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Quantum algebraic approach to refined topological vertex

H. Awata,[a] **B. Feigin**[b,c,d] **and J. Shiraishi**[e]

[a] *Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan*
[b] *Landau Institute for Theoretical Physics, Chernogolovka, 142432, prosp. Akademika Semenova, 1a, Russia*
[c] *Higher School of Economics, 101000, Myasnitskaya ul., 20, Moscow, Russia*
[d] *Independent University of Moscow, 119002, Bol'shoi Vlas'evski per., 11, Moscow, Russia*
[e] *Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan*

E-mail: awata@math.nagoya-u.ac.jp, borfeigin@gmail.com, shiraish@ms.u-tokyo.ac.jp

Abstract: We establish the equivalence between the refined topological vertex of Iqbal-Kozcaz-Vafa and a certain representation theory of the quantum algebra of type $W_{1+\infty}$ introduced by Miki. Our construction involves trivalent intertwining operators Φ and Φ^* associated with triples of the bosonic Fock modules. Resembling the topological vertex, a triple of vectors $\in \mathbb{Z}^2$ is attached to each intertwining operator, which satisfy the Calabi-Yau and smoothness conditions. It is shown that certain matrix elements of Φ and Φ^* give the refined topological vertex $C_{\lambda\mu\nu}(t,q)$ of Iqbal-Kozcaz-Vafa. With another choice of basis, we recover the refined topological vertex $C_{\lambda\mu}{}^{\nu}(q,t)$ of Awata-Kanno. The gluing factors appears correctly when we consider any compositions of Φ and Φ^*. The spectral parameters attached to Fock spaces play the role of the Kähler parameters.

Keywords: Quantum Groups, Topological Strings, Conformal and W Symmetry, Supersymmetric gauge theory

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Contents

1 Introduction

The aim of the present paper is to study the refined topological vertex $C_{\lambda\mu\nu}(t,q)$ of Iqbal, Kozcaz and Vafa [20] from the point of view of the quantum algebra of type $W_{1+\infty}$ introduced by Miki [22]. We also treat the vertex $C_{\lambda\mu}{}^{\nu}(q,t)$ considered by Awata and Kanno [3] in the same footing.

Let us first recall briefly the notion of the topological vertex [4, 18]. A trivalent graph plays an important role, since it encodes the information where the cycles of a T^2 fibration of a toric 3-fold degenerate. The Calabi-Yau threefold is then mapped to a Feynman graph with fixed Schwinger terms (Kähler classes of the threefold), and the topological vertex is associated with states in the threefold tensor product of bosonic Fock spaces. Each edges of the graph is an oriented straight line labeled by a vector $\mathbf{v} \in \mathbb{Z}^2$ corresponding to the generator of $H_1(T^2)$ (shrinking cycles). If all the edges are incoming, we have the condition $\sum_i \mathbf{v}_i = 0$ (Calabi-Yau condition), and $|\mathbf{v}_i \wedge \mathbf{v}_j| = 1$ for any pair of edges (smoothness condition). Together with a 'gluing rules,' on can calculate all genus amplitudes of the topological A-model for non-compact toric Calabi-Yau threefolds. The topological vertex $C_{\lambda\mu\nu}$ is represented by Okounkov, Reshetikhin and Vafa using the skew Schur functions [27, 28]

$$C_{\lambda\mu\nu}(q) = q^{\frac{\kappa(\mu)}{2}} s_{\nu'}(q^{-\rho}) \sum_{\eta} s_{\lambda'/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu'-\rho}), \tag{1.1}$$

where λ, μ, ν are partitions labeling the states in the threefold tensor of the Fock spaces, λ' denotes the transpose of λ, $\rho = (-1/2, -3/2, -5/2, \cdots)$, and $\kappa(\lambda) = \sum_i \lambda_i(\lambda_i + 1 - 2i)$.

In [20] a refined version of the topological vertex was introduced, based on the arguments of geometric engineering concerning the K-theoretic lift of the Nekrasov partition functions [15, 26]. See also [24, 25]. In this refined version, one more parameter t comes in and the theory seems to be deeply relate with the Macdonald functions $P_\lambda(x;q,t)$ [21]. The formula is

$$C^{(\mathrm{IKV})}_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{||\mu||^2}{2}} t^{\frac{\kappa(\mu)}{2}} q^{\frac{||\nu||^2}{2}} \widetilde{Z}_\nu(t,q) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda'/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu'} q^{-\rho}), \tag{1.2}$$

$$\widetilde{Z}_\nu(t,q) = \prod_{s\in\nu} (1 - q^{a_\nu(s)} t^{\ell_\nu(s)+1})^{-1} = t^{-\frac{||\nu'||^2}{2}} P_\nu(t^{-\rho};q,t), \tag{1.3}$$

where $||\lambda||^2 = \sum_i \lambda_i^2$. See [19] for recent development, and a remark on their notational convention.

There is another approach by Awata and Kanno [3], where Macdonald functions are used in some symmetric way

$$C_{\mu\lambda}{}^{\nu}(q,t) = P_\lambda(t^\rho;q,t) \sum_{\sigma} \iota P_{\mu'/\sigma'}(-t^{\lambda'} q^{\rho}; t, q) P_{\nu/\sigma}(q^\lambda t^\rho; q,t) (q^{1/2}/t^{1/2})^{|\sigma|-|\nu|} f_\nu(q,t)^{-1}. \tag{1.4}$$

(See section 4.1 as for the notations.) Here they incorporated the 'framing factor'

$$f_\lambda(q,t) = (-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2}, \tag{1.5}$$

which was introduced by Taki [33]. It has been recognized that these two different formulas give us essentially the same result, and the difference should be superficial. As for the preliminary version of the formula (1.4), see [2].

Now we turn to the quantum algebra side. The algebra we consider was first introduced by Miki in his study on the $W_{1+\infty}$ algebra. After the first discovery by Miki, essentially the same algebraic structure has been rediscovered by several authors. See [11–14, 16, 29–32]. This verifies the naturalness and the richness of the algebra. Because of this, it has been called by several different names, and there is no good choice at this moment than waiting for well established terminologies. In this paper, we denote the algebra by $\mathcal{U}$.

Motivated by the construction in (refined) topological vertex, we study a representation theory of the quantum algebra $\mathcal{U}$ which includes the following ingredients:

1. triple of the Fock spaces and associated intertwining operators,
2. trivalent vertex with edges labeled by vectors $\in \mathbb{Z}^2$ satisfying the Calabi-Yau and smoothness conditions,
3. spectral parameters playing the role of the Kähler parameters.

It has been recognized that the quantum algebra $\mathcal{U}$ has two central elements, and they obey a certain transformation formula with respect to the $\mathrm{SL}(2,\mathbb{Z})$ action [22, 31, 32]. Namely, the $\mathrm{SL}(2,\mathbb{Z})$ action preserves the structure of the algebra up to the shift in the central elements. As a consequence of the $\mathrm{SL}(2,\mathbb{Z})$ action, we have two types of the Fock representations of $\mathcal{U}$, one in [16] and the other in [13]. After fixing convention suitably, one can say that the former has level $(0,1)$ (vertical), and the latter has level $(1,0)$ (horizontal). The action of the T generator of the $\mathrm{SL}(2,\mathbb{Z})$ can be easily treated, and we can modify the 'horizontal' Fock representation to level $(1,N)$ with $N\in\mathbb{Z}$. We restrict ourselves only to the family of the Fock modules $\mathcal{F}_u^{(0,1)}$ and $\mathcal{F}_u^{(1,N)}$ $(N\in\mathbb{Z})$, where u is the spectral parameter. (See sections 2.3, 2.4.) Note if one of the edges (the preferred edge) is labeld by $(0,1)$, then from the Calabi-Yau and the smoothness condition the rest should be $(1,N)$ and $(-1,-N-1)$ where $N\in\mathbb{Z}$.

Consider the intertwining operators of $\mathcal{U}$-modules associated with three Fock modules of the forms $\Phi = \Phi\left[\begin{smallmatrix}\mathbf{v}_3,u_3\\ \mathbf{v}_1,u_1;\mathbf{v}_2,u_2\end{smallmatrix}\right] : \mathcal{F}_{u_1}^{\mathbf{v}_1}\otimes\mathcal{F}_{u_2}^{\mathbf{v}_2}\to\mathcal{F}_{u_3}^{\mathbf{v}_3}$ and $\Phi^* = \Phi^*\left[\begin{smallmatrix}\mathbf{v}_2,u_2;\mathbf{v}_1,u_1\\ \mathbf{v}_3,u_3\end{smallmatrix}\right] : \mathcal{F}_{u_3}^{\mathbf{v}_3}\to\mathcal{F}_{u_2}^{\mathbf{v}_2}\otimes\mathcal{F}_{u_1}^{\mathbf{v}_1}$. The following particular cases are essential in our construction:

$$\Phi : \mathcal{F}_v^{(0,1)}\otimes\mathcal{F}_u^{(1,N)} \longrightarrow \mathcal{F}_{-vu}^{(1,N+1)}, \qquad a\Phi = \Phi\Delta(a) \qquad (\forall a\in\mathcal{U}), \tag{1.6}$$

$$\Phi_\lambda(\alpha) = \Phi(P_\lambda\otimes\alpha) \qquad (\forall P_\lambda\otimes\alpha\in\mathcal{F}_v^{(0,1)}\otimes\mathcal{F}_u^{(1,N)}), \quad \Phi_\emptyset(1) = 1+\cdots, \tag{1.7}$$

$$\Phi^* : \mathcal{F}_{-vu}^{(1,N+1)} \longrightarrow \mathcal{F}_v^{(1,N)}\otimes\mathcal{F}_u^{(0,1)}, \qquad \Delta(a)\Phi^* = \Phi^* a \qquad (\forall a\in\mathcal{U}), \tag{1.8}$$

$$\Phi^*(\alpha) = \sum_\lambda \Phi^*_\lambda(\alpha)\otimes Q_\lambda \qquad (\forall\alpha\in\mathcal{F}_{-vu}^{(1,N+1)}), \quad \Phi^*_\emptyset(1) = 1+\cdots, \tag{1.9}$$

where Φ_λ and Φ^*_λ are normalized components of Φ and Φ^*. We prove that such (normalized) intertwining operators Φ and Φ^* exist uniquely (theorems 3.3, 3.6).

Let $S_\lambda(q,t)$'s be the dual of the Schur function s_μ's with respect to the Macdonald scalar product in (2.8) satisfying $\langle S_\lambda(q,t), s_\mu\rangle_{q,t} = \delta_{\lambda,\mu}$. We show that the refined topological vertex $C^{(\mathrm{KIV})}_{\lambda\mu\nu}(t,q)$ coincides with the matrix element $\langle S_\mu(q,t)|\,\Phi^*_\nu\,|s_{\lambda'}\rangle$ up to a simple factor (proposition 4.4). If we use the bases (ιP_λ) and (ιQ_μ), then the refined topological vertex $C_{\lambda\mu}{}^{\nu}$ arises as the matrix element $\langle \iota P_\nu|\,\Phi^*_\lambda\,|\iota Q_\mu\rangle$ (proposition 4.7).

We cheek that any types of the compositions of the intertwining operators Φ and Φ^* produces contractions of topological vertices involving correct gluing factors (see definition 4.9). Thereby proving the equivalence of the topological vertex and our representation theory (theorem 4.10). Since the discovery of Alday, Gaiotto and Tachikawa [6], it has been intensively studied that we have the representation theory of the Virasoro and W algebras playing a profound role in the Nekrasov instanton partition function [5, 10, 17, 23]. As for the K-theoretic version, see [1, 7, 8]. Our quantum algebraic approach extends this idea in such a way that the topological A-model and the topological vertex are involved. We hope this will give us better understandings both in string theory side and in quantum integrable system side.

We remark that the intertwining operator proposed in [1] can be explicitly constructed by composing our Φ's and Φ^*'s in a suitable manner. However it still remains unclear how to describe the the structure of the 'integral basis' proposed there. Therefore we do not go in this direction here.

This paper is organized as follows. In section 2, we recall our notations for the algebra $\mathcal{U}$ and introduce the family of the Fock modules $\mathcal{F}^{(0,1)}_u$ and $\mathcal{F}^{(1,N)}_u$ ($N\in\mathbb{Z}$). In section 3, the trivalent intertwining operators Φ and Φ^* are defined (definitions 3.1 and 3.4). The existence and the uniqueness of them are stated in theorems 3.3 and 3.6. Section 4 is devoted to establishing the equivalence with topological vertex of Iqbal-Kozcaz-Vafa, and Awata-Kanno. For this purpose, we calculate the matrix elements of Φ and Φ^* (propositions 4.4 and 4.7). Then we check the gluing rules for all the possible compositions (propositions 4.11, 4.12, 4.13, 4.14 and 4.15). Our main theorem is stated in theorem 4.10. In section 5, we present two examples of calculations which involve Nekrasov partition functions, and investigate the meaning of the spectral parameters. Proofs of theorems 3.3 and 3.6 are given in section 6. A proof of proposition 5.1 is stated in section 7.

2 Preliminaries

2.1 Algebra $\mathcal{U}$

Let q and t be independent indeterminates, and set $\mathbb{F}=\mathbb{Q}(q,t)$. Set also $\widetilde{\mathbb{F}}=\mathbb{Q}(q^{1/4},t^{1/4})$. We sometimes work over the field $\widetilde{\mathbb{F}}$ to keep the notation considerably symmetric for our dual constructions. We briefly recall our notation for the algebra $\mathcal{U}$. We follow the notation in [13] which is based on [9]. Let

$$g(z)=\frac{G^+(z)}{G^-(z)}\in\mathbb{Q}(q,t)[[z]],\qquad G^\pm(z)=(1-q^{\pm1}z)(1-t^{\mp1}z)(1-q^{\mp1}t^{\pm1}z).\qquad(2.1)$$

Definition 2.1. Let $\mathcal{U}$ be a unital associative algebra over $\mathbb{F}$ generated by the Drinfeld currents $x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x^{\pm}_n z^{-n}, \psi^{\pm}(z) = \sum_{\pm n\in\mathbb{N}} \psi^{\pm}_n z^{-n}$, and the central element $\gamma^{\pm 1/2}$, satisfying the defining relations

$$\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \qquad \psi^{+}(z)\psi^{-}(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^{-}(w)\psi^{+}(z), \tag{2.2}$$

$$\psi^{+}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^{+}(z), \tag{2.3}$$

$$\psi^{-}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z), \tag{2.4}$$

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\Big(\delta(\gamma^{-1}z/w)\psi^{+}(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^{-}(\gamma^{-1/2}w)\Big), \tag{2.5}$$

$$G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w) = G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z), \tag{2.6}$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Proposition 2.2. The algebra $\mathcal{U}$ has a Hopf algebra structure defined by the coproduct Δ:

$$\begin{aligned}
&\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}\otimes\gamma^{\pm 1/2},\\
&\Delta(x^{+}(z)) = x^{+}(z)\otimes 1 + \psi^{-}(\gamma_{(1)}^{1/2}z)\otimes x^{+}(\gamma_{(1)}z),\\
&\Delta(x^{-}(z)) = x^{-}(\gamma_{(2)}z)\otimes\psi^{+}(\gamma_{(2)}^{1/2}z) + 1\otimes x^{-}(z),\\
&\Delta(\psi^{\pm}(z)) = \psi^{\pm}(\gamma_{(2)}^{\pm 1/2}z)\otimes\psi^{\pm}(\gamma_{(1)}^{\mp 1/2}z),
\end{aligned}$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2}\otimes 1$ and $\gamma_{(2)}^{\pm 1/2} = 1\otimes\gamma^{\pm 1/2}$. We omit the counit ε and the antipode a since we do not need them here.

