

Klein Foams

ANTONIO F. COSTA, SABIR M. GUSEIN-ZADE &
SERGEY M. NATANZON

ABSTRACT. Klein foams are analogues of Riemann and Klein surfaces with one-dimensional singularities. We prove that the field of dianalytic functions on a Klein foam Ω coincides with the field of dianalytic functions on a Klein surface K_Ω . We construct the moduli space of Klein foams, and we prove that the set of classes of topologically equivalent Klein foams form an analytic space homeomorphic to \mathbb{R}^n/Mod , where Mod is a discrete group.

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1. INTRODUCTION

Foams are surfaces with one-dimensional singularities. Topological foams are exploited in different fields of mathematical physics [6, 13, 22, 25] and topology [11, 12, 16, 24, 26]. A topological foam is constructed from finitely many ordinary surfaces with boundaries (“patches”) by gluing them along segments of their boundaries. The glued boundaries of the surfaces form a “seamed graph”, which is the singular part of the complex.

Here we consider Klein foams that are analogues of Riemann and Klein surfaces for foams. Thus we consider a foam with concordant complex structures on its patches. “Concordant” means that there exists a dianalytic map from the foam to the complex disk D . Maps of this type on a Klein foam Ω we call dianalytic functions on Ω .

In Section 2 we prove that the field of dianalytic functions on a Klein foam Ω coincides with the field of dianalytic functions on a Klein surface K_Ω . Moreover, there exists a dianalytic map $\varphi_\Omega : \Omega \rightarrow K_\Omega$ such that any dianalytic function on Ω is of the form $f\varphi_\Omega$ where f is a dianalytic function on K_Ω .

We say that Klein foams Ω and Ω' are topologically equivalent if there exist homeomorphisms $f_\Omega : \Omega \rightarrow \Omega'$ and $f_K : K_\Omega \rightarrow K_{\Omega'}$ such that $\varphi_{\Omega'}f_\Omega = \varphi_\Omega f_K$. In Section 3 we prove that any class M of topological equivalence of Klein foams (i.e., the set of Klein foams with a fixed topological type) has a natural analytic structure. It is connected and homeomorphic to $\mathbb{R}^n / \text{Mod}$, where Mod is a discrete group. This gives a topological description of the moduli space of Klein foams.

A motivation to study the moduli spaces of Klein foams is the string theory and 2D gravity [6, 9, 15, 25]. In Subsection 2.3 we prove that our definition of Klein foams is compatible with the cyclic foam topological field theory [17] which is a rough topological approach to the corresponding version of the string theory.

2. FOAMS

2.1. Topological foams. We shall consider *generalized graphs*, that is, one-dimensional spaces consisting of (finitely many) vertices and edges, where edges are either segments (connecting *different* vertices) or (isolated) circles (without vertices on them). Some pairs of vertices may be connected by several edges.

A *topological foam* Ω is a triple (S, Δ, φ) , where

- $S = S(\Omega)$ is a compact surface (2-manifold, possibly non-connected and non-orientable) with boundary ∂S (which consists of pair-wise non-intersecting circles);
- $\Delta = \Delta(\Omega)$ is a generalized graph;
- $\varphi = \varphi_\Omega : \partial S \rightarrow \Delta$ is *the gluing map*, that is, a map such that
 - (a) $\text{Im } \varphi = \Delta$;
 - (b) on each connected component of the boundary ∂S , φ is a homeomorphism on a circle in Δ ;
 - (c) for an edge ℓ of Δ , any connected component of S contains at most one connected component of $\varphi^{-1}(\ell \setminus \partial \ell)$.

The result of the gluing (i.e., $S \cup \Delta$ with x and $\varphi(x)$ identified for $x \in \partial S$) will be denoted by $\tilde{\Omega}$. Let Ω_b be the set of vertices of the graph Δ . We say that a foam Ω is *normal* if, for any vertex v of the graph Δ , its punctured neighbourhood in $\tilde{\Omega}$ is connected.

