Denote the set of such points of $\operatorname{Teich}^\eta$ by $\operatorname{Teich}^\eta_{\mathrm{gen}}$. It follows from Theorem 2.4 that, for any $I \in \operatorname{Teich}^\eta_{\mathrm{gen}}$, either η or $-\eta$ is a Kähler class on (M,I).

Consider a connected component $\operatorname{Teich}^{\eta,I}$ of $\operatorname{Teich}^\eta$. Changing the sign of η if necessary, we may assume that η is Kähler on (M,I). By Kodaira's theorem about stability of Kähler classes, η is Kähler in some neighborhood $U \subset \operatorname{Teich}^{\eta,I}$ of I. Therefore, the sets

$$V_+ := \bigl\{I \in \operatorname{Teich}^\eta_{\mathrm{gen}} \bigm| \eta \text{ is Kähler on } (M,I)\bigr\}$$

and

$$V_- := \bigl\{I \in \operatorname{Teich}^\eta_{\mathrm{gen}} \bigm| -\eta \text{ is Kähler on } (M,I)\bigr\}$$

are open in $\operatorname{Teich}^\eta_{\mathrm{gen}}$. It is easy to see that $\operatorname{Teich}^\eta_{\mathrm{gen}}$ is a complement to a union of countably many divisors in $\operatorname{Teich}^\eta$ corresponding to the points $I' \in \operatorname{Teich}^\eta$ with $\operatorname{rk}\operatorname{Pic}(M, I') > 1$. Therefore, for any connected open subset $U \subset \operatorname{Teich}^\eta$, the intersection $U \cap \operatorname{Teich}^\eta_{\mathrm{gen}}$ is connected. Since $\operatorname{Teich}^\eta_{\mathrm{gen}}$ is represented as a disjoint union of open sets $V_+ \sqcup V_-$, every connected component of $\operatorname{Teich}^\eta_{\mathrm{gen}}$ and of $\operatorname{Teich}^\eta$ is contained in V_+ or in V_-. We obtained the following corollary.

Corollary 2.8. *Let $\eta \in H^2(M)$ be a positive integer vector,* $\operatorname{Teich}^\eta$ *the corresponding divisor in the Teichmüller space, and* $\operatorname{Teich}^{\eta,I}$ *a connected component of* $\operatorname{Teich}^\eta$ *containing a complex structure I. Assume that η is Kähler on (M, I). Then η is Kähler for all $I' \in \operatorname{Teich}^{\eta,I}$ which satisfy $\operatorname{rk} H^{1,1}(M, \mathbb{Q}) = 1$.* $\square$

We call the set $\operatorname{Teich}^\eta_{\mathrm{pol}}$ of all $I \in \operatorname{Teich}^\eta$ for which η is Kähler the **polarized Teichmüller space**, and η its **polarization**. From the above arguments it is clear that the polarized Teichmüller space $\operatorname{Teich}^\eta_{\mathrm{pol}}$ is open and dense in $\operatorname{Teich}^\eta$.

The quotient $\mathcal{M}_\eta$ of $\operatorname{Teich}^\eta_{\mathrm{pol}}$ by the subgroup of the mapping class group fixing η is called the **moduli of polarized hyperkähler manifolds**. It is known (due to the general theory which goes back to Viehweg and Grothendieck) that $\mathcal{M}_\eta$ is Hausdorff and quasiprojective (see e.g. [36] and [14]).

Remark 2.9. We conclude that there are countably many quasiprojective divisors $\mathcal{M}_\eta$ immersed in the moduli space Mod of hyperkähler manifolds. Moreover, every algebraic complex structure belongs to one of these divisors. However, these divisors need not be closed. Indeed, as proven in [1], each of $\mathcal{M}_\eta$ is dense in Mod.

In [1, Theorem 1.7], the following theorem was proven.

Theorem 2.10. *Let M be a compact, simple hyperkähler manifold,* Teich^I *a connected component of its Teichmüller space, and* $\operatorname{Teich}^I \xrightarrow{\Psi} \operatorname{Teich}^I / \Gamma^I = \operatorname{Mod}$ *its projection to the moduli space of complex structures. Consider a positive or negative vector $\eta \in H^2(M, \mathbb{Z})$, and let* $\operatorname{Teich}^{I,\eta}$ *be the corresponding connected component of the polarized Teichmüller space. Assume that $b_2(M) > 3$. Then the image $\Psi(\operatorname{Teich}^{I,\eta})$ is dense in* Mod.

