BASS' TRIANGULABILITY PROBLEM

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Abstract. Exploring Bass' Triangulability Problem on unipotent algebraic subgroups of the affine Cremona groups, we prove a triangulability criterion, the existence of nontriangulable connected solvable affine algebraic subgroups of the Cremona groups, and stable triangulability of such subgroups; in particular, in the stable range we answer Bass' Triangulability Problem is the affirmative. To this end we prove a theorem on invariant subfields of 1-extensions. We also obtain a general construction of all rationally triangulable subgroups of the Cremona groups and, as an application, classify rationally triangulable connected one-dimensional unipotent affine algebraic subgroups of the Cremona groups up to conjugacy.

1. Introduction

We assume given an algebraically closed field k of arbitrary characteristic which serves as domain of definition for each of the varieties considered below. In this paper, "variety" means "algebraic variety" and it is identified with its set k-rational points.

Recall that the *Cremona group* (*over* k) *of rank* n is the group

$$\mathcal{C}_n := \mathrm{Aut}_k\, k(\mathbf{A}^n),$$

and $\mathrm{Aut}_k\, k[\mathbf{A}^n]$ is the *affine Cremona group* (*over* k) *of rank* n. The group $\mathrm{Bir}\,\mathbf{A}^n$ of rational self-maps of $\mathbf{A}^n$ (resp. the group $\mathrm{Aut}\,\mathbf{A}^n$) is identified with $\mathcal{C}_n$ (resp. $\mathrm{Aut}_k\, k[\mathbf{A}^n]$) by means of the isomorphism $\varphi \mapsto (\varphi^*)^{-1}$. For $n > 1$, we identify $k(\mathbf{A}^{n-1})$ with the subfield of $k(\mathbf{A}^n)$ by means of the natural embedding $k(\mathbf{A}^{n-1}) \hookrightarrow k(\mathbf{A}^n)$ determined by the projection $\mathbf{A}^n \to \mathbf{A}^{n-1}$, $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{n-1})$. This makes $k[\mathbf{A}^{n-1}]$ the subalgebra of $k[\mathbf{A}^n]$. We put $k(\mathbf{A}^0) = k[\mathbf{A}^0] := k$.

Let $x_i\colon \mathbf{A}^n \to k$, $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_i$, be the ith standard coordinate function. We have

$$k(\mathbf{A}^n) = k(x_1, \ldots, x_n), \quad k[\mathbf{A}^n] = k[x_1, \ldots, x_n]. \tag{1}$$

The group $\mathcal{C}_{n-1}$ is identified with the subgroup of $\mathcal{C}_n$ by means of the embedding

$$\mathcal{C}_{n-1} \hookrightarrow \mathcal{C}_n,\ \varphi \mapsto \widetilde{\varphi},\ \text{where } \widetilde{\varphi}(x_i) := \varphi(x_i) \text{ for } i < n \text{ and } \widetilde{\varphi}(x_n) := x_n. \tag{2}$$

This makes $\mathrm{Aut}_k\, k[\mathbf{A}^{n-1}]$ the subgroup of $\mathrm{Aut}_k\, k[\mathbf{A}^n]$.

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Although the groups $\mathcal{C}_n$ and $\operatorname{Aut}_k k[\mathbf{A}^n]$ are infinite-dimensional for $n > 1$ (see [Ra 1964], [Po 2014]), the analogies between them and algebraic groups catch the eye: they have the Zariski topology, algebraic subgroups, tori, roots, the Weyl groups, ... (see [Po 2013_1], [Po 2013_2] and references therein). The de Jonquières subgroup

$$\mathcal{J}_n := \{\varphi \in \operatorname{Aut}_k k[\mathbf{A}^n] \mid \varphi(x_i) = \alpha_i x_i + h_i,\ \alpha_i \in k^{\times}, h_i \in k[\mathbf{A}^{i-1}]\}$$

is viewed by some authors as an analog of Borel subgroup for $\operatorname{Aut}_k k[\mathbf{A}^n]$ (see, e.g., [Ba 1984]). Supporting this viewpoint, [Po 2013_1, Thm. 3.1] implies that every algebraic subgroup of $\mathcal{J}_n$ is affine solvable. Having in mind conjugacy of Borel subgroups in every finite dimensional affine algebraic group, one leads to the question whether it is true that every connected solvable affine algebraic subgroup G of $\operatorname{Aut}_k k[\mathbf{A}^n]$ is conjugate in $\operatorname{Aut}_k k[\mathbf{A}^n]$ to a subgroup of $\mathcal{J}_n$. In particular, Problem III in [Ba 1984] asks whether it is true for unipotent G. In [Ba 1984] Bass answered the latter question in the negative for $n = 3$ and $G = k^+$, the one-dimensional additive group. In [Po 1987] was then elaborated a simple general method yielding negative answers for $G = k^+$ and all $n > 2$ (this method, in the form of usage of so called replicas, became the crucial instrument in the recent studies on infinite transitivity of automorphism groups of algebraic varieties [Ka 2012]). Given these developments, Bass formulated for $\operatorname{char} k = 0$ the following

Bass' Triangulability Problem ([Ba 1984, Question 4])**.** "If a unipotent group G acts on $\mathbf{A}^n$, can the action be rationally triangularized, i.e., can we write $k(x_1, \dots, x_n) = k(y_1, \dots, y_n)$ so that each subfield $k(y_1, \dots, y_i)$ is G-invariant?"

