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SHORT COMMUNICATIONS

Conditions for the Discreteness of Extremal Probability Measures (the Finite-Dimensional Case)

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In this paper, we establish conditions for the discreteness of extremal probability measures on finite-dimensional spaces. This problem appears in Choquet theory [1]–[6], stochastic financial mathematics [7]–[9], in the construction of examples of the solution of the Monge–Kantorovich problem [10].

The proof the main result of the paper is not based on these papers.

1. Recall some notation and definitions. Let $(E, \mathscr{E})$ be a measurable space, and let $M(E)$ be the set of probability measures on $(E, \mathscr{E})$. Let $f\colon E \to \mathbb{R}^+$ be any $\mathscr{E}$-measurable positive bounded function. We shall use the following notation:

i) $\operatorname{extr} A$ is the set of extreme points of the set A;

ii) $I^\mu \triangleq |\operatorname{supp}\mu|$, $\mu \in M(E)$.

2. To state the main assertion, we shall need the following definition and auxiliary statement.

Definition 1. A probability measure μ^* on $(E, \mathscr{E})$ is said to be *extremal* with respect to the set of probability measures $\mathfrak{R} \subseteq M(E)$ if, for any $\mathscr{E}$-measurable bounded function $f\colon E \to \mathbb{R}^+$, the following relation holds:

$$\sup_{\mu \in \mathfrak{R}} \int_E f(x)\,\mu(dx) = \int_E f(x)\,\mu^*(dx). \tag{1}$$

Proposition 1. *There exists an extremal probability measure μ^* with respect to the set $\mathfrak{R} \subseteq M(E)$ if and only if $\mathfrak{R}$ is a weakly relatively compact set.*

Remark 1. The sufficiency of the condition in Proposition 1 is a consequence of the fact that $\mathfrak{R}$ is a a weakly relatively compact set, of the definition of an upper bound, and of the Dunford–Pettis theorem. The proof of this statement repeats almost word-for-word the proof of Theorem 5 from [9]. The necessity is obvious.

Remark 2. Let $\mathfrak{R} \subseteq M(E)$ satisfy the inequality $\sup_{\mu\in\mathfrak{R}} \int_E |x|\,\mu(dx) < \infty$. It is well known [11, Sec. 2, Chap. III] that, in this case, $\mathfrak{R}$ is a weakly relatively compact set and the extremal (with respect to it) probability measure μ^* and the finite Lebesgue integral $m^* \triangleq \int_E x\,\mu^*(dx)$ exist.

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3. Let

$$D \triangleq \{\gamma \in \mathbb{R}^d : \int_E e^{f(x)-(\gamma,x-m^*)}\,\mu^*(dx) < \infty\}.$$

Obviously, $D \neq \varnothing$. The following statement is the main result of the present paper.

Theorem. *The following assertions are valid*:

1) *Let* $E = \mathbb{R}^d$, $d < \infty$. *The extremal (with respect to the set* $\mathfrak{R}$*) probability measure* μ^* *is discrete if and only if there exists a* $\gamma^* \in D$ *such that the following inequality holds*:

$$\int_E e^{f(x)-(\gamma^*,x-m^*)}\,\mu^*(dx) \leq \int_E e^{f(x)-(\gamma,x-m^*)}\,\mu^*(dx), \tag{2}$$

where γ *is any one from* D *and* $I^{\mu^*} \leq d+1$.

2) *Let* E *be a* d*-dimensional compact set,* $d < \infty$. *Then there exists a discrete probability measure* $\mu^* \in M(E)$, $I^{\mu^*} \leq d+1$, *such that the following relations hold*:

$$\sup_{\mu\in\mathfrak{R}} \int_E f(x)\,\mu(dx) = \int_E f(x)\,\mu^*(dx) = \sum_{i=1}^{I^{\mu^*}} c_i f(x_i), \tag{3}$$

where $x_i \in \operatorname{extr} E$, $1 \leq i \leq I^{\mu^*}$; *further,* $c_i \triangleq \mu^*(\{x_i\}) > 0$ *and* $\sum_{i=1}^{I^{\mu^*}} c_i x_i = m^*$, $\sum_{i=1}^{I^{\mu^*}} c_i = 1$.