For a surface S with boundary, the *double* of S is an oriented surface \hat{S} defined in the following way. Let \tilde{S} be the “orientation” 2-fold covering over S . (The surface \tilde{S} consists of pairs (x, δ) , where x is a point of S and δ is a local orientation of S at the point x . If the surface S is orientable, \tilde{S} is the union of 2 copies of S with different orientations. The surface \tilde{S} is oriented in the natural way. If S is a smooth surface, the surface \tilde{S} is also smooth.) The surface \tilde{S} possesses a natural involution σ , $\tilde{S}/\sigma = S$. The double \hat{S} of the surface S is the surface \tilde{S} with the points (x, δ) and $(x, -\delta)$ identified for $x \in \partial S$ ($-\delta$ is the local orientation opposite to the orientation δ).

A *morphism* f of topological foams $\Omega' \rightarrow \Omega''$ ($\Omega' = (S', \Delta', \varphi')$, $\Omega'' = (S'', \Delta'', \varphi'')$) is a pair (f_S, f_Δ) of (continuous) maps $f_S : \hat{S}' \rightarrow \hat{S}''$ and $f_\Delta : \Delta' \rightarrow \Delta''$ such that f_S is an orientation-preserving ramified covering commuting with the natural involutions on \hat{S}' and \hat{S}'' , $\varphi'' \circ f_S = f_\Delta \circ \varphi'$, and $f_\Delta|_{\Delta' \setminus \Omega'_b}$ is a local homeomorphism $\Delta' \setminus \Omega'_b \rightarrow \Delta'' \setminus \Omega''_b$.

2.2. Dianalytic foams. A *Klein surface* [5] (see also [18]) is a surface S (possibly with boundary and/or non-orientable) with a class of equivalence of *dianalytic atlases*. A *dianalytic atlas* consists of charts $\{(U_\alpha, \psi_\alpha) \mid \alpha \in \mathcal{A}\}$, where $\Omega = \bigcup_\alpha U_\alpha$, $\psi_\alpha : U_\alpha \rightarrow D \subset \mathbb{C}$ is a homeomorphism on $\psi_\alpha(U_\alpha)$ and $\psi_\alpha \psi_\beta^{-1}$ is a holomorphic or an anti-holomorphic map on $\psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta)$ for any $\alpha, \beta \in \mathcal{A}$. (Two dianalytic atlases are called equivalent if their union is also a dianalytic atlas.)

One can see that a Klein surface is a surface S with a complex analytic structure on the double \hat{S} such that the natural involution on \hat{S} is anti-holomorphic. A *morphism* of Klein surfaces is an analytic map between their doubles which commutes with the natural involutions. Moduli space of Klein surfaces were studied in [19, 20, 23]. The category of compact Klein surfaces is isomorphic to the category of real algebraic curves [5].

A normal topological foam $\Omega = (S, \Delta, \varphi)$, where S is a Klein surface, is called *dianalytic*. A *morphism* f of dianalytic foams $\Omega' \rightarrow \Omega''$ ($\Omega' = (S', \Delta', \varphi')$, $\Omega'' = (S'', \Delta'', \varphi'')$) is a morphism (f_S, f_Δ) of the corresponding topological foams such that f_S is a morphism of Klein surfaces.

Let Ω_0 be the dianalytic foam $(D, \partial D, I_D)$, where D is the unit disk in the complex line and I_D is the tautological map. A *dianalytic function* on a dianalytic foam Ω is a morphism of Ω to Ω_0 .

Define now a *Klein foam* as a dianalytic foam $\Omega = (S, \Delta, \varphi)$ admitting an everywhere locally non-constant dianalytic function f_0 . Any Klein surface K can be considered as a Klein foam in a natural way.

