Proc. of the Second Intl. Conf. on Advances in Management, Economics and Social Science - MES 2015
Copyright © Institute of Research Engineers and Doctors, USA .All rights reserved.
ISBN: 978-1-63248-046-0 doi: 10.15224/ 978-1-63248-046-0-53

# The Nonlinear Differential Dynamics of Interdependent Branches of Industry

[ Victor Dmitriev, Svetlana Maltseva, Andrey Dmitriev ]

***Abstract* — Nonlinear differential dynamic model of the relation between the branches of production was proposed. Mathematically, this model is expressed as a system of first-order ODE. Dynamic variables of the model – the value of the output of each branch of production. Each differential equation of the system includes independent growth and diminution of finished goods; growth and decline of production related to the production of allied industries. Two models were proposed: a model with Malthusian products growth (model with no restrictions on the amount of product), the model with the Verhulst limiting of the growth of output. The equilibrium points of dynamical systems, system stability were determined as well as the qualitative analysis of dynamic systems was made.**

***Keywords* — interdependent industries, busines dynamics, dynamical systems, qualitative analysis, Lyapunov first approximation.**

## I. Introduction

Traditionally, economic systems are studied mainly through the optimization theory and game theory. However, in the past two decades, methods of the theory of dynamical systems is one of the most effective methods for modeling economic systems and processes [1,2], and business dynamics [3]. Application of the dynamical chaos theory to the model of nonequilibrium market and the crisis detection presented in our recent works [4,5].

The mathematical basis of the continuous dynamic systems theory is the systems of ordinary differential equations theory. Greatest interest for our research is the first order autonomous system of differential equations:

$$\dot{X} = f(X, \boldsymbol{\alpha}) \quad (1)$$

where $X = (x_1(t), x_2(t), \dots, x_n(t))$ - the vector of state variables of the system, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ - the vector of system parameters, $\dot{X}$ - the first order time-derivative, $f$ - generally a nonlinear real function of n-variables.

There are two approaches to the solution of the system (1): quantitative and qualitative analysis.

Victor Dmitriev, Svetlana Maltseva, Andrey Dmitriev
National Research University Higher School of Economics
Russia
a.dmitriev@hse.ru

The first approach is the solution of the Cauchy problem. If $f$ is an analytical function in a neighborhood of the point $X_0 = (x_1(0), x_2(0), \dots, x_n(0))$, find a solution $X(t, X_0) = \left(x_1(t, x_1(0)), x_2(t, x_2(0)), \dots, x_n(t, x_n(0))\right)$ of the system (1) with initial conditions $X_0 = (x_1(0), x_2(0), \dots, x_n(0))$. For most non-linear functions *f* fails obtain an analytical solution of the Cauchy problem.

In the second approach is formulated and solved the problem of finding the equilibrium points of the system and the analysis of their stability. In most cases, mathematical modeling of economic systems is sufficient the qualitative analysis [1].

By definition, the equilibrium points $X^*$ are the roots of an algebraic equation:

$$f(X, \boldsymbol{\alpha}) = 0 \quad (2)$$

To study the stability of the rough equilibrium points, there are two Lyapunov theorems.

By the Lyapunov stability theorem in the first approximation, if all the Jacobian (*J*) eigenvalues $\lambda_i$ have negative real parts, then the equilibrium point $X^*$ of the original system (1) and the linearized system $\dot{X} = JX$ is an asymptotically stable.

The Jacobian of the system (1) is $J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$.

By the Lyapunov instability theorem in the first approximation, if at least one Jacobian (*J*) eigenvalues $\lambda_i$ have positive real parts, then the equilibrium point $X^*$ of the original system (1) and the linearized system $\dot{X} = JX$ is an asymptotically unstable.

These theorems allow us to study the stability of the rough equilibrium points. These points have nonzero real parts of the $\lambda_i$. Lyapunov method in the first approximation is limited to study of the rough systems or structurally stable systems. For second-order systems, there are only three types of rough equilibrium points: a node, a spiral node and a saddle point. Figures 1-5 are shown the phase portraits of these systems. In these cases, the phase portrait is a plot of $x_2 = g(x_1)$, with different initial conditions.

