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Degenerate twistor spaces for hyperkähler manifolds

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ABSTRACT

Let M be a hyperkähler manifold, and η a closed, positive $(1, 1)$-form with $\operatorname{rk}\eta < \dim M$. We associate to η a family of complex structures on M, called a degenerate twistor family, and parametrized by a complex line. When η is a pullback of a Kähler form under a Lagrangian fibration L, all the fibers of degenerate twistor family also admit a Lagrangian fibration, with the fibers isomorphic to that of L. Degenerate twistor families can be obtained by taking limits of twistor families, as one of the Kähler forms in the hyperkähler triple goes to η.

1. Introduction

1.1. Complex structures obtained from non-degenerate closed 2-forms

The degenerate twistor spaces (Definition 3.17) are obtained through the following construction.

Definition 1.1. A complex-valued 2-form Ω on a real manifold M is called *non-degenerate* if $\Omega(v, \cdot) \neq 0$ for any non-zero tangent vector $v \in T_mM$. Complex structures on M can be obtained from complex sub-bundles $B = T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$B \oplus \overline{B} = TM \otimes_{\mathbb{R}} \mathbb{C}, \quad [B, B] \subset B \tag{1.1}$$

(Claim 3.3).

To obtain such B, take a non-degenerate (Definition 1.1), closed 2-form $\Omega \in \Lambda^2(M, \mathbb{C})$, satisfying $\Omega^{n+1} = 0$, where $4n = \dim_{\mathbb{R}} M$. Then $\ker \Omega := \{v \in T_mM \otimes_{\mathbb{R}} \mathbb{C} \mid \Omega(v, \cdot) = 0\}$ satisfies the conditions of (1.1) (see Theorem 3.5).

Degenerate twistor spaces are obtained by constructing a family Ω_t of such 2-forms, parametrized by $t \in \mathbb{C}$, on hyperkähler manifolds. The relation $\Omega_t^{n+1} = 0$ follows from the properties of cohomology of hyperkähler manifolds, most notably the Fujiki formula, computation of cohomology performed in [1], and positivity (see Section 3.5).

1.2. Degenerate twistor families and Teichmüller spaces

In this subsection, we provide a motivation for the term “degenerate twistor family”. We introduce the twistor families of complex structures on hyperkähler manifolds and the corresponding rational curves in the moduli, called *the twistor lines*.

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A degenerate twistor family is a family $\mathcal{Z}$ of deformations of a holomorphically symplectic manifold (M, Ω) associated with a positive, closed, semidefinite form η satisfying $\eta^{n-i} \wedge \Omega^{i+1} = 0$, for all $i = 0, 1, \dots, n$, where $\dim_{\mathbb{C}} M = 2n$ (Theorem 3.10). In this subsection, we define a twistor family of a hyperkähler manifold, and explain how these families can be obtained as limits of twistor deformations.

Throughout this paper, *a hyperkähler manifold* is a compact, holomorphically symplectic manifold M of Kähler type. It is called *simple* (Definition 2.3) if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$. We shall (sometimes silently) assume that all hyperkähler manifolds we work with are simple.

A hyperkähler metric is a metric g compatible with three complex structures I, J, K satisfying the quaternionic relations $IJ = -JI = K$, which is Kähler with respect to I, J, K. By the Calabi–Yau theorem, any compact, holomorphically symplectic manifold of Kähler type admits a hyperkähler metric, which is unique in each Kähler class (Theorem 2.2).

A hyperkähler structure is a hyperkähler metric g together with the compatible quaternionic action, that is, a triple of complex structures satisfying the quaternionic relations and Kähler. For any $(a, b, c) \in S^2 \subset \mathbb{R}^3$, the quaternion $L := aI + bJ + cK$ defines another complex structure on M, also Kähler with respect to g. This can be seen because the Levi-Civita connection ∇ of (M, g) preserves I, J, K, hence $\nabla L = 0$, and this implies integrability and Kählerness of L.

Such a complex structure is called *induced complex structure*. The $\mathbb{C}P^1$-family of induced complex structures obtained this way is in fact holomorphic (Section 2.1). It is called *the twistor deformation*. The twistor families can be described in terms of periods of hyperkähler manifolds as follows.

Definition 1.2. Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (also known as *the group of isotopies*). Denote by Comp the space of complex structures on M, equipped with topology induced from the C^∞-topology on the space of all tensors, and let $\operatorname{Teich} := \mathsf{Comp}/\operatorname{Diff}_0(M)$. We call it *the Teichmüller space*.

Definition 1.3. Let

$$\mathsf{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$$

map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per is called *the period map*.

For a simple hyperkähler manifold, an important bilinear symmetric form $q \in \operatorname{Sym}^2 H^2(M, \mathbb{Q})^*$ is defined, called *Bogomolov–Beauville–Fujiki form* (Definition 2.6). This form is a topological invariant of the manifold M, allowing one to describe deformations of a complex structure very explicitly. Recall that two points x, y on a topological space are called *non-separable*, if all their neighborhoods $U_x \ni x$, $U_y \ni y$ intersect. We denote the corresponding symmetric relation in Teich by $x \sim y$. D. Huybrechts has shown that $x \sim y$ for $x, y \in \operatorname{Teich}$ implies that the corresponding complex manifolds (M, x) and (M, y) are bimeromorphic [2]. In [3] it was shown that $\sim$ defines an equivalence relation on Teich; the corresponding quotient space $\operatorname{Teich}/\sim$ is called *the birational Teichmüller space*, and denoted by Teich_b.

Define the *period space* $\mathbb{P}\mathsf{er}$ as

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}(H^2(M, \mathbb{C})) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

The global Torelli theorem [3] can be stated as follows.

Theorem 1.4. *Let M be a simple hyperkähler manifold,* Teich_b *the birational Teichmüller space, and* $\mathsf{Per} : \operatorname{Teich}_b \longrightarrow \mathbb{P}(H^2(M, \mathbb{C}))$ *the period map. Then* Per *maps* Teich_b *to* $\mathbb{P}\mathsf{er}$, *inducing a diffeomorphism of each connected component of* Teich_b *with* $\mathbb{P}\mathsf{er}$.