Theorem 2.1. *For any Klein foam Ω , there exist a Klein surface $K = K_\Omega$ and a dianalytic morphism $\varphi_\Omega : \Omega \rightarrow K$ such that the correspondence $f \mapsto f\varphi_\Omega$ is an isomorphism between the sets (fields) of dianalytic functions on K and on Ω respectively, i.e., any dianalytic function $f : \Omega \rightarrow \Omega_0$ is the composition $f_{(K)}\varphi_\Omega$, where $f_{(K)} : K \rightarrow \Omega_0$ is a dianalytic function, and vice versa, for any dianalytic function $f_{(K)}$ on K , the composition $f_{(K)}\varphi_\Omega$ is a dianalytic function on the foam Ω .*

Proof. Let us call points q_1 and q_2 of the surface \hat{S} pre-equivalent if there exist paths $\gamma_j(t)$ on \hat{S} , $j = 1, 2$, $t \in [0, 1]$, such that

- $\varphi(\gamma_1(0)) = \varphi(\gamma_2(0))$ is an inner point of an edge of the graph Δ (and therefore not a ramification point of the map (dianalytic function) f^0);
- $f^0(\gamma_1(t)) = f^0(\gamma_2(t))$ for $t \in [0, 1]$;
- $\gamma_j(1) = q_j$ for $j = 1, 2$;
- $f^0(\gamma_j(t))$ is not a ramification point of the map f^0 for any $t \in [0, 1)$, $j = 1, 2$.

The values of a dianalytic function f on the foam Ω at pre-equivalent points coincide. This follows from the uniqueness of the analytic continuation taking into account the fact that f^0 can be considered as a local coordinate for all points $\gamma_j(t)$ with $t \in [0, 1)$ and the restrictions of f to neighborhoods of the points $\gamma_j(0)$ in ∂S_Ω , $j = 1, 2$, coincide as functions of f_0 .

Let us call points q' and q'' from \hat{S} equivalent if there exists a sequence of points $q_j \in \hat{S}$, $j = 0, 1, \dots, n$, such that $q_0 = q'$, $q_n = q''$, and the point q_{j-1} is pre-equivalent to the point q_j for $j = 1, 2, \dots, n$. Outside of (the preimage of) the set of ramification points, the described equivalence relation identifies some points in the preimages in \hat{S} (with respect to the function f_0) of points of \hat{D} . If two non-ramification points are identified, they have neighborhoods which are identified with each other by this equivalence relation, and the identification is a homeomorphism. This means that the factor K_Ω by this equivalence relation inherits the structure of a (closed) complex analytic curve with an anti-holomorphic involution and with a map to \bar{D} . The factorization map φ_Ω is a (usual) covering outside of the set of ramification points of the map f^0 . A dianalytic function on the foam Ω induces a dianalytic function on K_Ω . This implies that it is induced from a dianalytic function on K_Ω .

For a dianalytic function $f_{(K)}$ on K_Ω , the composition $f_{(K)}\varphi_\Omega$ may fail to be a dianalytic function on the foam Ω only if there exist two points q_1 and q_2 of ∂S_Ω such that they map to the same vertex of the graph Δ but are not equivalent. In particular, this means that no point of ∂S_Ω of a neighborhood of the point q_1 is equivalent to one of a neighborhood of the point q_2 , which contradicts the requirement of normality of the foam Ω . This proves the statement. \square

We call $\varphi_\Omega : \Omega \rightarrow K_\Omega$ the *canonical morphism*. An isomorphism $f : \Omega' \rightarrow \Omega''$ induces an equivalence of the canonical morphisms $\varphi_{\Omega'} : \Omega' \rightarrow K'_\Omega$ and $\varphi_{\Omega''} : \Omega'' \rightarrow K''_\Omega$.

Corollary 2.2. *Let $f : \Omega' \rightarrow \Omega''$ be an isomorphism of Klein foams. There exists an isomorphism of Klein surfaces $f_K : \Omega'_K \rightarrow \Omega''_K$ such that $f_K \varphi_{\Omega'} = \varphi_{\Omega''} f$.*

We say that Klein foams $\Omega' = (S', \Delta', \varphi')$ and $\Omega'' = (S'', \Delta'', \varphi'')$ have *the same topological type* if there exist isomorphisms of topological foams $f : \Omega' \rightarrow \Omega''$ and $f_K : \Omega'_K \rightarrow \Omega''_K$ such that $f_K \varphi_{\Omega'} = \varphi_{\Omega''} f$.