Proc. of the Second Intl. Conf. on Advances in Management, Economics and Social Science - MES 2015
Copyright © Institute of Research Engineers and Doctors, USA .All rights reserved.
ISBN: 978-1-63248-046-0 doi: 10.15224/ 978-1-63248-046-0-53

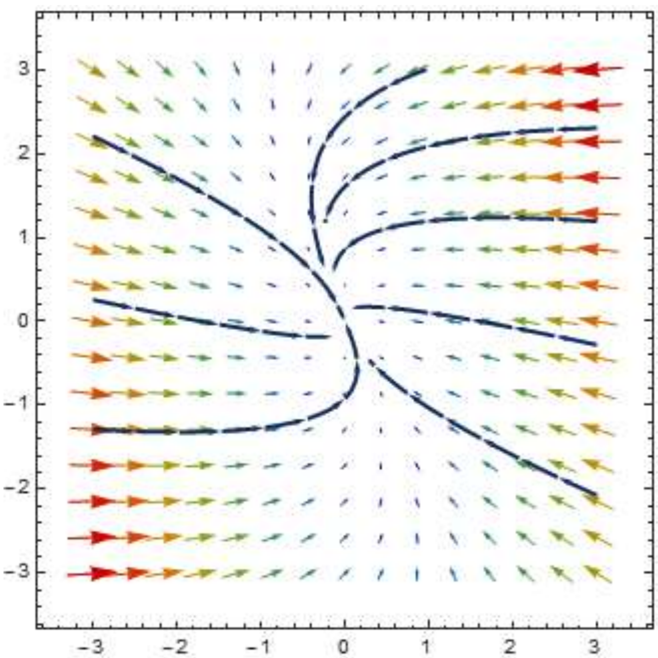



Figure 1. The stable node.

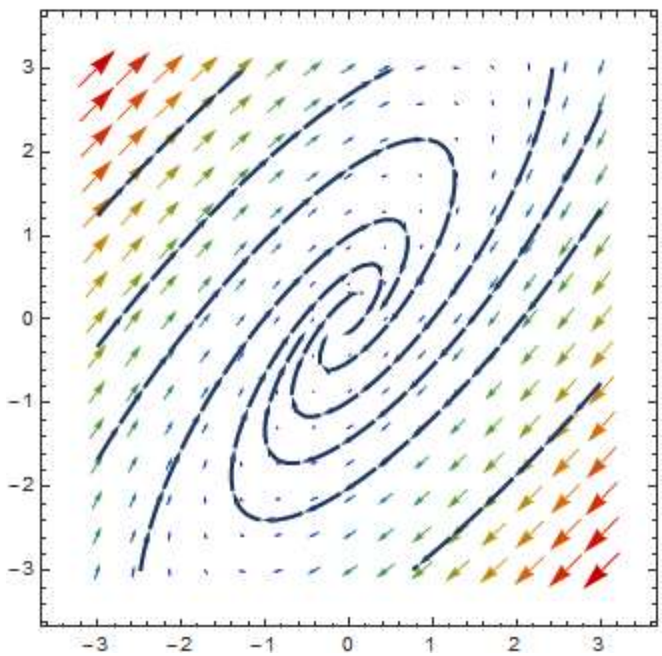



Figure 3. The stable spiral node.

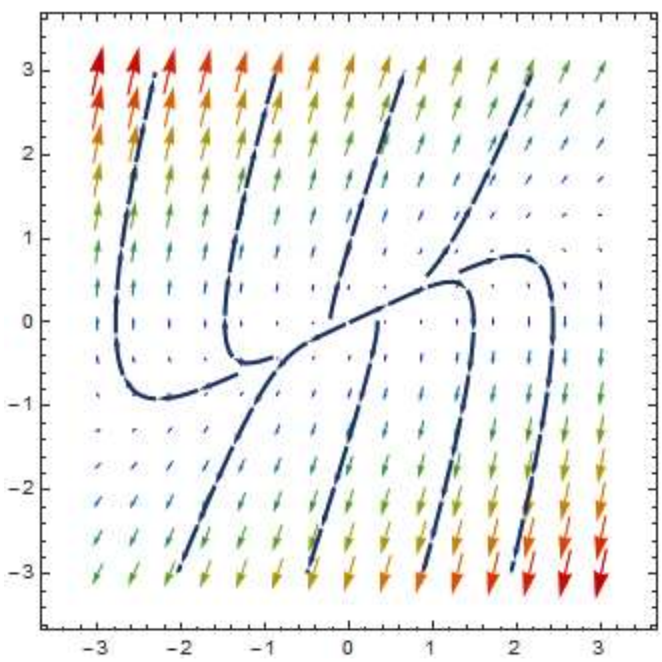



Figure 2. The unstable node.