Proof. See [3]. ∎

Remark 1.5. The period space $\mathbb{P}\mathsf{er}$ is equipped with a transitive action of $SO(H^2(M, \mathbb{R}))$. Using this action, one can identify $\mathbb{P}\mathsf{er}$ with the Grassmann space of 2-dimensional, positive, oriented planes $\operatorname{Gr}_{+,+}(H^2(M, \mathbb{R})) = SO(b_2-3, 3)/SO(2) \times SO(b_2-3, 1)$. Indeed, for each $l \in \mathbb{P}H^2(M, \mathbb{C})$, the space generated by $\langle \operatorname{Im} l, \operatorname{Re} l\rangle$ is 2-dimensional, because $q(l, l) = 0, q(l, \bar{l}) \neq 0$ implies that $l \cap H^2(M, \mathbb{R}) = 0$. This produces a point of $\operatorname{Gr}_{+,+}(H^2(M, \mathbb{R}))$ from $l \in \mathbb{P}\mathsf{er}$. To obtain the converse correspondence, notice that for any 2-dimensional positive plane $V \in H^2(M, \mathbb{R})$, the quadric $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0\}$ consists of two lines $l \in \mathbb{P}\mathsf{er}$. A choice of one of two lines is determined by the orientation in V.

We shall describe the Teichmüller space and the moduli of hyperkähler structures in the same spirit, as follows.

Recall that any hyperkähler structure (M, I, J, K, g) defines a triple of Kähler forms $\omega_I, \omega_J, \omega_K \in \Lambda^2(M)$ (Section 2.1). A hyperkähler structure on a simple hyperkähler manifold is determined by a complex structure and a Kähler class (Theorem 2.2).

We call hyperkähler structures *equivalent* if they can be obtained by a homothety and a quaternionic reparametrization:

$$(M, I, J, K, g) \sim (M, hIh^{-1}, hJh^{-1}, hKh^{-1}, \lambda g),$$

for $h \in \mathbb{H}^*$, $\lambda \in \mathbb{R}^{>0}$. Let $\operatorname{Teich}^{\mathcal{H}}$ be the set of equivalence classes of hyperkähler structures up to the action of $\operatorname{Diff}_0(M)$, and $\operatorname{Teich}^{\mathcal{H}}_b$ its quotient by $\sim$ (the non-separability relation).

Theorem 1.6. *Consider the period map*

$$\mathsf{Per}_{\mathcal{H}}: \ \mathrm{Teich}_b^{\mathcal{H}} \longrightarrow \mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$$

associating the plane $\langle \omega_I, \omega_J, \omega_K\rangle$ in the Grassmannian of 3-dimensional positive oriented planes to an equivalence class of hyperkähler structures. Then $\mathsf{Per}_{\mathcal{H}}$ is injective, and defines an open embedding on each connected component of $\mathrm{Teich}_b^{\mathcal{H}}$.

Proof. As follows from global Torelli theorem (Theorem 1.4) and Remark 1.5, a complex structure is determined (up to diffeomorphism and a birational equivalence) by a 2-plane $V \in \mathrm{Gr}_{+,+}(H^2(M,\mathbb{R})) = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$, where $V = \langle \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega\rangle$, and Ω a holomorphically symplectic form (defined uniquely up to a multiplier). Let $\omega \in H^{1,1}(M,I) = V^\perp$ be a Kähler form. The corresponding hyperkähler structure gives an orthogonal triple of Kähler forms $\omega_J, \omega_K \in V$, $\omega_I := \omega \in V^\perp$ satisfying $q(\omega_I,\omega_I) = q(\omega_J,\omega_J) = q(\omega_K,\omega_K) = C$. The group $SU(2)\times\mathbb{R}^{>0}$ acts on the set of such orthogonal bases transitively. Therefore, a hyperkähler structure is determined (up to equivalence of hyperkähler structures and non-separability) by a 3-plane $W = \langle \omega_I, \omega_J, \omega_K\rangle \subset H^2(M,\mathbb{R})$.

We have shown that $\mathsf{Per}_{\mathcal{H}}$ is injective. To finish the proof of Theorem 1.6, it remains to show that $\mathsf{Per}_{\mathcal{H}}$ is an open embedding. However, for a sufficiently small $v \in \langle \omega_J, \omega_K\rangle^\perp = H^{1,1}_{\mathbb{R}}(M,I)$, the form $v + \omega_I$ is also Kähler (the Kähler cone is open in $H^{1,1}_{\mathbb{R}}(M,I)$), hence $W' = \langle \omega_I + v, \omega_J, \omega_K\rangle$ also belongs to an image of $\mathsf{Per}_{\mathcal{H}}$. This implies that the differential $D(\mathsf{Per}_{\mathcal{H}})$ is surjective. ■

Every hyperkähler structure induces a whole 2-dimensional sphere of complex structures on M, as follows. Consider a triple $a, b, c \in \mathbb{R}$, $a^2+b^2+c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponding quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence L is an almost complex structure. Since I, J, K are Kähler, they are parallel with respect to the Levi-Civita connection. Therefore, L is also parallel. Any parallel complex structure is integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ a *complex structure induced by the hyperkähler structure*. The corresponding complex manifold is denoted by (M, L). There is a holomorphic family of induced complex structures, parametrized by $S^2 = \mathbb{C}P^1$. The total space of this family is called the *twistor space* of a hyperkähler manifold; it is constructed as follows.

Let M be a hyperkähler manifold. Consider the product $\mathrm{Tw}(M) = M \times S^2$. Embed the sphere $S^2 \subset \mathbb{H}$ into the quaternion algebra $\mathbb{H}$ as the set of all quaternions J with $J^2 = -1$. For every point $x = m \times J \in X = M \times S^2$ the tangent space $T_x\mathrm{Tw}(M)$ is canonically decomposed $T_xX = T_mM \oplus T_JS^2$. Identify S^2 with $\mathbb{C}P^1$, and let $I_J : T_JS^2 \to T_JS^2$ be the complex structure operator. Consider the complex structure $I_m : T_mM \to T_mM$ on M induced by $J \in S^2 \subset \mathbb{H}$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x\mathrm{Tw}(M) \to T_x\mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}} \circ I_{\mathrm{Tw}} = -1$. It depends smoothly on the point x, hence it defines an almost complex structure on $\mathrm{Tw}(M)$. This almost complex structure is known to be integrable (see e.g. [4,5]).