Here we explore this problem in the broader context of connected solvable affine algebraic groups G and arbitrary $\operatorname{char} k$. To formulate our results we first introduce two definitions.

Definition 1. If a field K is a simple transcendental extension of a field L (i.e., K is a purely transcendental field extension of L of degree 1), then, for brevity, we say that K is a 1-*extension* of L.

Definition 2. A subgroup G of $\mathcal{C}_n$ is called *rationally triangulable* if there is a flag

$$k(\mathbf{A}^n) =: K_n \supset K_{n-1} \supset \cdots \supset K_1 \supset K_0 := k \tag{3}$$

of G-stable subfields of $k(\mathbf{A}^n)$ such that K_i/K_{i-1} is a 1-extension for every $i > 0$. A rational action of an algebraic group on $\mathbf{A}^n$ is called *rationally triangulable* if the image of this group under the homomorphism to $\mathcal{C}_n$ determined by this action is rationally triangulable.

Now we shall formulate our results.

We start with proving the following theorem that is heavily used in the proofs of our next results.

Theorem 1 (Invariant subfields of 1-extensions). *Let Q be a finitely generated field extension of k and let P be a 1-extension of Q. Let G be a connected solvable affine algebraic subgroup of* $\mathrm{Aut}_k(P)$ *such that Q is G-stable. Then*

(i) $P^G = Q^G$ *if* $Q^G = Q$;
(ii) P^G *is a* 1*-extension of* Q^G *if* $Q^G \subsetneq Q$.

Using Theorem 1, we obtain the following triangulability criterion:

Theorem 2 (Triangulability criterion). *The following properties of a connected solvable affine algebraic subgroup G of the Cremona group $\mathcal{C}_n$ are equivalent:*

(i) $k(\mathbf{A}^n)^G$ *is purely transcendental over k;*
(ii) *G is rationally triangulable.*

Theorem 2 generalizes [DF 1991, Thm. 3.1], where the claim is proved for one-dimensional unipotent algebraic subgroups of $\mathrm{Aut}_k \mathbf{A}^n$ in case $\mathrm{char}\, k = 0$.

Corollary 1. *A connected solvable affine algebraic subgroup G of $\mathcal{C}_n$ is rationally triangulable in either of the following cases:*

(i) $\mathrm{tr\,deg}_k k(\mathbf{A}^n)^G \leqslant 1$;
(ii) $\mathrm{tr\,deg}_k k(\mathbf{A}^n)^G = 2$ *and* $\mathrm{char}\, k = 0$;
(iii) $G \subset \mathrm{Aut}\, \mathbf{A}^n$ *and*
  (a) $\dim G \cdot x \geqslant n-1$ *for some point* $x \in \mathbf{A}^n$, *or*
  (b) $\mathrm{char}\, k = 0$ *and* $\dim G \cdot x = n-2$ *for some point* $x \in \mathbf{A}^n$.

Corollary 2. *If* $\mathrm{char}\, k = 0$, *then every connected solvable affine algebraic subgroup of* $\mathrm{Aut}\, \mathbf{A}^3$ *is rationally triangulable.*

Corollary 2 generalizes [DF 1991, Cor. 3.2], where the claim is proved for one-dimensional unipotent algebraic subgroups of $\mathrm{Aut}\, \mathbf{A}^3$ and $\mathrm{char}\, k = 0$.

Corollary 3. *The following properties of an affine algebraic torus T in the Cremona group $\mathcal{C}_n$ are equivalent:*

(i) *T is rationally triangulable,*
(ii) *T is linearizable, i.e., conjugate in $\mathcal{C}_n$ to a subgroup of* GL_n.

Next we show that the nontriangulable connected solvable affine algebraic subgroups of $\mathcal{C}_n$ do exist. In particular, the following theorem implies that in case (ii) of Corollary 1 it is not possible to replace 2 by a bigger integer.

Theorem 3 (Nontriangulable subgroups). *Let n be an integer $\geqslant 5$. Every $(n-3)$-dimensional connected solvable affine algebraic group G is isomorphic to a rationally nontriangulable algebraic subgroup of the Cremona group $\mathcal{C}_n$.*

As far as we do not claim that the subgroup in the formulation of Theorem 3 lies in $\mathrm{Aut}_k\, k[\mathbf{A}^n]$, this theorem does not furnish the negative answer to Bass' Triangulability Problem. However, its proof demonstrates intimate interrelation between triangulability and Zariski's Cancellation Problem: it shows that if there is a *nonrational* variety Z such that $\mathbf{A}^n$ is isomorphic to $\mathbf{A}^s \times Z$, then the answer to Bass' Triangulability Problem is negative (in view of this it is worth to recall that in positive characteristic Zariski's Cancellation Problem is solved in the negative in [Gu 2014]); by Theorems 5, 6, the converse it true on birational level.