Remark 2.3. The $\psi_0^{\pm}$ are central elements in $\mathcal{U}$.

Definition 2.4. Let M be a left $\mathcal{U}$-module over $\widetilde{\mathbb{F}}$. If we have

$$\gamma^{1/2}\alpha = (t/q)^{l_1/4}\alpha, \qquad (\psi_0^{+})^{-1}\psi_0^{-}\alpha = (t/q)^{l_2}\alpha \tag{2.7}$$

for any $\alpha\in M$ and for some fixed $l_1, l_2\in\mathbb{Z}$, we call M of level (l_1, l_2).

2.2 Macdonald symmetric functions and Fock space $\mathcal{F}$

We basically follow [21] for the notations. A partition λ is a series of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ with finitely many nonzero entries. We use the following symbols: $|\lambda| := \sum_{i\geq 1}\lambda_i$, $n(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$. If $\lambda_l > 0$ and $\lambda_{l+1} = 0$, we write $\ell(\lambda) := l$ and call it the length of λ. The conjugate partition of λ is denoted by λ' which corresponds to the transpose of the diagram λ. The empty sequence is denoted by $\emptyset$. The dominance ordering is defined by $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\sum_{k=1}^{i}\lambda_k \geq \sum_{k=1}^{i}\mu_k$ for all $i = 1, 2, \ldots$.

We also follow [21] for the convention of the Young diagram. Namely, the first coordinate i (the row index) increases as one goes downwards, and the second coordinate j (the column index) increases as one goes rightwards. We denote by $\square = (i,j)$ the box located at the coordinate (i,j). For a box $\square = (i,j)$ and a partition λ, we use the following notations:

$$i(\square) := i, \quad j(\square) := j, \quad a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda'_j - i.$$

Let Λ be the ring of symmetric functions in $x = (x_1, x_2, \ldots)$ over $\mathbb{Z}$, and let $\Lambda_{\mathbb{Q}(q,t)} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}(q,t)$. Let m_λ be the monomial symmetric functions. Denote the power sum function by $p_n = \sum_{i\geq 1} x_i^n$. For a partition λ, we write $p_\lambda = \prod_i p_{\lambda_i}$. Macdonald's scalar product on $\Lambda_{\mathbb{F}}$ is

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \qquad z_\lambda = \prod_{i\geq 1} i^{m_i} \cdot m_i!, \tag{2.8}$$

Here we denote by m_i the number of entries in λ equal to i.

Fact 2.5. The Macdonald symmetric function $P_\lambda(x;q,t)$ is uniquely characterized by the conditions [21, chapter VI, (4.7)].

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu \qquad (u_{\lambda\mu} \in \mathbb{Q}(q,t)),$$
$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \qquad (\lambda \neq \mu).$$

Denote $Q_\lambda := P_\lambda / \langle P_\lambda, P_\lambda \rangle_{q,t}$. Then (Q_λ) and (P_λ) are dual bases of $\Lambda_{\mathbb{F}}$. We have $\langle P_\lambda, P_\lambda \rangle_{q,t} = c'_\lambda / c_\lambda$ where

$$c_\lambda := \prod_{\square\in\lambda} (1 - q^{a_\lambda(\square)} t^{\ell_\lambda(\square)+1}), \quad c'_\lambda := \prod_{\square\in\lambda} (1 - q^{a_\lambda(\square)+1} t^{\ell_\lambda(\square)}). \tag{2.9}$$

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are two infinite sets of independent indeterminates. The skew Macdonald polynomials $P_{\lambda/\mu}$ satisfy $P_\lambda(x,y) = \sum_\mu P_\mu(x) P_{\lambda/\mu}(y)$ [21, chapter VI, (7.9′)].

Let $\mathcal{H}$ be the Heisenberg algebra over $\mathbb{F}$ with generators $\{a_n \mid n \in \mathbb{Z}\}$ satisfying

$$[a_m, a_n] = m \frac{1-q^{|m|}}{1-t^{|m|}} \delta_{m+n,0}\, a_0.$$

Let $|0\rangle$ be the vacuum state satisfying the annihilation conditions for the positive Fourier modes $a_n|0\rangle = 0$ $(n \in \mathbb{Z}_{>0})$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we denote $|a_\lambda\rangle = a_{-\lambda_1} a_{-\lambda_2} \cdots |0\rangle$ for short. Denote by $\mathcal{F}$ the Fock space having the basis $(|a_\lambda\rangle)$.

As graded vector spaces, the space of the symmetric functions $\Lambda_{\mathbb{F}}$ and the Fock space $\mathcal{F}$ are isomorphic, and we may identify them:

$$\mathcal{F} \xrightarrow{\sim} \Lambda_{\mathbb{F}}, \quad |a_\lambda\rangle \mapsto p_\lambda. \tag{2.10}$$

We give an $\mathcal{H}$-module structure on $\Lambda_{\mathbb{F}}$ by setting $a_0 v = v$ and

$$a_{-n} v = p_n v, \quad a_n v = n \frac{1-q^n}{1-t^n} \frac{\partial v}{\partial p_n}, \qquad (n > 0, v \in \Lambda_{\mathbb{F}}).$$

Let $\langle 0|$ be the dual vacuum satisfying $\langle 0|\, a_n = 0$ $(n \in \mathbb{Z}_{<0})$, and $\langle a_\lambda| = \langle 0|\, a_{\lambda_1} a_{\lambda_2} \cdots$. The dual Fock space $\mathcal{F}^*$ has the basis $(\langle a_\lambda|)$. We identify symmetric functions with states in $\mathcal{F}$ (or $\mathcal{F}^*$) when it is convenient, and write $|P_\lambda\rangle$ (or $\langle P_\lambda|$) for P_λ for example. With this notation we have $\langle P_\lambda|\, \mathcal{O}\, |P_\mu\rangle = \langle P_\lambda, \mathcal{O} P_\mu \rangle_{q,t}$ for any $\mathcal{O} \in U(\mathcal{H})$.

2.3 Level $(0,1)$ module $\mathcal{F}_u^{(0,1)}$

Definition 2.6. Let $\lambda=(\lambda_1,\lambda_2,\ldots)$ be a partition, and $i\in\mathbb{Z}_{\geq 0}$. Set $A^{\pm}_{\lambda,i}\in\mathbb{Q}(q,t)$, $B^{\pm}_{\lambda}(z)\in\mathbb{Q}(q,t)[[z]]$ by

$$A^{+}_{\lambda,i}=(1-t)\prod_{j=1}^{i-1}\frac{(1-q^{\lambda_i-\lambda_j}t^{-i+j+1})(1-q^{\lambda_i-\lambda_j+1}t^{-i+j-1})}{(1-q^{\lambda_i-\lambda_j}t^{-i+j})(1-q^{\lambda_i-\lambda_j+1}t^{-i+j})}, \tag{2.11}$$

$$A^{-}_{\lambda,i}=(1-t^{-1})\frac{1-q^{\lambda_{i+1}-\lambda_i}}{1-q^{\lambda_{i+1}-\lambda_i+1}t^{-1}}\prod_{j=i+1}^{\infty}\frac{(1-q^{\lambda_j-\lambda_i+1}t^{-j+i-1})(1-q^{\lambda_{j+1}-\lambda_i}t^{-j+i})}{(1-q^{\lambda_{j+1}-\lambda_i+1}t^{-j+i-1})(1-q^{\lambda_j-\lambda_i}t^{-j+i})}, \tag{2.12}$$

$$B^{+}_{\lambda}(z)=\frac{1-q^{\lambda_1-1}tz}{1-q^{\lambda_1}z}\prod_{i=1}^{\infty}\frac{(1-q^{\lambda_i}t^{-i}z)(1-q^{\lambda_{i+1}-1}t^{-i+1}z)}{(1-q^{\lambda_{i+1}}t^{-i}z)(1-q^{\lambda_i-1}t^{-i+1}z)}, \tag{2.13}$$

$$B^{-}_{\lambda}(z)=\frac{1-q^{-\lambda_1+1}t^{-1}z}{1-q^{-\lambda_1}z}\prod_{i=1}^{\infty}\frac{(1-q^{-\lambda_i}t^{i}z)(1-q^{-\lambda_{i+1}+1}t^{i-1}z)}{(1-q^{-\lambda_{i+1}}t^{i}z)(1-q^{-\lambda_i+1}t^{i-1}z)}. \tag{2.14}$$

Note that if $\lambda_i=\lambda_{i-1}$ then $A^{+}_{\lambda,i}=0$, and if $\lambda_i=\lambda_{i+1}$ then $A^{-}_{\lambda,i}=0$. If $\lambda_i<\lambda_{i-1}$, we may obtain a new partition by adding one box to the i-th row, and we denote it by $\lambda+\mathbf{1}_i=(\lambda_1,\lambda_2,\ldots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},\ldots)$ for simplicity. If $\lambda_i>\lambda_{i+1}$, we may obtain a new partition by removing one box from the i-th row, and we write $\lambda-\mathbf{1}_i=(\lambda_1,\lambda_2,\ldots,\lambda_{i-1},\lambda_i-1,\lambda_{i+1},\ldots)$.

Proposition 2.7. Let u be an indeterminate. We can endow a left $\mathcal{U}$-module structure over $\widetilde{\mathbb{F}}$ to $\mathcal{F}$ by setting

$$\gamma^{1/2}P_\lambda=P_\lambda, \tag{2.15}$$

$$x^{+}(z)P_\lambda=\sum_{i=1}^{\ell(\lambda)+1}A^{+}_{\lambda,i}\,\delta(q^{\lambda_i}t^{-i+1}u/z)P_{\lambda+\mathbf{1}_i}, \tag{2.16}$$

$$x^{-}(z)P_\lambda=q^{1/2}t^{-1/2}\sum_{i=1}^{\ell(\lambda)}A^{-}_{\lambda,i}\,\delta(q^{\lambda_i-1}t^{-i+1}u/z)P_{\lambda-\mathbf{1}_i}, \tag{2.17}$$

$$\psi^{+}(z)P_\lambda=q^{1/2}t^{-1/2}B^{+}_{\lambda}(u/z)P_\lambda, \tag{2.18}$$

$$\psi^{-}(z)P_\lambda=q^{-1/2}t^{1/2}\,B^{-}_{\lambda}(z/u)P_\lambda. \tag{2.19}$$

This is a level $(0,1)$ module. We denote this $\mathcal{U}$-module by $\mathcal{F}_u^{(0,1)}$.

This was obtained in [11, 16].

2.4 Level $(1, N)$ module $\mathcal{F}_u^{(1,N)}$

Definition 2.8. Set

$$\eta(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^n\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^n}{n} a_n z^{-n}\Big), \tag{2.20}$$

$$\xi(z) = \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} q^{-n/2} t^{n/2} a_{-n} z^n\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^n}{n} q^{-n/2} t^{n/2} a_n z^{-n}\Big), \tag{2.21}$$

$$\varphi^+(z) = \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^n}{n} (1-t^n q^{-n}) q^{n/4} t^{-n/4} a_n z^{-n}\Big), \tag{2.22}$$

$$\varphi^-(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^n q^{-n}) q^{n/4} t^{-n/4} a_{-n} z^n\Big). \tag{2.23}$$

Proposition 2.9. Let u be an indeterminate, and $N \in \mathbb{Z}$. We can endow a left $\mathcal{U}$-module structure over $\widetilde{\mathbb{F}}$ to $\mathcal{F}$ by setting

$$\gamma^{1/2} P_\lambda = (t/q)^{1/4} P_\lambda, \tag{2.24}$$

$$x^+(z) P_\lambda = u z^{-N} q^{-N/2} t^{N/2} \eta(z) P_\lambda, \tag{2.25}$$

$$x^-(z) P_\lambda = u^{-1} z^N q^{N/2} t^{-N/2} \xi(z) P_\lambda, \tag{2.26}$$

$$\psi^+(z) P_\lambda = q^{N/2} t^{-N/2} \varphi^+(z) P_\lambda, \tag{2.27}$$

$$\psi^-(z) P_\lambda = q^{-N/2} t^{N/2} \varphi^-(z) P_\lambda. \tag{2.28}$$

This is a level $(1, N)$ module. We denote this $\mathcal{U}$-module by $\mathcal{F}_u^{(1,N)}$.

This is obtained as an easy modification of the representation constructed in [13].

3 Trivalent intertwining operators Φ and Φ^*

3.1 Intertwining operator Φ

Let $N \in \mathbb{Z}$ and u, v, w be independent indeterminates.

Definition 3.1. Let $\Phi = \Phi\left[\begin{smallmatrix} (1,N+1),w \\ (0,1),v\,;\,(1,N),u \end{smallmatrix}\right]$ be the trivalent intertwining operator satisfying the conditions

$$\Phi : \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,N)} \longrightarrow \mathcal{F}_w^{(1,N+1)}, \tag{3.1}$$

$$a\Phi = \Phi\Delta(a) \qquad (\forall a \in \mathcal{U}). \tag{3.2}$$

Introduce the components Φ_λ by setting

$$\Phi_\lambda(\alpha) = \Phi(P_\lambda \otimes \alpha) \qquad (\forall P_\lambda \otimes \alpha \in \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,N)}). \tag{3.3}$$

We normalize Φ by requiring $\Phi_\emptyset(1) = 1 + \cdots$.

Lemma 3.2. The intertwining relations (3.2) read

$$\sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}\,\delta(q^{\lambda_i}t^{-i+1}v/z)\Phi_{\lambda+\mathbf{1}_i} + q^{-1/2}t^{1/2}B^-_\lambda(z/v)\Phi_\lambda x^+(z) = x^+(z)\Phi_\lambda, \tag{3.4}$$

$$q^{1/2}t^{-1/2}\sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}\delta(q^{\lambda_i-1}t^{-i+1}v/z)\Phi_{\lambda-\mathbf{1}_i}\psi^+(q^{1/4}t^{-1/4}z)+$$
$$\Phi_\lambda x^-(q^{1/2}t^{-1/2}z) = x^-(q^{1/2}t^{-1/2}z)\Phi_\lambda, \tag{3.5}$$

$$q^{1/2}t^{-1/2}B^+_\lambda(v/z)\Phi_\lambda\psi^+(q^{1/4}t^{-1/4}z) = \psi^+(q^{1/4}t^{-1/4}z)\Phi_\lambda, \tag{3.6}$$

$$q^{-1/2}t^{1/2}B^-_\lambda(z/v)\Phi_\lambda\psi^-(q^{-1/4}t^{1/4}z) = \psi^-(q^{-1/4}t^{1/4}z)\Phi_\lambda. \tag{3.7}$$

Theorem 3.3. The normalized intertwining operator Φ exists only when $w = -vu$. In this case, it is determined uniquely and written in terms of the Heisenberg generators as

$$\Phi_\lambda\begin{bmatrix} (1,N+1),-vu \\ (0,1),v\,;\,(1,N),u\end{bmatrix} = t(\lambda,u,v,N)\widetilde{\Phi}_\lambda(v), \tag{3.8}$$

$$t(\lambda,u,v,N) = (-vu)^{|\lambda|}(-v)^{-(N+1)|\lambda|}f_\lambda^{-N-1}. \tag{3.9}$$

Here we have used the notations

$$\widetilde{\Phi}_\lambda(v) = \frac{q^{n(\lambda')}}{c_\lambda} :\Phi_\emptyset(v)\eta_\lambda(v):, \tag{3.10}$$

$$\widetilde{\Phi}_\emptyset(v) = \exp\Big(-\sum_{n=1}^\infty \frac{1}{n}\frac{1}{1-q^n}a_{-n}v^n\Big)\exp\Big(-\sum_{n=1}^\infty \frac{1}{n}\frac{q^n}{1-q^n}a_n v^{-n}\Big), \tag{3.11}$$

$$\eta_\lambda(v) =: \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\eta(q^{j-1}t^{-i+1}v):, \tag{3.12}$$

where the symbol $:\cdots:$ denotes the usual normal ordering, and f_λ is Taki's framing factor (1.5).