4. This section is devoted to the proof of the theorem, which is based on the solution of the auxiliary problem considered below.

4.1. Consider the auxiliary problem

$$\int_E e^{f(x)-(\gamma,x-m^*)}\,\mu^*(dx) \to \inf_{\gamma\in D}. \tag{4}$$

Definition 2. By a *solution of problem* (4) we mean a vector $\gamma^* \in D$ such that

$$v \triangleq \inf_{\gamma\in D} \int_E e^{f(x)-(\gamma,x-m^*)}\,\mu^*(dx) = \int_E e^{f(x)-(\gamma^*,x-m^*)}\,\mu^*(dx). \tag{5}$$

4.2. The proof of the theorem is based on the solvability of problem (4). We shall need some auxiliary statements.

Proposition 2. *The solution of problem* (4) *exists if and only if inequality* (2) *holds.*

Proof. The proof of the assertion of Proposition 2 is obvious. □

The validity of assertion 1) of the theorem follows from the following proposition.

Proposition 3. *The following assertions are equivalent*:

1) γ^* *is a solution of problem* (4);

2) *any measurable bounded function* $f(x)$ *admits the unique representation*

$$f(x) = \int_E f(x)\,\mu^*(dx) + (\gamma^*, x-m^*); \tag{6}$$

3) μ^* *is discrete and* $I^{\mu^*} \leq d+1$.

Proof of Proposition 3. *The implication* 1)⇒2). Let $g\colon E \to \mathbb{R}^+ \setminus \{0\}$ be any $\mathscr{E}$-measurable bounded function. For any $A \in \mathscr{E}$, set

$$\mu(A) \triangleq \int_A \frac{g(x)}{\int_E g(x)\,\mu^*(dx)}\,\mu^*(dx). \tag{7}$$

Obviously,

i) $\mu \neq \mu^*$;

ii) $\mu \in \mathfrak{R}$.

Note that the measure μ^* dominates any measure $\widetilde{\mu} \in \mathfrak{R}$, and hence $\widetilde{\mu} \ll \mu^*$. Therefore, the following inequality holds:

$$v \triangleq \int_E e^{f(x)-(\gamma^*,x-m^*)}\,\mu^*(dx) \geq \int_E e^{f(x)-(\gamma^*,x-m^*)}\,\mu(dx),$$

which, in view of (7), can be rewritten as

$$\int_E g(x)\exp\{f(x)-(\gamma^*,x-m^*)-\ln v\}\,\mu^*(dx) \leq \int_E g(x)\,\mu^*(dx). \tag{8}$$

Since the measurable function $g(x)$ from (8) is arbitrary, we obtain the inequality

$$f(x)-(\gamma^*,x-m^*)-\ln v \leq 0 \quad \mu^*\text{-a.e.}$$

On the other hand, from (5) we find the equality

$$\int_E \exp\{f(x)-(\gamma^*,x-m^*)-\ln v\}\,\mu^*(dx) = 1.$$

Therefore, μ^*-a.e.

$$f(x) = \ln v + (\gamma^*,x-m^*). \tag{9}$$

Let us integrate both sides of relation (9) with respect to the measure μ^*. We see that

$$\ln v = \int_E f(x)\,\mu^*(dx).$$

This implies (6).

Let us prove the uniqueness of the representation (6). The proof is argued by contradiction. Suppose that there exists a $\widetilde{\gamma} \in D$, $\widetilde{\gamma} \neq \gamma^*$, such that μ^*-a.e.

$$f(x) = \int_E f(x)\,\mu^*(dx) + (\widetilde{\gamma},x-m^*). \tag{10}$$

We subtract relation (6) from (10), obtaining $(\widetilde{\gamma}-\gamma^*,x-m^*)=0$ μ^*-a.e. It follows from the fact that $f(x)$ is bounded and from (6) that $(\gamma^*,x-m^*)^2$ is bounded. Therefore, there exists a covariance matrix K of the random vector x with respect to the measure μ^*. Therefore,

$$\int_E (\widetilde{\gamma}-\gamma^*,x-m^*)^2\,\mu^*(dx) = (\widetilde{\gamma}-\gamma^*,K(\widetilde{\gamma}-\gamma^*)) = 0. \tag{11}$$

It follows from the positive definiteness of the matrix K and relations (11) that we have a contradiction with the assumption $\widetilde{\gamma} \neq \gamma^*$. Hence our assumption is false and the representation (6) is unique.