On the other hand, in the stable range we do answer Bass' Triangulability Problem is the affirmative. Namely, the following theorem shows that despite the existence of rationally nontriangulable connected solvable affine algebraic subgroups of $\mathcal{C}_n$, every such subgroup is stably rationally triangulable. More precisely, the following holds true.

Theorem 4 (Stable triangulability)**.** *Every connected solvable affine algebraic subgroup G of the Cremona group $\mathcal{C}_n$ is rationally triangulable in the Cremona group $\mathcal{C}_m$ for every*

$$m \geqslant 2n - \operatorname{tr\,deg}_k k(\mathbf{A}^n)^G. \tag{4}$$

Theorem 4 generalizes [DF 1991, Thm. 3.1], where the statement is proved for one-dimensional unipotent algebraic subgroups of $\operatorname{Aut}_k \mathbf{A}^n$, assuming $\operatorname{char} k = 0$.

Next we obtain a general construction of all rationally triangulable subgroups of $\mathcal{C}_n$, see Theorem 6 in Section 3. As an application, it leads to the classification (given below in Corollary 4) of rationally triangulable one-dimensional connected unipotent algebraic subgroups in $\mathcal{C}_n$ up to conjugacy. In this classification we use the following terminology. A one-dimensional connected unipotent algebraic subgroup G of $\mathcal{C}_n$, identified with k^+ by means of an isomorphism $G \to k^+$, is called *standard* if $x_1, \ldots x_{n-1} \in k[\mathbf{A}^n]^G$ and, for every $t \in k^+$, the following holds:

(i) for $\operatorname{char} k = 0$, we have
$$t(x_n) = x_n + t$$
(in this case G is also called the *translation*);

(ii) for $\operatorname{char} k = p > 0$, we have
$$t(f_n) = f_n + c_1 t^{p^{i_1}} + \cdots + c_d t^{p^{i_d}},$$
where all c_j's are the nonzero elements of $k(x_1, \ldots, x_{n-1})$, and $i_1 < \cdots < i_d$ are the nonnegative integers.

Corollary 4 (One-dimensional rationally triangulable unipotent subgroups)**.** *A one-dimensional connected unipotent algebraic subgroup of $\mathcal{C}_n$ is rationally triangulable if and only if it is conjugate in $\mathcal{C}_n$ to a standard subgroup.*

Corollary 4 generalizes [DF 1991, Thm. 2.2], where the claim is proved for $\operatorname{char} k = 0$.

Corollary 5. *Let U be a one-dimensional connected unipotent algebraic subgroup of $\mathcal{C}_n$.*

(i) *If $n = 2$, then U is conjugate in $\mathcal{C}_2$ to a standard subgroup.*

(ii) *If $\operatorname{char} k = 0$ and $n = 3$, then U is conjugate in $\mathcal{C}_3$ to the translation.*

By [Re 1968], if $\operatorname{char} k = 0$, $n = 2$, and $U \subset \operatorname{Aut} \mathbf{A}^2$, then "in $\mathcal{C}_2$" in Corollary 5(i) may be replaced by "in $\operatorname{Aut} \mathbf{A}^2$".

By [Ka 2004], for $k = \mathbf{C}$, $n = 3$, $U \subset \operatorname{Aut} \mathbf{A}^3$, if U acts on $\mathbf{A}^3$ freely, then U is conjugate in $\operatorname{Aut} \mathbf{A}^3$ to translation. Corollary 5(ii) shows that allowing conjugation in $\mathcal{C}_3$, the "if" assumption in this result may be dropped, i.e., conjugacy to the translation becomes true for every U in $\mathcal{C}_3$.

The proofs of Theorems 1, 2, 3, 4 and Corollaries 1, 2, 3 are given in Section 2. Theorem 6 is formulated and proved, together with Corollaries 4, 5, in Section 3.

Notation and conventions. We use freely the standard notation and conventions of [PV 1994], [Bo 91], [Sp 1998] and refer to [Ro 1956], [Ro 1961], [Ro 1963], [PV 1994], [Po 2013$_1$] regarding the definitions and basic properties of rational and regular (morphic) actions of algebraic groups on varieties.

Unless otherwise specified, we will assume that every field appearing below contains k and every embedding of fields restricts to the identity map on k.

2. Proofs of Theorems 1, 2, 3, 4 and Corollaries 1, 2

Proof of Theorem 1.

1. We argue by induction on $\dim G$. Assuming the statement is proved for $\dim G = 1$, the inductive step is proved as follows.

Since G is connected solvable affine algebraic group, it contains a closed connected normal subgroup N such that $\dim G/N = 1$, see [Gr 1956, p. 6–03, Cor. 1]. We have

$$(P^N)^{G/N} = P^G \ \text{ and } \ (Q^N)^{G/N} = Q^G \tag{5}$$

By the inductive assumption one of the following holds:

(a) $P^N = Q^N$;

(b) P^N is a 1-extension of Q^N.

If (a) is fulfilled, then $P^G = Q^G$ in view of (5), i.e., (i) holds.

If (b) is fulfilled, then, since G/N is a one-dimensional connected solvable affine algebraic group, (i) follows from (5) and the assumption that the statement is proved for one-dimensional groups.