A proof of this will be given in section 6.

3.2 Intertwining operator Φ^*

Definition 3.4. Let $\Phi^* = \Phi^*\left[\begin{smallmatrix}(1,N),v\,;\,(0,1),u \\ (1,N+1),w\end{smallmatrix}\right]$ be the trivalent intertwining operator satisfying the conditions

$$\Phi^* : \mathcal{F}^{(1,N+1)}_w \longrightarrow \mathcal{F}^{(1,N)}_v \otimes \mathcal{F}^{(0,1)}_u, \tag{3.13}$$

$$\Delta(a)\Phi^* = \Phi^* a \qquad (\forall a \in \mathcal{U}). \tag{3.14}$$

Introduce the components Φ_λ by setting

$$\Phi^*(\alpha) = \sum_\lambda \Phi^*_\lambda(\alpha)\otimes Q_\lambda \qquad (\forall\alpha\in\mathcal{F}^{(1,N+1)}_w). \tag{3.15}$$

We normalize Φ^* by requiring $\Phi^*_\emptyset(1) = 1+\cdots$.

Lemma 3.5. The intertwining relations (3.14) read

$$\Phi_\lambda^* x^+(q^{1/2}t^{-1/2}z) = x^+(q^{1/2}t^{-1/2}z)\Phi_\lambda^* - \psi^-(q^{1/4}t^{-1/4}z)\sum_{i=1}^{\ell(\lambda)} qA_{\lambda,i}^-\,\delta(q^{\lambda_i-1}t^{-i+1}u/z)\Phi^*_{\lambda-\mathbf{1}_i}, \tag{3.16}$$

$$\Phi_\lambda^* x^-(z) = q^{1/2}t^{-1/2}B_\lambda^+(u/z)x^-(z)\Phi_\lambda^* - q^{1/2}t^{-1/2}\sum_{i=1}^{\ell(\lambda)+1} q^{-1}A_{\lambda,i}^+\,\delta(q^{\lambda_i}t^{-i+1}u/z)\Phi^*_{\lambda+\mathbf{1}_i}, \tag{3.17}$$

$$\Phi_\lambda^*\psi^+(q^{-1/4}t^{1/4}z) = q^{1/2}t^{-1/2}B_\lambda^+(u/z)\psi^+(q^{-1/4}t^{1/4}z)\Phi_\lambda^*, \tag{3.18}$$

$$\Phi_\lambda^*\psi^-(q^{1/4}t^{-1/4}z) = q^{-1/2}t^{1/2}B_\lambda^-(z/u)\psi^-(q^{1/4}t^{-1/4}z)\Phi_\lambda^*. \tag{3.19}$$

Theorem 3.6. The intertwining operator Φ^* exists uniquely only when $w=-vu$. In this case, it is written in terms of the Heisenberg generators as

$$\Phi_\lambda^*\begin{bmatrix}(1,N),v\,;\,(0,1),u\\ (1,N+1),-vu\end{bmatrix} = t^*(\lambda,u,v,N)\widetilde{\Phi}_\lambda^*(u), \tag{3.20}$$

$$t^*(\lambda,u,v,N) = (q^{-1}v)^{-|\lambda|}(-u)^{N|\lambda|}f_\lambda^N. \tag{3.21}$$

Here f_λ is given in (1.5), and

$$\widetilde{\Phi}_\lambda^*(u) = \frac{q^{n(\lambda')}}{c_\lambda} :\widetilde{\Phi}_\emptyset^*(u)\xi_\lambda(u):, \tag{3.22}$$

$$\widetilde{\Phi}_\emptyset^*(u) = \exp\Big(\sum_{n=1}^\infty \frac{1}{n}\frac{1}{1-q^n}q^{-n/2}t^{n/2}a_{-n}u^n\Big)\exp\Big(\sum_{n=1}^\infty \frac{1}{n}\frac{q^n}{1-q^n}q^{-n/2}t^{n/2}a_n u^{-n}\Big), \tag{3.23}$$

$$\xi_\lambda(u) =: \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\xi(q^{j-1}t^{-i+1}u):. \tag{3.24}$$

In section 6, we give a proof of this.

4 Identification with refined topological vertex

4.1 Notations

Let $s_\lambda(x)\in\Lambda_{\mathbb{Z}}$ be the Schur function, and $c_{\lambda\mu}^\nu$ be the Littlewood-Richardson coefficient determined by $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$. The skew Schur function is defined by $s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu$, and we have $s_\lambda(x,y) = \sum_\mu s_{\lambda/\mu}(x)s_\mu(y)$ [21, chapter I. (5.9)]. Let $S_\lambda(x;q,t)\in\Lambda_{\mathbb{F}}$ be the dual of s_λ with respect to the scalar product (2.8), namely $\langle S_\lambda(q,t), s_\mu\rangle_{q,t} = \delta_{\lambda,\mu}$. Set $S_{\lambda/\mu}(x;q,t) = \sum_\nu c_{\mu\nu}^\lambda S_\nu(x;q,t)$. We have $S_\lambda(x,y;q,t) = \sum_\mu S_{\lambda/\mu}(x;q,t)S_\mu(y;q,t)$.

Recall the $\mathbb{F}$-algebra endomorphism $\omega_{u,v}$ of Macdonald [21, chapter VI. (2.14)].

$$\omega_{u,v}(p_n) = -(-1)^n\frac{1-u^n}{1-v^n}p_n. \tag{4.1}$$

It is convenient to have two operations ι and $\varepsilon_\lambda^\pm$ acting on $\Lambda_{\mathbb{F}}$ introduced in [3]. The ι is defined to be the involution on $\Lambda_{\mathbb{F}}$ given by

$$\iota : \Lambda_{\mathbb{F}} \to \Lambda_{\mathbb{F}}, \qquad \iota(p_n) = -p_n \quad (n \in \mathbb{Z}_{>0}). \tag{4.2}$$

The $\varepsilon_\lambda^\pm = \varepsilon_{\lambda,q,t}^\pm$ is defined to be the algebra homomorphism

$$\varepsilon_\lambda^\pm : \Lambda_{\mathbb{F}} \to \mathbb{F}, \qquad \varepsilon_\lambda^\pm(p_n) = \sum_{i=1}^\infty (q^{\pm\lambda_i n} - 1)t^{\mp(i-1/2)n} + \frac{t^{\mp n/2}}{1-t^{\mp n}}, \tag{4.3}$$

For any symmetric functions, we shall use the shorthand notations such as

$$\varepsilon_\lambda^\pm(s_\mu) = s_\mu(q^{\pm\lambda}t^{\pm\rho}), \qquad \varepsilon_\lambda^\pm(\iota s_\mu) = \iota s_\mu(q^{\pm\lambda}t^{\pm\rho}), \tag{4.4}$$

since we may have the interpretation $\rho = (-1/2, -3/2, -5/2, \ldots)$ in mind.

We have

$$\iota s_\lambda(x) = s_{\lambda'}(-x) = (-1)^{|\lambda|} s_{\lambda'}(x), \tag{4.5}$$
$$S_\lambda(x;q,t) = \iota\, \omega_{t,q} s_\lambda(-x), \tag{4.6}$$

and

$$\varepsilon_{\lambda,q,t}^+(p_n(x)) = \varepsilon_{\lambda',t,q}^-\, \omega_{q,t}(p_n(-q^{-1/2}t^{1/2}x)). \tag{4.7}$$

Hence

$$\varepsilon_{\lambda,q,t}^+ \iota S_\mu(x;q,t) = (q^{1/2}t^{-1/2})^{-|\mu|} \varepsilon_{\lambda',t,q}^-\, s_\mu(x). \tag{4.8}$$

In the shorthand notation this is written as $\iota S_\mu(q^\lambda t^\rho; q,t) = (q^{1/2}t^{-1/2})^{-|\mu|} s_\mu(t^{-\lambda'}q^{-\rho})$.

4.2 Matrix elements of Φ and Φ^*

We have simple but important formulas which essentially control the property of our intertwining operators.

Proposition 4.1. We have

$$\begin{aligned} &: \Phi_\emptyset(q^{1/2}v)\eta_\lambda(q^{1/2}v) : && (4.9)\\ &= \exp\!\left(\sum_{n=1}^\infty \frac{1}{n}\frac{1-t^n}{1-q^n} a_{-n}(q^{1/2}t^{-1/2})^n v^n \varepsilon_\lambda^+(p_n)\right) \exp\!\left(-\sum_{n=1}^\infty \frac{1}{n}\frac{1-t^n}{1-q^n} a_n(q^{1/2}t^{-1/2})^n v^{-n} \varepsilon_\lambda^-(p_n)\right), \\ &: \Phi_\emptyset^*(q^{1/2}u)\xi_\lambda(q^{1/2}u) : && (4.10)\\ &= \exp\!\left(-\sum_{n=1}^\infty \frac{1}{n}\frac{1-t^n}{1-q^n} a_{-n}u^n \varepsilon_\lambda^+(p_n)\right) \exp\!\left(\sum_{n=1}^\infty \frac{1}{n}\frac{1-t^n}{1-q^n} a_n u^{-n} \varepsilon_\lambda^-(p_n)\right). \end{aligned}$$

Corollary 4.2. We have

$$\begin{aligned}
&\langle S_\nu(q,t)| : \widetilde{\Phi}_\emptyset(q^{1/2}v)\eta_\lambda(q^{1/2}v) : |s_\mu\rangle \qquad (4.11)\\
&= v^{|\nu|-|\mu|}(q^{1/2}t^{-1/2})^{|\nu|+|\mu|}\sum_\sigma S_{\nu/\sigma}(q^\lambda t^\rho;q,t)\iota s_{\mu/\sigma}(q^{-\lambda}t^{-\rho})(q^{1/2}t^{-1/2})^{-2|\sigma|},\\
&\langle S_\nu(q,t)| : \widetilde{\Phi}^*_\emptyset(q^{1/2}u)\xi_\lambda(q^{1/2}u) : |s_\mu\rangle \qquad (4.12)\\
&= u^{|\nu|-|\mu|}\sum_\sigma \iota S_{\nu/\sigma}(q^\lambda t^\rho;q,t)s_{\mu/\sigma}(q^{-\lambda}t^{-\rho}),
\end{aligned}$$

and

$$\begin{aligned}
&\langle P_\nu| : \widetilde{\Phi}_\emptyset(q^{1/2}v)\eta_\lambda(q^{1/2}v) : |P_\mu\rangle \qquad (4.13)\\
&= v^{|\nu|-|\mu|}(q^{1/2}t^{-1/2})^{|\nu|+|\mu|}\sum_\sigma P_{\nu/\sigma}(q^\lambda t^\rho)\iota P_{\mu/\sigma}(q^{-\lambda}t^{-\rho})\langle P_\sigma,P_\sigma\rangle_{q,t}(q^{1/2}t^{-1/2})^{-2|\sigma|},\\
&\langle P_\nu| : \widetilde{\Phi}^*_\emptyset(q^{1/2}u)\xi_\lambda(q^{1/2}u) : |P_\mu\rangle \qquad (4.14)\\
&= u^{|\nu|-|\mu|}\sum_\sigma \iota P_{\nu/\sigma}(q^\lambda t^\rho)P_{\mu/\sigma}(q^{-\lambda}t^{-\rho})\langle P_\sigma,P_\sigma\rangle_{q,t}.
\end{aligned}$$

Proof. From proposition 4.1 we have

$$\begin{aligned}
&\langle 0|\prod_i a_{\nu_i} : \widetilde{\Phi}_\emptyset(q^{1/2}v)\eta_\lambda(q^{1/2}v) : \prod_j a_{-\mu_j}|0\rangle\\
&\quad = \langle 0|\prod_i(a_{\nu_i}+(q^{1/2}t^{-1/2}v)^{\nu_i}\varepsilon^+_\lambda(p_{\nu_i}))\cdot\prod_j(a_{-\mu_j}-(q^{1/2}t^{-1/2}v^{-1})^{\mu_j}\varepsilon^-_\lambda(p_{\mu_j}))|0\rangle,\\
&\langle 0|\prod_i a_{\nu_i} : \widetilde{\Phi}^*_\emptyset(q^{1/2}u)\xi_\lambda(q^{1/2}u) : \prod_j a_{-\mu_j}|0\rangle\\
&\quad = \langle 0|\prod_i(a_{\nu_i}-u^{\nu_i}\varepsilon^+_\lambda(p_{\nu_i}))\cdot\prod_j(a_{-\mu_j}+u^{-\mu_j}\varepsilon^-_\lambda(p_{\mu_j}))|0\rangle.
\end{aligned}$$

Then (4.11), (4.12), (4.13) and (4.14) follow from the property of the skew functions. □

4.3 Topological vertex of Iqbal-Kozcaz-Vafa

Definition 4.3 (Iqbal-Kozcaz-Vafa)**.** The refined topological vertex is defined by

$$C^{(\mathrm{IKV})}_{\lambda\mu\nu}(t,q) = \left(\frac{q}{t}\right)^{\frac{||\mu||^2}{2}} t^{\frac{\kappa(\mu)}{2}} q^{\frac{||\nu||^2}{2}} \frac{1}{c_\lambda}\sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda'/\eta}(t^{-\rho}q^{-\nu})s_{\mu/\eta}(t^{-\nu'}q^{-\rho}), \qquad (4.15)$$

where c_λ is defined in (2.9), $||\lambda||^2=\sum_i\lambda_i^2$, $\kappa(\lambda)=\sum_i\lambda_i(\lambda_i+1-2i)$.

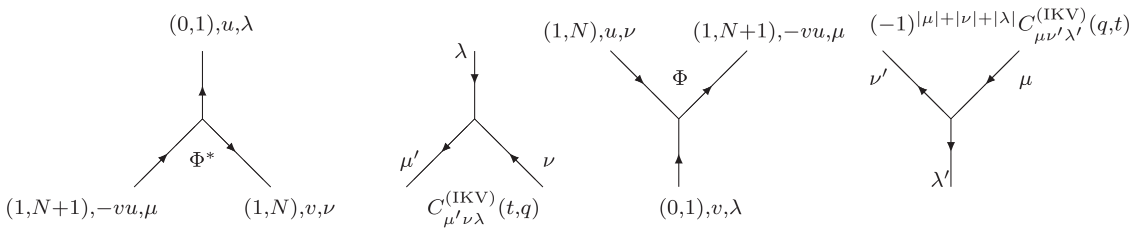

Figure 1. Comparison between Φ, Φ^* and $(-1)^{|\mu|+|\nu|+|\lambda|}C^{(\mathrm{IKV})}_{\mu\nu'\lambda'}(q,t), C^{(\mathrm{IKV})}_{\mu'\nu\lambda}(t,q)$.