2.3. Strongly oriented foams. Klein Topological Field Theories describe a rough topological approach to the corresponding versions of string theory and Hurwitz numbers [1–4, 8, 14]. From [17] it follows that one can extend Klein Topological Field Theory to *oriented foams*. An oriented foam is a topological foam (S, Δ, φ) with a special coloring of S . A special coloring of S exists if and only if

- vertices of any connected component of Δ allow a cyclic order that agrees with an orientation of ∂S by φ ;
- φ maps different connected components of the boundary of any connected component of S to different connected components of Δ .

A special coloring defines (via φ) orientations of all edges of Δ compatible with the order of vertices. An oriented foam (S, Δ, φ) is *strongly oriented* if the orientation of ∂S is induced by an orientation of S .

Lemma 2.3. *Let $\Omega = (S, \Delta, \varphi)$ be a strongly oriented foam. Then there exists a morphism (f_S, f_Δ) to the disk (i.e., to the topological foam $(D, \partial D, I_D)$) which is a homeomorphism on any connected component of ∂S .*

Proof. Since S is oriented, its double \hat{S} consists of S and of its copy with the other orientation (glued along the boundary ∂S). The double \hat{D} of the disk D is the 2-sphere S^2 (with the equator ∂D). To construct the map $f_S : \hat{S} \rightarrow S^2$, one can construct a map $f : S \rightarrow S^2$ sending the boundary ∂S to ∂D which is a local homeomorphism in a neighborhood of ∂S and a ramified covering inside S . The map f with the canonical involutions on \hat{S} and on \hat{D} define the map f_S . Let us fix orientation-preserving maps $f' : \partial S \rightarrow \partial D$ and $f_\Delta : \Delta \rightarrow \partial D$ (such that $f' = \varphi f_\Delta$) compatible with the orientations of ∂S and of the edges of Δ . Gluing a disk to each component of the boundary ∂S , one obtains an oriented closed surface S' . Now existence of the map f with the described properties follows from the fact that there exists a map from the surface S' to the 2-sphere S^2 which is a ramified covering of high order: greater or equal to the number d of components of the boundary ∂S . Moreover, one can choose this map so that disk neighborhoods of d fixed points are sent to a fixed disk in S^2 by prescribed homeomorphisms. To obtain the desired map, one should use the map of the added disks to the copy of D with the inverse orientation which is the radial extension of the map f' . \square

Theorem 2.4. *Any strongly oriented foam allows a dianalytic structure, turning such a foam into a Klein foam.*

Proof. Consider a strongly oriented foam $\Omega = (S, \Delta, \varphi)$. From Lemma 2.3 it follows that there exists a topological morphism (f_S, f_Δ) of Ω to the dianalytic disk $\Omega_0 = (D, \partial D, I_D)$. Therefore there exists a (unique) dianalytic structure on S such that f_S is a dianalytic map [10]. \square

3. MODULI SPACES

3.1. Moduli of Klein surfaces. In this and the next subsections we assume all surfaces to be connected and to have finitely generated fundamental groups. The fundamental group \check{Y}_+ of an orientable surface K_+ has a co-presentation of the form

$$\left\langle \begin{array}{l} a_1, b_1, \dots, a_{g_+}, b_{g_+}, h_1, \dots, h_m, x_1, \dots, x_r, e_1, \dots, e_k, c_{i1}, \dots, c_{ib_i}, \\ i = 1, \dots, k \mid c_{ij}^2 = 1, \prod [a_j, b_j] \prod x_j h_j \prod e_j = 1 \end{array} \right\rangle.$$

