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Determining the Term Structure of Interest Rates

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Abstract—The term structure of interest rates described by the zero-coupon yield curve is considered in developed countries as the main indicator of the financial market condition, one of the most important macroeconomic parameters, and a reference standard for security pricing in other sectors of the fixed-income market financial instruments. In this article, a new numerical method for constructing the term structure of interest rates from the quotes of a group of bonds of the same credit quality and liquidity is proposed.

Key words: yield curve, exponential-sinusoidal splines, discontinuous change of time, the Pontryagin maximum principle.

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1. INTRODUCTION

The term structure of interest rates (the zero-coupon yield curve), forming the basis of the theory of pricing fixed-income instruments and being one of the most arguable points of pursuing the effective debt and monetary policy, is a subject of intensive studies abroad for more than 30 years. In Russia, a modified yield-to-maturity curve is used, in which the term to maturity is replaced with the duration, corresponding to a flat term structure of interest rates. Hitherto, no generally accepted model for constructing the zero-coupon yield curve exists in the world [1]. Nevertheless, for a number of applications, it is the zero-coupon yield curve that is required. For example, it is necessary for estimating the capital investment projects; for estimating the current value of cash flow; and for pricing of some financial instruments such as bonds, bills, interest rate swaps, etc.

A bond is understood as a promissory note with the payment schedule known in advance (i.e., we consider fixed-coupon bonds), which circulates at the secondary market. In this article, we analyze the curves constructed from a group of bonds with equal degree of risk (the credit risk associated with a default on commitments by the issuer) and liquidity. As simplifying but generally accepted assumptions, we will consider a perfect market, at which bonds with all possible nominal values and times to maturity with the same degree of risk and liquidity as those of the given ones are used.

Let us consider a zero-coupon bond with bullet payoff of the principal debt with time to maturity T and a nominal value of 1. We denote its price at the current point of time by $d(T)$. This dependence is called the discount function.

The problem of determining the term structure of interest rates is directly associated with determining the discount function. It is assumed that the value of any coupon bond equals that of a portfolio of zero-coupon bonds with corresponding terms to maturity. Then, price P of a bond depends on terms t_i, $i = 0, \ldots, n$, and payments F_i, $i = 0, \ldots, n$, as $P = \sum_{i=0}^{n} d(t_i)F_i$; where $d(t)$ is the discount function. Assume that N instruments with prices P_k, $k = 1, \ldots, N$ and payments $F_{i,k}$, $i = 0, \ldots, n$ and $k = 1, \ldots, N$, are traded at the market at time points t_i, $i = 0, \ldots, n$, where points t_i are common for all instruments. In this case, it is required that the prices of instruments predicted via the obtained discount function in accordance with the pricing formula be approximately equal to the current market prices. In addition, this function must be sufficiently smooth. Thus, we come to the following problem: it is required to find a discount function $d(t)$, $t \geq 0$, satisfying the following conditions:

(1) $d(t)$ does not increase in the entire domain of definition;

(2) $d(t) > 0$ and $d(0) = 1$;

(3) $\sum_{i=0}^{n} d(t_i)F_{i,k} = P_k$ for all $k = 1, \ldots, N$.

The relation between the yield curve and the discount function is determined by the interest compounding; the most convenient is the continuous compounding. In this case, yield curve $r(t)$ and discount function $d(t)$ are coupled by the relationship $d(t) = \exp(-r(t)t)$. Correspondingly, instant forward interest rates are determined by the relationship $r(t) = \int_0^t f(\tau)d\tau$.

In such a formulation, this problem is ill-posed. Indeed, from the conditions listed above, one can determine only the values of the discount function at the points of division, i.e., determine $d(t_i)$. Moreover, this can be done only in the case when the system in condition 3 is determined, which can be guaranteed a priori only in the trivial case when none of the bonds has coupon payoffs (the bond maturity times usually do not coincide). Otherwise, a specially formed selection of bonds is required, which strongly restricts the generality of this method. Thus, in the general case, even the values of $d(t_i)$ cannot be uniquely determined, not to mention the values of $d(t)$ at intermediate points. Besides, it is easy to show that the problem is unstable even at fixed points.

The principles of solving the ill-posed problems are described in [19]. In accordance with these principles and in view of the fact that smoothness of the term structure is a requirement reasonable from the viewpoint of economics [2, 10, 11, 15], we will impose, for regularization, the requirement of maximum smoothness.