Proposition 4.4. The matrix elements of the intertwining operators Φ and Φ^* are written in terms of the refined topological vertex as

$$\frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \langle S_\mu(q,t)|\, \Phi_\lambda \begin{bmatrix} (1,N+1),-vu \\ (0,1),v;(1,N),u \end{bmatrix} |s_\nu\rangle \tag{4.16}$$

$$= \left(\frac{q^{-1/2}u}{(-v)^N}\right)^{|\lambda|} f_\lambda^{-N} \cdot \ (-q^{-1/2}v)^{-|\nu|} f_\nu \cdot (t^{-1/2}v)^{|\mu|} \cdot (-1)^{|\mu|+|\nu|+|\lambda|}\ C^{(\mathrm{IKV})}_{\mu\nu'\lambda'}(q,t),$$

$$\langle S_\nu(q,t)|\, \Phi^*_\lambda \begin{bmatrix} (1,N),v;(0,1),u \\ (1,N+1),-vu \end{bmatrix} |s_\mu\rangle \tag{4.17}$$

$$= \left(\frac{(-u)^N}{q^{-1/2}v}\right)^{|\lambda|} f_\lambda^N \cdot \ (-q^{-1/2}u)^{|\nu|} f_\nu^{-1} \cdot (t^{-1/2}u)^{-|\mu|} \cdot \ C^{(\mathrm{IKV})}_{\mu'\nu\lambda}(t,q).$$

Proof. Using corollary 4.2 and (4.8) we have the results. □

Remark 4.5. Note that in the formulation of Iqbal-Kozcaz-Vafa, the transpose of the partition is assigned to each outgoing edge. To identify the refined topological vertices with vertices for Φ, Φ^*, all the arrows should be reversed as shown in figure 1.

4.4 Topological vertex of Awata-Kanno

Definition 4.6 (Awata-Kanno)**.** The refined topological vertices $C_{\mu\lambda}{}^{\nu}(q,t), C^{\mu\lambda}{}_{\nu}(q,t)$ are defined by

$$C_{\mu\lambda}{}^{\nu}(q,t) = P_\lambda(t^\rho;q,t)\sum_\sigma \iota P_{\mu'/\sigma'}(-t^{\lambda'}q^\rho;t,q)P_{\nu/\sigma}(q^\lambda t^\rho;q,t)(q^{1/2}/t^{1/2})^{|\sigma|-|\nu|} f_\nu(q,t)^{-1}, \tag{4.18}$$

$$C^{\mu\lambda}{}_{\nu}(q,t) = (-1)^{|\lambda|+|\mu|+|\nu|} C_{\mu'\lambda'}{}^{\nu'}(t,q) \tag{4.19}$$

where f_λ being defined in (1.5).

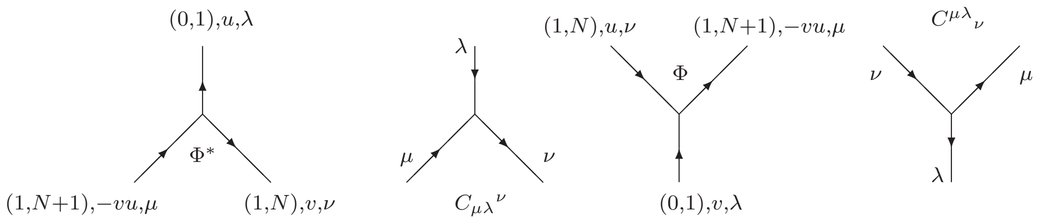

Figure 2. Comparison between Φ, Φ^* and $C^{\mu\lambda}{}_{\nu}, C_{\mu\lambda}{}^{\nu}$.

Proposition 4.7. The matrix elements of the intertwining operators Φ and Φ^* are written in terms of the refined topological vertices as

$$\begin{aligned} &\frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \langle \iota P_\mu | \Phi_\lambda \begin{bmatrix} (1,N+1), -vu \\ (0,1), v; (1,N), u \end{bmatrix} | \iota Q_\nu \rangle \\ &\quad = \left(\frac{-t^{1/2}u}{q(-v)^N} \right)^{|\lambda|} f_\lambda^{-N} \ (t^{-1/2}v)^{|\mu|-|\nu|} f_\nu^{-1} \ C^{\mu\lambda}{}_{\nu}(q,t), \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\langle \iota P_\nu | \Phi^*_\lambda \begin{bmatrix} (1,N), v; (0,1), u \\ (1,N+1), -vu \end{bmatrix} | \iota Q_\mu \rangle \\ &\quad = \left(\frac{q(-u)^N}{-t^{1/2}v} \right)^{|\lambda|} f_\lambda^{N} \ (t^{-1/2}u)^{-|\mu|+|\nu|} f_\nu \ C_{\mu\lambda}{}^{\nu}(q,t). \end{aligned} \tag{4.21}$$

Proof. Note that we have

$$\begin{aligned} C_{\mu\lambda}{}^{\nu}(q,t) = &\frac{(-1)^{|\lambda|} q^{n(\lambda')} t^{|\lambda|/2}}{c_\lambda} (q^{1/2}t^{-1/2})^{|\mu|-|\nu|} f_\nu^{-1} \frac{1}{\langle P_\mu, P_\mu \rangle_{q,t}} \\ &\times \sum_\sigma P_{\nu/\sigma}(q^\lambda t^\rho) \iota P_{\mu/\sigma}(q^{-\lambda} t^{-\rho}) \langle P_\sigma, P_\sigma\rangle_{q,t}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} C^{\mu\lambda}{}_{\nu}(q,t) = &\frac{q^{|\lambda|/2} t^{n(\lambda)}}{c'_\lambda} (q^{1/2}t^{-1/2})^{2|\nu|} f_\nu \frac{1}{\langle P_\nu, P_\nu \rangle_{q,t}} \\ &\times \sum_\sigma \iota P_{\mu/\sigma}(q^\lambda t^\rho) P_{\nu/\sigma}(q^{-\lambda} t^{-\rho}) \langle P_\sigma, P_\sigma\rangle_{q,t} (q^{1/2}t^{-1/2})^{-2|\sigma|}. \end{aligned} \tag{4.23}$$

Using corollary 4.2 we have the results. □

Remark 4.8. We note that all the vertical arrows should be get reversed to establish a correspondence between Φ, Φ^* and $C^{\mu\lambda}{}_{\nu}, C_{\mu\lambda}{}^{\nu}$ as seen in figure 2.

4.5 Gluing rules

Consider a trivalent vertex with edges, say, i, j and k, with two component vectors $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k \in \mathbb{Z}^2$ attached respectively (see figure 3 (a)). Here we regard all the vectors being outgoing. We assume that they satisfy the (Calabi-Yau and smoothness) conditions

$$\mathbf{v}_i + \mathbf{v}_j + \mathbf{v}_k = \mathbf{0}, \qquad \mathbf{v}_i \wedge \mathbf{v}_j = 1, \tag{4.24}$$

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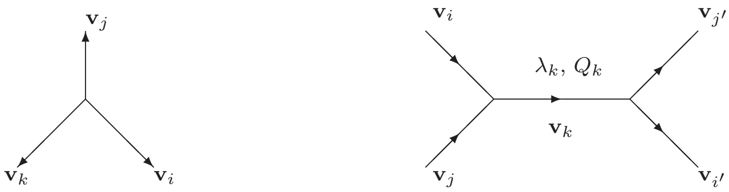

vertex

(a)

edge

(b)

Figure 3. Gluing rules.

where we have used the notation $(a,b)\wedge(c,d) = ad - bc$. Note that these mean $\mathbf{v}_j \wedge \mathbf{v}_k = \mathbf{v}_k \wedge \mathbf{v}_i = 1$.

Definition 4.9. Let $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_{i'}, \mathbf{v}_{j'} \in \mathbb{Z}^2$, and consider a graph as in figure 3 (b). Let λ_k be a partition, and Q_k be a parameter (Kähler parameter). To the internal edge, with the data $\mathbf{v}_k$, λ_k, Q_k attached, we associate the 'gluing factor'

$$Q_k^{|\lambda_k|}(f_{\lambda_k})^{\mathbf{v}_i \wedge \mathbf{v}_{i'}}. \tag{4.25}$$

The refined topological vertices are contracted by multiplying the gluing factor and making summation with respect to the repeated indices.

4.6 Check of gluing rules

Consider any intertwining operator of the $\mathcal{U}$ modules $\mathcal{F}_{u_1}^{\mathbf{v}_1} \otimes \cdots \otimes \mathcal{F}_{u_m}^{\mathbf{v}_m} \to \mathcal{F}_{u'_1}^{\mathbf{v}'_1} \otimes \cdots \otimes \mathcal{F}_{u'_n}^{\mathbf{v}'_n}$ obtained by composing the trivalent intertwining operators Φ and Φ^* in a certain way. The matrix elements can be evaluated by virtue of proposition 4.4 or proposition 4.7. Then we have a (not necessarily connected) graph with trivalent vertices, with the following structure associated:

1. a spectral parameter and a vector $\in \mathbb{Z}^2$ is attached to each edge,
2. the condition (4.24) is satisfied with respect to every vertex,
3. to eace vertex a refined topological vertex is associated (propositions 4.4, 4.7),
4. each internal edge gives a contraction of refined topological vertices.

Hence if it is shown that the correct gluing factor (4.25) appears to every internal edge in the matrix element, our approach from the representation theory of the algebra $\mathcal{U}$ precisely reproduces the quantity derived from the refined topological vertex, up to a factor depending on the data attached to the external edges. One can show that this is the case by checking it for all the possible (local) processes stated in propositions 4.11, 4.12, 4.13, 4.14, 4.15 below. We demonstrate these for the case of Awata-Kanno construction, since our notation gets a little simpler. The calculations for the topological vertex of Iqbal-Kozcaz-Vafa goes exactly the same way, and we omit them.

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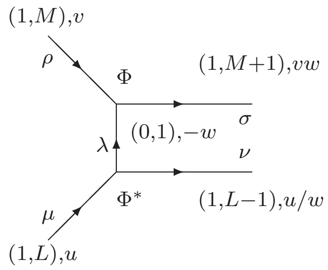

Figure 4. Case 1

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Theorem 4.10. Suppose we choose the preferred direction to be vertical $(0,1)$ in the web diagram. The matrix element of the composition of the trivalent intertwining operators Φ and Φ^*, and the corresponding quantity derived from the theory of the refined topological vertex of Iqbal-Kozcaz-Vafa or Awata-Kanno coincide, up to a factor depending on the data attached to external edges.

Proposition 4.11. The matrix element of the composition (see Fig 4)

$$\mathcal{F}_u^{(1,L)} \otimes \mathcal{F}_v^{(1,M)} \xrightarrow{\Phi^*\otimes \mathrm{id}} \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{-w}^{(0,1)} \otimes \mathcal{F}_v^{(1,M)} \xrightarrow{\mathrm{id}\otimes\Phi} \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{vw}^{(1,M+1)},$$

with respect to $\langle \iota P_\nu \otimes \iota P_\sigma|$ and $|\iota Q_\mu \otimes \iota Q_\rho\rangle$ is

$$(-t^{-1/2}w)^{-|\mu|+|\nu|-|\rho|+|\sigma|} f_\nu f_\rho^{-1} \sum_\lambda \left(w^{L-M}v/u\right)^{|\lambda|} f_\lambda^{L-M-1} C_{\mu\lambda}{}^{\nu} C^{\sigma\lambda}{}_{\rho}.$$

Recall we should reverse the vertical arrow, and apply the rule for calculating the gluing factor (4.25). We have $(1,M)\wedge(1,L-1)=L-M-1$, and thus the factor $\left(w^{L-M}v/u\right)^{|\lambda|} f_\lambda^{L-M-1}$ agrees with the gluing factor (4.25).

Proof. We have

$$\iota Q_\mu \otimes \iota Q_\rho \mapsto \sum_\lambda \Phi_\lambda^*(\iota Q_\mu) \otimes Q_\lambda \otimes \iota Q_\rho \mapsto \frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \sum_\lambda \Phi_\lambda^*(\iota Q_\mu) \otimes \Phi_\lambda(\iota Q_\rho).$$

Hence the matrix element is

$$\sum_\lambda \frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \langle \iota P_\nu | \, \Phi_\lambda^* \, | \iota Q_\mu\rangle \, \langle \iota P_\sigma | \, \Phi_\lambda \, | \iota Q_\rho\rangle \, .$$

□

Proposition 4.12. The matrix element of the composition (see figure 5)

$$\mathcal{F}_{-y}^{(0,1)} \otimes \mathcal{F}_u^{(1,L)} \xrightarrow{\mathrm{id}\otimes\Phi^*} \mathcal{F}_{-y}^{(0,1)} \otimes \mathcal{F}_{u/x}^{(1,L-1)} \otimes \mathcal{F}_{-x}^{(0,1)} \xrightarrow{\Phi\otimes\mathrm{id}} \mathcal{F}_{uy/x}^{(1,L)} \otimes \mathcal{F}_{-x}^{(0,1)},$$

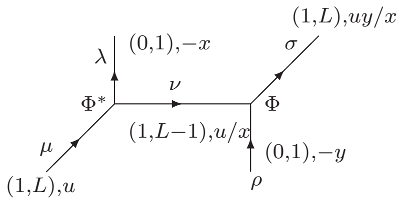

Figure 5. Case 2

with respect to $\langle \iota P_\sigma \otimes P_\lambda|$ and $|Q_\rho \otimes \iota Q_\mu\rangle$ is

$$\left(\frac{qx^{L-1}}{-t^{1/2}u/x}\right)^{|\lambda|} f_\lambda^{L-1} \left(\frac{-t^{1/2}u/x}{qy^{L-1}}\right)^{|\rho|} f_\rho^{-L+1} \\ \times (-t^{-1/2}x)^{-|\mu|}(-t^{-1/2}y)^{|\sigma|}\sum_\nu (x/y)^{|\nu|} C_{\mu\nu}{}^\lambda C^{\sigma\rho}{}_\nu. \tag{4.26}$$

Note that we have $(1,L)\wedge(1,L)=0$, and the factor $(x/y)^{|\nu|}$ agrees with (4.25).

Proof. We have

$$Q_\rho \otimes \iota Q_\mu \mapsto \sum_\nu Q_\rho \otimes \Phi_\nu^*(\iota Q_\mu)\otimes Q_\nu \mapsto \frac{1}{\langle P_\rho, P_\rho\rangle_{q,t}}\sum_\nu \Phi_\rho\Phi_\nu^*(\iota Q_\mu)\otimes Q_\nu.$$

Hence the matrix element is

$$\frac{1}{\langle P_\rho, P_\rho\rangle_{q,t}} \langle \iota P_\sigma|\,\Phi_\rho\Phi_\lambda^*\,|\iota Q_\mu\rangle = \sum_\nu \frac{1}{\langle P_\rho, P_\rho\rangle_{q,t}} \langle \iota P_\sigma|\,\Phi_\rho\,|\iota Q_\nu\rangle\,\langle \iota P_\nu|\,\Phi_\lambda^*\,|\iota Q_\mu\rangle\,.$$

□

Proposition 4.13. The matrix element of the composition (see figure 6)

$$\mathcal{F}_{-x}^{(0,1)} \otimes \mathcal{F}_v^{(1,M)} \xrightarrow{\Phi} \mathcal{F}_{vx}^{(1,M+1)} \xrightarrow{\Phi^*} \mathcal{F}_{vx/y}^{(1,M)} \otimes \mathcal{F}_{-y}^{(0,1)},$$

with respect to $\langle \iota P_\sigma \otimes P_\rho|$ and $|Q_\lambda \otimes \iota Q_\mu\rangle$ is

$$\left(\frac{-t^{1/2}v}{qx^M}\right)^{|\lambda|} f_\lambda^{-M} \left(\frac{qy^M}{-t^{1/2}vx/y}\right)^{|\rho|} f_\rho^{M} \\ \times (-t^{-1/2}x)^{-|\mu|}(-t^{-1/2}y)^{|\sigma|} f_\mu^{-1} f_\sigma \sum_\nu (x/y)^{|\nu|} C_{\nu\rho}{}^\sigma C^{\nu\lambda}{}_\mu. \tag{4.27}$$

Note that we have $(1,M)\wedge(1,M)=0$, and the factor $(x/y)^{|\nu|}$ agrees with (4.25).