Definition 1.7. The space $\mathrm{Tw}(M)$ constructed above is called *the twistor space* of a hyperkähler manifold.

The twistor space defines a family of deformations of a complex structure on M, called *the twistor family*; the corresponding curve in the Teichmüller space is called *the twistor line*.

Let (M, I, J, K) be a hyperkähler structure, and $W = \langle \omega_I, \omega_J, \omega_K\rangle$ the corresponding 3-dimensional plane. The twistor family gives a rational line $\mathbb{C}P^1 \subset$ Teich, which can be recovered from W as follows. Recall that by the global Torelli theorem, each component of Teich is identified (up to gluing together non-separable points) with the Grassmannian $\mathrm{Gr}_{+,+}(H^2(M,\mathbb{R}))$. There is a $\mathbb{C}P^1$ of oriented 2-dimensional planes in W; this family is precisely the twistor family associated with the hyperkähler structure corresponding to W.

In the present paper, we consider what happens if one takes a 3-dimensional plane $W \subset H^2(M,\mathbb{R})$ with a degenerate metric of signature $(+,+,0)$. Instead of a $\mathbb{C}P^1$ worth of complex structures, as happens when W is positive, the set of positive 2-planes in $W \subset H^2(M,\mathbb{R})$ is parametrized by $\mathbb{C} = \mathbb{R}^2$. It turns out that the corresponding family can be constructed explicitly from an appropriate semipositive form on a manifold, whenever such a form exists. Moreover, this family (called *a degenerate twistor family*; see Definition 3.17) is holomorphic and has a canonical smooth trivialization, just as the usual twistor family.

1.3. Semipositive (1, 1)-forms, degenerate twistor families and SYZ conjecture

Let (M, I, Ω) be a simple holomorphically symplectic manifold of Kähler type (that is, a hyperkähler manifold), and $\eta \in \Lambda^{1,1}(M, I)$ a real, positive, closed $(1,1)$-form. By Fujiki formula, either η is strictly positive somewhere, or at least half of the eigenvalues of η vanish (Proposition 3.9). In the latter case, the form $\Omega_t := \Omega + t\eta$ is non-degenerate and satisfies the assumption $\Omega_t^{n+1} = 0$ for all t, hence defines a complex structure (Theorem 3.10).

This is used to define the degenerate twistor space (Theorem 3.18).

Positive, closed forms $\eta \in \Lambda^{1,1}(M)$ with $\int_M \eta^{\dim_{\mathbb{C}} M} = 0$ are called *semipositive*. Such forms necessarily lie in the boundary of a Kähler cone; this implies that their cohomology classes are nef (Definition 3.8).

Notice that we exclude strictly positive forms from this definition.

Remark 1.8. The conventions for positivity of differential forms and currents are intrinsically confusing. Following the French tradition, one says "positive form" meaning really "non-negative", and "strictly positive" meaning "positive definite". On top of it, for $(n-k, n-k)$ forms on n-manifold, with $2 \leqslant k \leqslant n-2$, there are two notions of positive forms, called "strongly positive" and "weakly positive"; this creates monsters such that "strictly weakly positive" and "non-strictly strongly positive". The various notions of positivity in this paper are taken (mostly) from [6], following the French conventions as explained.

The study of nef classes which satisfy $\int_M \eta^{\dim_{\mathbb{C}} M} = 0$ (such classes are called *parabolic*) is one of the central themes of hyperkähler geometry. One of the most important conjectures in this direction is the so-called hyperkähler SYZ conjecture, due to Tyurin–Bogomolov–Hassett–Tschinkel–Huybrechts–Sawon ([7–9]; for more history, please see [10]). This conjecture postulates that any rational nef class η on a hyperkähler manifold is semiample, that is, associated with a holomorphic map $\varphi : M \longrightarrow X$, $\eta = \varphi^*\omega_X$, where ω_X is a Kähler class on X. For nef classes which satisfy $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$ (such nef classes are known as *big*), semiampleness follows from the Kawamata base point free theorem [11], but for parabolic classes it is quite non-trivial.

If a parabolic class η is semiample, it can obviously be represented by a smooth, semipositive differential form. The converse implication is not proven. However, in [10] it was shown that whenever a rational parabolic class can be represented by a semipositive form, it is $\mathbb{Q}$-effective (that is, represented by a rational effective divisor).

Existence of a smooth semipositive form in a given nef class is a separate (and interesting) question of hyperkähler geometry. The following conjecture is supported by empirical evidence obtained by S. Cantat and Dinh–Sibony ([12], [13, Theorem 5.3], [14, Corollary 3.5]).

Conjecture 1.9. *Let η be a parabolic nef class on a hyperkähler manifold. Then η can be represented by a semipositive closed form with mild (say, Hölder) singularities.*

Notice that η can be represented by a closed, positive current by compactness of the space of positive currents with bounded mass; however, there is no clear way to understand the singularities of this current.

If this conjecture is true, a cohomology class is $\mathbb{Q}$-effective whenever it is nef and rational [10,15]; this would prove a part of SYZ conjecture.

One of the ways of representing a nef class by a semipositive form is based on reverse-engineering the construction of degenerate twistor spaces. Let η be a parabolic nef class on a hyperkähler manifold (M, I), Ω its holomorphic symplectic form, and $W := \langle \eta, \operatorname{Re} \Omega, \operatorname{Im} \Omega\rangle$ the corresponding 3-dimensional subspace in $H^2(M, \mathbb{R})$. Clearly, the Bogomolov–Beauville–Fujiki form on W is degenerate of signature $(+, +, 0)$. The set S of positive, oriented 2-dimensional planes $V \subset W$ is parametrized by $\mathbb{C}$. Identifying the Grassmannian $\operatorname{Gr}_{++}(H^2(M, \mathbb{R}))$ with a component of Teich_b as in Theorem 1.6, we obtain a deformation $\mathcal{Z} \longrightarrow S$; as explained in Section 1.2, this family can be obtained as a limit of twistor families. The twistor families are split as smooth manifolds: $\operatorname{Tw}(M) = M \times \mathbb{C}P^1$; this gives an Ehresmann connection ∇ on the twistor family $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$. This connection satisfies $\nabla\Omega_t = \lambda\omega_I$, that is, a derivative of a holomorphically symplectic form is proportional to a Kähler form. If this connection converges to a smooth connection ∇_0 on the limit family $\mathcal{Z} \longrightarrow \mathbb{C}$, we would obtain $\nabla\Omega_t = \lambda\eta$, where η is a limit of Kähler forms, hence semipositive. This was the original motivation for the study of degenerate twistor spaces.

