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Thermodynamical Properties of a Superconducting Quantum Cylinder

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Abstract. The thermodynamical potential of a superconducting quantum cylinder is calculated. The dependence of the critical temperature and the heat capacity of a superconducting system of the surface concentration of electrons and on the radius of the nanotube is studied.

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1. INTRODUCTION

Great attention has recently been paid to the study of the electronic properties of curved low-dimensional systems [1, 2]. The manufacturing of graphene, which is an isolated planar system in three-dimensional space whose width is one atom, and also the production of quasi-one-dimensional structures, namely, carbon nanotubes, stimulated a new splash of interest for the problem of surface superconductivity posed in the investigations of V. L. Ginzburg and D. A. Kirzhnits [3]. The papers [4] and [5] report on the observation of the superconductivity phenomenon at the critical temperature $T_c \sim 1K$ in one-layer carbon nanotubes of radius $R = 5A$ and the superconductivity phenomenon with $T_c = 16K$ in nanotubes of radius $R = 2A$. Since no superconductivity is observed in graphite, the authors of [6] conjectured that the superconductivity of carbon nanotubes is related to the very curvature of the graphene layer. It should be noted that there is also another point of view [6] according to which the superconductivity must also be observed in an ideal graphite which can be regarded as a system of parallel plane sheets of graphene connected by van der Waals forces. Anyway, the results of the research in [4, 5] testify to the validity of Ginzburg's idea claiming that, in general, the two-dimensionality and even one-dimensionality of a system do not prevent the existence of superconducting properties [3].

In connection with the well-known delay in the discovery of superconductivity in two-dimensional electronic systems and in view of the existing information concerning the observation of surface superconductivity in carbon nanotubes [4, 5], and also on surfaces of metals and even dielectrics, it seems to be urgent to theoretically investigate the thermodynamical properties of these structures. The theory of planar two-dimensional superconductors with regard to the spin-orbital interaction was suggested in [7, 8, 9]. The superconductivity of a planar sheet of graphene using the solutions of the Dirac equation was studied in [10]. In our paper [11], using the BCS-model and the method of $(u - v)$ Bogolyubov transformations for Fermi operations, a microscopic superconductivity theory for an electron gas of a quantum cylinder was constructed and the dependence of the width of the energy gap under zero temperature on the radius of the nanostructure and on the Aharonov–Bohm parameter was studied.

The present paper is devoted to studying thermodynamical properties of a superconducting quantum cylinder. Using the computation of the thermodynamical potential of the system, we investigate the dependence of the width of the energy gap on the geometric sizes of the nanotube and on the temperature, and also find the value of the heat capacity jump of the structure when the critical temperature is achieved. As far as the authors know, the above properties of superconducting one-layer nanotubes were not considered earlier in the literature.

2. THERMODYNAMICAL POTENTIAL AND HEAT CAPACITY OF A SUPERCONDUCTING QUANTUM CYLINDER

We carry out the investigation of superconducting properties of a nanostructure by using the BCS model which correctly describes many properties of superconductors both qualitatively and quantitatively, despite the fact that this model is comparatively simple. The main interaction in the BCS Hamiltonian in the three-dimensional case is the interaction of pairs of electrons with opposite momenta and spins. On the surface of a quantum cylinder, the stationary state of an electron is determined by the azimuthal quantum number $n = 0, \pm 1, \dots$, the momentum of the longitudinal motion p_z, and the projection of the spin to the z axis directed along the axis of the cylinder.

The energy of the stationary states of an electron is defined by the formula [12]

$$E(n, p_3) = \varepsilon n^2 + \frac{p_3^2}{2m}, \tag{1}$$

where $\varepsilon = 1/(2mR^2)$ stands for the energy of the dimensional confinement, R for the radius of the cylinder, and m for the electron mass.

Denote by the symbol (p, s) the family of quantum numbers defining the stationary state of an electron, where $p = (n, p_3)$ and $s = \pm 1$ stands for the spin quantum number defining the projection of the electron spin in the units $\hbar/2$.