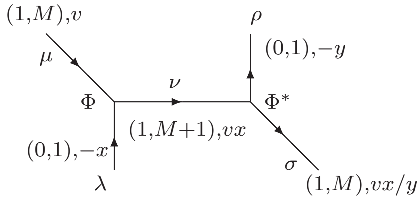

Figure 6. Case 3

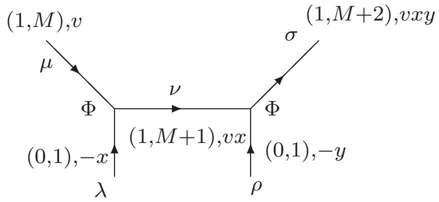

Figure 7. Case 4

Proof. We have

$$Q_\lambda \otimes \iota Q_\mu \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \Phi_\lambda(\iota Q_\mu) \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \sum_\nu \Phi^*_\nu \Phi_\lambda(\iota Q_\mu) \otimes Q_\nu .$$

Hence the matrix element is

$$\frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle \iota P_\sigma | \, \Phi^*_\rho \Phi_\lambda \, | \iota Q_\mu \rangle = \sum_\nu \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle \iota P_\sigma | \, \Phi^*_\rho \, | \iota Q_\nu \rangle \, \langle \iota P_\nu | \, \Phi_\lambda \, | \iota Q_\mu \rangle \, .$$

□

Proposition 4.14. The matrix element of the composition (see figure 7)

$$\mathcal{F}^{(0,1)}_{-y} \otimes \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,M)}_{v} \xrightarrow{\mathrm{id} \otimes \Phi} \mathcal{F}^{(0,1)}_{-y} \otimes \mathcal{F}^{(1,M+1)}_{vx} \xrightarrow{\Phi} \mathcal{F}^{(1,M+2)}_{vxy},$$

with respect to $\langle \iota P_\sigma |$ and $| Q_\rho \otimes Q_\lambda \otimes \iota Q_\mu \rangle$ is

$$\begin{aligned} &\left(\frac{-t^{1/2} v}{q x^M} \right)^{|\lambda|} f_\lambda^{-M} \left(\frac{-t^{1/2} v x}{q y^{M+1}} \right)^{|\rho|} f_\rho^{-M-1} \\ &\qquad\qquad \times (-t^{-1/2} x)^{-|\mu|} (-t^{-1/2} y)^{|\sigma|} f_\mu^{-1} \sum_\nu (x/y)^{|\nu|} f_\nu^{-1} C^{\sigma\rho}{}_{\nu} C^{\nu\lambda}{}_{\mu} . \end{aligned} \tag{4.28}$$

Note that we have $(1, M) \wedge (0, -1) = -1$, and the factor $(x/y)^{|\nu|} f_\nu^{-1}$ agrees with (4.25).

Proof. We have

$$Q_\rho \otimes Q_\lambda \otimes \iota Q_\mu \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} Q_\rho \otimes \Phi_\lambda(\iota Q_\mu) \mapsto \frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \frac{1}{\langle P_\rho, P_\rho \rangle_{q,t}} \Phi_\rho \Phi_\lambda(\iota Q_\mu).$$

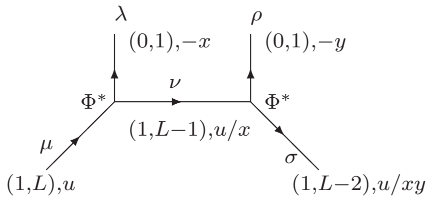

Figure 8. Case 5

Hence the matrix element is

$$\begin{aligned}&\frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}}\frac{1}{\langle P_\rho, P_\rho\rangle_{q,t}}\langle \iota P_\sigma|\,\Phi_\rho\Phi_\lambda\,|\iota Q_\mu\rangle\\&\quad=\sum_\nu \frac{1}{\langle P_\rho, P_\rho\rangle_{q,t}}\langle \iota P_\sigma|\,\Phi_\rho\,|\iota Q_\nu\rangle\frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}}\langle \iota P_\nu|\,\Phi_\lambda\,|\iota Q_\mu\rangle\,.\end{aligned}$$

$\square$

Proposition 4.15. The matrix element of the composition (see figure 8)

$$\mathcal{F}_u^{(1,L)}\xrightarrow{\Phi^*}\mathcal{F}_{u/x}^{(1,L-1)}\otimes\mathcal{F}_{-x}^{(0,1)}\xrightarrow{\Phi^*\otimes\mathrm{id}}\mathcal{F}_{u/xy}^{(1,L-2)}\otimes\mathcal{F}_{-y}^{(0,1)}\otimes\mathcal{F}_{-x}^{(0,1)},$$

with respect to $\langle\iota P_\sigma\otimes P_\rho\otimes P_\lambda|$ and $|\iota Q_\mu\rangle$ is

$$\begin{aligned}&\left(\frac{qx^{L-1}}{-t^{1/2}u/x}\right)^{|\lambda|}f_\lambda^{L-1}\left(\frac{qy^{L-2}}{-t^{1/2}u/xy}\right)^{|\rho|}f_\rho^{L-2}\\&\qquad\times(-t^{-1/2}x)^{-|\mu|}(-t^{-1/2}y)^{|\sigma|}f_\sigma\sum_\nu(x/y)^{|\nu|}f_\nu C_{\mu\lambda}{}^\nu C_{\nu\rho}{}^\sigma.\end{aligned}\tag{4.29}$$

Note that we have $(1,L)\wedge(0,1)=1$, and the factor $(x/y)^{|\nu|}f_\nu$ agrees with (4.25).

Proof. We have

$$\iota Q_\mu\mapsto\sum_\nu\Phi_\nu^*(\iota Q_\mu)\otimes Q_\nu\mapsto\sum_{\nu,\rho}\Phi_\rho^*\Phi_\nu^*(\iota Q_\mu)\otimes Q_\rho\otimes Q_\nu.$$

Hence the matrix element is

$$\langle\iota P_\sigma|\,\Phi_\rho^*\Phi_\lambda^*\,|\iota Q_\mu\rangle=\sum_\nu\langle\iota P_\sigma|\,\Phi_\rho^*\,|\iota Q_\nu\rangle\langle\iota P_\nu|\,\Phi_\lambda^*\,|\iota Q_\mu\rangle\,.$$

$\square$

5 Examples of compositions of intertwining operators

We have shown in theorem 4.10 that our construction based on the intertwining operators Φ,Φ^* derives the same result as the one from the theory of the refined topological vertex. Based on the findings in [3, 20, 33], it explains clearly the reason why the Nekrasov partition functions appear from matrix elements of intertwining operators of the algebra $\mathcal{U}$. In this section, we try to have an interpretation of the spectral parameters attached to our Fock modules by looking at two examples of the Nekrasov partition functions [15, 26].

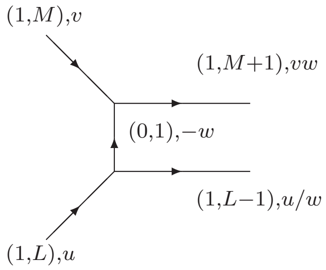

Figure 9. Four point operator.

5.1 Pure SU(N_c) partition function

Recall the formula of the instanton part of the (K-theoretic) partition function Z_m^{inst} of the pure SU(N_c) gauge theory on $\mathbb{R}^4 \times S^1$ with eight supercharges, associated with the m-th power $\mathcal{L}^{\otimes m}$ of the line bundle $\mathcal{L}$ over the instanton moduli space $M(N_c, k)$

$$Z_m^{\mathrm{inst}}(\mathbf{e}_1, \cdots, \mathbf{e}_{N_c}, \Lambda; q, t) = \sum_{\lambda^{(1)},\ldots,\lambda^{(N_c)}} \frac{\prod_{\alpha=1}^{N_c}((q^{1/2}t^{-1/2})^{-N_c}\Lambda^{2N_c}(-\mathbf{e}_\alpha)^{-m})^{|\lambda^{(\alpha)}|} f_{\lambda^{(\alpha)}}^{-m}}{\prod_{\alpha,\beta=1}^{N_c} N_{\lambda^{(\alpha)},\lambda^{(\beta)}}(\mathbf{e}_\alpha/\mathbf{e}_\beta)}, \tag{5.1}$$

where the notation

$$N_{\lambda,\mu}(u) = \prod_{(i,j)\in\lambda} (1 - uq^{-\mu_i+j-1}t^{-\lambda'_j+i}) \cdot \prod_{(k,l)\in\mu} (1 - uq^{\lambda_k-l}t^{\mu'_l-k+1}) \tag{5.2}$$

$$= \prod_{\square\in\lambda} (1 - uq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)}) \cdot \prod_{\blacksquare\in\mu} (1 - uq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}), \tag{5.3}$$

has been used. We demonstrate how Z_m^{inst} appears from our construction.

Let $L, M \in \mathbb{Z}$ and u, v, w be indeterminates. Consider the four point operator (see figure 9)

$$\Phi \begin{bmatrix} (1, L-1), u/w; (1, M+1), vw \\ (1, L), u; (1, M), v \end{bmatrix} : \mathcal{F}_u^{(1,L)} \otimes \mathcal{F}_v^{(1,M)} \longrightarrow \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{vw}^{(1,M+1)}, \tag{5.4}$$

defined by the composition of the trivalent intertwining operators (which we already considered in proposition 4)

$$\mathcal{F}_u^{(1,L)} \otimes \mathcal{F}_v^{(1,M)} \xrightarrow{\Phi^*\otimes\mathrm{id}} \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{-w}^{(0,1)} \otimes \mathcal{F}_v^{(1,M)} \xrightarrow{\mathrm{id}\otimes\Phi} \mathcal{F}_{u/w}^{(1,L-1)} \otimes \mathcal{F}_{vw}^{(1,M+1)}. \tag{5.5}$$

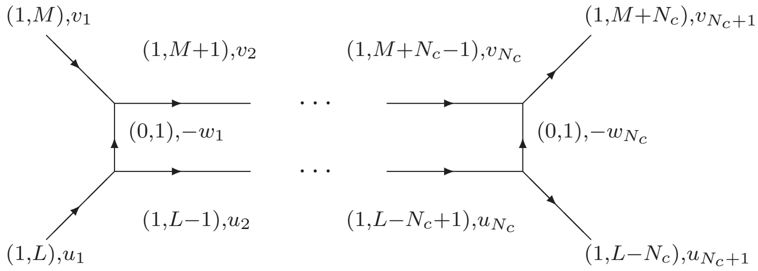

Figure 10. Web diagram for pure SU(N) gauge partition function.

For any $\alpha \otimes \beta \in \mathcal{F}_u^{(1,L)} \otimes \mathcal{F}_v^{(1,M)}$, we have

$$\begin{aligned}
&\Phi\begin{bmatrix}(1,L-1),u/w;(1,M+1),vw\\(1,L),u;(1,M),v\end{bmatrix}(\alpha\otimes\beta) \qquad (5.6)\\
&=\sum_\lambda \frac{1}{\langle P_\lambda,P_\lambda\rangle_{q,t}}\Phi^*_\lambda\begin{bmatrix}(1,L-1),u/w;(0,1),-w\\(1,L),u\end{bmatrix}(\alpha)\otimes\Phi_\lambda\begin{bmatrix}(1,M+1),vw\\(0,1),-w;(1,M),v\end{bmatrix}(\beta)\\
&=\sum_\lambda \frac{(q^{-1/2}t^{1/2}u^{-1}vw^{L-M})^{|\lambda|}f_\lambda^{L-M-1}}{N_{\lambda,\lambda}(1)}\Big(:\widetilde{\Phi}^*_\emptyset(-w)\xi_\lambda(-w):\alpha\Big)\otimes\Big(:\widetilde{\Phi}_\emptyset(-w)\eta_\lambda(-w):\beta\Big)
\end{aligned}$$

from theorems 3.3, 3.6 and the formula (7.3).

Let $w_1, w_2, \dots, w_{N_c}$ be a set of indeterminates. Set

$$u_i = u\prod_{k=1}^{i-1} w_k^{-1}, \qquad v_i = v\prod_{k=1}^{i-1} w_k, \qquad (i=1,2,\dots,N_c+1), \tag{5.7}$$

for simplicity. Define the four point operator (see figure 10)

$$\Phi\begin{bmatrix}(1,L-N_c),u_{N_c+1};(1,M+N_c),v_{N_c+1}\\(1,L),u_1;(1,M),v_1\end{bmatrix}:\mathcal{F}_{u_1}^{(1,L)}\otimes\mathcal{F}_{v_1}^{(1,M)}\longrightarrow\mathcal{F}_{u_{N_c+1}}^{(1,L-N_c)}\otimes\mathcal{F}_{v_{N_c+1}}^{(1,M+N_c)}, \tag{5.8}$$

as the composition

$$\begin{aligned}
&\Phi\begin{bmatrix}(1,L-N_c),u_{N_c+1};(1,M+N_c),v_{N_c+1}\\(1,L),u_1;(1,M),v_1\end{bmatrix} \qquad (5.9)\\
&=\Phi\begin{bmatrix}(1,L-N_c),u_{N_c+1};(1,M+N_c),v_{N_c+1}\\(1,L-N_c+1),u_{N_c};(1,M+N_c-1),v_{N_c}\end{bmatrix}\cdots\Phi\begin{bmatrix}(1,L-1),u_2;(1,M+1),v_2\\(1,L),u_1;(1,M),v_1\end{bmatrix}.
\end{aligned}$$

Proposition 5.1. When we identify parameters as

$$\mathbf{e}_i = -w_i, \qquad \Lambda^{2N_c} = \frac{v}{u}\prod_{i=1}^{N_c} w_i, \qquad m = -L+M+N_c, \tag{5.10}$$

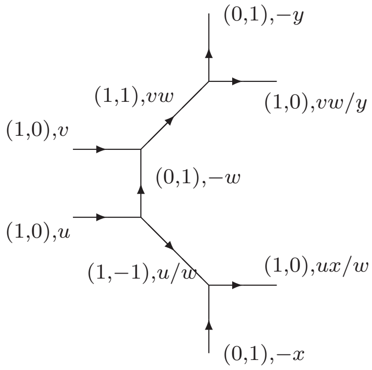

Figure 11. Six point operator.

we have

$$\begin{aligned}&\langle P_\emptyset \otimes P_\emptyset | \, \Phi \begin{bmatrix} (1, L-N_c), u_{N_c+1}; (1, M+N_c), v_{N_c+1} \\ (1,L), u_1; (1,M), v_1 \end{bmatrix} | P_\emptyset \otimes P_\emptyset \rangle \\ &\qquad = \prod_{1 \le i < j \le N} \mathcal{G}(\mathbf{e}_i/\mathbf{e}_j) \mathcal{G}(qt^{-1}\mathbf{e}_i/\mathbf{e}_j) \cdot Z_m^{\text{inst}}(\mathbf{e}_1, \cdots, \mathbf{e}_{N_c}, \Lambda; q, t),\end{aligned} \tag{5.11}$$

where

$$\mathcal{G}(u) = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{1}{(1-q^n)(1-t^{-n})} u^n \right) \in \mathbb{Q}(q,t)[[u]]. \tag{5.12}$$

A proof of this will be given in section 7.