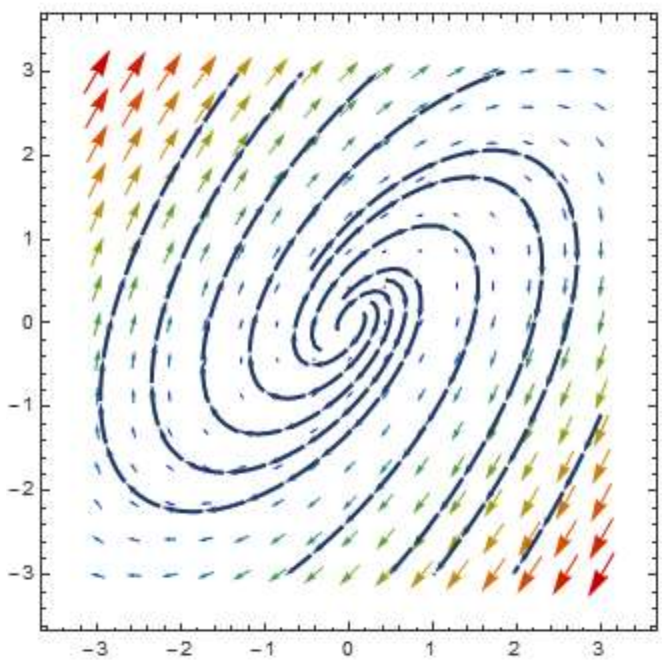



Figure 4. The unstable spiral node.

Proc. of the Second Intl. Conf. on Advances in Management, Economics and Social Science - MES 2015
Copyright © Institute of Research Engineers and Doctors, USA .All rights reserved.
ISBN: 978-1-63248-046-0 doi: 10.15224/ 978-1-63248-046-0-53

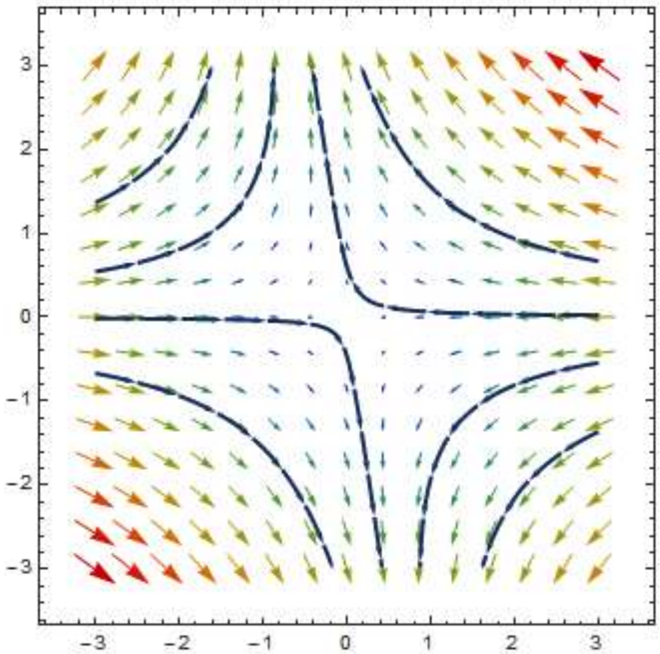
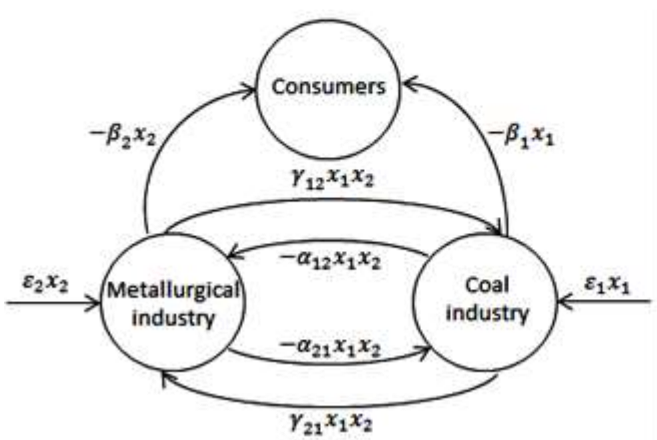


Figure 5. The unstable saddle point.