The fundamental group \check{Y}_- of a non-orientable surface K_- has a co-presentation of the form

$$\left\langle \begin{array}{l} d_1, \dots, d_{g_-}, h_1, \dots, h_m, x_1, \dots, x_r, e_1, \dots, e_k, c_{i1}, \dots, c_{ib_i}, \\ i = 1, \dots, k \mid c_{ij}^2 = 1, \prod d_j^2 \prod h_j \prod x_j \prod e_j = 1 \end{array} \right\rangle.$$

The collection $\check{t} = (\varepsilon \in \{+, -\}, g_\varepsilon, m, r, k, b_1, \dots, b_k)$ is the *topological type* of the surface K . Here g is the (geometric) genus of K ; m is the number of holes; r is the number of interior punctures; k is the number of boundary components; and b_i , $i = 1, 2, \dots, k$, is the number of punctures on the boundary component with the number i .

Any Klein surface \check{K} is isomorphic to $\check{K}^\psi = \Lambda / \psi(\check{Y})$, where $\check{Y} \in \{\check{Y}_+, \check{Y}_-\}$, $U = D \setminus \partial D$ and $\psi : \check{Y} \rightarrow \text{Aut}(U)$ is a monomorphism to the group $\text{Aut}(U)$ of dianalytic automorphisms of U [7]. Moreover, the monomorphism ψ satisfies the following conditions (see [21, Chapter 2.2]):

- $\psi(c_{ij})$ and $\psi(d_j)$ are antiholomorphic, and images of all other generators are holomorphic.
- The automorphisms $\psi(x_i)$ are parabolic, and the automorphisms $\psi(a_i)$, $\psi(b_i)$, $\psi(c_i)$, $\psi(h_i)$, $\psi(d_i^2)$ are hyperbolic.
- The hyperbolic and the parabolic automorphisms together form a sequential set (in the sense of [21, Chapter 1.1]).

These conditions imply that the group $\psi(\check{Y})$ is discrete.

Monomorphisms ψ satisfying these conditions will be called *admissible*. An admissible monomorphism ψ always generates a discrete subgroup of $\text{Aut}(U)$ and thus a Klein surface K^ψ . The Klein surfaces $K^{\psi'}$ and $K^{\psi''}$ are isomorphic if and only if the groups $\psi'(\check{Y})$ and $\psi''(\check{Y})$ are conjugate in $\text{Aut}(U)$, i.e., $\psi'(\check{Y}) = A\psi''(\check{Y})A^{-1}$ for an automorphism $A \in \text{Aut}(U)$.

The group of homotopy classes of autohomeomorphisms of K^ψ is naturally isomorphic to the group $\text{Mod}_\check{t} = \widehat{\text{Mod}}(\check{Y}) / \text{Aut}_0(\check{Y})$, where $\widehat{\text{Mod}}(\check{Y})$ is a subgroup

of $\text{Aut}(\check{Y})$ and $\text{Aut}_0(\check{Y}) \subset \widehat{\text{Mod}}(\check{Y})$ is the subgroup of interior automorphisms of \check{Y} .

Therefore the moduli space $M_{\check{t}}$ of Klein surfaces of topological type \check{t} is $T_{\check{t}}/\text{Mod}_{\check{t}}$, where $T_{\check{t}}$ is the set of conjugacy classes (with respect to $\text{Aut}(U)$) of admissible monomorphisms, and $\text{Mod}_{\check{t}}$ acts discretely. A description of geometric properties of admissible monomorphisms in terms of coordinates on $\text{Aut}(U)$ gives a homeomorphism $T_{\check{t}} \leftrightarrow \mathbb{R}^{6g_{\varepsilon}+3m+3k+2r+b_1+\dots+b_k-6}$.

For compact Klein surfaces ($r = b_1 = \dots = b_k = 0$) and for oriented surfaces without boundary ($\varepsilon = +, k = 0$), all these facts are proved in [19–21]. A proof for an arbitrary topological type is obtained by simple modifications of [19–21]. Thus we have the following theorem.