The proof relies on a more general proposition about lattices.

Proposition 2.11. *(See [1, Proposition 3.2, Remark 3.12].) Let V be an $\mathbb{R}$-vector space equipped with a non-degenerate symmetric form of signature (s_+, s_-) with $s_+ \geq 3$ and $s_- \geq 1$. Consider a lattice $L \subset V$. Let Γ be a subgroup of finite index in $\mathrm{O}(L)$, and $l \in L$. Then $\Gamma \cdot \mathrm{Gr}_{++}(l^\perp)$ is dense in $\mathrm{Gr}_{++}(V)$.*

Remark 2.12. Since the proof of this statement is symmetric in s_+ and s_-, the same proposition is valid if we assume that $s_+ \geq 1$ and $s_- \geq 3$.

3. Main results

3.1. The moduli of manifolds with Lagrangian fibrations

Here we assume that $b_2(M) \geqslant 7$ as we need it for our proof of Theorem 3.4. The authors conjecture that the result is valid for smaller Betti numbers as well.

Definition 3.1. Let L be a holomorphic line bundle on a hyperkähler manifold. We call L **Lagrangian** if it is parabolic and semiample.

Definition 3.2. Let M be a hyperkähler manifold. Fix a parabolic class $L \in H^2(M, \mathbb{Z})$. We denote by Teich_L the Teichmüller space of all complex structures I of hyperkähler type on M such that L is of type $(1,1)$ on (M, I). Clearly, Teich_L is a divisor in the whole Teichmüller space of M. The space Teich_L is called **the Teichmüller space of hyperkähler manifolds with parabolic class**.

Matsushita proves the following openness result in [25, Theorem 1.1]:

Theorem 3.3. *Let* $\mathrm{Teich}_L^\circ \subset \mathrm{Teich}_L$ *be the set of all* $I \in \mathrm{Teich}_L$ *for which* L *is Lagrangian. Then* Teich_L° *is open in* Teich_L*.* □

The main results of the present paper are the following two theorems.

Theorem 3.4. *The subspace* $\mathrm{Teich}_L^\circ \subset \mathrm{Teich}_L$ *is dense and open in* Teich_L *under the condition that* L *or* $-L$ *gives a Lagrangian fibration for some deformation of* M*.*

Proof. Fix a positive class $\eta \in H^2(M, \mathbb{Z})$ and define $\mathrm{Teich}_{L,\eta}^\circ$ to be the open subset of Teich_L° for which η is a polarization. Consider the projection Ψ to the moduli space Mod as defined in Theorem 2.10. Since $\mathcal{M}_\eta$ is quasiprojective (see [36]), then $\Psi(\mathrm{Teich}_{L,\eta}^\circ)$ is Zariski open, and therefore dense in $\Psi(\mathrm{Teich}_{L,\eta})$.

Fix a negative vector $L' \in H^2(M, \mathbb{Z})$ such that the sublattice $\langle L, L' \rangle$ is of rank 2. Notice that $\Psi(\mathrm{Teich}_L) = \{l \in \mathbb{P}H^2(M, \mathbb{Z}) \mid q(l,l) = 0, q(l, \bar{l}) > 0, q(L, l) = 0\}/\Gamma_L$ and $\Psi(\mathrm{Teich}_{L,\eta}) = \{l \in \mathbb{P}H^2(M, \mathbb{Z}) \mid q(l,l) = 0, q(l, \bar{l}) > 0, q(L, l) = 0, q(\eta, l) = 0\}/\Gamma_{L,\eta}$. Applying Proposition 2.11 to the quotient $H^2(M, \mathbb{Z})/\langle L, L' \rangle$, we see that $\Psi(\mathrm{Teich}_{L,L',\eta})$ is dense in $\Psi(\mathrm{Teich}_{L,L'})$ for any L'. Here we needed to assume $b_2 \geqslant 7$, because $H^2(M, \mathbb{Z})$ is of signature $(3, b_2 - 3)$ and the quotient $H^2(M, \mathbb{Z})/\langle L, L' \rangle$ is of signature $(2, b_2 - 4)$. This satisfies the conditions of Proposition 2.11 since $b_2 - 4 \geqslant 3$.

