

Automorphism groups of rigid geometries on leaf spaces of foliations

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April 13, 2018

Abstract

We introduce a category of rigid geometries on singular spaces which are leaf spaces of foliations and are considered as leaf manifolds. We single out a special category \mathfrak{F}_0 of leaf manifolds containing the orbifold category as a full subcategory. Objects of \mathfrak{F}_0 may have non-Hausdorff topology unlike the orbifolds. The topology of some objects of \mathfrak{F}_0 does not satisfy the separation axiom T_0 . It is shown that for every $\mathcal{N} \in \text{Ob}(\mathfrak{F}_0)$ a rigid geometry ζ on \mathcal{N} admits a desingularization. Moreover, for every such \mathcal{N} we prove the existence and the uniqueness of a finite dimensional Lie group structure on the automorphism group $\text{Aut}(\zeta)$ of the rigid geometry ζ on \mathcal{N} .

Key words: leaf space; leaf manifold; rigid geometry; automorphism group; orbifold

MSC: 53C12; 57R30; 18F15

Bibliography: 25 names.

1 Introduction and the Main results

Singular spaces and differential geometry on them are used in many branches of mathematics and physics (see for example [4], [20], [9]). Orbifold, forming the full subcategory of studied in this paper category of leaf manifolds of foliations, used in string theory and in theory of deformation quantization. Famous results of Thurston on the classification of closed 3-manifolds use the classification of 2-dimensional orbifolds. Orbifolds were being used by physicists in the study of conformal field theory, an overview of this aspect of orbifold history can be found in [1].

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Different approaches to investigation of additional structures on singular spaces of leaves of foliations are known [10]. Grotendieck presented an approach founded on consideration of the leaf space M/F of a foliation (M, F) as a topos $Sh(M/F)$ formed by all sheaves of M which are invariant under holonomy diffeomorphisms of (M, F) . Haefliger [11] constructed and used a classifying space $B\Gamma^n$ for foliations of codimension n . Connes introduced a concept of C^* -algebra of complex valued functions with compact supports on the holonomy groupoid of a foliation (M, F) [7]. This C^* -algebra may be considered as a desingularization of the leaf space M/F . Losik developed of some ideas of the "formal differential geometry" of Gel'fand [14] and applied them to the introduction new characteristic classes on singular leaf manifolds of foliations [13], [15]. At present the usage of holonomy groupoids and, in particular, of étale groupoids as models of leaf spaces of foliations takes central place [8].

As it was observed by Losik [15], a singular leaf space with a poor topology may have a rich differential geometry. Our work confirms this assertion.

We investigate rigid geometrical structures on singular spaces which are leaf spaces of some class of smooth foliations of an arbitrary codimension n on $(m+n)$ -dimensional manifolds, where $n > 0$, $m > 0$.

The rigid geometrical structures in sense of [22] include large classes of geometries such as Cartan, parabolic, conformal, projective, pseudo-Riemannian, Lorentzian, Riemannian, Weyl and affine connection geometries, rigid geometries in the sense of [2] and also G -structures of finite type.

In this work we introduce a concept of rigid geometries on singular spaces which are leaf spaces of foliations and investigate their automorphism groups.

Following Losik [13], we define a smooth structure on the leaf space M/F of a foliation (M, F) by an atlas (Section 4.1). This smooth structure is called induced by (M, F) . Smooth leaf spaces are called by us *leaf manifolds*. The codimension of a foliation is called the dimension of the induced leaf manifold. Leaf manifolds form a category \mathfrak{F} .

Further we assume that the foliations under consideration admit Ehresmann connections in the sense of Blumenthal and Hebda [5], unless otherwise specified. An Ehresmann connection for a foliation (M, F) of codimension n is an n -dimensional distribution \mathfrak{M} transverse to (M, F) which has the property of vertical-horizontal homotopy (we recall the exact definition in Section 3.1). An Ehresmann connection has the global differentially topological character.

Definition 1 *For a given leaf manifold \mathcal{N} , a smooth foliation (M, F) admitting an Ehresmann connection is called associated with \mathcal{N} if the leaf space M/F with the induced smooth structure becomes an object of the category \mathfrak{F} which is isomorphic to \mathcal{N} in \mathfrak{F} .*

A rigid geometry on a manifold T (disconnected in general) is a pair $\xi = (P(T, H), \beta)$ consisting of an H -bundle $P(T, H)$ over T , where P is equipped

with a non-degenerate \mathbb{R}^k -valued 1-form β agreed with the action of the group H on P . We say that $\mathcal{N} \in Ob(\mathfrak{F})$ has a rigid geometry ζ modelled on ξ if there exists an associated foliation (M, F) admitting ξ as a transverse structure. Note that for a given leaf manifold \mathcal{N} , there are a lot of associated foliations of different dimensions. We show that the definition of ζ is correct, i.e. it does not depend on the choice of foliation (M, F) modelled on ξ which is associated with \mathcal{N} and we prove the following theorem.

Theorem 1 *Let \mathcal{N} be a leaf manifold and (M, F) be an associated foliation. Assume that (M, F) is a foliation which has a transverse rigid geometry $\xi = (P(T, H), \omega)$ and admits an Ehresmann connection. Then the rigid geometry $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$ on \mathcal{N} and a structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(\zeta)$ are defined, where $\mathcal{R}_{\mathcal{F}}$ is the leaf manifold of the lift foliation $(\mathcal{R}, \mathcal{F})$ for (M, F) with the induced locally free action of the Lie group H on $\mathcal{R}_{\mathcal{F}}$ such that $\mathcal{R}_{\mathcal{F}}/H \cong \mathcal{N}$, α is the induced non-degenerate \mathbb{R}^k -valued 1-form on $\mathcal{R}_{\mathcal{F}}$, and the Lie algebra \mathfrak{g}_0 coincides with the structural Lie algebra of (M, F) .*

The category of rigid geometries on leaf manifolds from \mathfrak{F} is denoted by $\mathfrak{R}\mathfrak{F}$. The group of all automorphisms of $\xi \in \mathfrak{F}$ is denoted by $Aut(\xi)$ and called by the *automorphism group of ξ* .

Let $\mathfrak{R}\mathfrak{F}_0$ be the full subcategory of $\mathfrak{R}\mathfrak{F}$ objects of which have zero structural Lie algebra. Let $\mathcal{K} : \mathfrak{R}\mathfrak{F} \rightarrow \mathfrak{F}$ be the covariant functor which forgets a rigid geometry. Put $\mathfrak{F}_0 = \mathcal{K}(\mathfrak{R}\mathfrak{F}_0)$ and note that \mathfrak{F}_0 is a full subcategory of \mathfrak{F} .

Emphasize that any n -dimensional orbifold belongs to $Ob(\mathfrak{F}_0)$, and \mathfrak{F}_0 is a great expansion of the orbifold category. In particular, every leaf manifold $\mathcal{N} \in Ob(\mathfrak{F})$ admitting a rigid geometry and satisfying the separation axiom T_0 , belongs to $Ob(\mathfrak{F}_0)$. Moreover, there are $\mathcal{N} \in Ob(\mathfrak{F}_0)$ which do not satisfy the separation axiom T_0 .

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We prove the following two theorems.