Theorem 3.1. *The space of Klein surfaces of any given topological type is connected and homeomorphic to \mathbb{R}^n/Mod , where Mod is a discrete group.*

3.2. Spaces of morphisms of Klein surfaces. Two morphisms between Klein surfaces $f' : S' \rightarrow K'$ and $f'' : S'' \rightarrow K''$ are called *isomorphic* (respectively, *topologically equivalent*), if there exist isomorphisms (respectively, homeomorphisms) $f_S : S' \rightarrow S''$ and $f_K : K' \rightarrow K''$ such that $f_S f' = f'' f_K$. The space of isomorphic classes of morphisms (with the natural topology) is called *the moduli space of morphisms*. A class of topological equivalence of morphisms is called the *topological type of a morphism* of the class.

Now let us describe a class of morphisms of degree d with target a compact Klein surface of type $t = (\varepsilon, g, 0, k, 0, \dots, 0)$. Let \check{Y} be the fundamental group of a Klein surface of type $\check{t} = (\varepsilon, g, r, k, b_1, \dots, b_k)$, and let $\tilde{Y} \subset \check{Y}$ be a subgroup of index d . Consider $\psi \in T_{\check{t}}$ and $\check{K}^{\psi} = U/\psi(\check{Y})$, $\check{S}_{\tilde{Y}}^{\psi} = U/\psi(\tilde{Y})$. The natural embedding $\psi(\tilde{Y}) \subset \psi(\check{Y})$ induces a morphism $\check{f}_{\tilde{Y}}^{\psi} : \check{S}_{\tilde{Y}}^{\psi} \rightarrow \check{K}^{\psi}$ of degree d . After patching on $\check{S}_{\tilde{Y}}^{\psi}$ and \check{K}^{ψ} the punctures generated by parabolic shifts, we obtain a morphism $f_{\tilde{Y}}^{\psi} : S_{\tilde{Y}}^{\psi} \rightarrow K^{\psi}$ to a Klein surface of topological type t .

One can show that

- each non-constant morphism of Klein surfaces is isomorphic to a morphism of the form $f_{\tilde{Y}}^{\psi}$;
- the morphisms $f_{\tilde{Y}}^{\psi}$ and $f_{\tilde{Y}}^{\hat{\psi}}$ have the same topological type for any $\psi, \hat{\psi} \in T_{\check{t}}$; moreover, they are isomorphic if and only if $A\psi(z)A^{-1} = \hat{\psi}(z)$ for an automorphism $A \in \text{Aut}(U)$ and any $z \in \check{Y}$;
- the morphisms $f_{\tilde{Y}}^{\psi}$ and $f_{\tilde{Y}}^{\hat{\psi}}$ have the same topological type if and only if $\hat{Y} = \alpha(\tilde{Y})$, where $\alpha \in \widehat{\text{Mod}}_{\check{t}}$.

For orientable Riemann surfaces ($\varepsilon = +, k = 0$), all these facts are proved in [21, Section 1.6]. A proof for an arbitrary topological type is obtained by simple modifications of [21]. These arguments and Theorem 3.1 give the statement below.

Theorem 3.2. *The space of morphisms of Klein surfaces of any given topological type is connected and homeomorphic to $\mathbb{R}^n / \text{Mod}$, where Mod is a discrete group.*

3.3. Construction of Klein foams. Now we shall apply the previous section to connected Klein foams. We shall assume that the type t of \check{Y} is such that $k, b_1, \dots, b_k > 0$.

Let $\check{Y}^1, \dots, \check{Y}^n \subset \check{Y}$ be subgroups of finite indices. Consider conjugate classes $C_{i,j} = \bigcup_{g \in \check{Y}} g c_{i,j} g^{-1}$ and let $C_{i,j}^\ell := C_{i,j} \cap \check{Y}^\ell$. The group of interior automorphisms of \check{Y}^ℓ splits $C_{i,j}^\ell$ into orbits $C_{i,j}^{\ell,1}, \dots, C_{i,j}^{\ell,m_\ell}$. A set $(C_{i,j}^{\ell^1,m^1}, \dots, C_{i,j}^{\ell^s,m^s})$ is called an *edge* of $(\check{Y}; \check{Y}^1, \dots, \check{Y}^n)$ if $\ell^v \neq \ell^w$ for $v \neq w$. A *foam system* is a collection $\Gamma = \{\check{Y}; \check{Y}^1, \dots, \check{Y}^n; H\}$, where H is a set of mutually disjoint edges of $(\check{Y}; \check{Y}^1, \dots, \check{Y}^n)$ containing all $C_{i,j}^\ell$.