The purpose of our work is the qualitative analysis of dynamic systems of interdependent branches of industry. It's the finding an equilibrium product and their stability in the first approximation.




Without loss of generality, we construct a dynamic system of interdependent branches of industry on the example of coal industry and metallurgical industry. In these branches product flows are shown in the figure 6.

Figure 6. The product flows.

In this case the dynamical system has the following form:

$$\begin{cases} \dot{x}_1 = \varepsilon_1 x_1 - \beta_1 x_1 - \alpha_{12} x_1 x_2 + \gamma_{12} x_1 x_2 \\ \dot{x}_2 = \varepsilon_2 x_2 - \beta_2 x_2 - \alpha_{21} x_1 x_2 + \gamma_{21} x_1 x_2 \end{cases}, \quad (3)$$

where $x_1(t) \geq 0$ - the amount of coal, $x_2(t) \geq 0$ - the amount of steel, $\varepsilon_1 x_1$ - the coal growth associated with its production, $\varepsilon_2 x_2$ - the steel growth associated with its production, $-\beta_1 x_1$ - the decline of coal for the needs of the consumer, $-\beta_2 x_2$ - the decline of steel for the needs of the consumer, $-\alpha_{12} x_1 x_2$ - the decline of coal for steel production, $-\alpha_{21} x_1 x_2$ - the decline of steel for coal production, $\gamma_{12} x_1 x_2$ - the coal growth associated with the production of steel, $\gamma_{21} x_1 x_2$ - the steel growth associated with the production of coal. All parameters of the system (3) are positive numbers.

# II. Model of Unlimited Production and Consumption

The system (3) does not have limitations. For example, the production and consumption of products are unlimited. The basis of the model is Malthusian model of growth product [6]. In this case $\varepsilon_1$ and $\varepsilon_2$ are Malthusian parameters.

## A. *The First Approximation of the qualitative analysis of the system*

Reduce the number of the system (3) parameters:

$$\begin{cases} \dot{x}_1 = \Delta_1 x_1 - \delta_{12} x_1 x_2 \\ \dot{x}_2 = \Delta_2 x_2 - \delta_{21} x_1 x_2 \end{cases}, \quad (4)$$

where $\Delta_1 \equiv \varepsilon_1 - \beta_1$ , $\Delta_2 \equiv \varepsilon_2 - \beta_2$ , $\delta_{12} \equiv \alpha_{12} - \gamma_{12}$ , $\delta_{21} \equiv \alpha_{21} - \gamma_{21}$.

Equilibrium points of the system are the roots of the algebraic system:

$$\begin{cases} \Delta_1 x_1 - \delta_{12} x_1 x_2 = 0 \\ \Delta_2 x_2 - \delta_{21} x_1 x_2 = 0 \end{cases}. \quad (5)$$

The system (4) has two equilibrium points:

- $O = (0,0)$ is zero equilibrium point,
- $E = \left(\frac{\Delta_2}{\delta_{21}}, \frac{\Delta_1}{\delta_{12}}\right)$ is non-zero equilibrium point as $\frac{\Delta_1}{\delta_{12}} > 0, \frac{\Delta_2}{\delta_{21}} > 0$.

Jacobian of the system is $J = \begin{pmatrix} \Delta_1 - \delta_{12} x_2 & -\delta_{12} x_1 \\ -\delta_{21} x_2 & \Delta_2 - \delta_{21} x_1 \end{pmatrix}$.

The Jacobian $J_O = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}$ eigenvalues are $\lambda_1 = \Delta_1$ and $\lambda_2 = \Delta_2$.

There are the following cases:

- if $\Delta_1 < 0$, $\Delta_2 < 0$, $\delta_{12} < 0$, $\delta_{21} < 0$, then $O = (0,0)$ is the asymptotically stable node (fig. 1),
- if $\Delta_1 < 0$, $\Delta_2 > 0$, $\delta_{12} < 0$, $\delta_{21} > 0$, then $O = (0,0)$ is the unstable saddle point (fig. 5),
- if $\Delta_1 > 0$, $\Delta_2 < 0$, $\delta_{12} > 0$, $\delta_{21} < 0$, then $O = (0,0)$ is the unstable saddle point (fig. 5),
- if $\Delta_1 > 0$, $\Delta_2 > 0$, $\delta_{12} > 0$, $\delta_{21} > 0$, then $O = (0,0)$ is unstable node (fig. 2).