Theorem 2 *Let $\zeta \in Ob(\mathfrak{R}\mathfrak{F}_0)$ be a rigid geometry on n -dimensional leaf manifold $\mathcal{N} \in Ob(\mathfrak{F}_0)$. Let (M, F) be an associated foliation, $(\mathcal{R}, \mathcal{F})$ be its lifted foliation with the projection $\pi : \mathcal{R} \rightarrow M$ of the H -bundle and ω be the induced \mathbb{R}^k -valued 1-form on \mathcal{R} . Then the rigid geometry $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$ on \mathcal{N} has the following properties:*

- (i) 1) the leaf manifold $\mathcal{R}_{\mathcal{F}} \cong W$ is a smooth manifold with a smooth locally free action of the structural Lie group H such that \mathcal{N} is the orbit space W/H , 2) the canonical projections $\pi_b : \mathcal{R} \rightarrow \mathcal{R}/\mathcal{F} \cong W$, $\pi_{\mathcal{F}} : W \rightarrow W/H \cong \mathcal{N}$ and $r : M \rightarrow M/F \cong W/H$ satisfy the equality $\pi_{\mathcal{F}} \circ \pi_b = r \circ \pi$, 3) α is \mathbb{R}^k -valued non-degenerate 1-form on W such that $\pi_b^* \alpha = \omega$, where π_b^* is the codifferential of π_b ;

(ii) the automorphism group $Aut(\zeta)$ of ζ admits a structure of a finite dimension Lie group, and its Lie group structure is defined uniquely;

(iii) the dimension of $Aut(\zeta)$ satisfies the inequality

$$\dim Aut(\zeta) \leq \dim W = k,$$

and $k = n + s$, where s is the dimension of the Lie group H .

Thus a rigid geometry on every $\mathcal{N} \in Ob(\mathfrak{F}_0)$ admits desingularization indicated in Statement (i) of Theorem 2.

Using Theorem 2 we prove the following.

Theorem 3 *Let \mathcal{N} be a leaf manifold. Suppose that the underlying topological space of \mathcal{N} satisfies the separation axiom T_0 and \mathcal{N} admits a rigid geometry ζ . Then:*

1) the pair (\mathcal{N}, ζ) satisfies Theorem 2;

2) there exists an open dense subset \mathcal{N}_0 of \mathcal{N} such that \mathcal{N}_0 with induced smooth structure is isomorphic in the category \mathfrak{F} to an n -dimensional manifold, which is not necessarily connected and not necessarily Hausdorff.

It is well known that for any smooth orbifold \mathcal{N} there exists a Riemannian foliation (M, F) with an Ehresmann connection for which it is the leaf space (see, for example [24]). This fact implies that \mathcal{N} is a leaf manifold having (M, F) as the associated foliation, and (M, F) is a proper foliation with only closed leaves. Therefore for any rigid geometry ζ on an orbifold \mathcal{N} it is necessarily $\mathfrak{g}_0 = 0$, hence orbifolds form a full subcategory of \mathfrak{F}_0 .

The application Theorem 2 gives the following two statements.

Theorem 4 *Let \mathcal{N} be an n -dimensional orbifold equipped with a rigid geometry $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$. Then the automorphism group $Aut(\zeta)$ of ζ admits a structure of a finite dimension Lie group, and its Lie group structure is defined uniquely, the dimension of $Aut(\zeta)$ satisfies the inequality*

$$\dim Aut(\zeta) \leq \dim W = n + s,$$

where s is the dimension of the structural Lie group H of ζ .

Corollary 1 ([3, Theorem 1]) *Let $Aut(\zeta)$ be the automorphism group of a G -structure ζ of finite type and order m on a smooth n -dimensional orbifold \mathcal{N} . Then the group $Aut(\zeta)$ admits a unique topology and a unique smooth structure that makes it into a Lie group, and the dimension of $Aut(\zeta)$ satisfies the inequality*

$$\dim Aut(\zeta) \leq \dim W = n + \dim \mathfrak{g} + \dim \mathfrak{g}_1 + \dots + \dim \mathfrak{g}_{m-1},$$

where \mathfrak{g}_i is the i -th prolongation of the Lie algebra \mathfrak{g} of the group G .

The classical theorems of Myers and Steenrod, Nomizu, Hano and Morimoto, Ehresmann on the existence of a Lie group structure in the full automorphism groups of Riemannian, affine connection geometries and of a finite type structure on manifolds, respectively, follow from Theorem 4.

Assumptions Throughout this paper we assume for simplicity that all manifolds and maps are smooth of the class C^r , $r \geq 1$, and r is large enough which is necessary for a suitable rigid geometry. All neighborhoods are assumed to be open and all manifolds are assumed to be Hausdorff unless otherwise specified.

Notations Let $\mathfrak{X}(T)$ denote the module of smooth vector fields over the ring of smooth functions on a manifold T . If \mathfrak{M} is a smooth distribution on M and $f : K \rightarrow M$ is a submersion, then let $f^*\mathfrak{M}$ be the distribution on the manifold K such that $(f^*\mathfrak{M})_z = \{X \in T_zK \mid f_{*z}(X) \in \mathfrak{M}_{f(z)}\}$, where $z \in K$. Let $\mathfrak{X}_{\mathfrak{M}}(M) = \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \quad \forall u \in M\}$. Let id_M be the identity mapping of a manifold M . Denote by \mathcal{Fol} the foliation category in which morphisms are smooth maps transforming leaves to leaves.

The symbol \cong will denote the isomorphism of objects in the corresponding category.

Acknowledgements. I express my gratitude to Anton Galaev who drew my attention to the works of M. V. Losik.

The publication was prepared within the framework of the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2016–2018 (grant No 16-01-0010) and by the Russian Academic Excellence Project "5-100".

2 Rigid geometries

2.1 Rigid structures

A manifold that admits an e -structure is called parallelizable. In other words, a parallelizable manifold is a pair (P, β) , where P is a k -dimensional smooth manifold and β is a smooth non-degenerate \mathbb{R}^k -valued 1-form β on P , i.e., $\beta_u : T_uP \rightarrow \mathbb{R}^k$ is an isomorphism of the vector spaces for each $u \in P$.

Denote by $P(T, H)$ a principal H -bundle with the projection $p : P \rightarrow T$. Suppose that the action of H on P is a right action and R_a is the diffeomorphism of P corresponding to an element $a \in H$.

Two principal bundles $P(T, H)$ and $\tilde{P}(\tilde{T}, \tilde{H})$ are called *isomorphic* if $H = \tilde{H}$ and there exists a diffeomorphism $\Gamma : P \rightarrow \tilde{P}$ such that $\Gamma \circ R_a = \tilde{R}_a \circ \Gamma \quad \forall a \in H$, where \tilde{R}_a is the transformation of \tilde{P} defined by an element $a \in H$.

Definition 2 Let $P(T, H)$ be a principal H -bundle and (P, β) be a parallelizable manifold satisfying the following condition:

(S) there is an inclusion $\mathfrak{h} \subset \mathbb{R}^k$ of the vector space of the Lie algebra \mathfrak{h} of the Lie group H into the vector space \mathbb{R}^k such that $\beta(A^*) = A \forall A \in \mathfrak{h}$, where A^* is the fundamental vector field on P corresponding to A .