The implication 2)⇒3). First, let us make several remarks. Since μ^* is a probability measure, it follows that it admits the unique decomposition (see [12, Sec. 9]):

$$\mu^* = \alpha\mu^{*\mathrm{c}} + (1-\alpha)\mu^{*\mathrm{d}}, \qquad \alpha \in [0,1],$$

where $\mu^{*\mathrm{c}}$ is a continuous probability measure and $\mu^{*\mathrm{d}}$ is a discrete probability measure. Therefore, the measure $\mu^{*\mathrm{d}}$ is concentrated at the atoms of the measure μ^* whose number is finite or countable. Since the measures $\mu^{*\mathrm{d}}$ and $\mu^{*\mathrm{c}}$ are singular ([12, Sec. 9]), it follows that there exist $\mathscr{E}$-measurable sets B and $\overline{B}$ such that:

i) $B \cup \overline{B} = E$;

ii) $B \cap \overline{B} = \varnothing$;

iii) $\mu^{*\mathrm{c}}(B) = 1\,(\mu^{*\mathrm{d}}(B) = 0)$, $\mu^{*\mathrm{d}}(\overline{B}) = 1\,(\mu^{*\mathrm{c}}(\overline{B}) = 0)$.

Therefore, for any $A \in \mathscr{E}$, the following relations hold:

$$\alpha\mu^{*\mathrm{c}}(A) = \mu^*(A \cap B), \qquad (1-\alpha)\mu^{*\mathrm{d}}(A) = \mu^*(A \cap \overline{B}), \tag{12}$$

where $\alpha = \mu^*(B)$ and $1 - \alpha = \mu^*(\overline{B})$. To prove this assertion, it suffices to show that, for any $A \in \mathscr{E}$, the following equality holds:

$$\mu^*(A) = \mu^{*\mathrm{d}}(A). \tag{13}$$

In view of the assumption, the indicators $1_A(x)$, $1_{\overline{B}}(x)$, and $1_{A\cap\overline{B}}(x)$ admit the following unique representations with respect to the measures μ^*:

$$1_A(x) = \mu^*(A) + (\gamma^A, x - m^*), \qquad 1_{\overline{B}}(x) = \mu^*(\overline{B}) + (\gamma^{\overline{B}}, x - m^*), \tag{14}$$

$$1_{A\cap\overline{B}}(x) = \mu^*(A \cap \overline{B}) + (\gamma^{A\cap\overline{B}}, x - m^*), \tag{15}$$

respectively. Since

$$1_A(x)1_{\overline{B}}(x) = 1_{A\cap\overline{B}}(x),$$

it follows from (14) and (15) that μ^*-a.e.

$$\mu^*(A \cap \overline{B}) + (\gamma^{A\cap\overline{B}}, x - m^*) = \mu^*(A)\mu^*(\overline{B}) + \mu^*(A)(\gamma^{\overline{B}}, x - m^*) + \mu^*(\overline{B})(\gamma^A, x - m^*) + (\gamma^A, x - m^*)(\gamma^{\overline{B}}, x - m^*). \tag{16}$$

Relation (16) holds for all $x \in \operatorname{supp}\mu^*$. Since m^* is the barycenter of the measure μ^*, it follows that it belongs to the relative interior $\operatorname{supp}\mu^*$. Therefore, we set $x = m^*$. Then, using (16), we obtain the relation

$$\mu^*(A \cap \overline{B}) = \mu^*(A)\mu^*(\overline{B}).$$

This equality and (12) imply (13).