Let $\psi \in T_t$. From Subsection 3.2 it follows that the subgroups $\check{Y}^1, \dots, \check{Y}^n$ generate a morphism $f^\psi : S^\psi \rightarrow K^\psi$, where $S^\psi = S_{\check{Y}^1}^\psi \amalg \dots \amalg S_{\check{Y}^n}^\psi$. The parabolic points of \check{Y} divide the boundary of S into segments bijectively corresponding to the sets $C_{i,j}^\ell \subset C_{i,j}$. Any element from $C_{i,j}$ is the reflection in a straight line ℓ in the Poincaré model of Lobachevsky geometry on U . The group \check{Y} acts transitively on the set $\{\ell\}$ of these lines. The line corresponding to $C_{i,j}^\ell \subset C_{i,j}$ forms (under the natural projection) a segment on ∂S^ℓ . Let $(C_{i,j}^{\ell^1,m^1}, \dots, C_{i,j}^{\ell^s,m^s}) \in H$ be an edge of H . Let us glue the corresponding segments of $\partial S^{\ell^1}, \dots, \partial S^{\ell^s}$ by the action of \check{Y} on $\{\ell\}$. The continuation of the gluing to the closures of the segments gives a graph Δ and a map $\varphi : \partial S \rightarrow \Delta$. Let $\Omega_t^\psi = (S, \Delta, \varphi)$.

Lemma 3.3. *The triple Ω_t^ψ is a Klein foam.*

Proof. First let us prove that (S, Δ, φ) is a topological foam. The first two conditions and point (a) of the last condition follow directly from the construction. Points (b) and (c) follow from the definition of an edge of $(\check{Y}; \check{Y}^1, \dots, \check{Y}^n)$ (the requirement that $\ell^v \neq \ell^w$ for $v \neq w$) and the fact that the restriction of the natural projection to any line from $\{\ell\}$ is a homeomorphism. From Subsection 3.2, it follows that f^ψ induces a morphism of Klein foams $f : S \rightarrow K^\psi$. Moreover, the Klein surface K^ψ admits a morphism to the unit disk [5]. \square

Now we shall construct a class of foam systems corresponding to any connected Klein foam $\Omega = (S, \Delta, \varphi)$. Let us consider the Klein surface $K = K_\Omega \in M_t$ and the canonical morphism $\varphi_\Omega : \Omega \rightarrow K$ from Theorem 2.1. Let $B \subset K$ be the set of critical values of $\varphi = \varphi_\Omega$ and $\check{K} = K \setminus B \in M_t$. Let $\check{S} := \varphi^{-1}(\check{K})$.

From Subsection 3.1 it follows that $\check{K} = \Lambda/\psi(\check{Y})$ for $\psi \in T_t$, where \check{t} is the type of \check{Y} . From Subsection 3.2 it follows that the restrictions of φ to connected components of \check{S} are isomorphic to morphisms generated by a family of subgroups $\check{Y}^1, \dots, \check{Y}^n \subset \check{Y}$. The edges of Δ generate the family of edges H of $(\check{Y}; \check{Y}^1, \dots, \check{Y}^n)$ and thus a foam system $\Gamma = \{\check{Y}; \check{Y}^1, \dots, \check{Y}^n; H\}$. From Subsection 3.3 it follows

that the Klein foams $\Omega_{[\Gamma]}^\psi$ and Ω are isomorphic. Thus we have proved the following result.