Such $\xi = (P(T, H), \beta)$ is called a rigid structure on the manifold T . A pair (T, ξ) is called a rigid geometry. The Lie group H is called the structural Lie group of ξ .

Definition 3 Let $\xi = (P(T, H), \beta)$ and $\xi' = (P'(T', H), \beta')$ be two rigid structures with the projections $p : P \rightarrow T$ and $p' : P' \rightarrow T'$. An isomorphism $\Gamma : P \rightarrow P'$ of the H -bundles $P(T, H)$ and $P'(T', H)$ satisfying the equality $\Gamma^* \beta' = \beta$ is called an isomorphism of the rigid structures ξ and ξ' .

Such isomorphism Γ defines a map $\gamma : T \rightarrow T'$ satisfying the equality $p' \circ \Gamma = \gamma \circ p$, and γ is a diffeomorphism of T onto T' . The projection γ is called an isomorphism of the rigid geometries (T, ξ) and (T', ξ') .

2.2 Induced rigid geometries

Let $\xi = (P(T, H), \beta)$ be a rigid structure on a manifold T with the projection $p : P \rightarrow T$. Let V be an arbitrary open subset of the manifold T , let $P_V := p^{-1}(V)$ and $\beta_V := \beta|_{P_V}$. Then $\xi_V := (P_V(V, H), \beta_V)$ is also a rigid structure.

Definition 4 The pair (V, ξ_V) defined above is called the induced rigid geometry on the open subset V of T .

2.3 Effectiveness of rigid geometries

Let $Aut(\xi)$ be the automorphism group of a rigid structure $\xi = (P(T, H), \beta)$. It is a Lie group as a closed subgroup of the Lie automorphism group $Aut(P, \beta)$ of the parallelizable manifold (P, β) . Denote by $Aut(T, \xi)$ the group of all automorphisms of the geometry (T, ξ) , i.e., $Aut(T, \xi) := \{\gamma \in \text{Diff}(T) \mid \exists \Gamma \in Aut(\xi) : p \circ \Gamma = \gamma \circ p\}$. Consider the group epimorphism $\chi : Aut(\xi) \rightarrow Aut(T, \xi) : \Gamma \mapsto \gamma$, where γ is the projection of Γ with respect to $p : P \rightarrow T$.

Definition 5 Let $\xi = (P(T, H), \beta)$ be a rigid structure on a manifold T and let $p : P \rightarrow T$ be the projection. The group $\text{Gauge}(\xi) := \{\Gamma \in Aut(\xi) \mid p \circ \Gamma = p\}$ is called the group of gauge transformations of the rigid structure ξ .

Remark that $\text{Gauge}(\xi)$ is a closed normal Lie subgroup of the Lie group $Aut(\xi)$ as the kernel of the group epimorphism $\chi : Aut(\xi) \rightarrow Aut(T, \xi)$.

Definition 6 A rigid structure $\xi = (P(T, H), \omega)$ is called effective if for any open connected subset V in T the induced rigid structure $\xi_V = (P_V(V, H), \beta_V)$ has the trivial group of gauge transformations, i.e., $\text{Gauge}(\xi_V) = \{\text{id}_{P_V}\}$. A rigid geometry (T, ξ) is said to be effective if ξ is an effective rigid structure.

We want to emphasize that effective Cartan geometries ([19], [6]), G -structures of finite type and rigid structures in the sense of [2] are examples of effective rigid geometries.

2.4 Pseudogroup of local automorphisms

Let (T, ξ) be a rigid geometry, and the topological space of T may be disconnected. For arbitrary open subsets $V, V' \subset T$ an isomorphism $V \rightarrow V'$ of the induced rigid geometries (V, ξ_V) and $(V', \xi_{V'})$ is called a *local automorphism* of (T, ξ) . The family \mathcal{H} of all local automorphisms of a rigid geometry (T, ξ) forms a pseudogroup of local automorphisms. Denote it by $\mathcal{H} = \mathcal{H}(T, \xi)$. Recall that a pseudogroup \mathcal{H} of local diffeomorphisms of manifold T is called *quasi-analytic* if the existence of an open subset $V \subset T$ and an element $\gamma \in \mathcal{H}$ such that $\gamma|_V = \text{id}_V$ implies that $\gamma|_{D(\gamma)} = \text{id}_{D(\gamma)}$ in the entire (connected) domain $D(\gamma)$ on which γ is defined.

The following statement is important in the future.

Proposition 1 *A pseudogroup $\mathcal{H} = \mathcal{H}(T, \xi)$ of all local automorphisms of an effective rigid geometry (T, ξ) is quasi-analytic.*

Proof: Let $\gamma \in \mathcal{H}$ be defined on an open connected subset $D(\gamma)$ of T . Assume that there exists an open subset $V \subset D(\gamma)$ such that $\gamma|_V = \text{id}_V$. Since γ is the projection of an local automorphism $\Gamma : P_{D(\gamma)} \rightarrow P_{D(\gamma)}$ of the rigid geometry ξ , it is necessary $\Gamma \in \text{Gauge}(\xi_V)$. According to Definition 6 of effectiveness of the rigid geometry ξ we have $\Gamma_{P_V} = \text{id}_{P_V}$. As $P_V \subset P_{D(\gamma)}$ and $D(\gamma)$ is connected, then each connected component $P_{D(\gamma)}^c$ of $P_{D(\gamma)}$ contains some connected component P_V^c of P_V . Note that Γ preserves $P_{D(\gamma)}^c$. It is well known that every automorphism of a connected parallelizable manifold is uniquely determined by its value at a single point. Since $\Gamma(w) = w$ for $w \in P_V^c$, then $\Gamma = \text{id}_{P_{D(\gamma)}^c}$, where $P_{D(\gamma)}^c$ is an arbitrary connected component of $P_{D(\gamma)}$. Hence $\Gamma = \text{id}_{P_{D(\gamma)}}$. This implies $\gamma = \text{id}_{D(\gamma)}$. \diamond

3 Foliations with transverse rigid geometries. Foliated bundles

3.1 Ehresmann connections for foliations

The notion of an Ehresmann connection for foliations was introduced by Blumenthal and Hebda [5]. We use the terminology from [21]. Let (M, F) be a smooth foliation of codimension $n \geq 1$ and \mathfrak{M} be an n -dimensional transversal distribution on M . All maps and curves considered here are assumed to be

piecewise smooth. The curves in the leaves of the foliation are called vertical; the distribution \mathfrak{M} and its integral curves are called horizontal.

A map $H : I_1 \times I_2 \rightarrow M$, where $I_1 = I_2 = [0, 1]$, is called a vertical-horizontal homotopy if for each fixed $t \in I_2$, the curve $H_{|I_1 \times \{t\}}$ is horizontal, and for each fixed $s \in I_1$, the curve $H_{|\{s\} \times I_2}$ is vertical. The pair of curves $(H_{|I_1 \times \{0\}}, H_{|\{0\} \times I_2})$ is called the base of H .