1.4. Degenerate twistor spaces and Lagrangian fibrations

The main source of examples of degenerate twistor families comes from Lagrangian fibrations.

Let (M, Ω) be a simple holomorphically symplectic Kähler manifold, and $\varphi : M \longrightarrow X$ a surjective holomorphic map, with $0 < \dim X < \dim M$. Matsushita (Theorem 2.9) has shown that φ is a Lagrangian fibration, that is, the fibers of φ are Lagrangian subvarieties in M, and all smooth fibers of φ are Lagrangian tori. It is not hard to see that X is projective [16]. Let ω_X be the Kähler form on X. Then $\eta := \varphi^*\omega_X$ is a semipositive form, and Theorem 3.10 together with Theorem 3.5 implies the existence of a degenerate twistor family $\mathcal{Z} \longrightarrow \mathbb{C}$, with the fibers holomorphically symplectic manifolds $(M, \Omega + t\eta)$, $t \in \mathbb{C}$. For each fiber $Y := \varphi^{-1}(y)$, the restriction $\eta|_Y$ vanishes, because $\eta = \varphi^*\omega_X$. Therefore, the complex structure induced by $\Omega_t = \Omega + t\eta$ on Y does not depend on t. This implies that the fibers of φ remain holomorphic and independent from $t \in \mathbb{C}$.

Theorem 1.10. *Let M be a simple hyperkähler manifold equipped with a Lagrangian fibration $\varphi : M \longrightarrow X$, and (M_t, Ω_t) the degenerate twistor deformation associated with the family of non-degenerate 2-forms $\Omega + t\eta$, $\eta = \varphi^*\omega_X$ as in Theorem 3.10. Then the fibration $M_t \xrightarrow{\varphi_t} X$ is also holomorphic, and for any fixed $x \in X$, the fibers of φ_t are naturally isomorphic: $\varphi_t^{-1}(x) \cong \varphi^{-1}(x)$ for all $t \in \mathbb{C}$.*

Proof. The complex structure on M_t is determined from $T^{0,1}M_t = \ker \Omega_t$. Let $Z := \varphi^{-1}(x)$. Since $\eta(v, \cdot) = 0$ for each $v \in T_zZ$, one has $TZ \cap \ker \Omega_t = T^{0,1}Z$, hence the complex structure on Z is independent from t. Since Z is Lagrangian in M_t, its normal bundle is dual to TZ and trivial when Z is a torus (that is, for all smooth fibers of φ). Therefore, the complex structure on NZ is independent from $t \in \mathbb{C}$. This implies that the projection $M_t \xrightarrow{\varphi} X$ is holomorphic in the smooth locus of φ for all $t \in \mathbb{C}$. To extend it to the points where φ is singular, we notice that a map is holomorphic whenever its differential is complex linear, and complex linearity of a given tensor needs to be checked only in an open dense subset. ■

Remark 1.11. In [17], Eyal Markman considered the following procedure. One starts with a Lagrangian fibration π on a hyperkähler manifold and takes a 1-cocycle on the base of π taking values in fiberwise automorphisms of the fibration. Twisting the π by such a cocycle, one obtains another Lagrangian fibration with the same base and the respective fibers isomorphic to that of π. Markman calls this procedure “the Tate–Shafarevich twist”. In this context, degenerate twistor deformations associated with semipositive forms η, $[\eta] \in H^2(M, \mathbb{Z})$, occur very naturally; Markman calls them “Tate–Shafarevich lines”. One can view $\eta = \varphi^*\omega_X$ as lying in

$$\varphi^* H^{1,1}(X) = \varphi^* H^1(X, \Omega^1 X) \subset H^1(M, \varphi^*\Omega^1 X) = H^1(M, T_{M/X}),$$

where $T_{M/X}$ is the fiberwise tangent bundle, and $\varphi^*\Omega^1 X = T_{M/X}$ because $M \longrightarrow X$ is a Lagrangian fibration. Of course, this cocycle comes from X so it is constant in the fiber direction; it describes the deformation infinitesimally. Integrating the vector field then gives a 1-cocycle on X taking values in the bundle of fibrewise automorphisms. This is the 1-cocycle giving the “Tate–Shafarevich twist”.

Remark 1.12. The degenerate twistor family constructed in Theorem 3.18 consists of a family of complex structures, but it is not proven that all fibers, which are complex manifolds, are also Kähler (hence hyperähler). As is, the Kähler property is known only over a small open subset in the base (affine line), since the condition of being Kähler is open. We expect all members of the degenerate twistor family to be Kähler, but there is no obvious way to prove this. However, it is easy to show that the set of points on the base affine line corresponding to non-Kähler complex structures is closed and countable.

2. Basic notions of hyperkähler geometry

2.1. Hyperkähler manifolds

Definition 2.1. Let (M, g) be a Riemannian manifold, and I, J, K endomorphisms of the tangent bundle TM satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_{TM}.$$

The triple (I, J, K) together with the metric g is called *a hyperkähler structure* if I, J and K are integrable and Kähler with respect to g.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \qquad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \qquad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot). \tag{2.1}$$

An elementary linear-algebraic calculation implies that the 2-form

$$\Omega := \omega_J + \sqrt{-1}\omega_K \tag{2.2}$$

is of Hodge type $(2, 0)$ on (M, I). This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word “hyperkähler” is essentially synonymous with “holomorphically symplectic”, due to the following theorem, which is implied by Yau’s solution of Calabi conjecture [18,19].