However, $\bigcup_{L'} \Psi(\mathrm{Teich}_{L,L'})$ is dense in $\Psi(\mathrm{Teich}_L)$, and $\bigcup_{L'} \Psi(\mathrm{Teich}_{L,L',\eta})$ is dense in $\Psi(\mathrm{Teich}_{L,\eta})$. Therefore, $\Psi(\mathrm{Teich}_{L,\eta})$ is dense in $\Psi(\mathrm{Teich}_L)$ and Teich_L° is dense in Teich_L. □

Remark 3.5. Together with Proposition 2.11, Theorem 3.4 implies that the set of manifolds with Lagrangian fibrations is dense within the deformation space of a hyperkähler manifold M, if M admits a Lagrangian fibration.

Theorem 3.6. *Consider the action of the monodromy group Γ_I on $H^2(M,\mathbb{Z})$, and let $S \subset H^2(M,\mathbb{Z})$ be the set of all classes which are parabolic and primitive. Then there are only finitely many orbits of Γ_I on S.*

Proof. In the proof we use Nikulin's technique of discriminant-forms described in [27].

Denote by Λ the lattice $(H^2(M,\mathbb{Z}),q)$. It is a free $\mathbb{Z}$-module of finite rank together with a non-degenerate symmetric bilinear form q with values in $\mathbb{Z}$. If $\{e_i\}_{i\in I}$ is a basis of the lattice Λ, its *discriminant* is defined to be $\operatorname{discr}(\Lambda) = \det(e_i \cdot e_j)$. There is a canonical embedding $\Lambda \hookrightarrow \Lambda^* = \operatorname{Hom}(\Lambda,\mathbb{Z})$ using the bilinear form of Λ. The *discriminant group* $A_\Lambda = \Lambda^*/\Lambda$ is a finite Abelian group of order $|\operatorname{discr}(\Lambda)|$. One can extend the bilinear form to Λ^* with values in $\mathbb{Q}$ and define the *discriminant-bilinear form* of the lattice $b_\Lambda : A_\Lambda \times A_\Lambda \to \mathbb{Q}/\mathbb{Z}$. It is a finite non-degenerate form. A subgroup $H \subset A_\Lambda$ is *isotropic* if $q_\Lambda|_H = 0$, where q_Λ is the quadratic form corresponding to b_Λ. Given any subset $K \subset \Lambda$, its *orthogonal complement* is $K^\perp = \{v \in \Lambda \mid (v,K) = 0\}$.

An embedding of lattices $\Lambda_1 \hookrightarrow \Lambda_2$ is *primitive* if Λ_2/Λ_1 is a free $\mathbb{Z}$-module. Take a primitive vector $v \in \Lambda$ with $q(v) = 0$. We can choose a vector $f \in \Lambda$ with minimal positive quadratic intersection $\alpha = q(v,f)$. Then $0 < \alpha \leqslant |\operatorname{discr}(\Lambda)|$. It is implied by the following lemma:

Lemma 3.7. The minimal positive intersection α divides $\operatorname{discr}(\Lambda)$.

Proof. Since v is primitive, we can choose a free $\mathbb{Z}$-basis $\{v_1 = v, v_2, \dots, v_n\}$ of Λ, where $n = \operatorname{rk}(\Lambda)$. If $\alpha = \min\{q(v,f) \mid f \in \mathbb{Z}^n\}$, then $\alpha\mathbb{Z}$ is an ideal generated by $\{q(v,v_i), i = 1,\dots,n\}$. For every $i = 1,\dots,n$, $q(v,v_i) = \alpha \cdot a_i$ for some $a_i \in \mathbb{Z}$. Thus the matrix $[q(v_j,v_i)]$ has first column divisible by α. Then $\det[q(v_j,v_i)] = \operatorname{discr}(\Lambda)$ is divisible by α. $\square$

Let K be the primitive sublattice of Λ spanned by v and f. The intersection matrix of $\operatorname{Span}(v,f)$ has determinant $q(v,v)q(f,f) - q(v,f)^2 = -\alpha^2$ which is bounded: $-|\operatorname{discr}(\Lambda)|^2 \leqslant -\alpha^2 < 0$. Since $\operatorname{rk}(K) = 2$, K has at most four primitive isotropic vectors $(2\operatorname{rk}(K) = 4)$.

An *overlattice* of Λ is a lattice embedding $i : \Lambda \to \Lambda'$ with Λ and Λ' of the same rank, or equivalently, such that $H_{\Lambda'} = \Lambda'/\Lambda$ is a finite Abelian group. Note that we have the inclusions: $\Lambda \hookrightarrow \Lambda' \hookrightarrow \Lambda'^* \hookrightarrow \Lambda^*$. Therefore, $H_{\Lambda'} \subset \Lambda'^*/\Lambda \subset \Lambda^*/\Lambda = A_\Lambda$.