A pair of curves (σ, h) with a common starting point $\sigma(0) = h(0)$, where $\sigma : I_1 \rightarrow M$ is a horizontal curve, and $h : I_2 \rightarrow M$ is a vertical curve, is called admissible. If for each admissible pair of curves (σ, h) there exists a vertical-horizontal homotopy with the base (σ, h) , then the distribution \mathfrak{M} is called an *Ehresmann connection for the foliation* (M, F) . Note that there exists at most one vertical-horizontal homotopy with a given base.

3.2 Foliations with transverse rigid geometries

Let (T, ξ) be a rigid geometry on an n -dimensional manifold T , and the topological space of T may be disconnected. A foliation (M, F) of codimension n on an $(m+n)$ -dimensional manifold M has a transverse rigid geometry (T, ξ) if (M, F) is defined by a cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}_{i,j \in J}\}$ modelled on (T, ξ) , i.e.,

- 1) $\{U_i | i \in J\}$ is an open covering of M ;
- 2) $f_i : U_i \rightarrow T$ are submersions with connected fibres;
- 3) $\gamma_{ij} \circ f_j = f_i$ on $U_i \cap U_j$, where

$$\gamma_{ij} : (f_j(U_i \cap U_j), \xi_{f_j(U_i \cap U_j)}) \rightarrow (f_i(U_i \cap U_j), \xi_{f_i(U_i \cap U_j)})$$

is a local automorphism of (T, ξ) .

Without loss of generality, we will suppose that $T = \cup_{i \in J} f_i(U_i)$ and the family $\{(U_i, f_i)\}_{i \in J}$ is maximal as it is generally used in the manifold theory.

For short (M, F) is referred to as a foliation with TRG (i.e. with a transverse rigid geometry).

Recall that the pseudogroup generated by local diffeomorphisms γ_{ij} , $i, j \in J$, is referred to as the *holonomy pseudogroup* of (M, F) . It is denoted by $\mathcal{H} = \mathcal{H}(M, F)$.

Definition 7 *The cocycle η modelled on (T, ξ) is said to be an (T, ξ) -cocycle. It is said also that (M, F) is modelled on the rigid geometry (T, ξ) .*

Note that an e -foliation (or a transversally parallelizable foliation) is a foliation admitting a transverse rigid geometry with the trivial structure Lie group H , i.e. $H = \{e\}$.

3.3 The lifted foliation

We use the construction of the lifted foliation $(\mathcal{R}, \mathcal{F})$ for a foliation (M, F) with TRG from [22]. It generalizes a similar construction for an effective Cartan foliation [21] and for a Riemannian foliation [16]. For a given foliation (M, F) with TRG one may construct a principle H -bundle $\mathcal{R}(M, H)$ (called a foliated bundle) with the projection $\pi : \mathcal{R} \rightarrow M$, an H -invariant transversally parallelizable foliation $(\mathcal{R}, \mathcal{F})$ such that π is a morphism of $(\mathcal{R}, \mathcal{F})$ onto (M, F) in the category of foliations $\mathcal{F}ol$. Moreover, there exists a \mathbb{R}^k -valued 1-form ω on \mathcal{R} having the following properties:

- (i) $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field corresponding to A ;
- (ii) for any $u \in \mathcal{R}$, the map $\omega_u : T_u\mathcal{R} \rightarrow \mathbb{R}^k$ is surjective with the kernel $\ker \omega = T\mathcal{F}$, where $T\mathcal{F}$ is the tangent distribution to the foliation $(\mathcal{R}, \mathcal{F})$;
- (iii) the Lie derivative $L_X\omega$ is zero for any vector field X tangent to the leaves of $(\mathcal{R}, \mathcal{F})$.

The foliation $(\mathcal{R}, \mathcal{F})$ is called the *lifted foliation*.

The restriction $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow L$ of π to a leaf \mathcal{L} of $(\mathcal{R}, \mathcal{F})$ is a regular covering map onto the corresponding leaf L of (M, F) , and the group of deck transformation of $\pi|_{\mathcal{L}}$ is isomorphic to the germ holonomy group of L which is usually used in the foliation theory.

If \mathcal{R} is disconnected, then we consider a connected component of \mathcal{R} and denote it also \mathcal{R} . Without loss of generality we assume that the same Lie group H acts on \mathcal{R} and P .

Let (M, F) be defined by a (T, ξ) -cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$. Effectiveness of ξ guarantees the existence of a unique isomorphism Γ_{ij} of the induced rigid structures $\xi_{f_j(U_i \cap U_j)}$ and $\xi_{f_i(U_i \cap U_j)}$, whose projection coincides with γ_{ij} . Hence, in the case $U_i \cap U_j \cap U_k \neq \emptyset$, the equality $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ implies the equality $\Gamma_{ij} \circ \Gamma_{jk} = \Gamma_{ik}$. The following two equalities are direct corollaries of the previous equality and the effectiveness of η : $\Gamma_{ii} = \text{id}_{P_i}$ and $\Gamma_{ij} = (\Gamma_{ji})^{-1}$.

Remark that the holonomy pseudogroup of $(\mathcal{R}, \mathcal{F})$ is generated by Γ_{ij} , $i, j \in J$.

3.4 The structural Lie algebra of an e -foliation with an Ehresmann connection

At first we prove the following theorem.

Theorem 5 *Let (M, F) be an e -foliation with an Ehresmann connection on a connected manifold M . Then the closures of its leaves are fibers of a locally trivial fibration $\pi_b : M \rightarrow W$ over the manifold W . On every fiber \bar{L} , $L \in F$, of this fibration the induced foliation $(\bar{L}, F|_{\bar{L}})$ is a Lie foliation with dense leaves. The structural Lie algebra \mathfrak{g}_0 of the Lie foliation $(\bar{L}, F|_{\bar{L}})$ does not depend on the choice of $L \in F$ and \mathfrak{g}_0 is called the structural Lie algebra of (M, F) .*

Proof: Since an e -foliation (M, F) is a Riemannian one and it admits an Ehresmann connection, according to [23, Proposition 2] the holonomy group $\mathcal{H} = \mathcal{H}(M, F)$ is complete. Therefore we may apply the results of Salem [18]. According to [18], the closure \overline{L} of a leaf L is a smooth manifold and the induced foliation $(\overline{L}, F|_{\overline{L}})$ is a Lie foliation with dense leaves. Let \mathfrak{M} be an Ehresmann connection for (M, F) , then $\mathfrak{N} := \mathfrak{M} \cap \overline{L}$ is an Ehresmann connection for $(\overline{L}, F|_{\overline{L}})$. By [23, Proposition 2], the pseudogroup of $(\overline{L}, F|_{\overline{L}})$ is complete. Therefore the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(\overline{L}, F|_{\overline{L}})$ of the Lie foliation $(\overline{L}, F|_{\overline{L}})$ is defined [18]. Since the foliation (M, F) admits an Ehresmann connection, the automorphism group of (M, F) in the foliation category \mathcal{Fol} acts transitively on the set of its leaves. This means that for every leaves L and L' there is an automorphism $f : M \rightarrow M$ of (M, F) such that $f(L) = L'$. The diffeomorphism f has the property $f(\overline{L}) = \overline{L}'$. Therefore, $f|_{\overline{L}}$ is an isomorphism of the Lie foliations $(\overline{L}, F|_{\overline{L}})$ and $(\overline{L}', F|_{\overline{L}'})$ in \mathcal{Fol} . It is well known that the structural Lie algebra of a Lie foliation with dense leaves and with the complete holonomy pseudogroup is an invariant in the category \mathcal{Fol} [12], [18]. Therefore $\mathfrak{g}_0(\overline{L}, F|_{\overline{L}}) = \mathfrak{g}_0(\overline{L}', F|_{\overline{L}'})$ and the definition $\mathfrak{g}_0(M, F) := \mathfrak{g}_0(\overline{L}, F|_{\overline{L}})$ is correct.