2. Thus the proof is reduced to the case where $\dim G = 1$. We now assume that this equality holds.

The assumptions on Q and P imply that there is an irreducible variety X such that for

$$\begin{gathered} Y := \mathbf{A}^1 \times X, \\ \rho\colon Y \to X, \ \ (a, x) \mapsto x, \end{gathered} \tag{6}$$

we may (and shall) identify P with $k(Y)$:

$$P = k(Y), \tag{7}$$

and Q with the image of embedding $\rho^*\colon k(X) \hookrightarrow k(Y)$:

$$Q = \rho^*(k(X)). \tag{8}$$

The actions of G on Q and P determine the rational actions of G on X and Y. The action on Y is faithful and the morphism ρ is equivariant with respect to these actions.

By Weil's regularization theorem [We 1955] (see also [Ro 1956, Thm. 1]) there are

— irreducible algebraic varieties $\widetilde{X}$ and $\widetilde{Y}$;
— nonempty open subsets X_0 and Y_0 in, respectively, X and Y;
— open embeddings $\iota_1\colon X_0 \hookrightarrow \widetilde{X}$, $\iota_2\colon Y_0 \hookrightarrow \widetilde{Y}$

such that the induced rational actions of G on $\widetilde{X}$ and $\widetilde{Y}$ are regular (morphic). The action of G on $\widetilde{Y}$ is faithful since that on Y is. We identify X_0 and Y_0 with the images of, respectively, ι_1 and ι_2, that yields the natural identifications

$$k(X) = k(\widetilde{X}), \quad k(Y) = k(\widetilde{Y}). \tag{9}$$

By Rosenlicht's theorem on generic quotients [Ro 1956, Thm. 2], replacing $\widetilde{X}$, X_0, $\widetilde{Y}$, and Y by the appropriate open subsets, we may (and shall) assume that for the actions of G on $\widetilde{X}$ and $\widetilde{Y}$ respectively there are the geometric quotients

$$\pi_{\widetilde{X},G}\colon \widetilde{X} \to \widetilde{X}/G, \qquad \pi_{\widetilde{Y},G}\colon \widetilde{Y} \to \widetilde{Y}/G.$$

In particular, $\pi_{\widetilde{X},G}$ and $\pi_{\widetilde{Y},G}$ are equidimensional morphisms, their fibers are G-orbits, and, in view of (7), (8), (9),

$$\rho^* \circ \pi^*_{\widetilde{X},G}\colon k(\widetilde{X}/G) \xrightarrow{\simeq} Q^G, \qquad \pi^*_{\widetilde{Y},G}\colon k(\widetilde{Y}/G) \xrightarrow{\simeq} P^G. \tag{10}$$

Since $\dim G = 1$ and G acts on $\widetilde{Y}$ faithfully, every fiber of $\pi_{\widetilde{Y},G}$ is one-dimensional; in view of (6), this yields

$$\dim \widetilde{Y}/G = \dim Y - 1 = \dim X. \tag{11}$$

The morphism ρ induces a G-equivariant dominant rational map

$$\widetilde{\rho}\colon \widetilde{Y} \dashrightarrow \widetilde{X}.$$

Since its domain of definition is G-stable, replacing the varieties again by the appropriate open subsets we may (and shall) assume that $\widetilde{\rho}$ is a G-equivariant surjective morphism.

Thus we obtain the following commutative diagram

$$\begin{tikzcd}
Y \arrow[d, "\rho"'] & Y_0 \arrow[l, hook', "\iota_4"'] \arrow[r, hook, "\iota_2"] \arrow[d, "\rho_0"] & \widetilde{Y} \arrow[r, "\pi_{\widetilde{Y},G}"] \arrow[d, "\widetilde{\rho}"] & \widetilde{Y}/G \\
X & X_0 \arrow[l, hook', "\iota_3"'] \arrow[r, hook, "\iota_1"] & \widetilde{X} \arrow[r, "\pi_{\widetilde{X},G}"] & \widetilde{X}/G
\end{tikzcd}, \tag{12}$$

where $\rho_0 := \rho|_{Y_0} = \widetilde{\rho}|_{Y_0}$ and ι_3, ι_4 are the identical embeddings. Finally, replacing X_0 and Y_0 by the appropriate open subsets, we may (and shall) assume that $\rho_0(Y_0) = X_0$.

Now we shall consider separately two arising possibilities.

3. First, consider the case

$$Q^G = Q. \tag{13}$$

By (8), condition (13) is equivalent to triviality of the action of G on $\widetilde{X}$.

From $Q \subset P$ and (13) we obtain

$$Q \subseteq P^G \subset P, \tag{14}$$

From (7), (9), (11) we infer $\operatorname{tr}\deg_{P^G} P = 1$, and (6), (7), (9) yield $\operatorname{tr}\deg_Q P = 1$. Whence by (14) we obtain $\operatorname{tr}\deg_Q P^G = 0$. Since P is a 1-extension of Q, by Lüroth's theorem (see, e.g., [vdW 1971, §73]) the latter equality implies that $P^G = Q$. Thus, by (13), in the case under consideration we have $P^G = Q^G$. This proves claim (i) of Theorem 1.