Lemma 3.4. *For any Klein foam Ω , there exist a foam system Γ and $\psi \in T_{\check{\Gamma}}$ such that Ω and Ω_{Γ}^ψ are isomorphic.*

3.4. Moduli of Klein foams. Let Ω' and Ω'' be Klein foams with canonical morphisms $\varphi_{\Omega'} : \Omega' \rightarrow \Omega'_K$ and $\varphi_{\Omega''} : \Omega'' \rightarrow \Omega''_K$. We say that Ω' and Ω'' have the same topological type if there exist an isomorphism (f_S, f_Δ) of the topological foams, $f_S : \tilde{S}' \rightarrow \tilde{S}''$, $f_\Delta : \tilde{\Delta}' \rightarrow \tilde{\Delta}''$, and a homeomorphism $f_K : \tilde{K}' \rightarrow \tilde{K}''$ such that $\varphi_{\Omega''} f_S = f_K \varphi_{\Omega'}$.

The space of isomorphic classes of Klein foams (with the natural topology) is called the *moduli space of Klein foams*. A class of topological equivalence of morphisms is called the *topological type of a Klein foam*.

An element $(\alpha^1, \dots, \alpha^n) \in \widehat{\text{Mod}}(\check{\mathcal{Y}})^{\otimes n}$ acts on the set $\{\Gamma\}$ of foam systems sending $\tilde{\gamma}^\ell$ to $\alpha^\ell(\tilde{\gamma}^\ell)$ and $C_{i,j}^\ell$ to $\alpha^i(C_{i,j}^\ell)$ ($\ell = 1, \dots, n$). The orbit $[\Gamma]$ of a foam system Γ under the action of $\widehat{\text{Mod}}(\check{\mathcal{Y}})^{\otimes n}$ is called the *class of Γ* . Subsections 3.1 and 3.2 imply the following statement.

Lemma 3.5. *The Klein foams Ω_{Γ}^ψ and $\Omega_{\hat{\Gamma}}^{\hat{\psi}}$ have the same topological type for any $\psi, \hat{\psi} \in T_{\check{\Gamma}}$. Moreover, they are isomorphic if $A\psi(z)A^{-1} = \hat{\psi}(z)$ for an automorphism $A \in \text{Aut}(U)$ and any $z \in \check{\mathcal{Y}}$. The morphisms $f_{\check{\mathcal{Y}}}^\psi$ and $f_{\hat{\mathcal{Y}}}^{\hat{\psi}}$ have the same topological type if and only if $\hat{\mathcal{Y}} = \alpha(\check{\mathcal{Y}})$, where $\alpha \in \widehat{\text{Mod}}_{\check{\Gamma}}$.*

Thus the topological type of Ω_{Γ}^ψ is defined by $[\Gamma]$. Let

$$\text{Mod}_{[\Gamma]} := \widehat{\text{Mod}}(\check{\mathcal{Y}})^{\otimes n} / \text{Aut}_0(\check{\mathcal{Y}}).$$

The set of all Klein foams of topological type $[\Gamma]$ is in one-to-one correspondence with $T_t / \text{Mod}_{[\Gamma]}$. Therefore we have the following theorem.

Theorem 3.6. *The space of Klein foams of a given topological type is connected and homeomorphic to $\mathbb{R}^n / \text{Mod}$, where Mod is a discrete group.*

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ANTONIO F. COSTA:

Departamento de Matemáticas Fundamentales
Facultad de Ciencias, UNED
C/ Senda del rey 9
28040 Madrid, Spain
E-MAIL: acosta@mat.uned.es

SABIR M. GUSEIN-ZADE:

Moscow State University
GSP-1, Leninskie Gory
Moscow 119991, Russia
E-MAIL: sabir@mccme.ru

SERGEY M. NATANZON:

National Research University
Higher School of Economics
Vavilova Str. 7, 117312 Moscow, Russia
A.N. Belozersky Institute
Moscow State University
Independent University of Moscow
Institute of Theoretical and Experimental Physics (ITEP), Moscow
E-MAIL: natanzons@mail.ru

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