Observe that the foliation (M, \overline{F}) formed by closures of leaves of (M, F) is a regular Riemannian foliation with an induced Ehresmann connection, and all its leaves are closed. By analogy with proof of [16, Theorem 4.2'] we show that a leaf $\overline{L} \in \overline{F}$ has a saturated neighborhood \mathcal{U} such that $(\mathcal{U}, \overline{F}|_{\mathcal{U}})$ is e -foliation. This implies that leaves of (M, \overline{F}) are fibers of a locally trivial fibration which is denoted by $\pi_b : M \rightarrow W$. \diamond

For a complete e -foliations a similar theorem is proved in [16, Theorem 4.2'].

3.5 The structural Lie algebra of a foliation with TRG admitting an Ehresmann connection

We use notations introduced in Section 3.3. Let (M, F) be a foliation with TRG having an Ehresmann connection \mathfrak{M} . Let $(\mathcal{R}, \mathcal{F})$ be the lifted e -foliation and $\pi : \mathcal{R} \rightarrow M$ be the projection of H -bundle $\mathcal{R}(M, H)$. Observe that the distribution $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ is an Ehresmann connection for the foliation $(\mathcal{R}, \mathcal{F})$. Applying Theorem 5 to e -foliation $(\mathcal{R}, \mathcal{F})$ admitting an Ehresmann connection we obtain the following statement.

Theorem 6 *Let (M, F) be a foliation with TRG admitting an Ehresmann connection and let $(\mathcal{R}, \mathcal{F})$ be its lifted e -foliation. Then:*

- (i) *the closures of the leaves of the foliation \mathcal{F} are fibers of a certain locally trivial fibration $\pi_b : \mathcal{R} \rightarrow W$;*

- (ii) the foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ induced on the closure $\overline{\mathcal{L}}$ is a Lie foliation with dense leaves with the structure Lie algebra \mathfrak{g}_0 , that is the same for any $\mathcal{L} \in \mathcal{F}$.

According to Theorem 6 the following definition is correct.

Definition 8 *The structural Lie algebra \mathfrak{g}_0 of the Lie foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ is called the structural Lie algebra of the foliation (M, F) with TRG admitting an Ehresmann connection and is denoted by $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$.*

Remark 1 *If (M, F) is a Riemannian foliation on a compact manifold, this notion coincides with the notion of a structural Lie algebra in the sense of Molino [16].*

Definition 9 *The fibration $\pi_b: \mathcal{R} \rightarrow W$ satisfying Theorem 5 is called a basic fibration for (M, F) .*

Remark 2 *Under stronger conditions of completeness of (M, F) a similar theorem is obtained in [22, Theorem 2]. The advantage of Theorem 6 in comparing with [22, Theorem 2] is also that the condition of the existence of an Ehresmann connection for (M, F) is defined on M in contrast to the completeness of (M, F) with TRG which is defined on the space of the H -bundle over M .*

3.6 Foliations with the zero structural Lie algebra

The following proposition is proved using Theorem 6 by an analogy with [22, Proposition 7].

Proposition 2 *Let (M, F) be a foliation with TRG admitting an Ehresmann connection. Suppose that $\mathfrak{g}_0(M, F) = 0$. Let $\pi_b: \mathcal{R} \rightarrow W$ be the basic fibration. Then:*

- (i) *the map $\Phi^W: W \times H \rightarrow W: (w, a) \mapsto \pi_b(R_a(u)) \quad \forall (w, a) \in W \times H,$
 $\forall u \in \pi_b^{-1}(w)$ defines a smooth locally free action of the Lie group H on the basic manifold W ;*
- (ii) *there is a homeomorphism $s: M/F \rightarrow W/H$ between the leaf space M/F and the orbit space W/H satisfying the equality $q \circ \pi_b = s \circ r \circ \pi$, where $q: W \rightarrow W/H$ and $r: M \rightarrow M/F$ are the quotient maps;*
- (iii) *the equality $\pi_b^* \omega^W = \omega$ defines an \mathbb{R}^k -valued non-degenerate 1-form ω^W on W such that $\omega^W(A_W^*) = A$, where A_W^* is the fundamental vector field on W defined by an element $A \in \mathfrak{h} \subset \mathbb{R}^k$.*

4 Rigid geometries on leaf manifolds

We want to emphasize that Sections 4.1 and 4.2 (M, F) is any smooth foliation. We do not assume the existence of an Ehresmann connection for (M, F) .

4.1 Generalized manifolds of leaf spaces of foliations

Consider a smooth foliation (M, F) of codimension n on $(n + m)$ -dimensional manifold M . Denote by $r : M \rightarrow M/F$ the quotient map onto the leaf space. Referring to [13] let us consider the category \mathbb{R}_n with objects open submanifolds of \mathbb{R}^n , where morphisms $Hom(U, V)$ are diffeomorphisms $f : U \rightarrow V$ onto $f(U) \subset V$.

At every point $x \in M$ there exists a chart (V, φ) of M adapted to (M, F) . This means that $\varphi(V) = W \times U \subset \mathbb{R}^{m+n} \cong \mathbb{R}^m \times \mathbb{R}^n$, where W and U are open subsets in \mathbb{R}^m and \mathbb{R}^n respectively and $\varphi(x) = (y, z) \in W \times U$. Denote by $pr : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ the canonical projection. The fibers of the submersion $pr \circ \varphi : V \rightarrow U$ belong to leaves of the foliation (M, F) .

Let $r : M \rightarrow M/F$ be the projection onto the leaf space M/F of (M, F) . For a fixed $y \in W$ denote by $j : U \rightarrow W \times U$ the embedding such that $j(u) = (y, u) \in W \times U \quad \forall u \in U$. The pair (U, k) , where $k := r|_U \circ \varphi^{-1} \circ j : U \rightarrow M/F$, is called a \mathbb{R}_n -chart (or chart) on M/F .

Definition 10 *Two charts (U', k') and (U'', k'') on M/F for which $k'(U') \cap k''(U'') \neq \emptyset$, are called compatible if for each point $z \in k'(U') \cap k''(U'')$ there exists a chart (U, k) , $z \in k(U)$, with two morphisms $h' : U \rightarrow U'$ and $h'' : U \rightarrow U''$ in the category \mathbb{R}_n satisfying the following conditions $k' \circ h' = k$ and $k'' \circ h'' = k$.*

Definition 11 *A smooth atlas on M/F is a family of charts $\mathcal{A} = \{(U_i, k_i) \mid i \in J\}$ satisfying the following two conditions:*

- 1) *the set $\{k_i(U_i) \mid i \in J\}$ is a covering of M/F , i.e. $\cup_{i \in J} k_i(U_i) = M/F$;*
- 2) *every two charts from \mathcal{A} are compatible.*

A smooth atlas \mathcal{A} is maximal if it is maximal relatively inclusion.