2. REVIEW OF EXISTING METHODS

The approaches traditionally used for determining the term structure of interest rates (finding the discount function) can be divided into three groups: naive or engineering methods, parametric estimation, and interpolation methods.

The naive methods include, for example, the use of the yield of a similar bond, the use of the yield-to maturity curve, the method of stepwise determination of interest rates (bootstrapping) [2], and the kernel method [3].

The parametric approach was originally described in [4]. Today, a wide acceptance is gained by the Nelson–Siegel [5] and Svensson [6] methods. Models with arbitrarily specified number of parameters have been proposed in [7] (exponential functions) and [8] (exponentially–polynomial functions). Russian market also employs the G-curve, which, from the viewpoint of parameterization, is a variant of the Nelson–Siegel method [9].

Among the interpolation methods, we may distinguish the spline approach, originally proposed in [10]. Quadratic splines were used in [10], and cubic splines were used in [11, 12]. In [13], B-splines were used for the first time, and exponential splines were first used in [14]. In [15, 16], the importance of solution smoothness was noticed and, therefore, smoothing splines were employed. Later on, in [17], smoothness was put in dependence on the term (a segment of the curve). By combining exponential splines with the requirement of smoothness for regularization of the problem, exponentially–sinusoidal splines were obtained in [18].[1]

Parametric methods have several drawbacks: economically contradictive results (the use of parametric methods, in particular, the widely known Nelson–Siegel and Svensson methods, sometimes leads to the results absurd from the economic point of view), incorrect results in the absence of deals with short-term or long-term instruments, and numerical instability of the result (very often, a discard of one instrument substantially changes the form of the curve, especially at its short end, near $t = 0$).

Another drawback of the spline (and all nonparametric) methods is the complexity of obtained models and the ensuing difficulties of estimating the model parameters. Also, very often, the result is difficult to interpret from the economic point of view.

The approach proposed in this paper belongs to the spline methods and may be considered as a development of the approach described in [16], although it proceeds from other premises. Its distinction from the traditional spline methods is that no assumption about the shape of the function is made but the condition of smoothness of the discount function is imposed. This approach is economically justified, because it enables one to account for the continuity of expectation of the participants of the market, which nearly always takes place in practice. In this approach, the search for the solution in the form of a spline is a mathematically justified result.

[1] This result was de facto obtained by Smirnov and Zakharov in 2003 and was presented as a report at a session of the European Bond Commission (EFFAS-EBC).

3. NEW APPROACH TO THE PROBLEM

Using the idea proposed in [16], we make the change of variables $d(t) = \exp(-z(t))$ and consider the system

$$\begin{cases} \dot{z} = f(x) \\ \dot{x} = u. \end{cases} \tag{1}$$

Here, $u(t)$ is a new unknown function and f is a specified function. In order to satisfy conditions 1 and 2, we will require that $f(x) \geq 0$ and $z(0) = 0$. To satisfy condition 3, we will require that the solution is a minimum of the functional

$$J(u) = \sum_{k=1}^{N} \frac{w_k}{2}\left(\sum_{i=0}^{n} e^{-z(t_i)} F_{i,k} - P_k\right)^2.$$

For regularization, according to [19], we complement this functional with a term responsible for smoothness of the solution, $\frac{\alpha^2}{2}\int_0^T u^2(\tau)d\tau$, where α is the regularization parameter regulating the relationship between smoothness and the desired accuracy of approximation.

Then, the problem can be reformulated as follows:

$$\begin{cases} \dot{z} = f(x) \\ \dot{x} = u \\ J(u) = \frac{\alpha^2}{2}\int_0^T u^2(\tau)d\tau + \frac{1}{2}\sum_{k=1}^{N} w_k\left(\sum_{i=0}^{n} e^{-z(t_i)} F_{i,k} - P_k\right)^2 \longrightarrow \min. \end{cases}$$

Introduce new variables $y_k(t) = \sum_{i=0}^{t_i \leq t} F_{i,k} e^{-z(t_i)}$, $k = 1, \ldots, N$. Then, the problem can be written in the form

$$\begin{cases} \dot{z} = f(x), \quad z(0) = 0 \\ \dot{x} = u, \quad x(0) = x_0 \\ \dot{y}_k = \sum_{i=0}^{n} F_{i,k} e^{-z} \delta(t - t_i), \quad y_k(0) = 0 \\ J(x_0, u) = \frac{\alpha^2}{2}\int_0^T u^2(\tau)d\tau + \frac{1}{2}\sum_{k=1}^{N} w_k (y_k(T) - P_k)^2 \longrightarrow \min, \end{cases}$$