We further assume that the orbital angular momenta of the electrons in a Cooper pair on the surface of the cylinder are directed oppositely ($n_1 = -n_2 = n$), as well as the momenta of the longitudinal motion and the projections of the electron spins to the z axis ($p_{1z} = -p_{2z} = p_z$ and $s_{1z} = -s_{2z} = s_z$). In the representation of second quantization, the simplest model Hamiltonian in the superconductivity theory of a quantum cylinder (in which the interaction of electrons with values of quantum numbers of opposite signs is preserved) can be represented in the following form ([13, 14, 15]):

$$H = \sum_{p,s} E(p) a^+(p, s) a(p, s) - \frac{1}{S} \sum_{p,p' \neq p} J(p, p') a^+(-p, \downarrow) a^+(p, \uparrow) a(p', \uparrow) a(-p', \downarrow). \tag{2}$$

Here $S = 2\pi RL$ stands for the area of the surface of the cylinder and $J(p, p')$ for the quantity describing the effective interaction of the electrons in a Cooper pair, and also the following notation is used:

$$\sum_{p,s} f(n, p_3) \equiv 2 \sum_n \frac{L}{2\pi} \int dp_3 f(n, p_3), \tag{3}$$

$$E(p) = E(n, p_3) - \mu, \tag{4}$$

where μ stands for the chemical potential of the electron gas.

As in BCS theory, we assume that the quantity $J(p, p')$ is equal to some constant g concentrated near the Fermi surface [13],

$$J(p, p') = \begin{cases} g & \text{if } |E(n, p_3) - E_F| < \omega, \\ 0 & \text{if } |E(n, p_3) - E_F| > \omega. \end{cases} \tag{5}$$

With regard to (5), the summation over n and the integration over p_3 in formula (3) should be carried out over the quantum numbers belonging to the energy layer of width ω,

$$E_F - \omega < E(n, p_3) < E_F + \omega. \tag{6}$$

Using further the $(u - v)$ Bogolyubov transformation for Fermi operators and the Bogolyubov statistical variational principle, we can represent the thermodynamical potential of the superconducting electron gas of the quantum cylinder in the form

$$\Omega = \left(\frac{L}{2\pi}\right) \sum_n \int dp_3 \left[E - \sqrt{E^2 + \Delta^2} - 2T \log\left(1 + e^{-\frac{\sqrt{E^2+\Delta^2}}{T}}\right)\right] + \frac{S}{g}\Delta^2, \tag{7}$$

where

$$E \equiv E(n, p_3, \mu) = \frac{p_3^2}{2m} + \varepsilon n^2 - \mu,$$

T stands for the temperature, and Δ is the width of the energy gap, which can be found from the equation

$$1 = \frac{g}{2S} \sum_n \int dp_3 \Big(\frac{L}{2\pi}\Big) \frac{1}{\sqrt{E^2 + \Delta^2}} \operatorname{th} \frac{\sqrt{E^2 + \Delta^2}}{2T}, \tag{8}$$

whereas the summation is carried out over states whose energies belong to the ω-neighborhood of the Fermi energy.

As in the three-dimensional case, we begin the investigation of the thermodynamical properties of a nanostructure from the study of the temperature dependence of the value of the gap, i.e., of Δ. Under the zero temperature, it follows from formula (8) that

$$\Delta_0 = 2\omega \exp\Big[-\frac{4\pi R}{g}\Big(\frac{\partial N_L}{\partial \mu}\Big)^{-1}\Big]. \tag{9}$$

Here N_L stands for the linear concentration of the degenerate electron gas; H_L is related to the Fermi momentum

$$p_F = \sqrt{2mE_F}$$

by the formula

$$N_L = Rp_F^2 + \frac{2}{\pi} p_F \sum_{k=1}^{\infty} \frac{J_1(2\pi k p_F R)}{k}, \tag{10}$$

where $J_1(x)$ stands for the first-order Bessel function.

Transforming equation (8) further to the form

$$-1 + \frac{g}{2S} \sum_n \Big(\frac{L}{2\pi}\Big) \int dp_3 \frac{1}{\sqrt{E^2 + \Delta^2}} = \frac{g}{S} \sum_n \Big(\frac{L}{2\pi}\Big) \int dp_3 \frac{1}{\sqrt{E^2 + \Delta^2}} \frac{1}{e^{\frac{\sqrt{E^2+\Delta^2}}{T}} + 1}, \tag{11}$$

we can then use the approach presented in [14] in the three-dimensional superconducting case.

We finally obtain the equation

$$\log \frac{\Delta_0}{\Delta(T)} = 2F\Big(\frac{\Delta}{T}\Big), \tag{12}$$

where

$$F(t) = \int_0^{\infty} \frac{dx}{\sqrt{t^2 + x^2}[\exp\sqrt{t^2 + x^2} + 1]}. \tag{13}$$

Thus, the thermodynamical properties of a superconducting gas of quasiparticles are evaluated by using formula (7) for the thermodynamical potential with regard to formulas (9)–(10) and (12)–(13). Here the dependence of Ω (the potential) on the geometric sizes of the nanotube and on the concentration of electrons is determined by the width of the gap under zero temperature, whereas the corresponding dependence on the temperature is reflected by formula (13) whose form is just like that in three-dimensional superconductivity theory ([13, 14]).