Definition 12 *A pair $(M/F, \mathcal{A})$, where \mathcal{A} is a maximal atlas on M/F is referred to as a leaf manifold. This leaf manifold is called induced by the foliation (M, F) and it is denoted by \mathcal{N} . The number n is called the dimension of \mathcal{N} .*

Any atlas \mathcal{A} defines the maximal atlas $\widehat{\mathcal{A}}$ as the set of charts which are compatible with all charts from \mathcal{A} , hence \mathcal{A} defines a smooth structure on M/F .

Definition 13 Let $(\mathcal{N}_1, \mathcal{A}_1)$ and $(\mathcal{N}_2, \mathcal{A}_2)$ be two n -dimensional leaf manifolds. A morphism of \mathcal{N}_1 to \mathcal{N}_2 is a map $h : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ such that for each chart $(U, k) \in \mathcal{A}_1$ the pair $(U, h \circ k)$ is a chart from \mathcal{A}_2 .

Denote by \mathfrak{F} the category of leaf manifolds of a fixed dimension n .

Definition 14 A foliation (M', F') is referred to as associated with a leaf manifold \mathcal{N} if (M', F') induces a leaf manifold \mathcal{N}' on the leaf space M'/F' , and \mathcal{N}' is isomorphic to \mathcal{N} in the category \mathfrak{F} .

Remark 3 We emphasize that Definition 12 is equivalent to the definition a smooth structure on a leaf space in the sense of [14]. It is known [14, Theorem 2] that for every foliation there exists a smooth atlas on the leaf space.

Remark 4 On a leaf manifold \mathcal{N} the tangent bundle $T\mathcal{N}$, differentials and codifferentials of smooth maps, vector-valued forms are defined [13], [15].

4.2 The pseudogroup approach to leaf manifolds

Let $(\mathcal{N}, \mathcal{A})$ be a leaf manifold and (M, F) be an associated foliation. Then the topological space of \mathcal{N} is the leaf space M/F . Denote by $\mathcal{H} = \mathcal{H}(M, F)$ the holonomy pseudogroup of (M, F) . If the foliation (M, F) is given by an T -cocycle $\{U_i, f_i, \{\gamma_{ij}\}_{i,j \in J}\}$, then \mathcal{H} is generated by local diffeomorphisms γ_{ij} , and $T = \cup_{i \in J} f_i(U_i)$. Let T/\mathcal{H} be the quotient space. with the quotient map $q : T \rightarrow T/\mathcal{H}$. It is easy to show that $q : T \rightarrow T/\mathcal{H}$ is continuous and open map. There exists a homeomorphism $\theta : M/F \rightarrow T/\mathcal{H}$ defined by the equality $\theta([L]) := [\mathcal{H}.v]$, where $[L]$ is a leaf $L = L(x)$ considered as a point of M/F and $[\mathcal{H}.v] \in T/\mathcal{H}$ is the orbit $\mathcal{H}.v$ of a point $v = f_i(x)$ for some submersion f_i from the T -cocycle. Let us identify through θ the topological spaces M/F and T/\mathcal{H} .

Consider a chart (V, ψ) of the manifold T . Then $U = \psi(V)$ is an open subset in \mathbb{R}^n . It is easy to see that the set \mathcal{B} formed by the charts (U, k) where $U = \psi(V)$ and $k = q \circ \psi^{-1} : U \rightarrow M/F$ is an atlas on M/F , and \mathcal{B} is compatible with the atlas \mathcal{A} of \mathcal{N} induced by the foliation (M, F) .

Now let us show that the atlas $\mathcal{B} = \{(U_i, k_i) \mid i \in J\}$ defines a pseudogroup $\tilde{\mathcal{H}}$ of local diffeomorphisms of the n -dimensional manifold $\tilde{T} = \coprod_{i \in J} U_i$. Let (U_i, k_i) and (U_j, k_j) be two charts from \mathcal{A} and $z \in k_i(U_i) \cap k_j(U_j)$. According to the compatibility of these charts there exist a chart (U, k) such that $z \in k(U)$ and two morphisms $h_i : U \rightarrow U_i$, $h_j : U \rightarrow U_j$ in the category \mathbb{R}_n satisfying the equalities $k_i \circ h_i = k$ and $k_j \circ h_j = k$. Note that $h_i(U)$ and $h_j(U)$ are open subsets of \tilde{T} . Therefore the map $\tilde{\gamma}_{ij} := h_i \circ h_j^{-1}|_{h_j(U)} : h_j(U) \rightarrow h_i(U)$ is a local diffeomorphism of \tilde{T} . The set $\{\tilde{\gamma}_{ij} \mid i, j \in J\}$ generates a pseudogroup $\tilde{\mathcal{H}}$ of local diffeomorphisms of \tilde{T} . This pseudogroup is equivalent to the holonomy

pseudogroup of any foliation associated with the leaf manifold \mathcal{N} in the terms of the work [18]. Since the holonomy pseudogroup of a foliation is defined up to an equivalence, we have the following statement.

Proposition 3 *Any leaf manifold \mathcal{N} with an associated foliation (M, F) may be defined by the indicated above atlas \mathcal{B} defined by the holonomy pseudogroup $\mathcal{H}(M, F)$ of (M, F) .*

Every two foliations (M, F) and (M', F') associated with \mathcal{N} have the same holonomy pseudogroup.

Thus without loss of generality we consider the pseudogroup on \tilde{T} defined by the atlas \mathcal{B} as the holonomy pseudogroup of an associated foliation (M, F) and use notations $\tilde{T} = T$, $\tilde{\mathcal{H}} = \mathcal{H} = \mathcal{H}(\mathcal{N}) = \mathcal{H}(M, F)$.

In the case when the smoothness is C^∞ , we may define the algebra of C^∞ -smooth functions $C^\infty(\mathcal{N})$ on \mathcal{N} as the algebra of \mathcal{H} -invariant C^∞ -smooth functions on T . The Lie algebra of C^∞ -smooth vector fields $\mathfrak{X}(\mathcal{N})$ on \mathcal{N} is defined as the Lie algebra of all derivations of the algebra of functions $C^\infty(\mathcal{N})$.

4.3 Rigid geometries on leaf manifolds and their structural Lie algebras

The proof of Theorem 1 Consider any n -dimensional leaf manifold \mathcal{N} . Let (M, F) be an associated foliation, then M/F is the topological space of \mathcal{N} . Assume that (M, F) admits a transverse rigid geometry $\xi = (P(T, H), \beta)$. This is equivalent to the existence of a lifted foliation $(\mathcal{R}, \mathcal{F})$, where $\mathcal{R}(M, H)$ is a principal H -bundle with the projection $\pi : \mathcal{R} \rightarrow M$, and the \mathbb{R}^k -valued 1-form ω on \mathcal{R} satisfying the conditions (i) – (iii) from Section 3.3. Denote by $\mathcal{R}_{\mathcal{F}}$ the leaf manifold induced by $(\mathcal{R}, \mathcal{F})$.