5.2 SU(N_c) with $N_f = 2N_c$.

Next, we turn to the case with $N_f = 2N_c$ fundamental matters. Let u, v, w, x, y be indeterminates. Consider the six point operator

$$\begin{aligned}&\Phi \begin{bmatrix} (1,0), ux/w; (1,0), vw/y; (0,1), -y \\ (0,1), -x; (1,0), u; (1,0), v \end{bmatrix} \\ &: \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,0)}_{u} \otimes \mathcal{F}^{(1,0)}_{v} \longrightarrow \mathcal{F}^{(1,0)}_{ux/w} \otimes \mathcal{F}^{(1,0)}_{vw/y} \otimes \mathcal{F}^{(0,1)}_{-y},\end{aligned} \tag{5.13}$$

defined by the composition of the intertwining operators

$$\begin{aligned}&\mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,L)}_{u} \otimes \mathcal{F}^{(1,M)}_{v} \xrightarrow{\text{id}\otimes\Phi^*\otimes\text{id}} \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,L-1)}_{u/w} \otimes \mathcal{F}^{(0,1)}_{-w} \otimes \mathcal{F}^{(1,M)}_{v} \\ &\xrightarrow{\text{id}^{\otimes 2}\otimes\Phi} \mathcal{F}^{(0,1)}_{-x} \otimes \mathcal{F}^{(1,L-1)}_{u/w} \otimes \mathcal{F}^{(1,M+1)}_{vw} \xrightarrow{\Phi\otimes\Phi^*} \mathcal{F}^{(1,L)}_{ux/w} \otimes \mathcal{F}^{(1,M)}_{vw/y} \otimes \mathcal{F}^{(0,1)}_{-y}.\end{aligned} \tag{5.14}$$

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For any $P_\lambda \otimes \alpha \otimes \beta \in \mathcal{F}_{-x}^{(0,1)} \otimes \mathcal{F}_u^{(1,0)} \otimes \mathcal{F}_v^{(1,0)}$, we have

$$
\begin{aligned}
&\Phi \begin{bmatrix} (1,0), ux/w; (1,0), vw/y; (0,1), -y \\ (0,1), -x; (1,0), u; (1,0), v \end{bmatrix} (P_\lambda \otimes \alpha \otimes \beta) \\
&\quad = \sum_{\mu,\nu} \frac{1}{\langle P_\mu, P_\mu \rangle_{q,t}} \\
&\qquad \times \Phi_\lambda \begin{bmatrix} (1,0), ux/w \\ (0,1), -x; (1,-1), u/w \end{bmatrix} \Phi_\mu^* \begin{bmatrix} (1,-1), u/w; (0,1), -w \\ (1,0), u \end{bmatrix} (\alpha) \\
&\qquad \otimes \Phi_\nu^* \begin{bmatrix} (1,0), vw/y; (0,1), -y \\ (1,1), vw \end{bmatrix} \Phi_\mu \begin{bmatrix} (1,1), vw \\ (0,1), -w; (1,0), v \end{bmatrix} (\beta) \otimes Q_\nu \\
&\quad = \sum_{\mu,\nu} \frac{q^{n(\lambda')} (ux/w)^{|\lambda|} q^{n(\nu')} (qy/vw)^{|\nu|}}{c_\lambda c_\nu} \frac{(q^{1/2} t^{-1/2})^{-|\mu|} (v/u)^{|\mu|} f_\mu^{-1}}{N_{\mu,\mu}(1)} \\
&\qquad \times \Big(: \widetilde{\Phi}_\emptyset(-x) \eta_\lambda(-x) :: \widetilde{\Phi}_\emptyset^*(-w) \xi_\mu(-w) : \alpha \Big) \\
&\qquad \otimes \Big(: \widetilde{\Phi}_\emptyset^*(-y) \xi_\nu(-y) :: \widetilde{\Phi}_\emptyset(-w) \eta_\mu(-w) : \beta \Big) \otimes Q_\nu,
\end{aligned}
\tag{5.15}
$$

Restricting this six point operator, introduce the four point operator

$$
\Phi \begin{bmatrix} (1,0), ux/w; (1,0), vw/y \\ (1,0), u; (1,0), v \end{bmatrix} : \mathcal{F}_u^{(1,0)} \otimes \mathcal{F}_v^{(1,0)} \longrightarrow \mathcal{F}_{ux/w}^{(1,0)} \otimes \mathcal{F}_{vw/y}^{(1,0)},
$$

defined by specifying the action on any $\alpha \otimes \beta \in \mathcal{F}_u^{(1,0)} \otimes \mathcal{F}_v^{(1,0)}$ as

$$
\begin{aligned}
&\Phi \begin{bmatrix} (1,0), ux/w; (1,0), vw/y \\ (1,0), u; (1,0), v \end{bmatrix} (\alpha \otimes \beta) \\
&\quad = \mathrm{id} \otimes \mathrm{id} \otimes \langle P_\emptyset, \bullet \rangle_{q,t} \circ \Phi \begin{bmatrix} (1,0), ux/w; (1,0), vw/y; (0,1), -y \\ (0,1), -x; (1,0), u; (1,0), v \end{bmatrix} (P_\emptyset \otimes \alpha \otimes \beta) \\
&\quad = \sum_{\mu} \frac{(q^{1/2} t^{-1/2})^{-|\mu|} (v/u)^{|\mu|} f_\mu^{-1}}{N_{\mu,\mu}(1)} \\
&\qquad \times \Big(\widetilde{\Phi}_\emptyset(-x) : \widetilde{\Phi}_\emptyset^*(-w) \xi_\mu(-w) : \alpha \Big) \otimes \Big(\widetilde{\Phi}_\emptyset^*(-y) : \widetilde{\Phi}_\emptyset(-w) \eta_\mu(-w) : \beta \Big),
\end{aligned}
\tag{5.16}
$$

where we have used the shorthand notation $\langle P_\emptyset, \bullet \rangle_{q,t} \circ Q_\nu = \langle P_\emptyset, Q_\nu \rangle_{q,t}$.

Let $u, v, w_1, \ldots, w_{N_c}, x_1, \ldots, x_{N_c}, y_1, \ldots, y_{N_c}$ be a set of indeterminates. Set

$$
u_i = u \prod_{k=1}^{i-1} x_k / w_k, \qquad v_i = v \prod_{k=1}^{i-1} w_k / y_k, \qquad (i = 1, 2, \ldots, N_c + 1), \tag{5.17}
$$

for simplicity.

Proposition 5.2. Set

$$
\mathbf{e}_i = -w_i, \quad \mathbf{e}_i' = -q^{1/2} t^{-1/2} y_i, \quad \mathbf{e}_i'' = -q^{-1/2} t^{1/2} x_i, \quad \Lambda^{2N_c} = (q^{1/2} t^{-1/2})^{-N} \frac{v}{u} \prod_{i=1}^{N_c} \frac{w_i}{y_i}. \tag{5.18}
$$

We have

$$\langle P_\emptyset \otimes P_\emptyset | \, \Phi \begin{bmatrix} (1,0), u_{N_c+1}; (1,0), v_{N_c+1} \\ (1,0), u_{N_c}; (1,0), v_{N_c} \end{bmatrix} \cdots \Phi \begin{bmatrix} (1,0), u_2; (1,0), v_2 \\ (1,0), u_1; (1,0), v_1 \end{bmatrix} | P_\emptyset \otimes P_\emptyset \rangle \tag{5.19}$$

$$= \prod_{k=1}^{N_c} \frac{1}{\mathcal{G}(\mathbf{e}_k/\mathbf{e}''_k)\mathcal{G}(qt^{-1}\mathbf{e}_k/\mathbf{e}'_k)} \cdot \prod_{1\le i<j\le N_c} \frac{\mathcal{G}(\mathbf{e}_i/\mathbf{e}_j)\mathcal{G}(qt^{-1}\mathbf{e}_i/\mathbf{e}_j)\mathcal{G}(qt^{-1}\mathbf{e}''_i/\mathbf{e}''_j)\mathcal{G}(\mathbf{e}'_i/\mathbf{e}'_j)}{\mathcal{G}(\mathbf{e}_i/\mathbf{e}''_j)\mathcal{G}(qt^{-1}\mathbf{e}''_i/\mathbf{e}_j)\mathcal{G}(qt^{-1}\mathbf{e}_i/\mathbf{e}'_j)\mathcal{G}(\mathbf{e}'_i/\mathbf{e}_j)}$$

$$\times \sum_{\lambda^{(1)},\cdots,\lambda^{(N_c)}} \prod_{k=1}^{N_c} \Lambda^{2N_c|\lambda^{(k)}|} \prod_{1\le i,j,\le N_c} \frac{N_{\emptyset,\lambda^{(j)}}(\mathbf{e}'_i/\mathbf{e}_j) N_{\lambda^{(i)},\emptyset}(\mathbf{e}_i/\mathbf{e}''_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(\mathbf{e}_i/\mathbf{e}_j)}.$$

Since our proof of this goes in a parallel way as the one for proposition 5.1, we omit it.

6 Proofs of theorems 3.3 and 3.6

6.1 Some formulas concerning $A^\pm_{\lambda,i}$, $B^\pm_\lambda(z)$

Lemma 6.1. Let $c_\lambda, c'_\lambda, A^+_{\lambda,i}, A^-_{\lambda,i}$ be as in (2.9), (2.11), (2.12). We have

$$\frac{c'_{\lambda+\mathbf{1}_i}}{c'_\lambda}\frac{c_\lambda}{c_{\lambda+\mathbf{1}_i}} A^+_{\lambda,i} = -qA^-_{\lambda+\mathbf{1}_i,i}, \qquad \frac{c'_{\lambda-\mathbf{1}_i}}{c'_\lambda}\frac{c_\lambda}{c_{\lambda-\mathbf{1}_i}} A^-_{\lambda,i} = -q^{-1}A^+_{\lambda-\mathbf{1}_i,i}. \tag{6.1}$$

Hence the action of $\mathcal{U}$ is written in terms of the basis (Q_λ) as

$$\gamma Q_\lambda = Q_\lambda, \tag{6.2}$$

$$x^+(z)Q_\lambda = -\sum_{i=1}^{\ell(\lambda)+1} qA^-_{\lambda+\mathbf{1}_i,i}\,\delta(q^{\lambda_i}t^{-i+1}u/z)Q_{\lambda+\mathbf{1}_i}, \tag{6.3}$$

$$x^-(z)Q_\lambda = -q^{1/2}t^{-1/2}\sum_{i=1}^{\ell(\lambda)} q^{-1}A^+_{\lambda-\mathbf{1}_i,i}\,\delta(q^{\lambda_i-1}t^{-i+1}u/z)Q_{\lambda-\mathbf{1}_i}, \tag{6.4}$$

$$\psi^+(z)Q_\lambda = q^{1/2}t^{-1/2}B^+_\lambda(u/z)Q_\lambda, \tag{6.5}$$

$$\psi^-(z)Q_\lambda = q^{-1/2}t^{1/2}\,B^-_\lambda(z/u)Q_\lambda. \tag{6.6}$$

Proof. From the definitions of c_λ, c'_λ, it immediately follows that

$$\frac{c_{\lambda+\mathbf{1}_k}}{c_\lambda} = (1-q^{\lambda_k}t^{\ell(\lambda)-k+1})\prod_{i=1}^{k-1}\frac{1-q^{\lambda_i-\lambda_k-1}t^{k-i+1}}{1-q^{\lambda_i-\lambda_k-1}t^{k-i}}\prod_{j=k+1}^{\ell(\lambda)}\frac{1-q^{\lambda_k-\lambda_j}t^{j-k}}{1-q^{\lambda_k-\lambda_j}t^{j-k+1}},$$

$$\frac{c'_{\lambda+\mathbf{1}_k}}{c'_\lambda} = (1-q^{\lambda_k+1}t^{\ell(\lambda)-k})\prod_{i=1}^{k-1}\frac{1-q^{\lambda_i-\lambda_k}t^{k-i}}{1-q^{\lambda_i-\lambda_k}t^{k-i-1}}\prod_{j=k+1}^{\ell(\lambda)}\frac{1-q^{\lambda_k-\lambda_j+1}t^{j-k-1}}{1-q^{\lambda_k-\lambda_j+1}t^{j-k}}.$$

Noting that

$$A^-_{\lambda,k} = (1-t^{-1})\frac{1-q^{-\lambda_k}t^{-\ell(\lambda)+k}}{1-q^{-\lambda_k+1}t^{-\ell(\lambda)+k-1}}\prod_{j=k+1}^{\ell(\lambda)}\frac{1-q^{-\lambda_k+\lambda_j}t^{-j+k+1}}{1-q^{-\lambda_k+\lambda_j}t^{-j+k}}\frac{1-q^{-\lambda_k+\lambda_j+1}t^{-j+k-1}}{1-q^{-\lambda_k+\lambda_j+1}t^{-j+k}},$$

one obtains (6.1). $\square$

Lemma 6.2. Let $B^{\pm}_{\lambda}(z)$ be as in (2.13), (2.14). We have

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} g(q^{j-1}t^{-i+1}v/z)^{-1} = \frac{1-v/z}{1-q^{-1}tv/z}B^{+}(v/z), \tag{6.7}$$

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} g(q^{-j+1}t^{i-1}v/z) = \frac{1-z/v}{1-qt^{-1}z/v}B^{-}_{\lambda}(z/v), \tag{6.8}$$

where $g(z)$ is given in (2.1).

The following will also be needed.