where $\delta(x)$ is the Dirac delta function. In order to apply the Pontryagin maximum principle, we will use a discontinuous change of time. Let us introduce function $\nu(t)$ stopping the time and write the functional as one more state variable:

$$\begin{cases} \dot{z} = f(x)\nu(t), \quad z(0) = 0 \\ \dot{x} = u\nu(t), \quad x(0) = x_0 \\ \dot{y}_k = \sum_{i=0}^{n} F_{i,k} e^{-z}(1 - \nu_i(t)), \quad y_k(0) = -P_k \\ \dot{x}_0 = \frac{\alpha^2}{2} u^2 \nu(t) + \sum_{k=1}^{N} w_k (y_k - P_k) \sum_{i=0}^{n} F_{i,k} e^{-z}(1 - \nu_i(t)) \\ J(x_0, u) = x_0(T) \longrightarrow \min. \end{cases}$$

Here,

$$\nu(t) = \min_{i=0,\dots,n} \nu_i(t), \quad \nu_i(t) = \begin{cases} 0, & t \in [t_i + i, t_i + i + 1], \quad i = 0, \dots, n \\ 1, & \text{otherwise.} \end{cases}$$

The stopped time has the dynamics $\dot{\tau} = \nu(t)$. Below, we will show that, if a solution exists, it is unique and is represented by a spline of a special form. The question whether the solution to the problem exists in this statement and is unique remains open. However, under economically reasonable restrictions (in particular, the boundedness of instant forward interest rate $f(x(t))$), there is a unique solution to the problem.

The Hamiltonian of the system is

$$H(z, x, y, \psi, u, t) = -\frac{\alpha^2}{2}u^2\nu(t) - \sum_{k=1}^{N} w_k(y_k - P_k)\sum_{i=0}^{n} F_{i,k}e^{-z}(1 - \nu_i(t))$$
$$+ \psi_z f(x)\nu(t) + \psi_x u\nu(t) + \sum_{k=1}^{N}\psi_k\sum_{i=0}^{n} F_{i,k}e^{-z}(1 - \nu_i(t)).$$

Here, $\psi = (\psi_0, \psi_z, \psi_x, \psi_k)$ (we assumed that $\psi_0 = -1$).

According to the maximum principle for the optimal pair $((z, x, y), u^*)$, there are functions ψ_z, ψ_x, and ψ_k such that

$$H(z, x, y, \psi, u^*, t) = \max_u H(z, x, y, \psi, u, t),$$

$$\dot{\psi}_z = -\sum_{k=1}^{N} w_k(y_k - P_k)\sum_{i=0}^{n} F_{i,k}e^{-z}(1 - \nu_i(t)) - \sum_{k=1}^{N}\psi_k\sum_{i=0}^{n} F_{i,k}e^{-z}(1 - \nu_i(t)),$$

$$\dot{\psi}_x = -\psi_z f'(x)\nu(t),$$

$$\dot{\psi}_k = w_k\dot{y}_k.$$

The transversality conditions give

$$\psi_z(T) = \psi_x(0) = \psi_x(T) = \psi_k(T) = 0.$$

Taking this into account, we find u at $\nu(t) = 1$: $u = \frac{\psi_x}{\alpha^2}$. Obviously, the value of u at $\nu(t) = 0$ affects nothing.

In addition, let us try to find ψ. We introduce a new variable $q_k = y_k(T)$:

$$\psi_k = w_k(y_k - q_k),$$

$$\dot{\psi}_z = -\sum_{k=1}^{N} w_k(q_k - P_k)\sum_{i=0}^{n} F_{i,k}e^{-z}(1 - \nu_i(t)).$$

Taking into account the structure of $\nu_i(t)$ and the fact that $z, x = \text{const}$ at $\nu(t) = 0$, we obtain

$$\psi_z = \sum_{k=1}^{N} w_k(q_k - P_k)\sum_{i}^{t_i + i > t} F_{i,k}e^{-z(t_i)}.$$

Evidently, ψ_z is a piecewise constant function. Let $\psi_z = c_i$ in $[t_{i-1} + i, t_i + i)$, where

$$c_i = \sum_{k=1}^{N} w_k(q_k - P_k)\sum_{s=i}^{n} F_{s,k}e^{-z(t_s + s)}.$$

Solving the system, we find

$$\ddot{x} = -\frac{c_i}{\alpha^2} f'(x)\nu(t), \quad t \in [t_{i-1} + i, t_i + i).$$