A leaf L considered as a point of the leaf space is denoted by $[L]$. Let $r : M \rightarrow \mathcal{N}$ and $r_{\mathcal{F}} : \mathcal{R} \rightarrow \mathcal{R}_{\mathcal{F}}$ be the projections onto the leaf manifolds \mathcal{N} and $\mathcal{R}_{\mathcal{F}}$ respectively. Since $\pi : \mathcal{R} \rightarrow M$ is a morphism in the foliation category, the following map $\pi_{\mathcal{F}} : \mathcal{R}_{\mathcal{F}} \rightarrow \mathcal{N} : [\mathcal{L}] \mapsto [\pi(\mathcal{L})]$, $\mathcal{L} \in \mathcal{F}$, is defined and satisfies the equality $\pi_{\mathcal{F}} \circ r_{\mathcal{F}} = r \circ \pi$.

Due to H -invariance of $(\mathcal{R}, \mathcal{F})$, the map

$$\Phi : \mathcal{R}_{\mathcal{F}} \times H \rightarrow \mathcal{R}_{\mathcal{F}} : [\mathcal{L}] \mapsto [R_a(\mathcal{L})], \mathcal{L} \in \mathcal{F}, a \in H,$$

defines a right action of the Lie group H on $\mathcal{R}_{\mathcal{F}}$. The isotropy subgroup $H_{[\mathcal{L}]} = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ is a discrete subgroup of H as the deck transformation group of the regular covering $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow L$. Therefore the action Φ of H on $\mathcal{R}_{\mathcal{F}}$ is locally free, i.e. all isotropy groups are discrete subgroups of the Lie group H . The orbit space $\mathcal{R}_{\mathcal{F}}/H$ is homeomorphic to \mathcal{N} . Let us identify \mathcal{N} with $\mathcal{R}_{\mathcal{F}}/H$. We use the notation $\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H)$ for the quotient map $\pi_{\mathcal{F}} : \mathcal{R}_{\mathcal{F}} \rightarrow \mathcal{N} \cong \mathcal{R}_{\mathcal{F}}/H$.

Let (M, F) and (M', F') be two associated foliations with the same leaf manifold \mathcal{N} . According to Proposition 3 they have the common holonomy pseudogroup $\mathcal{H} = \{\gamma_{ij} \mid i, j \in J\}$ of local automorphisms of the rigid geometry $\xi = (P(T, H), \beta)$. Due to the efficiency of the rigid geometry ξ for each γ_{ij} there exists a unique local automorphism Γ_{ij} of ξ lying over γ_{ij} relative to the projection $p : P \rightarrow T$. Emphasize that $\mathcal{S} = \{\Gamma_{ij} \mid i, j \in J\}$ is the pseudogroup of the both lifted foliations $(\mathcal{R}, \mathcal{F})$ for (M, F) and $(\mathcal{R}', \mathcal{F}')$ for (M', F') . So we will use the following notation $\mathcal{S} = \mathcal{S}(\mathcal{N}, \xi)$. Since $\mathcal{R}_{\mathcal{F}} = P/\mathcal{S}$ is defined by \mathcal{S} , the leaf manifold $\mathcal{R}_{\mathcal{F}}$ is not depend on the choice of an associated foliation.

Every Γ_{ij} acts freely on P as a local automorphism of the parallelizable manifold (P, β) . The free action of the pseudogroup $\mathcal{S} = \{\Gamma_{ij} \mid i, j \in J\}$ on P implies the free action on TP of the pseudogroup $\mathcal{S}_* := \{\Gamma_{ij*} \mid i, j \in J\}$ formed by differentials of local transformations belonging to \mathcal{S} , and $T\mathcal{R}_{\mathcal{F}} = TP/\mathcal{S}_*$. Therefore the tangent space $T_z\mathcal{R}_{\mathcal{F}}$ is a vector space, and the dimension of $T_z\mathcal{R}_{\mathcal{F}}$ is equal to $k = \dim(P)$ at any point $z \in \mathcal{R}_{\mathcal{F}}$. As $\Gamma_{ij*}\beta = \beta$, $i, j \in J$, a non-degenerate \mathbb{R}^k -valued 1-form α is defined on $\mathcal{R}_{\mathcal{F}}$ and satisfies the equality $\mu^*\alpha = \beta$, where $\mu : P \rightarrow P/\mathcal{S} = \mathcal{R}_{\mathcal{F}}$ is the quotient map. We emphasize that α coincides with the 1-form defined by the following equality $\pi_{\mathcal{F}}^*\alpha = \omega$, where ω is the basic \mathbb{R}^k -valuated 1-form on \mathcal{R} satisfying the conditions (i) – (iii) in Section 3.3.

Since $(\mathcal{R}, \mathcal{F})$ is a Riemannian foliation with the Ehresmann connection $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ where \mathfrak{M} is an Ehresmann connection for (M, F) , according to [23, Proposition 2] the holonomy pseudogroup \mathcal{S} of $(\mathcal{R}, \mathcal{F})$ is complete. By [18, Theorem 3.1] the structural Lie algebra $\mathfrak{g}_0(\mathcal{S})$ is defined, and this Lie algebra is isomorphic to the structural Lie algebra $\mathfrak{g}_0(\mathcal{R}, \mathcal{F})$ of $(\mathcal{R}, \mathcal{F})$. According to Definition 8 $\mathfrak{g}_0(M, F) := \mathfrak{g}_0(\mathcal{R}, \mathcal{F})$. As $\mathcal{S} = \mathcal{S}(\mathcal{N}, \xi)$, then $\mathfrak{g}_0 = \mathfrak{g}_0(\mathcal{S})$ is not depend on the choice of an associated foliation (M, F) . \square

Definition 15 *The pair $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$ defined in the proof of Theorem 1 is called the rigid geometry on the leaf manifold \mathcal{N} modelled on the transverse rigid geometry $\xi = (P(N, H), \beta)$ of an associated foliation (M, F) . The structural Lie algebra \mathfrak{g}_0 of (M, F) is called by the structural Lie algebra of the rigid geometry ζ on \mathcal{N} and is denoted by $\mathfrak{g}_0 = \mathfrak{g}_0(\zeta)$.*

According to the proof of Theorem 1 Definition 15 is correct, i.e. \mathfrak{g}_0 does not depend on the choice of the associated foliation (M, F) .

We want to emphasize that the rigid geometry $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$ and the structural Lie algebra \mathfrak{g}_0 depend only on the pseudogroup \mathcal{H} of local automorphisms of the transverse geometry (T, ξ) on which (M, F) is modelled.

5 Automorphisms of rigid geometries on leaf manifolds

5.1 The category of rigid geometries on leaf manifolds

Definition 16 Let $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha)$ and $\zeta' = (\mathcal{R}'_{\mathcal{F}'}(\mathcal{N}', H'), \alpha')$ be two rigid geometries on n -dimensional leaf manifolds \mathcal{N} and \mathcal{N}' respectively. A map $\Gamma : \mathcal{R}_{\mathcal{F}} \rightarrow \mathcal{R}'_{\mathcal{F}'}$ is called a morphism $\zeta \rightarrow \zeta'$ if $H = H'$ and Γ satisfies the following two conditions:

$$1) \Gamma^* \alpha' = \alpha, \quad 2) \Gamma \circ R'_a = R_a \circ \Gamma \quad \forall a \in H.$$

Thus rigid geometries on leaf manifolds form the category $\mathfrak{R}\mathfrak{F}$. Let $\mathfrak{R}\mathfrak{F}_0$ be the full subcategory of $\mathfrak{R}\mathfrak{F}$ objects of which have zero structural Lie algebra.