Proposition 3.8. *(See [27, Proposition 1.4.1].) The correspondence $\Lambda' \to H_{\Lambda'}$ determines a bijection between overlattices of Λ and isotropic subgroups of A_Λ. Furthermore, $H_{\Lambda'}^\perp = \Lambda'^*/\Lambda$ and $H_{\Lambda'}^\perp/H_{\Lambda'} = A_{\Lambda'}$.*

Let $L = K^\perp$ be the orthogonal complement of K in Λ. Then $K \oplus L \subset \Lambda \subset K^* \oplus L^*$. Since $\det(L)$ is bounded, in view of Proposition 3.8, there are finitely many ways of expressing Λ as an overlattice of $\Lambda_K \doteq K \oplus K^\perp$ because A_Λ is finite of order $|\mathrm{discr}(\Lambda)|$ and there are finitely many isotropic subgroups.

Define the lattices Λ and Λ' to be *stably equivalent* if there exists a lattice M such that $\Lambda \oplus M \simeq \Lambda' \oplus M$. The following proposition is a reformulation of Theorem 1.1 in Chapter 9 of Cassels's book [8].

Proposition 3.9. *There exist only a finite number of lattices stably equivalent to Λ.* □

If we assume that there are infinitely many orbits of Γ_I, this would imply that there exist infinitely many non-isomorphic pairs of lattices $(K, K^\perp)$. Then for infinitely many of them $K^\perp$ would be stably equivalent to $K_1^\perp$ for another K_1 since there are only finitely many choices for K. This contradicts Proposition 3.9 and the result follows. □

Corollary 3.10. *For any hyperkähler manifold, there are only finitely many orbits of Γ_I on the set of all divisors* Teich_L *with a parabolic class.*

Combining Corollary 3.10 and Theorem 3.4, we obtain the following result.

Corollary 3.11. *Let M be a hyperkähler manifold. Then there are only finitely many deformation types of Lagrangian fibrations $(M, I) \longrightarrow S$, for all complex structures on M.*

Proof. By Remark 2.7 we can assume that $H^{1,1}(M, \mathbb{Q})$ is one-dimensional and generated by a parabolic class L. Since either L or $-L$ is nef, we can assume L to be nef. From Theorem 3.4 it follows that for each pair (M, L) there exists a unique deformation type of a fibration structure. We conclude finiteness of the deformation types of Lagrangian fibrations since there are finitely many orbits of Γ_I on the set Teich_L. □

3.2. Kobayashi hyperbolicity in hyperkähler geometry

Definition 3.12. A compact manifold M is called **Kobayashi hyperbolic** if any holomorphic map $\mathbb{C} \longrightarrow M$ is constant.

For an introduction to the hyperbolic geometry, please see [22].

As an application of Theorem 3.4, we obtain the following result.

Theorem 3.13. *Let M be an irreducible holomorphic symplectic manifold in one of 4 known classes known, that is, a deformation of a Hilbert scheme of points on K3, a deformation of generalized Kummer variety, or a deformation of one of two examples by O'Grady. Then M is not Kobayashi hyperbolic.*

Proof. From Brody's lemma it follows that a limit of non-hyperbolic manifolds is again non-hyperbolic. Therefore, it would suffice to find a dense set of non-hyperbolic manifolds within the moduli space. A hyperkähler manifold admitting a holomorphic Lagrangian fibration is non-hyperbolic, because it contains complex tori. As follows from Claim 1.20, all known types of hyperkähler manifolds admit a deformation which has a Lagrangian fibration. By Remark 3.5, such deformations are dense in the moduli. □

It is conjectured that all hyperkähler and Calabi–Yau manifolds are not hyperbolic. The strongest result about non-hyperbolicity of hyperkähler manifolds so far was due to F. Campana, who proved in [7] that any twistor family of a hyperkähler manifold has at least one fiber which is non-hyperbolic.[3]

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[3] After the completion of this paper, the second named author proved that all hyperkähler manifolds are Kobayashi non-hyperbolic in [35]. Moreover, both authors together with S. Lu proved vanishing of the Kobayashi pseudometric for K3 surfaces and for many classes of hyperkähler manifolds [20].

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