4. Now consider the case $Q^G \subsetneq Q$, i.e., G acts on $\widetilde{X}$ nontrivially. Every fiber of $\pi_{\widetilde{X},G}$ is then a one-dimensional G-orbit; whence

$$\dim \widetilde{X}/G = \dim X - 1. \tag{15}$$

In view of (7), (8), (9), (11), (15), we have

$$\operatorname{tr}\deg_{Q^G} P^G = 1. \tag{16}$$

Since G is a connected solvable affine algebraic group, by Rosenlicht's theorem on section [Ro 1956, Thm. 10] there is a rational section

$$\sigma\colon \widetilde{X}/G \dashrightarrow \widetilde{X}$$

of $\pi_{\widetilde{X},G}$, i.e. a rational map such that

$$\pi_{\widetilde{X},G} \circ \sigma = \mathrm{id}_{\widetilde{X}}. \tag{17}$$

Since $g \circ \sigma$ for every element $g \in G$ is also a rational section of $\pi_{\widetilde{X},G}$, we may (and shall) assume that $\sigma(\widetilde{X}/G) \cap X_0 \neq \varnothing$. This implies that there is a locally closed irreducible subvariety $S \subset X_0$ such that

$$\pi_{\widetilde{X},G}|_S\colon S \to \widetilde{X}/G \ \text{ is an open embedding.} \tag{18}$$

In view of (10), this means that

$$k(\widetilde{X})^G \to k(S),\ \ f \mapsto f|_S, \text{ is a well-defined isomorphsim.} \tag{19}$$

In particular, in view of (15), we have

$$\dim S = \dim X - 1. \tag{20}$$

From (6) we obtain that

$$\rho^{-1}(S) = \mathbf{A}^1 \times S. \tag{21}$$

Consider in $\widetilde{Y}$ the locally closed irreducible subvariety

$$Z := \rho^{-1}(S) \cap Y_0. \tag{22}$$

From (21) and (22) we infer that

$$k(Z) \text{ is a 1-extension of } K := \rho|_Z^*(k(S)), \tag{23}$$

and from (11), (20) that

$$\dim Z = \dim X = \dim \widetilde{Y}/G. \tag{24}$$

We claim that the morphism

$$\zeta := \pi_{\widetilde{Y},G}|_Z \colon Z \to \widetilde{Y}/G \tag{25}$$

is dominant. In view of (24), to prove this, it suffices to show that, for every point $z \in Z$, the fiber $\zeta^{-1}(\zeta(z))$ is finite. Assume the contrary. Since $\zeta^{-1}(\zeta(z)) = Z \cap \mathscr{O}$, where $\mathscr{O} := \pi^{-1}_{\widetilde{Y},G}(\pi_{\widetilde{Y},G}(z)$ is a certain G-orbit, we then infer from $\dim \mathscr{O} = 1$ that

$$\dim(Z \cap \mathscr{O}) = 1. \tag{26}$$

Since $\widetilde{\rho}$ is a G-equivariant morphism, $\mathscr{O}' := \widetilde{\rho}(\mathscr{O})$ is also a G-orbit. Hence $\dim \mathscr{O}' = 1$, because all G-orbits in $\widetilde{X}$ are one-dimensional. The latter equality and (26) imply that $\widetilde{\rho}(Z \cap \mathscr{O})$ is an open subset of $\mathscr{O}'$; in particular, it is infinite. On the other hand, (22) yields that $\widetilde{\rho}(Z \cap \mathscr{O})$ lies in $S \cap \mathscr{O}'$. Since, by (18), the latter is a single point, we obtain a contradiction. This proves the claim.

Thus, since ζ is dominant, it defines an embedding of fields $\zeta^* \colon k(\widetilde{Y}/G) \hookrightarrow k(Z)$. In view of (7), (9), this means that

$$\tau \colon P^G \hookrightarrow k(Z),\ \ f \mapsto f|_Z, \text{ is a well-defined embedding,} \tag{27}$$

and (8), (9), (19), (23) imply that

$$\tau \colon Q^G \hookrightarrow K \text{ is an isomorphism.} \tag{28}$$

Thus (27), (28) yield the following commutative diagram

$$\begin{tikzcd} k(Z) & \tau(P^G) \arrow[l, hook', "\mathrm{id}"'] & K \arrow[l, hook', "\mathrm{id}"'] \\ & P^G \arrow[u, "\tau", "\simeq"'] & Q^G \arrow[l, hook', "\mathrm{id}"'] \arrow[u, "\tau", "\simeq"'] \end{tikzcd}.$$

In view of (16), (23), this diagram and Lüroth's theorem imply that P^G is a 1-extension of Q^G. This completes the proof of claim (ii) of Theorem 1. $\square$

Proof of Theorem 2.

(i)⇒(ii) Assume that $k(\mathbf{A}^n)^G$ is a purely transcendental field extension of k. Then there is a flag

$$k(\mathbf{A}^n)^G =: I_t \supset I_{t-1} \supset \cdots \supset I_1 \supset I_0 := k \tag{29}$$

of the subfields of $k(\mathbf{A}^n)^G$ such that I_i/I_{i-1} is a 1-extension for every $i > 0$.