To complete the proof of this assertion, it remains to note that:

i) since μ^* is a discrete probability measure, it follows that it is a convex combination of a finite or countable number of Dirac measures [11, Sec. 2, Chap. II] concentrated at various isolated points $x_i \in \operatorname{supp}\mu^* \subseteq E$;

ii) there exists a one-to-one correspondence between the Dirac measures and the points $x_i \in \operatorname{supp}\mu^*$;

iii) it follows from Carathéodory's theorem [13, Sec. 17, Chap. IV] that $I^{\mu^*} \le d + 1$.

The implication 3) $\Rightarrow$ 1). Denote

$$\Phi(\gamma) \triangleq \int_E e^{f(x)-(\gamma,x-m^*)}\,\mu^*(dx).$$

Since the measure μ^* is discrete and $I^{\mu^*} \le d + 1$, it follows that m^* and $\Phi(\gamma)$ admit, respectively, the representations

$$m^* = \sum_{i=1}^{I^{\mu^*}} c_i x_i, \qquad \Phi(\gamma) = \sum_{i=1}^{I^{\mu^*}} c_i \exp\{f(x_i) - (\gamma, x_i - m^*)\}, \tag{17}$$

where $x_i \in \operatorname{supp}\mu^*$. The assumptions of the theorem also imply:

i) $D = \mathbb{R}^d$;

ii) for any $x \in E$, the following inequality holds: $|f(x)| \le c$.

Therefore, by Jensen's inequality, we have

$$\Phi(\gamma) \ge e^{-c} > 0. \tag{18}$$

Let us pass to the proof of the implication. Let $I^{\mu^*} = 1$. Then it follows from (17) that, for any bounded $\gamma \in D$, inequalities $\mu^*((\gamma, x - m^*) > 0) > 0$ and $\mu^*((\gamma, x - m^*) < 0) > 0$ hold. Therefore, any $\gamma \in \mathbb{R}^d$ is a solution of problem (4).

Suppose that $1 < I^{\mu^*} \le d + 1$, and $\Gamma(\mu^*)$ is the convex hull $\operatorname{supp} \mu^*$. It is known that m^* belongs to the relative interior $\Gamma(\mu^*)$. Therefore, it follows from (17) that there exist x_i, $x_j \in E$ such that $x_i < m^*$, $x_j > m^*$. Therefore, $\Phi(\gamma)$ is strictly convex and, by (18), it is a function bounded below; also, $\Phi(\gamma) \to \infty$ as $|\gamma| \to \infty$. Therefore, there exists a $\gamma^* \in \mathbb{R}^d$ such that $\inf_{\gamma \in D} \Phi(\gamma) = \Phi(\gamma^*)$. The proof is complete. □

Remark 3. The proof of the implication from 2) to 3) in Proposition 3 and the uniqueness of the representation (6) implies the uniqueness of the discrete measure μ^*.

4.3. Proof of assertion 2) of the theorem. It follows from the compactness of the set E that:

i) $\mathfrak{R} = M(E)$ is a compact set in the topology of weak convergence of probability measures;

ii) by Proposition 1, there exists a probability measure $\mu^* \in M(E)$ that dominates any other measure $\mu \in M(E)$; therefore, $\mu \ll \mu^*$, and hence μ^* is an extreme point of the set $M(E)$;

iii) there exists a $\gamma^* \in D$ that satisfies (2).

Therefore, it follows from the assertion of Proposition 3 that the measure μ^* is discrete. The discreteness of the measure μ^* implies that it is the convex hull of the Dirac measures. As was already noted in the proof of the implication from 2) to 3) in Proposition 3, there exists a one-to-one correspondence between the Dirac measures and the points $x_i \in \operatorname{supp} \mu^*$ Therefore, by Carathéodory's theorem, the measure μ^* is concentrated at no more than $d + 1$ points. Since the measure μ^* is extreme, it follows that its support is concentrated on $\operatorname{extr} E$. Hence the validity of (3) is established.

Remark 4. If $f(x)$ is a continuous function, then assertion 2) of the theorem is well known. In this case, its new proof may be of interest.

Remark 5. It follows from assertion 2) of the theorem that the support of the measure μ^* is concentrated on $\operatorname{extr} E$. We can easily verify that the converse of assertion 2) of the theorem is valid.

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