The result depends on the form of function $f(x)$, for which we assumed only nonnegativeness. For our further analysis, it is reasonable to set $f(x) = x^2$, as in [18].[2] Then, we have

$$\ddot{x} = -\frac{2c_i}{\alpha^2} x\nu(t),$$

$$(\ddot{x} - \lambda x)\nu(t) = 0, \quad \lambda = -\frac{2c_i}{\alpha^2}, \quad \dot{x}(0) = \dot{x}(T) = 0,$$

$$\dot{x} = 0 \quad \text{for} \quad \nu(t) = 0.$$

Solving this equation, we obtain

$$x = p_{i-1}\phi_{\lambda_i}(t_i - \tau) + p_i \phi_{\lambda_i}(\tau - t_{i-1}), \quad \tau \in [t_{i-1}, t_i), \tag{2}$$

where $\dot{\tau} = \nu(t)$, $\tau(0) = 0$, and

$$\phi_{\lambda_i}(\tau) = \begin{cases} \dfrac{\sinh\sqrt{\lambda_i}\tau}{\sinh\sqrt{\lambda_i}(t_i - t_{i-1})}, & \lambda_i > 0 \\ \dfrac{\sin\sqrt{-\lambda_i}\tau}{\sin\sqrt{-\lambda_i}(t_i - t_{i-1})}, & \lambda_i < 0 \\ \dfrac{\tau}{t_i - t_{i-1}}, & \lambda_i = 0. \end{cases} \tag{3}$$

It should be noted that, if there is an optimal solution, there is also an optimal solution satisfying the condition $x \geq 0$. Indeed, let us consider point t^1 at which $x = 0$ and $u < 0$. If there is no such a point, $x \geq 0$ everywhere, because $x(0) \geq 0$. Starting from this point ($t > t^1$), we set $u^1 = -u$ and $x^1 = -x$. The values of z and $\int u^2 d\tau$ will remain the same because they depend only on the squares of x and u. Then, we take the next point, t^2, at which $x = 0$ and $u < 0$. As a result, we obtain x^* and u^* such that $x^* \geq 0$.

This property implies that $\lambda_i \geq -\dfrac{\pi^2}{(t_i - t_{i-1})^2}$ (see expression (3)).

Now, we write the conditions of continuous differentiability for $x(t)$ (this property follows from the fact that $\ddot{x}$ is piecewise continuous):

$$-p_{i-1}\phi'_{\lambda_i}(0) + p_i\phi'_{\lambda_i}(t_i - t_{i-1}) = -p_i\phi'_{\lambda_{i+1}}(t_{i+1} - t_i) + p_{i+1}\phi'_{\lambda_{i+1}}(0).$$

This condition as well as the conditions $\dot{x}(0) = \dot{x}(T) = 0$ can be written as $B_\lambda p = B_c p = 0$, because λ_i and c_i are coupled by a one-to-one correspondence. However, matrix B_c is square, and the equation $B_c p = 0$ has a nontrivial solution. Hence, B_c has an incomplete rank, one variable (p_0) can be set free and the remaining variables can be expressed in terms of the free variable. The equation takes the form $A_c p = f_c p_0$, and the variables p are expressed as $p = p_0 A_c^{-1} f_c$, where the matrix A_c is triangular with nonzero elements in the diagonal.

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[2] However, it should be noted that it is not the only possible variant. For example, one can choose $f(x) = e^x$, which also gives an analytical solution.

Thus, we obtain $2n + 1$ equations with $2n + 1$ unknowns:

$$\begin{cases} c_i = \sum_{k=1}^{N} w_k(q_k - P_k)\sum_{s=i}^{n} F_{s,k}e^{-z(t_s)} \\ A_c p = f_c p_0 \\ \det A_c = 0. \end{cases} \quad (4)$$

The value of $z(t)$ is found from the known values of p and c_i, because $\dot{z}$ is expressed by formulas (1) and (2).

However, practice shows that solution of this system proceeds very slowly. Better characteristics were exhibited by the following method: the requirement of satisfying Eq. (4) must be introduced into a functional minimized in the method described in [18]. It should be noted that it is impossible to obtain Eq. (4) in the scope of the approach used for obtaining this method. At the same time, it was confirmed experimentally that this additional term substantially increases the rate of convergence of the iterative process.