Lemma 6.3. We have

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} f(q^{j-1}t^{-i+1}v/z) = \frac{1-v/z}{1-t^{-\ell(\lambda)}v/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}t^{-i}v/z}{1-q^{\lambda_i}t^{-i+1}v/z}, \tag{6.9}$$

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} f(q^{-j+1}t^{i-1}z/v) = \frac{1-qt^{-1}z/v}{1-qt^{\ell(\lambda-1)}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-1}z/v}{1-q^{-\lambda_i+1}t^{i-2}z/v}. \tag{6.10}$$

where

$$f(z) = \frac{(1-z)(1-qt^{-1}z)}{(1-qz)(1-t^{-1}z)}. \tag{6.11}$$

Lemma 6.4. We have

$$\frac{q^{n(\lambda')}}{c_\lambda}\left(\frac{1}{1-t^{-\ell(\lambda)}v/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}t^{-i}v/z}{1-q^{\lambda_i}t^{-i+1}v/z}+\frac{z}{v}\frac{1}{1-t^{\ell(\lambda)}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i}t^{i}z/v}{1-q^{-\lambda_i}t^{i-1}z/v}\right) \tag{6.12}$$
$$= \sum_{i=1}^{\ell(\lambda)+1}\frac{q^{n((\lambda+\mathbf{1}_i)')}}{c_{\lambda+\mathbf{1}_i}}A^{+}_{\lambda,i}\,\delta(q^{\lambda_i}t^{-i+1}v/z).$$

Proof. It follows from

$$\frac{1}{1-t^{-\ell(\lambda)}v/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}t^{-i}v/z}{1-q^{\lambda_i}t^{-i+1}v/z}+\frac{z}{v}\frac{1}{1-t^{\ell(\lambda)}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i}t^{i}z/v}{1-q^{-\lambda_i}t^{i-1}z/v}$$
$$=\sum_{i=1}^{\ell(\lambda)+1}q^{\lambda_i}t^{-i+1}\delta(q^{\lambda_i}t^{-i+1}v/z)\frac{1-t}{1-q^{\lambda_i}t^{\ell(\lambda)-i+1}}\prod_{j=1}^{i-1}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j}t^{j-i}}\prod_{j=i+1}^{\ell(\lambda)}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j}t^{j-i}},$$

and

$$\frac{c_{\lambda+\mathbf{1}_i}}{c_\lambda} = t^{i-1}(1-q^{\lambda_i}t^{\ell(\lambda)-i+1})\prod_{j=1}^{i-1}\frac{1-q^{\lambda_i-\lambda_j+1}t^{j-i-1}}{1-q^{\lambda_i-\lambda_j+1}t^{j-i}}\prod_{j=i+1}^{\ell(\lambda)}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i}}{1-q^{\lambda_i-\lambda_j}t^{j-i+1}},$$

$$n(\lambda') = \sum_{i\geq 0}\frac{\lambda_i(\lambda_i-1)}{2}, \qquad \frac{q^{n((\lambda+\mathbf{1}_i)')}}{q^{n(\lambda')}} = q^{\lambda_i}.$$

□

Lemma 6.5. We have

$$\frac{q^{n(\lambda')}}{c_\lambda}\Bigg((1-q^{-1}t^{-\ell(\lambda)+1}v/z)\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i-1}t^{-i+2}v/z}{1-q^{\lambda_i-1}t^{-i+1}v/z} \tag{6.13}$$
$$+q^{-1}t\frac{v}{z}(1-qt^{\ell(\lambda)-1}z/v)\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-2}z/v}{1-q^{-\lambda_i+1}t^{i-1}z/v}\Bigg)$$
$$=\sum_{i=1}^{\ell(\lambda)}\frac{q^{n((\lambda-\mathbf{1}_i)')}}{c_{\lambda-\mathbf{1}_i}}A^-_{\lambda,i}\delta(q^{\lambda_i-1}t^{-i+1}v/z).$$

6.2 Operator product formulas for $\widetilde{\Phi}_\lambda(v)$

Lemma 6.6. The operator product formulas between $\widetilde{\Phi}_\emptyset(v)$ and the generators of $\mathcal{U}$ are

$$\eta(z)\widetilde{\Phi}_\emptyset(v)=\frac{1}{1-v/z}:\eta(z)\widetilde{\Phi}_\emptyset(v):, \tag{6.14}$$
$$\widetilde{\Phi}_\emptyset(v)\eta(z)=\frac{1}{1-qt^{-1}z/v}:\eta(z)\widetilde{\Phi}_\emptyset(v):, \tag{6.15}$$
$$\xi(z)\widetilde{\Phi}_\emptyset(v)=(1-q^{-1/2}t^{1/2}v/z):\xi(z)\widetilde{\Phi}_\emptyset(v):, \tag{6.16}$$
$$\widetilde{\Phi}_\emptyset(v)\xi(z)=(1-q^{1/2}t^{-1/2}z/v):\xi(z)\widetilde{\Phi}_\emptyset(v):, \tag{6.17}$$
$$\varphi^+(q^{1/4}t^{-1/4}z)\widetilde{\Phi}_\emptyset(v)=\frac{1-q^{-1}tv/z}{1-u/z}\widetilde{\Phi}_\emptyset(v)\varphi^+(q^{1/4}t^{-1/4}z), \tag{6.18}$$
$$\varphi^-(q^{-1/4}t^{1/4}z)\widetilde{\Phi}_\emptyset(v)=\frac{1-qt^{-1}z/v}{1-z/v}\widetilde{\Phi}_\emptyset(v)\varphi^-(q^{-1/4}t^{1/4}z). \tag{6.19}$$

Proposition 6.7. We have

$$\varphi^+(q^{1/4}t^{-1/4}z)\widetilde{\Phi}_\lambda(v)\varphi^+(q^{1/4}t^{-1/4}z)^{-1}=B^+_\lambda(v/z)\widetilde{\Phi}_\lambda(v), \tag{6.20}$$
$$\varphi^-(q^{-1/4}t^{1/4}z)\widetilde{\Phi}_\lambda(v)\varphi^-(q^{-1/4}t^{1/4}z)^{-1}=B^-_\lambda(z/v)\widetilde{\Phi}_\lambda(v). \tag{6.21}$$

Proof. Note that

$$\varphi^+(q^{1/4}t^{-1/4}z)\eta(v)\varphi^+(q^{1/4}t^{-1/4}z)^{-1}=g(v/z)^{-1}\eta(v),$$
$$\varphi^-(q^{-1/4}t^{1/4}z)\eta(v)\varphi^-(q^{-1/4}t^{1/4}z)^{-1}=g(z/v)\eta(v).$$

Then (6.20), (6.21) follow from lemma 6.2 and (6.18), (6.19) in lemma 6.6. □

Lemma 6.8. We have

$$\eta(z)\widetilde{\Phi}_\lambda(v)=\frac{1}{1-t^{-\ell(\lambda)}v/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}t^{-i}v/z}{1-q^{\lambda_i}t^{-i+1}v/z}:\eta(z)\widetilde{\Phi}_\lambda(v):, \tag{6.22}$$
$$\widetilde{\Phi}_\lambda(v)\eta(z)=\frac{1}{1-qt^{\ell(\lambda)-1}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-1}z/v}{1-q^{-\lambda_i+1}t^{i-2}z/v}:\eta(z)\widetilde{\Phi}_\lambda(v):, \tag{6.23}$$
$$B^-_\lambda(z/v)\widetilde{\Phi}_\lambda(v)\eta(z)=\frac{1}{1-t^{\ell(\lambda)}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i}t^{i}z/v}{1-q^{-\lambda_i}t^{i-1}z/v}:\eta(z)\widetilde{\Phi}_\lambda(v):. \tag{6.24}$$

Proof. We have $\eta(z)\eta(v) = f(v/z) : \eta(z)\eta(v) :$. Hence from lemma 6.3

$$\eta(z)\eta_\lambda(v) = \frac{1-v/z}{1-t^{-\ell(\lambda)}v/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}t^{-i}v/z}{1-q^{\lambda_i}t^{-i+1}v/z} : \eta(z)\eta_\lambda(v) :,$$

$$\eta_\lambda(v)\eta(z) = \frac{1-qt^{-1}z/v}{1-qt^{\ell(\lambda)-1}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-1}z/v}{1-q^{-\lambda_i+1}t^{i-2}z/v} : \eta(z)\eta_\lambda(v) : .$$

Then (6.22), (6.23) follow from (6.14), (6.15) in lemma 6.6. $\square$

Proposition 6.9. We have

$$\eta(z)\widetilde{\Phi}_\lambda(v) + \frac{z}{v}B^-_\lambda(z/v)\widetilde{\Phi}_\lambda(v)\eta(z) = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}\,\widetilde{\Phi}_{\lambda+\mathbf{1}_i}(v)\delta(q^{\lambda_i}t^{-i+1}v/z). \tag{6.25}$$

Proof. It follows from lemma 6.4 and (6.22), (6.24) in lemma 6.8. $\square$

Lemma 6.10. We have

$$\xi(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda(v) = (1-q^{-1}t^{-\ell(\lambda)+1}v/z)\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i-1}t^{-i+2}v/z}{1-q^{\lambda_i-1}t^{-i+1}v/z} : \xi(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda(v) :, \tag{6.26}$$

$$\widetilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z) = (1-qt^{\ell(\lambda)-1}z/v)\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-2}z/v}{1-q^{-\lambda_i+1}t^{i-1}z/v} : \widetilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z) : . \tag{6.27}$$

Proof. Note that $\xi(q^{1/2}t^{-1/2}z)\eta(v) = f(q^{-1}tv/z)^{-1} : \xi(q^{1/2}t^{-1/2}z)\eta(v) :$, and $\eta(v)\xi(q^{1/2}t^{-1/2}z) = f(z/v)^{-1} : \xi(q^{1/2}t^{-1/2}z)\eta(v) :$. Thus from lemma 6.3, we have

$$\xi(q^{1/2}t^{-1/2}z)\eta_\lambda(v) = \frac{1-q^{-1}t^{-\ell(\lambda)+1}v/z}{1-q^{-1}tv/z}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i-1}t^{-i+2}v/z}{1-q^{\lambda_i-1}t^{-i+1}v/z} : \xi(q^{1/2}t^{-1/2}z)\eta_\lambda(v) :,$$

$$\eta_\lambda(v)\xi(q^{1/2}t^{-1/2}z) = \frac{1-qt^{\ell(\lambda)-1}z/v}{1-qt^{-1}z/v}\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{-\lambda_i+1}t^{i-2}z/v}{1-q^{-\lambda_i+1}t^{i-1}z/v} : \xi(q^{1/2}t^{-1/2}z)\eta_\lambda(v) : .$$

Then (6.26), (6.27) follow from (6.16), (6.17) in lemma 6.6. $\square$

Proposition 6.11. We have

$$\begin{aligned}&\xi(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda(v) + q^{-1}t\frac{v}{z}\widetilde{\Phi}_\lambda(v)\xi(q^{1/2}t^{-1/2}z)\\ &\quad= \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}\,\widetilde{\Phi}_{\lambda-\mathbf{1}_i}(v)\delta(q^{\lambda_i-1}t^{-i+1}v/z)\varphi^+(q^{1/4}t^{-1/4}z).\end{aligned} \tag{6.28}$$

Proof. It follows from lemmas 6.5, 6.10 and $: \xi(q^{1/2}t^{-1/2}z)\eta(z) := \varphi^+(q^{1/4}t^{-1/4}z)$. $\square$

6.3 Operator product formulas for $\widetilde{\Phi}^*_\lambda(u)$

Lemma 6.12. We have

$$\eta(z)\widetilde{\Phi}^*_\emptyset(u) = (1 - q^{-1/2}t^{1/2}u/z) : \eta(z)\widetilde{\Phi}^*_\emptyset(u) :, \tag{6.29}$$
$$\widetilde{\Phi}^*_\emptyset(u)\eta(z) = (1 - q^{1/2}t^{-1/2}z/u) : \eta(z)\widetilde{\Phi}^*_\emptyset(u) :, \tag{6.30}$$
$$\xi(z)\widetilde{\Phi}^*_\emptyset(u) = \frac{1}{1 - q^{-1}tu/z} : \xi(z)\widetilde{\Phi}^*_\emptyset(u) :, \tag{6.31}$$
$$\widetilde{\Phi}^*_\emptyset(u)\xi(z) = \frac{1}{1 - z/u} : \xi(z)\widetilde{\Phi}^*_\emptyset(u) :, \tag{6.32}$$
$$\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\widetilde{\Phi}^*_\emptyset(u) = \frac{1 - q^{-1}tu/z}{1 - u/z}\widetilde{\Phi}^*_\emptyset(u)\varphi^+(q^{-1/4}t^{1/4}z)^{-1}, \tag{6.33}$$
$$\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\widetilde{\Phi}^*_\emptyset(u) = \frac{1 - qt^{-1}z/u}{1 - z/u}\widetilde{\Phi}^*_\emptyset(u)\varphi^-(q^{1/4}t^{-1/4}z)^{-1}. \tag{6.34}$$

Proposition 6.13. We have

$$\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\widetilde{\Phi}^*_\lambda(u)\varphi^+(q^{-1/4}t^{1/4}z) = B^+_\lambda(u/z)\widetilde{\Phi}^*_\emptyset(u), \tag{6.35}$$
$$\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\widetilde{\Phi}^*_\lambda(u)\varphi^-(q^{1/4}t^{-1/4}z) = B^-_\lambda(z/u)\widetilde{\Phi}^*_\emptyset(u). \tag{6.36}$$

Proof. Note that

$$\varphi^+(q^{-1/4}t^{1/4}z)^{-1}\xi(u)\varphi^+(q^{-1/4}t^{1/4}z) = g(u/z)^{-1}\xi(u),$$
$$\varphi^-(q^{1/4}t^{-1/4}z)^{-1}\xi(u)\varphi^-(q^{1/4}t^{-1/4}z) = g(z/u)\xi(u).$$

Then (6.35), (6.36) follow from lemmas 6.2 and (6.33), (6.34) in lemma 6.12. □

Lemma 6.14. We have

$$\xi(z)\widetilde{\Phi}^*_\lambda(u) = \frac{1}{1 - q^{-1}t^{-\ell(\lambda)+1}u/z}\prod_{i=1}^{\ell(\lambda)}\frac{1 - q^{\lambda_i-1}t^{-i+1}u/z}{1 - q^{\lambda_i-1}t^{-i+2}u/z} : \xi(z)\widetilde{\Phi}^*_\lambda(u) :, \tag{6.37}$$
$$B^+_\lambda(u/z)\xi(z)\widetilde{\Phi}^*_\lambda(u) = \frac{1}{1 - t^{-\ell(\lambda)}u/z}\prod_{i=1}^{\ell(\lambda)}\frac{1 - q^{\lambda_i}t^{-i}u/z}{1 - q^{\lambda_i}t^{-i+1}u/z} : \xi(z)\widetilde{\Phi}^*_\lambda(u) :, \tag{6.38}$$
$$\widetilde{\Phi}^*_\lambda(u)\xi(z) = \frac{1}{1 - t^{\ell(\lambda)}z/u}\prod_{i=1}^{\ell(\lambda)}\frac{1 - q^{-\lambda_i}t^{i}z/u}{1 - q^{-\lambda_i}t^{i-1}z/u} : \xi(z)\widetilde{\Phi}^*_\lambda(u) : . \tag{6.39}$$

Proof. From $\xi(z)\xi(u) = f(q^{-1}tu/z) : \xi(z)\xi(u) :$, and lemma 6.3 we have

$$\xi(z)\xi_\lambda(u) = \frac{1 - q^{-1}tu/z}{1 - q^{-1}t^{-\ell(\lambda)+1}u/z}\prod_{i=1}^{\ell(\lambda)}\frac{1 - q^{\lambda_i-1}t^{-i+1}u/z}{1 - q^{\lambda_i-1}t^{-i+2}u/z} : \xi(z)\xi_\lambda(u) :,$$
$$\xi_\lambda(u)\xi(z) = \frac{1 - z/u}{1 - t^{\ell(\lambda)}z/u}\prod_{i=1}^{\ell(\lambda)}\frac{1 - q^{-\lambda_i}t^{i}z/u}{1 - q^{-\lambda_i}t^{i-1}z/u} : \xi(z)\xi_\lambda(u) : .$$

Then (6.37), (6.39) follow from (6.31), (6.32) in lemma 6.12. □

Proposition 6.15. We have

$$B_\lambda^+(u/z)\xi(z)\widetilde{\Phi}_\lambda^*(u) + \frac{z}{u}\widetilde{\Phi}_\lambda^*(u)\xi(z) = \sum_{i=1}^{\ell(\lambda)+1} A_{\lambda,i}^+ \, \widetilde{\Phi}_{\lambda+\mathbf{1}_i}^*(u)\delta(q^{\lambda_i}t^{-i+1}u/z). \tag{6.40}$$

Proof. It follows from lemma 6.4 and (6.38), (6.39) in lemma 6.14. □

Lemma 6.16. We have

$$\eta(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda^*(u) = (1-q^{-1}t^{-\ell(\lambda)+1}u/z)\prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i-1}t^{-i+2}u/z}{1-q^{\lambda_i-1}t^{-i+1}u/z} :\eta(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda^*(u):, \tag{6.41}$$

$$\widetilde{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z) = (1-qt^{\ell(\lambda)-1}z/u)\prod_{i=1}^{\ell(\lambda)} \frac{1-q^{-\lambda_i+1}t^{i-2}z/u}{1-q^{-\lambda_i+1}t^{i-1}z/u} :\widetilde{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z):. \tag{6.42}$$