Theorem 2.2. *Let M be a compact, Kähler, holomorphically symplectic manifold, ω its Kähler form,* $\dim_{\mathbb{C}} M = 2n$. *Denote by Ω the holomorphic symplectic form on M. Assume that $\int_M \omega^{2n} = \int_M (\mathrm{Re}\,\Omega)^{2n}$. Then there exists a unique hyperkähler metric g within the same Kähler class as ω, and a unique hyperkähler structure (I, J, K, g), with $\omega_J = \mathrm{Re}\,\Omega$, $\omega_K = \mathrm{im}\,\Omega$.* ■

2.2. The Bogomolov–Beauville–Fujiki form

Definition 2.3. A hyperkähler manifold M is called *simple* if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$. In the literature, such manifolds are often called *irreducible holomorphic symplectic*, or *irreducible symplectic varieties*.

This definition is motivated by the following theorem of Bogomolov [20].

Theorem 2.4 ([20]). *Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.* ■

Theorem 2.5 ([21]). *Let $\eta \in H^2(M)$, and* $\dim M = 2n$*, where M is a simple hyperkähler manifold. Then $\int_M \eta^{2n} = \lambda q(\eta, \eta)^n$, for some integer quadratic form q on $H^2(M)$, and $\lambda \in \mathbb{Q}$ a positive rational number.* ■

Definition 2.6. This form is called *Bogomolov–Beauville–Fujiki form*. It is defined by this relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville; [18], [9, 23.5])

$$\lambda q(\eta, \eta) = (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - (1 - n) \frac{\left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n\right)\left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\right)}{\int_M \Omega^n \wedge \overline{\Omega}^n}$$

where Ω is the holomorphic symplectic form, and λ a positive constant.

Remark 2.7. The form q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on the space $\langle \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega, \omega\rangle$ where ω is a Kähler form, as seen from the following formula

$$\mu q(\eta_1, \eta_2) = \int_X \omega^{2n-2} \wedge \eta_1 \wedge \eta_2 - \frac{2n-2}{(2n-1)^2} \frac{\int_X \omega^{2n-1} \wedge \eta_1 \cdot \int_X \omega^{2n-1} \wedge \eta_2}{\int_M \omega^{2n}}, \quad \mu > 0 \tag{2.3}$$

(see e.g. [1, Theorem 6.1], or [9, Corollary 23.9]).

Definition 2.8. Let $[\eta] \in H^{1,1}(M)$ be a real (1, 1)-class in the closure of the Kähler cone of a hyperkähler manifold M. We say that $[\eta]$ is *parabolic* if $q([\eta], [\eta]) = 0$.

2.3. The hyperkähler SYZ conjecture

Theorem 2.9 (*D. Matsushita, see [22]*)**.** *Let $\pi : M \longrightarrow X$ be a surjective holomorphic map from a simple hyperkähler manifold M to a complex variety X, with $0 < \dim X < \dim M$. Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian (this means that the symplectic form vanishes on the fibers).*[1]

Definition 2.10. Such a map is called *a holomorphic Lagrangian fibration*.

Remark 2.11. The base of π is conjectured to be rational. J.-M. Hwang [23] proved that $X \cong \mathbb{C}P^n$, if X is smooth and M projective. D. Matsushita [16] proved that it has the same rational cohomology as $\mathbb{C}P^n$ when M is projective.

Remark 2.12. The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be (conjecturally) used to determine the topology of M [24–26].

Remark 2.13. Matsushita's theorem is implied by the following formula of Fujiki. Let M be a hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $\eta_1, \ldots, \eta_{2n} \in H^2(M)$ cohomology classes. Then

$$C \int_M \eta_1 \wedge \eta_2 \wedge \cdots = \frac{1}{(2n)!} \sum_\sigma q(\eta_{\sigma_1}\eta_{\sigma_2}) q(\eta_{\sigma_3}\eta_{\sigma_4}) \cdots q(\eta_{\sigma_{2n-1}}\eta_{\sigma_{2n}}) \tag{2.4}$$

with the sum taken over all permutations, and C a positive constant, called *Fujiki constant*. An algebraic argument (see e.g. Corollary 2.15) allows to deduce from this formula that for any non-zero $\eta \in H^2(M)$, one would have $\eta^n \neq 0$, and $\eta^{n+1} = 0$, if $q(\eta, \eta) = 0$, and $\eta^{2n} \neq 0$ otherwise. Applying this to the pullback $\pi^*\omega_X$ of the Kähler class from X, we immediately obtain that $\dim_{\mathbb{C}} X = n$ or $\dim_{\mathbb{C}} X = 2n$. Indeed, $\omega_X^{\dim_{\mathbb{C}} X} \neq 0$ and $\omega_X^{\dim_{\mathbb{C}} X+1} = 0$. This argument was used by Matsushita in his proof of Theorem 2.9. The relation (2.4) is another form of Fujiki's theorem (Theorem 2.5), obtained by differentiation of $\int_M \eta^{2n} = \lambda q(\eta, \eta)^n$.

2.4. Cohomology of hyperkähler manifolds

Further on in this paper, some basic results about cohomology of hyperkähler manifolds will be used. The following theorem was proved in [1], using representation theory.

Theorem 2.14 (*[1]*)**.** *Let M be a simple hyperkähler manifold, and $H^*_r(M)$ the part of cohomology generated by $H^2(M)$. Then $H^*_r(M)$ is isomorphic to the symmetric algebra (up to the middle degree). Moreover, the Poincare pairing on $H^*_r(M)$ is non-degenerate.* ■

This brings the following corollary.

Corollary 2.15. *Let $\eta_1, \ldots \eta_{n+1} \in H^2(M)$ be cohomology classes on a simple hyperkähler manifold,* $\dim_{\mathbb{C}} M = 2n$*. Suppose that $q(\eta_i, \eta_j) = 0$ for all i, j. Then $\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{n+1} = 0$.*

Proof. See e.g. [15, Corollary 2.15]. This equation also follows from (2.4). ■

3. Degenerate twistor space

3.1. Integrability of almost complex structures and Cartan formula

An *almost complex structure* on a manifold is a section $I \in \mathrm{End}(TM)$ of the bundle of endomorphisms, satisfying $I^2 = -\mathrm{Id}$. It is called *integrable* if $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$, where $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ is the eigenspace of I, defined by

$$v \in T^{1,0}M \Leftrightarrow I(v) = \sqrt{-1}v.$$

[1] Here, as elsewhere, we silently assume that the hyperkähler manifold M is simple.