Proc. of the Second Intl. Conf. on Advances in Management, Economics and Social Science - MES 2015
Copyright © Institute of Research Engineers and Doctors, USA .All rights reserved.
ISBN: 978-1-63248-046-0 doi: 10.15224/ 978-1-63248-046-0-53

The Jacobian $J_E = \begin{pmatrix} 0 & -\frac{\delta_{12}}{\delta_{21}}\Delta_2 \\ -\frac{\delta_{21}}{\delta_{12}}\Delta_1 & 0 \end{pmatrix}$ eigenvalues are $\lambda_1 = -\sqrt{\Delta_1\Delta_2}$ and $\lambda_2 = \sqrt{\Delta_1\Delta_2}$. Hence $E = \left(\frac{\Delta_2}{\delta_{21}}, \frac{\Delta_1}{\delta_{12}}\right)$ is the unstable saddle point (fig. 5).

### B. *Discussion of the Results*

In the first approximation, the system (3) has only the zero asymptotically stable equilibrium products. Nonzero equilibrium product is unstable for all values of the parameters. This fact is a consequence of an approximate model. It's the unlimited production and unlimited consumption of products. We cannot exclude that the system (3) has non-zero stable equilibrium point, for example, the center. To find this point is insufficient analysis of the stability in the first approximation. In this case, we can use the method of Lyapunov functions. In our opinion, the main reason for the lack of non-zero asymptotically stable products is limited model. Hence the system (3) requires consideration of limits on the number of products.

## III. Model of Limited Production and Unlimited Consumption

### A. *The first approximation of the qualitative analysis of the system*

We use the Verhulst model [7] to account for the limitation productions. In this case, the system has the following form:

$$\begin{cases} \dot{x}_1 = \Delta_1 x_1 - k_1 x_1^2 - \delta_{12} x_1 x_2 \\ \dot{x}_2 = \Delta_1 x_2 - k_2 x_2^2 - \delta_{21} x_1 x_2 \end{cases}, \qquad (6)$$

where $k_1 \equiv \frac{\varepsilon_1}{K_1}$, $k_2 \equiv \frac{\varepsilon_2}{K_2}$, $K_i > 0$ - the maximum of products.

Equilibrium points of the system are the roots of the algebraic system:

$$\begin{cases} \Delta_1 x_1 - k_1 x_1^2 - \delta_{12} x_1 x_2 = 0 \\ \Delta_2 x_2 - k_2 x_2^2 - \delta_{21} x_1 x_2 = 0 \end{cases}. \qquad (7)$$

The system (6) has fourth equilibrium points:

- $O = (0{,}0)$ is zero equilibrium point,
- $E_1 = \left(0, \frac{\Delta_2}{k_2}\right)$ is non-zero equilibrium point as $\Delta_2 > 0$,
- $E_2 = \left(\frac{\Delta_1}{k_1}, 0\right)$ is non-zero equilibrium point as $\Delta_1 > 0$,
- $E_3 = \left(\frac{\Delta_1 k_2 - \Delta_2 \delta_{12}}{k_1 k_2 - \delta_{12}\delta_{21}}, \frac{\Delta_2 k_1 - \Delta_1 \delta_{21}}{k_1 k_2 - \delta_{12}\delta_{21}}\right)$ is non-zero equilibrium point.

$E_3$-point is a point of equilibrium in the following cases:

- $\Delta_1 > 0, \Delta_2 > 0, \frac{\delta_{21}}{k_2} \geq 0, \frac{\Delta_2}{k_2} > \frac{\Delta_1}{k_1}\frac{\delta_{21}}{k_2}, \frac{\delta_{12}}{k_1} < \frac{\Delta_1}{\Delta_2}\frac{k_2}{k_1};$
- $\Delta_1 > 0, \Delta_2 > 0, \frac{\delta_{21}}{k_2} \geq 0, 0 < \frac{\Delta_2}{k_2} < \frac{\Delta_1}{k_1}\frac{\delta_{21}}{k_2}, \frac{\delta_{12}}{k_1} > \frac{\Delta_1}{\Delta_2}\frac{k_2}{k_1};$
- $\Delta_1 > 0, \Delta_2 > 0, \frac{\delta_{21}}{k_2} < 0, \frac{k_2}{\delta_{21}} < \frac{\delta_{12}}{k_1} < \frac{\Delta_1}{\Delta_2}\frac{k_2}{k_1}.$

Jacobian of the system is $J = \begin{pmatrix} \Delta_1 - 2k_1x_1 - \delta_{12}x_2 & -\delta_{12}x_1 \\ -\delta_{21}x_2 & \Delta_2 - 2k_2x_2 - \delta_{21}x_1 \end{pmatrix}.$

The Jacobian $J_O$ eigenvalues are $\lambda_1 = \Delta_1 > 0$ and $\lambda_2 = \Delta_2 > 0$.