Since G is a connected solvable affine algebraic group, there is a flag of its closed connected normal subgroups

$$G =: G_0 \supset G_1 \supset \cdots \supset G_{s-1} \supset G_s := \{e\} \tag{30}$$

such that $\dim G_{i-1}/G_i = 1$ for every $i > 0$, see [Gr 1956, p. 6–03, Cor. 1]. From (30) we obtain the following flag of G-stable subfields of $k(\mathbf{A}^n)$:

$$k(\mathbf{A}^n) = K_s \supseteq K_{s-1} \supseteq \cdots \supseteq K_1 \supseteq K_0 = k(\mathbf{A}^n)^G, \quad K_i := k(\mathbf{A}^n)^{G_i}. \tag{31}$$

By construction,

$$K_{i-1} = k(\mathbf{A}^n)^{G_{i-1}} = (k(\mathbf{A}^n)^{G_i})^{G_{i-1}/G_i} = K_i^{G_{i-1}/G_i} \quad \text{for every } i > 0. \tag{32}$$

If the action of G_{i-1}/G_i on K_i is trivial, then (32) yields $K_i = K_{i-1}$. If it is nontrivial, then, since G_{i-1}/G_i is a one-dimensional connected solvable affine algebraic group, we infer from (32) and [Po 2015, Thm. 1] (or [Ma 1963, Thm. 1] if $\operatorname{char} k = 0$) that K_i/K_{i-1} is a 1-extension. Therefore, once repetitions in flag (31) are eliminated, we obtain a flag

$$k(\mathbf{A}^n) =: L_d \supset L_{d-1} \supset \cdots \supset L_1 \supset L_0 := k(\mathbf{A}^n)^G \tag{33}$$

of G-stable subfields of $k(\mathbf{A}^n)$ such that L_i/L_{i-1} is a 1-extension for every $i > 0$. From (33) and (29) we then obtain the flag

$$k(\mathbf{A}^n) =: L_d \supset L_{d-1} \supset \cdots \supset L_1 \supset L_0 = I_t \supset I_{t-1} \supset \cdots \supset I_1 \supset I_0 := k$$

of G-stable subfields of $k(\mathbf{A}^n)$ whose "levels" are 1-extensions. By Definition 2 the group G is then rationally triangulable.

(ii)⇒(i) Conversely, assume that the group G is rationally triangulable and consider flag (3) of G-stable subfields of $k(\mathbf{A}^n)$. Passing to the G-invariant subfields, we then obtain the following flag of subfields of $k(\mathbf{A}^n)^G$:

$$k(\mathbf{A}^n)^G = K_n^G \supseteq K_{n-1}^G \supseteq \cdots \supseteq K_1^G \supseteq K_0^G = k. \tag{34}$$

By Theorem 1 every "level" of this flag is either the equality or a 1-extension. Therefore, (34) implies that $k(\mathbf{A}^n)^G$ is a purely transcendental extension of k. This completes the proof. □

Proof of Corollary 1. In cases (i) and (ii), the claim follows, in view of Theorem 2, from, respectively, Lüroth's theorem and Castelnuovo's theorem (see, e.g., [Ha 1977, Chap. V, 6.2.1]). In case (iii), it follows from (i) and (ii), since, by [Ro 1956, Thm. 2], $\operatorname{tr\,deg}_k k(\mathbf{A}^n)^G = n - \max_{a \in \mathbf{A}^n} G \cdot a$. □

Proof of Corollary 2. This follows from Corollary 1. □

Proof of Corollary 3. In view of Theorem 2, this follows from [Po 2013$_1$, Thm. 2.4]. □

Proof of Theorem 3. By [S-B 2004] there exists a nonrational threefold X such that $\mathbf{A}^2 \times X$ is birationally isomorphic to $\mathbf{A}^5$. Hence we may (and shall)

$$\text{fix a birational isomorphism } \mathbf{A}^{n-3} \times X \dashrightarrow \mathbf{A}^n. \tag{35}$$

Since the underlying variety of G is rational (see [Gr 1958, p. 5-02, Cor.]), we also may (and shall) fix a birational isomorphism between it and $\mathbf{A}^{n-3}$.

We then obtain from the action of G on itself by left translations a faithful rational action of G on $\mathbf{A}^{n-3}$ such that

$$k(\mathbf{A}^{n-3})^G = k. \tag{36}$$

In turn, the latter action yields a faithful rational action of G on $\mathbf{A}^{n-3} \times X$ via the first factor. By [Bo 91, Cor. of Prop. II.6.6] and (36), for this action,

$$k(\mathbf{A}^{n-3} \times X)^G \text{ and } k(X) \text{ are isomorphic.} \tag{37}$$

Thus, given (35), we obtain a faithful rational action of G on $\mathbf{A}^n$ such that the field $k(\mathbf{A}^n)^G$ is isomorphic to $k(X)$, and hence is not purely transcendental over k according to the choice of X.

By Theorem 2 we then conclude that the algebraic subgroup of $\mathcal{C}_n$ determined by this action is isomorphic to G and rationally nontriangulable. □

Proof of Theorem 4. First, we shall introduce some notation. Given a rational action of an algebraic group H on an irreducible algebraic variety Z, we denote by $Z/\!\!/H$ a rational quotient of this action, i.e., an irreducible variety (uniquely defined up to birational isomorphism) such that there exists an isomorphism $k(Z/\!\!/H) \to k(Z)^H$. We shall write $X \approx Y$ if X and Y are birationally isomorphic irreducible varieties.