Equivalently, I is integrable if $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$, where $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ is a complex conjugate to $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$.

One of the ways of making sure a given almost complex structure is integrable is by using the Cartan formula expressing the de Rham differential through commutators of vector fields.

Proposition 3.1. *Let (M, I) be a manifold equipped with an almost complex structure, and $\Omega \in \Lambda^{2,0}(M)$ a non-degenerate $(2, 0)$-form (Definition 3.4). Assume that $d\Omega = 0$. Then I is integrable.*

Proof. Let $X \in T^{1,0}M$ and $Y, Z \in T^{0,1}(M)$. Since Ω is a $(2, 0)$-form, it vanishes on $(0, 1)$-vectors. Then Cartan formula together with $d\Omega = 0$ implies that

$$0 = d\Omega(X, Y, Z) = \Omega(X, [Y, Z]). \tag{3.1}$$

From the non-degeneracy of Ω we obtain that unless $[Y, Z] \in T^{0,1}(M)$, for some $X \in T^{1,0}M$, one would have $\Omega(X, [Y, Z]) \neq 0$. Therefore, (3.1) implies $[Y, Z] \in T^{0,1}(M)$, for all $Y, Z \in T^{0,1}(M)$, which means that I is integrable. ■

Remark 3.2. It is remarkable that the closedness of Ω is in fact unnecessary. The proof of Proposition 3.1 remains true if one assumes that $d\Omega \in \Lambda^{3,0}(M) \oplus \Lambda^{2,1}(M)$.

Notice that the sub-bundle $T^{1,0}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ uniquely determines the almost complex structure. Indeed, $I(x + y) = \sqrt{-1}x - \sqrt{-1}y$, for all $x \in T^{1,0}M$, $y \in T^{0,1}M = \overline{T^{1,0}M}$, and we have a decomposition $T^{1,0}M \oplus T^{0,1}M = TM \otimes_{\mathbb{R}} \mathbb{C}$. This decomposition is the necessarily and sufficient ingredient for the reconstruction of an almost complex structure:

Claim 3.3. *Let M be a smooth, $2n$-dimensional manifold. Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\dim_{\mathbb{C}} T^{0,1}M = n$ and $T^{0,1}M \cap TM = 0$ (the last condition means that there are no real vectors in $T^{1,0}M$).* ■

The last two statements allow us to define complex structures in terms of complex-valued 2-forms (see Theorem 3.5 below). For this theorem, any reasonable notion of non-degeneracy would suffice; for the sake of clarity, we state the one we would use.

Definition 3.4. Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a smooth, complex-valued 2-form on a $2n$-dimensional manifold. Ω is called *non-degenerate* if for any real vector $v \in T_mM$, the contraction $\Omega \lrcorner v$ is non-zero.

Theorem 3.5. *Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a smooth, complex-valued, non-degenerate 2-form on a $4n$-dimensional real manifold. Assume that $\Omega^{n+1} = 0$. Consider the bundle*

$$T^{0,1}_{\Omega}(M) := \{v \in TM \otimes \mathbb{C} \mid \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_{\Omega}(M)$ satisfies assumptions of Claim 3.3, hence defines an almost complex structure I_{Ω} on M. If, in addition, Ω is closed, I_{Ω} is integrable.

Proof. Integrability of I_{Ω} follows immediately from Proposition 3.1. Let $v \in TM$ be a non-zero real tangent vector. Then $\Omega \lrcorner v \neq 0$, hence $T^{0,1}_{\Omega}(M) \cap TM = 0$. To prove Theorem 3.5, it remains to show that $\operatorname{rk} T^{0,1}_{\Omega}(M) \geqslant 2n$. Clearly, Ω is non-degenerate on $\frac{TM\otimes\mathbb{C}}{T^{0,1}_{\Omega}(M)}$, hence its rank is equal to $4n - \operatorname{rk} T^{0,1}_{\Omega}(M)$. From $\Omega^{n+1} = 0$ it follows that rank of Ω cannot exceed $2n$, hence $\operatorname{rk} T^{0,1}_{\Omega}(M) \geqslant 2n$. ■

3.2. Semipositive (1, 1)-forms on hyperkähler manifold

Definition 3.6. Let $\eta \in \Lambda^{1,1}(M, \mathbb{R})$ be a real $(1, 1)$-form on a complex manifold (M, I). It is called *semipositive* if $\eta(x, Ix) \geqslant 0$ for any $x \in TM$, but it is nowhere positive definite.

Remark 3.7. Fix a Hermitian structure h on (M, I). Clearly, any semipositive $(1, 1)$-form is diagonal in some h-orthonormal basis in TM. The entries of its matrix in this basis are called *eigenvalues*; they are real, non-negative numbers. The maximal number of positive eigenvalues is called *the rank* of a semipositive $(1, 1)$-form.

Definition 3.8. A closed semipositive form η on a compact Kähler manifold (M, I, ω) is a limit of Kähler forms $\eta + \varepsilon\omega$, hence its cohomology class is *nef* (belongs to the closure of the Kähler cone). Its cohomology class $[\eta]$ is *parabolic*, that is, it satisfies $\int_M [\eta]^{\dim_{\mathbb{C}} M} = 0$. However, not every parabolic nef class can be represented by a closed semipositive form [27].