The heat capacity of the system can be found by differentiating the thermodynamical potential with respect to the temperature,

$$C = -T \frac{\partial^2 \Omega}{\partial T^2}. \tag{14}$$

With regard to the fact that the potential Ω depends on the temperature both explicitly and implicitly, via the width of the gap, we see that

$$C = \Big(\frac{L}{2\pi}\Big) \sum_n \int dp_3 \frac{1}{\operatorname{ch}^2 \frac{\sqrt{E^2+\Delta^2}}{2T}} \Big\{\frac{E^2 + \Delta^2}{2T^2} - \frac{1}{4T} \frac{\partial}{\partial T}(\Delta^2)\Big\}, \tag{15}$$

where the quantity E is given by formula (4) and $\Delta = \Delta(T)$ stands for the width of the gap.

In a neighborhood of the Fermi energy, we have

$$E \simeq v_F(n)|p_3 - p_F(n)|, \tag{16}$$

where the Fermi momentum of the longitudinal motion, for the nth subzone of the transversal energy, is equal to

$$p_F(n) = mv_F(n) = \sqrt{2m[E_F - \varepsilon n^2]}, \tag{17}$$

and the summation in (15) is carried out over the values of n for which the radicand is positive.

Passing from the variable p_3 to a new variable of integration, $\xi = v_F(n)(p_3 - p_F(n))$, we see, by (15), that

$$\frac{C}{L} = \frac{1}{2\pi}\Big(\sum_n \frac{1}{v_F(n)}\Big)\int_{-\infty}^{+\infty} d\xi \frac{1}{\operatorname{ch}^2 \frac{\sqrt{\xi^2+\Delta^2}}{2T}}\Big[\frac{E^2+\Delta^2}{2T^2} - \frac{1}{4T}\frac{\partial}{\partial T}(\Delta^2)\Big], \tag{18}$$

where, since the integral converges rapidly, we have extended the limits of integration with respect to the variable ξ to the entire axis from $-\infty$ to $+\infty$.

Using the asymptotic expressions for the quantity Δ in the domain of low temperatures ($T \ll \Delta_0$) and near the critical point T_c (as $T \to T_c - 0$), we can find the heat capacity of the superconducting electron gas of the quantum cylinder in these limit cases. If $T \ll \Delta_0$, then the main contribution to (18) is given by the first summand in the integrand, and in the essential domain we have $\xi^2 \ll \Delta_0^2$ and $\Delta \simeq \Delta_0$, where Δ_0 stands for the width of the gap under the zero temperature.

In this limit case, after integrating with respect to the variable ξ, we obtain

$$\frac{C}{L} = \sqrt{\frac{2}{\pi}}T\Big(\frac{\Delta_0}{T}\Big)^{\frac{5}{2}} e^{-\frac{\Delta_0}{T}}\Big(\sum_n \frac{1}{v_F(n)}\Big). \tag{19}$$

To evaluate the left limit value of the heat capacity at the critical temperature, the following formula should be used:

$$\Delta^2 = \frac{8\pi^2}{7\zeta(3)}T^2 \log\frac{\gamma\Delta_0}{\pi T}, \tag{20}$$

which follows from (12) with regard to the expansion (13) in a neighborhood of the transition point. Here $\gamma = e^C$, where C stands for the Euler constant and $\zeta(x)$ for the Riemann zeta function.

Finally, we obtain the following formula for the heat capacity:

$$\frac{C}{L} = \frac{1}{3}\pi T_c\Big[1 + \frac{12}{7\zeta(3)}\Big]\Big(\sum_n \frac{1}{v_F(n)}\Big), \qquad T \to T_c - 0, \tag{21}$$

and, conclude from (20) that the critical temperature is equal to

$$T_c = \Big(\frac{\gamma}{\pi}\Big)\exp\Big[-\frac{4\pi R}{g}\Big(\frac{\partial N_L}{\partial \mu}\Big)^{-1}\Big]2\omega, \tag{22}$$

provided that the linear concentration of electrons is given by formula (10).

Further, with regard to the fact that the heat capacity of the ideal Fermi gas at $T = T_c$ is

$$\frac{C_H}{L} = \frac{1}{3}\pi T_c\Big(\sum_n \frac{1}{v_F(n)}\Big), \tag{23}$$

we see that the difference between the superconducting and normal phase as $T \to T_c$ tends to the finite value

$$\frac{C - C_H}{L} = \frac{4\pi T_c}{7\zeta(3)}\sum_n \frac{1}{v_F(n)}. \tag{24}$$

Thus, when the temperature (22) is reached, the system is subjected to phase transition of the second kind from the superconducting state to the normal state of the electron gas, with the finite jump of heat conductivity given by (24).

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