Every $\Gamma \in Mor(\zeta, \zeta')$ defines the projection $\gamma : \mathcal{N} \rightarrow \mathcal{N}'$ such that $\pi'_{\mathcal{F}'} \circ \Gamma = \gamma \circ \pi_{\mathcal{F}}$, where $\pi_{\mathcal{F}} : \mathcal{R}_{\mathcal{F}} \rightarrow \mathcal{N} \cong \mathcal{R}_{\mathcal{F}}/H$ and $\pi'_{\mathcal{F}'} : \mathcal{R}'_{\mathcal{F}'} \rightarrow \mathcal{N}' \cong \mathcal{R}'_{\mathcal{F}'}/H$ are the quotient maps. There is a covariant functor $\mathcal{K} : \mathfrak{R}\mathfrak{F} \rightarrow \mathfrak{F}$ forgetting rigid geometries. Hence $\mathcal{K}(\zeta) = \mathcal{N}$ for each $\zeta \in Ob(\mathfrak{R}\mathfrak{F})$ and $\mathcal{K}(\Gamma) = \gamma$ for every $\Gamma \in Mor(\zeta, \zeta')$. Let $\mathfrak{F}_0 := \mathcal{K}(\mathfrak{R}\mathfrak{F}_0)$.

5.2 Proof of Theorem 2.

Consider a rigid geometry $\zeta = (\mathcal{R}_{\mathcal{F}}(\mathcal{N}, H), \alpha) \in \mathfrak{R}\mathfrak{F}_0$ on n -dimensional leaf manifold \mathcal{N} , then $\mathfrak{g}_0(\zeta) = 0$. In accordance with Proposition 2 in this case the leaves of the lifted foliation $(\mathcal{R}, \mathcal{F})$ are fibers of the locally trivial basic fibration $\pi_b : \mathcal{R} \rightarrow W$, and the leaf space \mathcal{R}/\mathcal{F} is W . Therefore there exists an isomorphism $f : \mathcal{R}_{\mathcal{F}} \rightarrow W$ in the category \mathfrak{F}_0 of the leaf manifold $\mathcal{R}_{\mathcal{F}}$ onto the manifold W of dimension $\dim(W) = n+s$, where $n = \dim(\mathcal{N})$ and $s = \dim(H)$. Moreover, $f^* \omega^W = \alpha$, hence f is an isomorphism of parallelizable manifolds $(\mathcal{R}_{\mathcal{F}}, \alpha)$ and (W, ω^W) . Let us identify $(\mathcal{R}_{\mathcal{F}}, \alpha)$ with (W, ω^W) through f , then $\mathcal{R}_{\mathcal{F}} = W$ and $\alpha = \omega^W$. Therefore the statement (i) of Theorem 3 follows from Propositions 2.

As it is well known, the automorphism group $A(W, \alpha) = \{h \in \text{Diff}(W) \mid f^* \alpha = \alpha\}$ of the parallelizable manifold (W, α) admits a Lie group structure, and its dimension is not greater than $\dim(W)$. According to Definition 16, the group $Aut(\zeta)$ of all automorphisms of ζ in the category \mathfrak{F} is equal to

$$Aut(\zeta) = \{h \in \text{Diff}(W) \mid h^* \alpha = \alpha, R_a^W \circ h = h \circ R_a^W \quad \forall a \in H\}.$$

Hence, $Aut(\zeta) = \{h \in A(W, \alpha) \mid R_a^W \circ h = h \circ R_a^W, a \in H\}$. This implies that $Aut(\zeta)$ is a closed subgroup of the Lie group $A(W, \alpha)$. It means that $Aut(\zeta)$ admits a Lie group structure, and its dimension is not greater than $\dim(W)$. Since $Aut(\zeta)$ is a transformation group, $Aut(\zeta)$ is equipped with the compact-open topology and a Lie group structure on $Aut(\zeta)$ is unique [17, Theorem VI]. Thus statements (ii) and (iii) of Theorem 2 are proved. \square

5.3 Proof of Theorem 3.

A smooth foliation (M, F) is called proper if each its leaf is an embedded submanifold in M . A subset M_0 of M is said to be saturated if it is a union of leaves of (M, F) .

Recall that a topological space satisfies the separation axiom T_0 if for any different points a and b there exists a neighborhood at least one of them contains no other point. As it is known [25, Lemma 4.1], a foliation is proper if and only if its leaf space satisfies the separation axiom T_0 . Assume that a leaf manifold \mathcal{N} satisfies the separation axiom T_0 . Hence according to the mentioned above lemma every associated foliation (M, F) is proper. Assume that (M, F) is a foliation with transversal rigid geometry ξ . According to Theorem 1 the induced rigid geometry ζ on \mathcal{N} is defined. Consider the lifted foliation $(\mathcal{R}, \mathcal{F})$ for (M, F) . By the assumption (M, F) admits an Ehresmann connection \mathfrak{M} . Preserving the notations used above we denote by $\pi : \mathcal{R} \rightarrow M$ the projection of the H -bundle $\mathcal{R}(M, H)$. Then $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ is an Ehresmann connection for $(\mathcal{R}, \mathcal{F})$.

It is known that any foliation has a leaf with the trivial germ holonomy group. Thus there is a proper leaf with the trivial germ holonomy group of the foliation (M, F) . This implies the existence of a proper leaf of the e -foliation $(\mathcal{R}, \mathcal{F})$ admitting an Ehresmann connection. Therefore $(\mathcal{R}, \mathcal{F})$ is also a proper foliation. Since the closure $\overline{\mathcal{L}}$ of a leaf $\mathcal{L} \in \mathcal{F}$ is a minimal set, in the case when $\overline{\mathcal{L}} \neq \mathcal{L}$, the closure $\overline{\mathcal{L}}$ contains only non-proper leaves. Hence it is necessary $\overline{\mathcal{L}} = \mathcal{L}$. This implies that the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(\zeta)$ is zero. Therefore the rigid geometry (\mathcal{N}, ζ) satisfies Theorem 2 and the statement 1) of Theorem 3 is proved.

Similarly to [25, Theorem 1.1] for proper Cartan foliations admitting Ehresmann connections, we prove the following statement.

Theorem 7 *Let (M, F) be an arbitrary proper foliation of codimension n with transverse rigid geometry $\xi = (P(T, H), \beta)$ admitting an Ehresmann connection.*

Then there exists a not necessarily connected, saturated, dense open subset M_0 of M such that the induced foliation $(M_0, F|_{M_0})$ is formed by the fibers of a locally trivial fibration $p : M_0 \rightarrow B$ with the standard fiber L_0 over a smooth n -dimensional (not necessarily Hausdorff) manifold B .

Now the statement 2) of Theorem 3 follows from Theorem 7. \square

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