Proposition 3.9. *On a simple hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$, any semipositive $(1, 1)$-form has rank 0 or $2n$.*

Proof. This assertion easily follows from Corollary 2.15. Indeed, if $q(\eta,\eta)\neq 0$, one has $\int_M \eta^{2n} = \lambda q(\eta,\eta)^n \neq 0$, hence its rank is $4n$. If $q(\eta,\eta)=0$, its cohomology class $[\eta]$ satisfies $[\eta]^n\neq 0$ and $[\eta]^{n+1}=0$ (Corollary 2.15). Since all eigenvalues of η are non-negative, its rank is twice the biggest number k for which one has $\eta^k\neq 0$. However, since η^k is a sum of monomials of an orthonormal basis with non-negative coefficients, $\int_M \eta^k\wedge\omega^{2n-k}=0 \Leftrightarrow \eta^k=0$ for any Kähler form ω on (M,I). Then $[\eta]^n\neq 0$ and $[\eta]^{n+1}=0$ imply that the rank of η is $2n$. ■

The main technical result of this paper is the following theorem.

Theorem 3.10. *Let (M,Ω) be a simple hyperkähler manifold, $\dim_{\mathbb{R}} M=4n$, and $\eta\in\Lambda^{1,1}(M,I)$ a closed, semipositive form of rank $2n$. Then the 2-form $\Omega+t\eta$ satisfies the assumptions of Theorem 3.5 for all $t\in\mathbb{C}$: namely, $\Omega+t\eta$ is non-degenerate, and $(\Omega+t\eta)^{n+1}=0$.*

Proof. Non-degeneracy of $\Omega_t:=\Omega+t\eta$ is clear. Indeed, let $u:=|t|t^{-1}$, and let $\omega_u:=\operatorname{Re} u\omega_K+\operatorname{im} u\omega_J$. Then ω_u is a Hermitian form associated with the induced complex structure $\operatorname{Im} uJ-\operatorname{Re} uK$, hence it is non-degenerate. However, the imaginary part of $u\Omega_t$ is equal to ω_u (see (2.1)). Then $\operatorname{Im}(\Omega_t \lrcorner v)\neq 0$ for each non-zero real vector $v\in TM$.

To see that $(\Omega+t\eta)^{n+1}=0$, we observe that this relation is true in cohomology; this is implied from [28] using the same argument as was used in the proof of Proposition 3.9.

Each Hodge component of $(\Omega+t\eta)^{n+1}$ is proportional to $\Omega^{n-p}\wedge\eta^{p+1}$, and it is sufficient to prove that $\Omega^{n-p}\wedge\eta^{p+1}=0$ for all p.

We deduce this from two observations, which are proved further on in this section.

Lemma 3.11. *Let (M,Ω), $\dim_{\mathbb{R}} M=4n$ be a holomorphically symplectic manifold, and $\eta\in\Lambda^{1,1}(M,I)$ a closed, semipositive form of rank $2n$. Assume that $\Omega^{n-p}\wedge\eta^{p+1}$ is exact. Then*

$$\Omega^{n-p}\wedge\overline{\Omega}^{n-p}\wedge\eta^{p+1}=0,$$

for all p.

Proof. See Section 3.3. ■

Lemma 3.12. *Let (M,Ω), $\dim_{\mathbb{R}} M=4n$, be a holomorphically symplectic manifold and $\rho\in\Lambda^{p+1,p+1}(M,I)$ a strongly positive form (Definition 3.13). Suppose that $\Omega^{n-p}\wedge\overline{\Omega}^{n-p}\wedge\rho=0$. Then $\Omega^{n-p}\wedge\rho=0$.*

Proof. See Section 3.4. ■

3.3. Positive (p,p)-forms

We recall the definition of a positive (p,p)-form (see e.g. [6]).

Definition 3.13. Recall that a real (p,p)-form η on a complex manifold is called *weakly positive* if for any complex subspace $V\subset TM$, $\dim_{\mathbb{C}} V=p$, the restriction $\rho|_V$ is a non-negative volume form. Equivalently, this means that

$$(\sqrt{-1})^p\rho(x_1,\bar{x}_1,x_2,\bar{x}_2,\dots,x_p,\bar{x}_p)\geqslant 0,$$

for any vectors $x_1,\dots,x_p\in T^{1,0}_xM$. A real (p,p)-form on a complex manifold is called *strongly positive* if it can be locally expressed as a sum

$$\eta=(\sqrt{-1})^p\sum_{i_1,\dots,i_p}\alpha_{i_1,\dots,i_p}\xi_{i_1}\wedge\bar{\xi}_{i_1}\wedge\dots\wedge\xi_{i_p}\wedge\bar{\xi}_{i_p},$$

running over some set of p-tuples $\xi_{i_1},\xi_{i_2},\dots,\xi_{i_p}\in\Lambda^{1,0}(M)$, with $\alpha_{i_1,\dots,i_p}$ real and non-negative functions on M.

The following basic linear algebra observations are easy to check (see [6]).

All strongly positive forms are also weakly positive. The strongly positive and the weakly positive forms form closed, convex cones in the space $\Lambda^{p,p}(M,\mathbb{R})$ of real (p,p)-forms. These two cones are dual with respect to the Poincare pairing

$$\Lambda^{p,p}(M,\mathbb{R})\times\Lambda^{n-p,n-p}(M,\mathbb{R})\longrightarrow\Lambda^{n,n}(M,\mathbb{R}).$$

For $(1,1)$-forms and $(n-1,n-1)$-forms, the strong positivity is equivalent to weak positivity. Finally, a product of a weakly positive form and a strongly positive one is always weakly positive (however, a product of two weakly positive forms may be not weakly positive).

Clearly, an exact weakly positive form η on a compact Kähler manifold (M,ω) always vanishes. Indeed, the integral $\int_M\eta\wedge\omega^{\dim M-p}$ is strictly positive for a non-zero weakly positive η, because the convex cones of weakly and strongly positive forms are dual, and $\omega^{\dim M-p}$ sits in the interior of the cone of strongly positive forms. However, by Stokes' formula, this integral vanishes whenever η is exact.

Now we are in position to prove Lemma 3.11. The form $\Omega^{n-p}\wedge\overline{\Omega}^{n-p}\wedge\eta^{p+1}$ is by assumption of this lemma exact, but it is a product of a weakly positive form $\Omega^{n-p}\wedge\overline{\Omega}^{n-p}$ and a strongly positive form η^{p+1}, hence it is weakly positive. Being exact, this form must vanish.