There are the following cases:

- if $\lambda_1 = \Delta_1 > 0$, $\lambda_2 = \Delta_2 > 0$, then $O = (0{,}0)$ is the unstable node (fig. 2),
- if $\lambda_1 = \Delta_1 < 0$, $\lambda_2 = \Delta_2 < 0$, then $O = (0{,}0)$ is the asymptotically stable node (fig. 1),
- if $\lambda_1 = \Delta_1 < 0$, $\lambda_2 = \Delta_2 > 0$, then $O = (0{,}0)$ is the unstable saddle point (fig. 5),
- if $\lambda_1 = \Delta_1 > 0$, $\lambda_2 = \Delta_2 < 0$, then $O = (0{,}0)$ is the unstable saddle point (fig. 5).

The Jacobian $J_{E_1}$ eigenvalues are $\lambda_1 = -\Delta_2 < 0$ and $\lambda_2 = \Delta_1 - \frac{\Delta_2}{k_2}\delta_{12}$.

There are the following cases:

- if $\Delta_1 - \frac{\Delta_2}{k_2}\delta_{12} < 0$, then $E_1 = \left(0, \frac{\Delta_2}{k_2}\right)$ is the asymptotically stable node (fig. 1),
- if $\Delta_1 - \frac{\Delta_2}{k_2}\delta_{12} > 0$, then $E_1 = \left(0, \frac{\Delta_2}{k_2}\right)$ is the unstable saddle point (fig. 5).

The Jacobian $J_{E_2}$ eigenvalues are $\lambda_1 = -\Delta_1 < 0$ and $\lambda_2 = \Delta_2 - \frac{\Delta_1}{k_1}\delta_{21}$.

There are the following cases:

- if $\Delta_2 - \frac{\Delta_1}{k_1}\delta_{21} < 0$, then $E_2 = \left(\frac{\Delta_1}{k_1}, 0\right)$ is the asymptotically stable node (fig. 1),
- if $\Delta_2 - \frac{\Delta_1}{k_1}\delta_{21} > 0$, then $E_2 = \left(\frac{\Delta_1}{k_1}, 0\right)$ is the unstable saddle point (fig. 5).

The Jacobian $J_{E_3}$ eigenvalues have the following form:

$$\lambda_{1,2} = \alpha \pm \sqrt{\beta},$$

where

$$\alpha = \frac{\Delta_1}{2} + \frac{\Delta_2}{2} - x_3^*\left(k_1 + \frac{\delta_{21}}{2}\right) - y_3^*\left(k_2 + \frac{\delta_{12}}{2}\right),$$

$$\beta = \frac{\alpha^2}{4} - (\Delta_1 - 2k_1x_3^* - \delta_{12}y_3^*)(\Delta_2 - 2k_2y_3^* - \delta_{21}x_3^*)$$
$$+\delta_{12}\delta_{21}x_3^*y_3^*,$$

$$x_3^* = \frac{\Delta_1 k_2 - \Delta_2 \delta_{12}}{k_1 k_2 - \delta_{12}\delta_{21}}$$

$$y_3^* = \frac{\Delta_2 k_1 - \Delta_1 \delta_{21}}{k_1 k_2 - \delta_{12}\delta_{21}}$$

There are the following cases:

Proc. of the Second Intl. Conf. on Advances in Management, Economics and Social Science - MES 2015
Copyright © Institute of Research Engineers and Doctors, USA .All rights reserved.
ISBN: 978-1-63248-046-0 doi: 10.15224/ 978-1-63248-046-0-53

- if $\beta > 0$, $\alpha \pm \sqrt{\beta} < 0$, then $E_3$ is the asymptotically stable node (fig. 1),
- if $\beta = 0$, $\alpha < 0$, then $E_3$ is the asymptotically stable node (fig. 1),
- if $\beta < 0$, $\alpha < 0$, then $E_3$ is the asymptotically stable spiral node (fig. 3),
- if $\beta > 0$, $\alpha \pm \sqrt{\beta} > 0$, then $E_3$ is the unstable node (fig. 2),
- if $\beta > 0$, $\alpha - \sqrt{\beta} > 0$, $\alpha + \sqrt{\beta} < 0$, then $E_3$ is the unstable saddle point (fig. 5),
- if $\beta > 0$, $\alpha - \sqrt{\beta} < 0$, $\alpha + \sqrt{\beta} > 0$, then $E_3$ is the unstable saddle point (fig. 5),
- if $\beta = 0$, $\alpha > 0$, then $E_3$ is the unstable node (fig. 2),
- if $\beta < 0$, $\alpha > 0$, then $E_3$ is the unstable spiral node (fig. 4).