4. CHOICE OF THE INITIAL APPROXIMATION

Let us also propose a method for finding the initial approximation for the iterative minimization algorithms. Introduce a new set of parameters, $d_i = d(t_i)$, $d = (d_0, \ldots, d_n)^T$, and impose the requirement of smoothness on function $d(t)$. Thus, we obtain the problem

$$\begin{cases} Fd = P \\ \int_{t_0}^{t_n} (d''(t))^2 dt \longrightarrow \min \\ d_0 = 1, \quad d_i \geq d_{i+1} > 0, \end{cases}$$

where F is the payment flow matrix, $F_{i,k}$ is the amount of payment for the kth instrument at time point t_i, and $P = (P_1, \ldots, P_k)^T$ is the instrument price vector. As above, we replace the condition of equality $Fd = P$ by the minimum discrepancy condition $\|Fd - P\|^2 \longrightarrow \min$. In addition, we replace the integral by an integral sum over a grid with nodes t_i and replace the derivative by its difference analog on the same grid.

It is clear that an adequate approximation can be attained only with an appropriate arrangement of nodes. A temptation arises to divide too long intervals between nodes by adding virtual nodes or introduce a new, uniform grid. However, in this case, it is necessary to recalculate matrix F. The advantage of a uniform grid is that the second derivatives are approximated with a higher accuracy than on a nonuniform grid. We pay for this advantage by retreat from reality, because the node choice problem arises. In the preceding analysis, this question was solved by itself; however, in this case, one has to find additional arguments in favor of one or another partition. An increase in the number of nodes also negatively affects the speed, because the dimensionality of the problem coincides with the number of nodes in the grid. Moreover, our problem requires just a reasonable initial approximation rather than the exact solution. For these purposes, the natural grid is quite sufficient.

Thus, we obtain the problem

$$\begin{cases} \|Fd - P\|^2 \longrightarrow \min \\ \|D^2 d\|^2 \longrightarrow \min \\ d_0 = 1, \quad d_i \geq d_{i+1} > 0, \end{cases}$$

which can be rewritten as the quadratic programming problem

$$\begin{cases} \|Hd - f\|^2 \longrightarrow \min \\ Ad \leq b. \end{cases}$$

There are highly efficient algorithms for solving this problem.

In order to use the solution to this problem as the initial approximation for minimization of J, it is necessary to reduce it to the form appropriate for the minimization methods described above.

For changing from variables d_i to variables p_i and λ_j, the following algorithm can be used.

1. Determine auxiliary values f_k from the expression

$$d_i = \exp\left\{-\sum_{k=1}^{i} f_k\right\}; \quad f_k = \ln\frac{d_{k-1}}{d_k}.$$

2. Fix arbitrary p_i, $i = 0, \ldots, n$.

3. Determine λ_i from the equalities

$$f_k = \int_{t_{i-1}}^{t_i} f(t)^2 dt = \int_{t_{i-1}}^{t_i} (p_{i-1}\phi_{\lambda_i}(t_i - t) + p_i\phi_{\lambda_i}(t - t_{i-1}))^2 dt.$$

4. Consider the functional

$$I(p_0, \ldots, p_n) = \sum_{i=1}^{n-1} (f'(t_i+0) - f'(t_i-0))^2 + f'(t_0+0)^2 + f'(t_n-0)^2,$$

where $f(t)$ is defined by expressions (2) and (3) with obtained $\lambda_1, \ldots, \lambda_n$ and solve the problem

$$I(p_0, \ldots, p_n) \longrightarrow \min_{p_0, \ldots, p_n}.$$

The obtained values of $p_0, \ldots, p_n$ and corresponding $\lambda_1, \ldots, \lambda_n$ will determine the function found by the approximate method.

The obtained values can be used as the initial values in the iterative minimization procedure.

5. MAIN RESULTS

In this study, we have performed one more analysis of the problem of determining the term structure of interest rates. The approach used here is new from the viewpoint of justification of the made assumptions and imposed a priori conditions. A new formalization of the problem has been proposed. This approach has enabled a deeper insight into the internal structure of the model and has provided a more reliable result as compared to the results of previous studies [16, 18], which are developed in this paper. Nevertheless, the proposed formalization fully corresponds to the approach of [18] and retains all its advantages: the economic relevancy of assumptions and economic meaning of the solution; in particular, the lack of growth of the discount function.

The obtained result is a closed-form solution to the problem. This means that the number of obtained equations equals the number of unknowns. Previous results [16, 18] proposed solution of minimization subproblems, because the number of equations was not sufficient. On the basis of the obtained solution, a numerical algorithm of finding the discount function from real data has been constructed and implemented.

The method proposed here can also be used in related applied problems requiring construction of discount functions, in particular, for determination of credit spreads.

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