By [Po 2015, Thm. 1] (or [Ma 1963, Thm. 1] if $\operatorname{char} k = 0$) we have

$$\mathbf{A}^n \approx \mathbf{A}^n/\!\!/G \times \mathbf{A}^s, \text{ where } s := n - \operatorname{tr}\deg_k k(\mathbf{A}^n)^G. \tag{38}$$

Take an integer m such that (4) holds and consider G as a subgroup of $\mathcal{C}_m$ via embedding (2). The arising rational action of G on $\mathbf{A}^m = \mathbf{A}^n \times \mathbf{A}^{m-n}$ is that through the first factor. Therefore

$$\left.\begin{aligned} \mathbf{A}^m/\!\!/G &= (\mathbf{A}^n/\!\!/G) \times \mathbf{A}^{m-n} \\ &\overset{(4)}{=} (\mathbf{A}^n/\!\!/G) \times \mathbf{A}^s \times \mathbf{A}^{m-n-s} \\ &\overset{(38)}{\approx} \mathbf{A}^n \times \mathbf{A}^{m-n-s} = \mathbf{A}^{m-s} \end{aligned}\right\} . \tag{39}$$

By (39) the field $k(\mathbf{A}^m)^G$ is purely transcendental over k. Hence by Theorem 2 the group G is rationally triangulable in $\mathcal{C}_m$. □

3. Construction ⊛

Now we shall give a general construction of all rationally triangulable subgroups of $\mathcal{C}_n$. It is prompted by the following result from [Po 2015] that yields a general construction of all connected solvable affine algebraic subgroups of $\mathcal{C}_n$:

Theorem 5 (Standard invariant open subsets [Po 2015, Thm. 3])**.** *Let X be an irreducible variety endowed with a regular action of a connected solvable affine algebraic group S. Then for the restriction of this action on a certain S-stable nonempty open subset Q of X there exist*

— *the geometric quotient $\pi_{S,Q}\colon Q \to Q/S$;*

— *an isomorphism* $\varphi\colon Q \to \mathbf{A}^{r,u} \times (Q/S)$, *where*

$$\mathbf{A}^{r,u} := \{(\alpha_1, \ldots, \alpha_{r+u}) \in \mathbf{A}^{r+u} \mid \alpha_i \neq 0 \textit{ for every } i \leqslant r\}, \quad r \geqslant 0, \ u \geqslant 0,$$

such that the natural projection $\mathbf{A}^{r,s} \times (Q/S) \to Q/S$ *is the geometric quotient of the regular action of* S *on* $\mathbf{A}^{r,u} \times (Q/S)$ *induced by* φ.

Theorem 5 leads to the following

Construction $\circledast$

Let S be a connected solvable affine algebraic group and let Z be an irreducible variety such that, for some nonnegative integers r, u,

$$\text{the variety } Z \times \mathbf{A}^{r,u} \text{ is rational.} \tag{40}$$

Consider a map

$$\varphi\colon S \times Z \to \operatorname{Aut} \mathbf{A}^{r,u}$$

that has the following properties:

(i) φ is an algebraic family [Ra 1964], [Po 2014], i.e.,

$$S \times Z \times \mathbf{A}^{r,u} \to \mathbf{A}^{r,u}, \quad (s, z, a) \mapsto \varphi(s, z)(a),$$

is a morphism;

(ii) for every point $z \in Z$, the algebraic family

$$\varphi_z\colon S \to \operatorname{Aut} \mathbf{A}^{r,u}, \quad s \mapsto \varphi(s, z),$$

is a homomorphism determining a transitive action of S on $\mathbf{A}^{r,u}$.

These data determine the regular action of S on $Z \times \mathbf{A}^{r,u}$ by the formula

$$S \times (Z \times \mathbf{A}^{r,u}) \to Z \times \mathbf{A}^{r,u}, \quad \big(s, (z, a)\big) \mapsto \big(z, \varphi(s, z)(a)\big). \tag{41}$$

By (ii), the orbits of this action are the fibers of the projection

$$\pi\colon Z \times \mathbf{A}^{r,u} \to Z, \quad (z, a) \mapsto z,$$

and, more precisely, for every point $z \in Z$, the variety $\mathbf{A}^{r,u}$ endowed with the G-action determined by φ_z is G-isomorphic to the fiber $\pi^{-1}(z)$. By [Bo 91, Prop. II.6.6] this implies that $k(Z \times \mathbf{A}^{r,u})^S \simeq k(Z)$.

In view of (40), for $n = \dim Z + r + u$, we may (and shall) fix a birational isomorphism

$$\gamma\colon Z \times \mathbf{A}^{r,u} \dashrightarrow \mathbf{A}^n.$$

Then γ and action (41) determine a rational action of S on $\mathbf{A}^n$. The image of the homomorphism $S \to \mathcal{C}_n$ determined by this rational action is a connected solvable affine algebraic subgroup G of $\mathcal{C}_n$, and for this G we have

$$k(\mathbf{A}^n)^G \simeq k(Z). \tag{42}$$

We say that G is a *subgroup of* $\mathcal{C}_n$ *obtained by Construction* $\circledast$.