Remark 3.14. A form is strongly positive if it is generated by products of $dz_i \wedge d\bar{z}_i$ with positive coefficients; hence η and all its powers are positive. The form $\Omega \wedge \overline{\Omega}$ and its powers are positive on all complex spaces of appropriate dimensions, which can be seen by using Darboux coordinates. This means that this form is weakly positive.

3.4. Positive (p, p)-forms and holomorphic symplectic forms

Now we shall prove Lemma 3.12. This is a linear-algebraic statement, which can be proven pointwise. Fix a complex vector space V, equipped with a non-degenerate complex linear 2-form Ω. Every strongly positive form ρ on V is a sum of monomials $(\sqrt{-1})^p \xi_{i_1} \wedge \overline{\xi}_{i_1} \wedge \cdots \wedge \xi_{i_p} \wedge \overline{\xi}_{i_p}$ with positive coefficients, and the equivalence

$$\Omega^{n-p} \wedge \rho \neq 0 \Leftrightarrow \Omega^{n-p} \wedge \overline{\Omega}^{n-p} \wedge \rho \neq 0$$

is implied by the following sublemma.

Sublemma 3.15. *Let V be a complex vector space, equipped with a non-degenerate complex linear 2-form $\Omega \in \Lambda^{2,0}V$. Then for any monomial $\rho = (\sqrt{-1})^p \xi_{i_1} \wedge \overline{\xi}_{i_1} \wedge \cdots \wedge \xi_{i_p} \wedge \overline{\xi}_{i_p}$ for which $\Omega^{n-p} \wedge \rho$ is non-zero, the form $\Omega^{n-p} \wedge \overline{\Omega}^{n-p} \wedge \rho$ is non-zero and weakly positive.*

Proof. Let $\xi_{j_1}, \xi_{j_1}, \dots, \xi_{j_{n-p}}$ be the elements of the basis in V complementary to $\xi_{i_1}, \xi_{i_1}, \dots, \xi_{i_p}$, and $W \subset V$ the space generated by $\xi_{j_1}, \xi_{j_1}, \dots, \xi_{j_{n-p}}$. Clearly, a form α is non-zero on W if and only if $\alpha \wedge \rho$ is non-zero, and positive on W if and only if $\alpha \wedge \rho$ is positive.

Now, Sublemma 3.15 is implied by the following trivial assertion: for any $(n-p)$-dimensional subspace $W \subset V$ such that $\Omega^{n-p}|_W$ is non-zero, the restriction $\Omega^{n-p} \wedge \overline{\Omega}^{n-p}|_W$ is non-zero and positive.

This proves Sublemma 3.15, and Lemma 3.12 follows as indicated. ■

As a corollary of the vanishing of the forms $\Omega^{n-p} \wedge \eta^{p+1}$, we prove the following statement, used further on.

Lemma 3.16. *Let (M, Ω) be a simple holomorphically symplectic manifold, $\dim_{\mathbb{R}} M = 4n$ and $\eta \in \Lambda^{1,1}(M, I)$ a closed, semi-positive form of rank $2n$. Let I_t be the complex structure on M defined by $\Omega + t\eta$, as in Theorem 3.10. Then $\eta \in \Lambda^{1,1}(M, I_t)$.*

Proof. By construction, (M, I_t) is a holomorphically symplectic manifold, with the holomorphic symplectic form $\Omega_t := \Omega + t\eta$. For a holomorphic symplectic manifold (M, Ω_t), $\dim_{\mathbb{R}} M = 4n$, there exists an elementary criterion allowing one to check whether a given 2-form η is of type $(1, 1)$: one has to have $\eta \wedge \Omega_t^n = 0$ and $\eta \wedge \overline{\Omega}_t^n = 0$. However, from Lemma 3.12 it follows immediately that $\eta \wedge \Omega_t^n = 0$ and $\eta \wedge \overline{\Omega}_t^n = 0$, hence η is of type $(1, 1)$. ■

3.5. Degenerate twistor space: a definition

Just as it is done with the usual twistor space, to define a degenerate twistor space we construct a certain almost complex structure, and then prove it is integrable. The proof of integrability is in fact identical to the argument which could be used to prove that the usual twistor space is integrable.

Definition 3.17. Let (M, Ω) be an irreducible holomorphically symplectic manifold, $\dim_{\mathbb{R}} M = 4n$ and $\eta \in \Lambda^{1,1}(M, I)$ a closed, semipositive form of rank $2n$. Consider the product $\mathrm{Tw}_\eta(M) := \mathbb{C} \times M$, equipped with the almost complex structure $\mathcal{I}$ acting on $T_t\mathbb{C} \oplus T_m M$ as $I_{\mathbb{C}} \oplus I_t$, where $I_{\mathbb{C}}$ is the standard complex structure on $\mathbb{C}$ and I_t is the complex structure recovered from the form $\Omega + t\eta$ using Theorems 3.10 and 3.5. The almost complex manifold $(\mathrm{Tw}_\eta(M), \mathcal{I})$ is called *a degenerate twistor space* of M.

Theorem 3.18. *The almost complex structure on a degenerate twistor space is always integrable.*

Proof. We introduce a dummy variable w, and consider a product $\mathrm{Tw}_\eta(M) \times \mathbb{C}$, equipped with the $(2, 0)$-form $\widetilde{\Omega} := \Omega + t\eta + dt \wedge dw$. Here, Ω is a holomorphic symplectic form on M lifted to $M \times \mathbb{C} \times \mathbb{C}$, and t and w are complex coordinates on $\mathbb{C} \times \mathbb{C}$. Clearly, $\widetilde{\Omega}$ is a non-degenerate $(2, 0)$-form. From Lemma 3.16 we obtain that $d\widetilde{\Omega} = \eta \wedge dt \in \Lambda^{2,1}(\mathrm{Tw}_\eta(M) \times \mathbb{C})$. Now, Remark 3.2 implies that $\widetilde{\Omega}$ defines an integrable almost complex structure on $\mathrm{Tw}_\eta(M) \times \mathbb{C}$. However, on $\mathrm{Tw}_\eta(M) \times \{w\}$ this almost complex structure coincides with the one given by the degenerate twistor construction. ■

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