## B. *Discussion of the Results*

The system (6) has three non-zero equilibrium points. There may be cases when there is a non-zero asymptotically stable equilibrium product. These are asymptotically stable node and asymptotically stable spiral node.

For example, if $\Delta_1 = \Delta_2 = 1$ , $k_1 = 2$ , $k_2 = 3$ , $\delta_{12} = \delta_{21} = 1$, then the system has

- the unstable node $O(0,0)$,
- the unstable saddle points $E_1(0,0.3)$ and $E_2(0.5,0)$,
- the asymptotically stable spiral node $E_3(0.4,0.2)$.

If $\Delta_1 = 1$, $\Delta_2 = 3$, $k_1 = 2$, $k_2 = 4$, $\delta_{12} = \delta_{21} = 1$, then the system has

- the unstable node $O(0,0)$,
- the unstable saddle points $E_1(0,0.7)$ and $E_2(0.5,0)$,
- the asymptotically stable node $E_3(0.1,0.7)$.

# IV. Conclusion

Our proposed model of nonlinear differential dynamics of interdependent industries makes it possible solve the problem of optimal control. By varying the control parameters of the system can be obtained asymptotically stable state of the system. In this case, it's non-zero asymptotically stable equilibrium products. In this work, we find constraints on the control parameters for which there exists an asymptotic stability of the system. Non-zero asymptotically stable equilibrium product exists only in a system with constraints on the produced product.

Further modification of the dynamic model of interdependent industries will be associated with the large number of industries. In addition, we plan to consider limits on the number consumed products, such as Holling-Tanner form [8]. These modifications will result to growth of the number of stable states, the stable and unstable limit cycle and dynamical chaos. Such studies cannot be made in the first approximation. We have to use the method of Lyapunov function and numerical modeling trajectories of the system.

## *References*


[1] W. B. Zhang, Senergetic Economics. Time and Change in Nonlinear Economics, Springer-Verlag, 1999.

[2] R. Shone, Economic Dynamics. Phase Diagrams and Their Economic Application, Cambridge University Press, 2002.

[3] J. D. Sterman, Business Dynamics. System Thinking and Modeling for a Complex World, McGraw-Hill, 2000.

[4] A. V. Dmitriev, S.V. Mfltseva, N. Markov, "Crisis forecasting in ask and bid prices formation," Proceeding Global Interdisciplinary Business-Economics Advancment Conference, 2014, FL, USA, pp. 11-18.

[5] S.V. Maltseva, A. V. Dmitriev, "Dynamic system crisis's using big data flow," Proceeding Performance Management Conference, 2014, Aarhus, Denmark, pp. 192-201.

[6] J.E. Cohen, How Many People Can the Earth Support?, Norton&Company, NY, 1995.

[7] N. A. Gershenfeld, The Nature of Mathematical Modeling, Cambridge University Press, 1999.


**Victor Dmitriev**, student of Business Informatics Bachelor's Program, Faculty of Business Informatics, National Research University Higher School of Economics. Victor Dmitriev's current research interests include the fields of dynamic modeling.

**Dr. Svetlana Maltseva**, professor of National Research University Higher School of Economics, Head of The Innovation and Business in Information Technologies Department, Acting Dean of The Faculty of Business Informatics, Member of the HSE Academic Council. She graduated from Moscow Institute of Electronic Engineering with a degree in Computer Science. IEEE member. Member of the Intellectual Property Committee of the Chamber of Commerce and Industry of Russian Federation. IBM Faculty Award 2013.

**Andrey Dmitriev**, Doctor of Sciences (Physics and Mathematics), is a Professor in the Department of Corporate Information Systems, Business Informatics Programme Academic Supervisor, High School of Economics. Dmitriev's current research interests include the fields of stochastic and dynamic modeling, data analysis, non-stationary time series analysis.