Theorem 5 (combined with Weil's regularization theorem [We 1955]) implies that this construction is universal, i.e., every connected solvable affine algebraic subgroup of $\mathcal{C}_n$ is obtained by Construction $\circledast$.

Example 1 (One-dimensional connected unipotent subgroups of $\mathcal{C}_n$)**.** Let G be the one-dimensional additive group k^+. In view of (ii), we then have $r=0$, $u=1$, i.e., $\varphi_z\colon G=k^+\to \operatorname{Aut}\mathbf{A}^1$ for every $z\in Z$. Since $\operatorname{Aut}\mathbf{A}^1=T\ltimes N$, where T is a one-dimensional torus and $N\simeq k^+$ is the subgroup consisting of all translations $\mathbf{A}^1\to\mathbf{A}^1$, $a\mapsto a+t$, $t\in k^+$, this means that every φ_z may be identified with a surjective homomorphism $k^+\to k^+$. What happens next depends on $\operatorname{char} k$, see [Sp 1998, Lemma 3.3.5].

Namely, a map $k^+\to k^+$ is a homomorphism if and only if it has the following form:

(i) case $\operatorname{char} k=0$: $t\mapsto ct$, where c is a fixed element of k,

(ii) case $\operatorname{char} k=p>0$: $t\mapsto \alpha_1 t^{p^{i_1}}+\cdots+\alpha_d t^{p^{i_d}}$, where $\alpha_1,\ldots,\alpha_d$ are the nonzero elements of k and $i_1,\ldots,i_d$ is an increasing sequence of nonnegative integers.

Since every one-dimensional connected unipotent algebraic group is isomorphic to k^+ (see, e.g., [Sp 1998, Thm. 3.4.9]), this yields the following general method of constructing connected one-dimensional unipotent algebraic subgroups of $\mathcal{C}_n$.

Take an irreducible variety Z such that $Z\times\mathbf{A}^1$ and $\mathbf{A}^n$ are birationally isomorphic. If $\operatorname{char} k=0$, fix a nonzero regular function $f\in k[Z]$. If $\operatorname{char} k=p>0$, fix a sequence of nonnegative integers $i_1<\ldots<i_d$ and a sequence of nonzero regular functions $f_1,\ldots,f_d\in k[Z]$. Consider the action of $S=k^+$ on $Z\times\mathbf{A}^1$ defined by the formula

$$
\begin{gathered}
S\times(Z\times\mathbf{A}^1)\to Z\times\mathbf{A}^1,\\
(s,(z,a))\mapsto\begin{cases}(z,a+f(z)s) & \text{if } \operatorname{char} k=0,\\ (z,a+f_1(z)s^{p^{i_1}}+\cdots+f_d(z)s^{p^{i_d}}) & \text{if } \operatorname{char} k=p>0.\end{cases}
\end{gathered}\tag{43}
$$

Then action (43) and a fixed birational isomorphism $\gamma\colon Z\times\mathbf{A}^1 \dashrightarrow \mathbf{A}^n$ determine a one-dimensional connected unipotent algebraic subgroup G of $\mathcal{C}_n$, and every such subgroup is obtained this way.

Theorem 6 (Structure theorem)**.** *The following properties of a connected solvable affine algebraic subgroup G of the Cremona group $\mathcal{C}_n$ are equivalent:*

(i) *G is rationally triangulable;*

(ii) *G is obtained by Construction ⊛, in which the variety Z is rational.*

Proof. This follows from (42) and Theorem 2. □

Proof of Corollary 4. Let G be a rationally triangulable one-dimensional connected unipotent algebraic subgroup of $\mathcal{C}_n$. By Theorem 6 and Example 1, G is obtained from the action of $S=k^+$ on $Z\times\mathbf{A}^1$ defined by formulas (43), where Z is a rational variety. Therefore there are functions $f_1,\ldots,f_n\in k(\mathbf{A}^n)$ such that $k(f_1,\ldots,f_n)=k(\mathbf{A}^n)$, $f_1,\ldots,f_{n-1}\in k(\mathbf{A}^n)^G$, and $t(f_n)$ for every element $t\in S$ is the following function:

(i) if $\operatorname{char} k=0$, then $t(f_n)=f_n+ct$, where $c\in k(f_1,\ldots,f_{n-1})$,

(ii) if $\operatorname{char} k = p > 0$, then

$$t(f_n) = f_n + b_1 t^{p^{i_1}} + \cdots + b_d t^{p^{i_d}},$$

where $b_j \in k(f_1, \ldots, f_{n-1})$, $b_j \neq 0$ for every j and $i_1 < \cdots < i_d$ are the nonzero integers.

In case (i), replacing f_n by f_n/c we may (and shall) assume that $c = 1$. Then, conjugating S by means of $\varphi \in \mathcal{C}_n$ such that $\varphi(f_i) = x_i$ for every i (see (1)), we obtain a standard subgroup.

The converse (that standard subgroups are rationally triangulable) is clear. $\square$

Proof of Corollary 5. This follows from Corollaries 1, 2, and 4. $\square$

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