Proof. Note that $\eta(q^{1/2}t^{-1/2}z)\xi(u) = f(q^{-1}tu/z)^{-1} :\eta(q^{1/2}t^{-1/2}z)\xi(u):$, and $\xi(u)\eta(q^{1/2}t^{-1/2}z) = f(z/u)^{-1} :\eta(q^{1/2}t^{-1/2}z)\xi(u):$. From lemma 6.3 we have

$$\eta(q^{1/2}t^{-1/2}z)\xi_\lambda(u) = \frac{1-q^{-1}t^{-\ell(\lambda)+1}u/z}{1-q^{-1}tu/z}\prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i-1}t^{-i+2}u/z}{1-q^{\lambda_i-1}t^{-i+1}u/z} :\eta(q^{1/2}t^{-1/2}z)\xi_\lambda(u):,$$

$$\xi_\lambda(u)\eta(q^{1/2}t^{-1/2}z) = \frac{1-qt^{\ell(\lambda)-1}z/u}{1-qt^{-1}z/u}\prod_{i=1}^{\ell(\lambda)} \frac{1-q^{-\lambda_i+1}t^{i-2}z/u}{1-q^{-\lambda_i+1}t^{i-1}z/u} :\eta(q^{1/2}t^{-1/2}z)\xi_\lambda(u):.$$

Then (6.41), (6.42) follow from (6.29), (6.30) in lemma 6.12. □

Proposition 6.17. We have

$$\begin{aligned}\eta(q^{1/2}t^{-1/2}z)\widetilde{\Phi}_\lambda^*(u) &+ q^{-1}t\frac{u}{z}\widetilde{\Phi}_\lambda^*(u)\eta(q^{1/2}t^{-1/2}z)\\ &= \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^- \, \widetilde{\Phi}_{\lambda-\mathbf{1}_i}^*(v)\delta(q^{\lambda_i-1}t^{-i+1}v/z)\varphi^-(q^{1/4}t^{-1/4}z)\end{aligned} \tag{6.43}$$

Proof. It follows from lemmas 6.5, 6.16 and $:\eta(q^{1/2}t^{-1/2}z)\xi(z): = \varphi^-(q^{1/4}t^{-1/4}z)$. □

6.4 Final step of proofs

The intertwining relations in lemma 3.2 are rewritten in terms of $\eta, \xi, \varphi^{\pm}$ as

$$\varphi^{+}(q^{1/4}t^{-1/4}z)\Phi_{\lambda}\varphi^{+}(q^{1/4}t^{-1/4}z)^{-1} = B_{\lambda}^{+}(v/z)\Phi_{\lambda}, \tag{6.44}$$

$$\varphi^{-}(q^{-1/4}t^{1/4}z)\Phi_{\lambda}\varphi^{-}(q^{-1/4}t^{1/4}z)^{-1} = B_{\lambda}^{-}(z/v)\Phi_{\lambda}, \tag{6.45}$$

$$\begin{aligned}&\eta(z)\Phi_{\lambda} - \frac{uz}{w}B_{\lambda}^{-}(z/v)\Phi_{\lambda}\eta(z) \\ &\quad= \sum_{i=1}^{\ell(\lambda)+1} w^{-1}(q^{1/2}t^{-1/2}q^{\lambda_i}t^{-i+1}v)^{N+1}A_{\lambda,i}^{+}\,\delta(q^{\lambda_i}t^{-i+1}v/z)\Phi_{\lambda+\mathbf{1}_i},\end{aligned} \tag{6.46}$$

$$\begin{aligned}&\xi(q^{1/2}t^{-1/2}z)\Phi_{\lambda} - q^{-1}t\frac{w}{uz}\Phi_{\lambda}\xi(q^{1/2}t^{-1/2}z) \\ &\quad= \sum_{i=1}^{\ell(\lambda)} w(q^{1/2}t^{-1/2}q^{\lambda_i-1}t^{-i+1}v)^{-N-1}A_{\lambda,i}^{-}\,\delta(q^{\lambda_i-1}t^{-i+1}v/z)\Phi_{\lambda-\mathbf{1}_i}\psi^{+}(q^{1/4}t^{-1/4}z).\end{aligned} \tag{6.47}$$

Proof of theorem 3.3. From (6.44) and (6.45), we must have that Φ_{λ} be proportional to $\widetilde{\Phi}_{\lambda}(v)$ by virtue of proposition 6.7. Write $\Phi_{\lambda} = t(\lambda, v, u, N)\widetilde{\Phi}_{\lambda}(v)$. Then in view of propositions 6.9, 6.11, we find that (6.46) and (6.47) may hold only in the case $w = -vu$ and when $t(\lambda, v, u, N)$ is given by (3.9). □

The intertwining relations in lemma 3.2 are rewritten in terms of $\eta, \xi, \varphi^{\pm}$ as

$$\varphi^{+}(q^{-1/4}t^{1/4}z)^{-1}\Phi_{\lambda}^{*}\varphi^{+}(q^{-1/4}t^{1/4}z) = B_{\lambda}^{+}(u/z)\Phi_{\lambda}^{*}, \tag{6.48}$$

$$\varphi^{-}(q^{1/4}t^{-1/4}z)^{-1}\Phi_{\lambda}^{*}\varphi^{-}(q^{1/4}t^{-1/4}z) = B_{\lambda}^{-}(z/u)\Phi_{\lambda}^{*}, \tag{6.49}$$

$$\begin{aligned}&B^{+}(u/z)\xi(z)\Phi_{\lambda}^{*} - \frac{vz}{w}\Phi_{\lambda}^{*}\xi(z) \\ &\quad= \sum_{i=1}^{\ell(\lambda)+1} q^{-1}v(q^{1/2}t^{-1/2}q^{\lambda_i}t^{-i+1}u)^{-N}A_{\lambda,i}^{+}\delta(q^{\lambda_i}t^{-i+1}u/z)\Phi_{\lambda+\mathbf{1}_i}^{*},\end{aligned} \tag{6.50}$$

$$\begin{aligned}&\eta(q^{1/2}t^{-1/2}z)\Phi_{\lambda}^{*} - q^{-1}t\frac{w}{vz}\Phi_{\lambda}^{*}\eta(q^{1/2}t^{-1/2}z) \\ &\quad= \varphi^{-}(q^{1/4}t^{-1/4}z)\sum_{i=1}^{\ell(\lambda)+1} qv^{-1}(q^{1/2}t^{-1/2}q^{\lambda_i-1}t^{-i+1}u)^{N}A_{\lambda,i}^{-}\delta(q^{\lambda_i-1}t^{-i+1}u/z)\Phi_{\lambda-\mathbf{1}_i}^{*}.\end{aligned} \tag{6.51}$$

Proof of theorem 3.6. From (6.48) and (6.49), we must have that Φ_{λ}^{*} be proportional to $\widetilde{\Phi}_{\lambda}^{*}(v)$ by virtue of proposition 6.13. Write $\Phi_{\lambda}^{*} = t^{*}(\lambda, v, u, N)\widetilde{\Phi}_{\lambda}^{*}(v)$. Then in view of propositions 6.15, 6.17, we find that (6.50) and (6.51) may hold only in the case $w = -vu$ and when $t^{*}(\lambda, v, u, N)$ is given by (3.21). □

7 Proof of proposition 5.1

7.1 Some formulas concerning $N_{\lambda,\mu}(u)$.

Lemma 7.1. We have

$$N_{\lambda,\mu}(u) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (uq^{-\mu_i+\lambda_{j+1}}t^{i-j};q)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\alpha=1}^{\ell(\mu)} \prod_{\beta=1}^{\ell(\mu)} (uq^{\lambda_\alpha-\mu_\beta}t^{-\alpha+\beta+1};q)_{\mu_\beta-\mu_{\beta+1}}, \tag{7.1}$$

$$N_{\lambda,\mu}(q^{1/2}t^{-1/2}x) = N_{\mu,\lambda}(q^{1/2}t^{-1/2}x^{-1})x^{|\lambda|+|\mu|}\frac{f_\lambda}{f_\mu}, \tag{7.2}$$

$$c_\lambda c'_\lambda = (-1)^{|\lambda|}q^{n(\lambda')+|\lambda|}t^{n(\lambda)}N_{\lambda,\lambda}(1). \tag{7.3}$$

Lemma 7.2. Let $\varepsilon^{\pm}_\lambda$ be the algebra homomorphism in (4.3). We have

$$\exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\left(\varepsilon^+_\lambda p_n\right)\left(\varepsilon^-_\mu p_n\right)u^n\right) = \mathcal{G}(u)^{-1}N_{\lambda,\mu}(u), \tag{7.4}$$

where $\mathcal{G}(u)$ being as in (5.12).

Proof. Fix an integer ℓ such that $\ell \geq \mathrm{Max}(\ell(\lambda),\ell(\mu))$. We have

$$\begin{aligned}
\text{l.h.s.} &= \exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}u^n\left(\frac{t^{-n}}{1-t^{-n}}+\sum_{i=1}^{\ell}(q^{\lambda_i n}-1)t^{-in}\right)\left(\frac{t^n}{1-t^n}+\sum_{j=1}^{\ell}(q^{-\mu_j n}-1)t^{jn}\right)\right)\\
&= \mathcal{G}(u)^{-1}\exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}u^n\left(\sum_{i,j=1}^{\ell}(q^{\lambda_i}t^{-i})^n(q^{-\mu_j}t^j)^n\right.\right.\\
&\qquad\qquad \left.\left.+\frac{t^{-(\ell+1)n}}{1-t^{-n}}\sum_{j=1}^{\ell}(q^{-\mu_j}t^j)^n+\frac{t^{(\ell+1)n}}{1-t^n}\sum_{i=1}^{\ell}(q^{\lambda_i}t^{-i})^n\right)\right)\\
&= \mathcal{G}(u)^{-1}\prod_{i=1}^{\ell}\prod_{j=1}^{\ell}\frac{(uq^{-\mu_i+\lambda_j}t^{i-j+1};q)_\infty}{(uq^{-\mu_i+\lambda_j}t^{i-j};q)_\infty}\cdot\prod_{k=1}^{\ell}\frac{(uq^{-\mu_k}t^{k-\ell};q)_\infty}{(uq^{\lambda_k}t^{-k+\ell+1};q)_\infty},
\end{aligned}$$

were we have used the notation

$$(u;q)_\infty = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{1-q^n}u^n\right) \in \mathbb{Q}(q)[[u]].$$

Note that $(u;q)_\infty/(q^n u;q)_\infty = (u;q)_n$ $(n = 0,1,2,\ldots)$, and use (7.1), then we have (7.4). □

Proposition 7.3. We have the operator product formulas

$$: \widetilde{\Phi}^*_\emptyset(z)\xi_\lambda(z) :: \widetilde{\Phi}^*_\emptyset(w)\xi_\mu(w) := \frac{\mathcal{G}(w/z)}{N_{\mu,\lambda}(w/z)} : \widetilde{\Phi}^*_\emptyset(z)\xi_\lambda(z)\widetilde{\Phi}^*_\emptyset(w)\xi_\mu(w) :, \tag{7.5}$$

$$: \widetilde{\Phi}_\emptyset(z)\eta_\lambda(z) :: \widetilde{\Phi}_\emptyset(w)\eta_\mu(w) := \frac{\mathcal{G}(qt^{-1}w/z)}{N_{\mu,\lambda}(qt^{-1}w/z)} : \widetilde{\Phi}_\emptyset(z)\eta_\lambda(z)\widetilde{\Phi}_\emptyset(w)\eta_\mu(w) :, \tag{7.6}$$

$$: \widetilde{\Phi}^*_\emptyset(z)\xi_\lambda(z) :: \widetilde{\Phi}_\emptyset(w)\eta_\mu(w) := \frac{N_{\mu,\lambda}(q^{1/2}t^{-1/2}w/z)}{\mathcal{G}(q^{1/2}t^{-1/2}w/z)} : \widetilde{\Phi}^*_\emptyset(z)\xi_\lambda(z)\widetilde{\Phi}_\emptyset(w)\eta_\mu(w) :, \tag{7.7}$$

$$: \widetilde{\Phi}_\emptyset(z)\eta_\lambda(z) :: \widetilde{\Phi}^*_\emptyset(w)\xi_\mu(w) := \frac{N_{\mu,\lambda}(q^{1/2}t^{-1/2}w/z)}{\mathcal{G}(q^{1/2}t^{-1/2}w/z)} : \widetilde{\Phi}_\emptyset(z)\eta_\lambda(z)\widetilde{\Phi}^*_\emptyset(w)\xi_\mu(w) : . \tag{7.8}$$

Proof. These follow from (4.9), (4.10) and (7.4). □

7.2 Proof of proposition 5.1

Using lemma 7.1 and proposition 7.3, we have

$$\begin{aligned}
\text{l.h.s. of (5.11)} = &\sum_{\lambda^{(1)},\cdots,\lambda^{(N_c)}} \prod_{k=1}^{N_c} \frac{\left(q^{-1/2}t^{1/2}v_i u_i^{-1} w_i^{L-M-2i+2}\right)^{|\lambda^{(i)}|}}{N_{\lambda^{(i)},\lambda^{(i)}}(1)} f_{\lambda^{(i)}}^{L-M-2i+1} \\
&\times \langle 0| : \widetilde{\Phi}^*_\emptyset(-w_{N_c})\xi_{\lambda^{(N_c)}}(-w_{N_c}) : \cdots : \widetilde{\Phi}^*_\emptyset(-w_1)\xi_{\lambda^{(1)}}(-w_1) : |0\rangle \\
&\times \langle 0| : \widetilde{\Phi}_\emptyset(-w_{N_c})\eta_{\lambda^{(N_c)}}(-w_{N_c}) : \cdots : \widetilde{\Phi}_\emptyset(-w_1)\eta_{\lambda^{(1)}}(-w_1) : |0\rangle \\
= &\sum_{\lambda^{(1)},\cdots,\lambda^{(N_c)}} \prod_{k=1}^{N_c} \frac{\left(q^{-1/2}t^{1/2}v_i u_i^{-1} w_i^{L-M-2i+2}\right)^{|\lambda^{(i)}|}}{N_{\lambda^{(i)},\lambda^{(i)}}(1)} f_{\lambda^{(i)}}^{L-M-2i+1} \\
&\times \prod_{1\le i<j\le N_c} \frac{\mathcal{G}(w_i/w_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(w_i/w_j)} \frac{\mathcal{G}(qt^{-1}w_i/w_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(qt^{-1}w_i/w_j)}.
\end{aligned}$$

Simplifying the factors by using lemma 7.4 below, we have the result. □

Lemma 7.4. We have

$$\prod_{1\le i<j\le N} (q^{1/2}t^{-1/2})^{-|\lambda^{(i)}|-|\lambda^{(j)}|} = \prod_{i=1}^{N} (q^{1/2}t^{-1/2})^{-(N-1)|\lambda^{(i)}|}, \tag{7.9}$$

$$\prod_{1\le i<j\le N} w_i^{-|\lambda^{(i)}|} w_j^{|\lambda^{(j)}|} = \prod_{i=1}^{N} w_i^{(-N+2i-1)|\lambda^{(i)}|}, \tag{7.10}$$

$$\prod_{1\le i<j\le N} w_i^{-|\lambda^{(j)}|} w_j^{|\lambda^{(i)}|} = \prod_{i=1}^{N} (w_1 w_2 \cdots w_N)^{|\lambda^{(i)}|} w_i^{-|\lambda^{(i)}|} (w_1 w_2 \cdots w_{i-1})^{-2|\lambda^{(i)}|}. \tag{7.11}$$

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