Modeling optimal social choice: matrix-vector representation of various solution concepts based on majority rule

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Abstract: Various Condorcet consistent social choice functions based on majority rule (tournament solutions) are considered in the general case, when ties are allowed: the core, the weak and strong top cycle sets, versions of the uncovered and minimal weakly stable sets, the uncaptured set, the untrapped set, classes of *k*-stable alternatives and *k*-stable sets. The main focus of the paper is to construct a unified matrix-vector representation of a tournament solution in order to get a convenient algorithm for its calculation. New versions of some solutions are also proposed.

Keywords: *solution concept, majority relation, tournament, matrix-vector representation, Condorcet winner, core, top cycle, uncovered set, weakly stable set, externally stable set, uncaptured set, untrapped set, k-stable alternative, k-stable set*

1. Introduction

More than two centuries have passed since marquise de Condorcet, a man of brilliant genius and deep insight, had proposed a social choice procedure now known as the choice of the Condorcet winner – an alternative preferred to any other one by the majority of voters.

However, it was Condorcet himself who constructed a counterexample that demonstrated how inconsistent this choice rule might be. He considered the case of three alternatives: *a*, *b*, *c*, and three voters, who were assumed to have the following preferences with respect to alternatives:

1^{st} voter: $a \succ b \succ c$,

2^{nd} voter: $c \succ a \succ b$,

3^{rd} voter: $b \succ c \succ a$.

The preferences were also assumed to be transitive, i.e. if $a \succ b$ and $b \succ c$, then $a \succ c$.

If one uses the simple majority rule to construct social preferences, i.e. if one says that *a* is preferred to *b* socially when at least 2 voters out of 3 prefer *a* to *b*, then the social preferences will cycle: $a \succ b$, $b \succ c$ and $c \succ a$, and the majority relation will not have a maximal element. This situation has been called "the Condorcet paradox".

Solution concepts based on majority relation (tournament solutions) were designed to resolve the problem the Condorcet paradox presents. As the works of 20^{th}-century social theorists have shown, Condorcet's idea to use majority rule to define "the will of the people" is normatively sound - when choices are to be made by a group, the only methods of aggregation of individual preferences that satisfy several important normative conditions (independence of irrelevant alternatives, Pareto efficiency, monotonicity, neutrality with respect to alternatives and anonymity with respect to voters) are different versions of the majority rule. Therefore social preferences are often modeled by a binary relation based on simple majority rule (majority relation). A major defect of this model is impossibility to define the best choice simply as a choice of maximal elements of a relation representing preferences, since the majority relation almost never possesses maximal elements. Over the last 50 years of research in the area numerous attempts to bypass the Condorcet paradox led to proliferation of alternative concepts of optimal social choice and related solutions, always nonempty and Condorcet consistent (i.e. picking up maximal elements of the majority relation whenever they exist).

In this paper we develop a unified matrix-vector representation of such solutions as the core, the uncovered, uncaptured, untrapped and minimal externally stable sets, the weak and strong top cycle sets, the classes of *k*-stable alternatives and k-stable sets. This representation determines convenient algorithms for their calculation. We also propose new versions of some tournament solutions.

The structure of the text is as follows. Basic definitions and notations are given in Section 2. In this section it is demonstrated how a relation and a subset of alternatives can be represented by the Boolean matrix and the Boolean characteristic vector, and the vector-matrix representation for the set of maximal elements of an arbitrary relation is obtained.

Section 3 contains matrix-vector representations for the following solution concepts: the Condorcet winner, the core, the fifteen versions of the uncovered set [1-6], the uncaptured set [7], the union of minimal externally stable sets [8-10], the weak and strong top cycle sets [11-16], the untrapped set [7]. These representations are obtained in the general case, when ties are allowed. Also in this section new versions of the uncovered set and a new version of the minimal weakly stable set, called weakly externally stable set, are proposed. A criterion to determine whether an alternative belongs to the union of minimal weakly externally stable sets is established. This criterion provides a connection between this solution and some versions of the covering relation.

Section 4 contains matrix-vector representations for the classes of k-stable alternatives and classes of k-stable sets introduced by Aleskerov and Subochev [17] (see also [10, 18]).

In Section 5 it is demonstrated how to use matrix-vector representations for calculation of such solutions as the weakly uncovered set and Levchenkov sets.

In Section 6 the results of the paper are summarized in the form of a theorem. The proof of Lemma 2 is given in Appendix.

2. Matrix-vector representation of sets and relations: basic definitions

A *decision* is modeled as a *choice* of a subset from a set A of available alternatives. We presume that A is finite, $|A|=n<\infty$. Alternatives from A are denoted by a unique natural number i, $1\leq i\leq n$. In computations a subset B, $B\subseteq A$, can be represented by the *characteristic* (n-component) *vector* $\mathbf{b}=[b_i]$: $b_i=1 \Leftrightarrow i\in B$

and b_i=0 $\Leftrightarrow$ $i\notin B$. The characteristic vectors of the set A and the set containing only one alternative $\{j\}$ will be denoted as **a** and **e**(j), respectively.

It is presumed that choices are guided by *preferences*. Preferences of a chooser are modeled by a *binary relation* ρ on A. Formally ρ is a set of ordered pairs from A, $\rho\subseteq A\times A$. A pair (i, j) that belongs to ρ is also denoted as $i\rho j$. In computations a binary relation on A can be uniquely represented by $(n\times n)$ *matrix* $\mathbf{R}=[r_{ij}]$: $r_{ij}=1 \Leftrightarrow (i, j)\in\rho$ and $r_{ij}=0 \Leftrightarrow (i, j)\notin\rho$. Matrix $\mathbf{E}=[e_{ij}]$: $e_{ij}=1$ if $i=j$, 0 otherwise, represents the relation of identity ε: $(i, j)\in\varepsilon \Leftrightarrow i=j$.[1]

If it is not specifically noted, all matrices and vectors are presumed to be Boolean ones. Therefore in all expressions, containing addition and/or multiplication of elements, these operations are understood as logical disjunction and conjunction, respectively. Addition and multiplication of matrices and vectors are defined and denoted in a standard way. $\mathbf{R}^{tr}$ denotes a transposed matrix: $\mathbf{Q}=\mathbf{R}^{tr} \Leftrightarrow q_{ij}=r_{ji}$. $\overline{\mathbf{R}}$ and $\overline{\mathbf{v}}$ denote a matrix and a vector obtained by logical inversion of values of all entries of the corresponding matrix **R** and vector **v**, $\overline{r}_{ij}=0 \Leftrightarrow r_{ij}=1$. If **v** is the characteristic vector for a set V, $V\subseteq A$, then $\overline{\mathbf{v}}$ is the characteristic vector for the set $A\backslash V$.

An idea of optimal choice is connected with the concept of *maximal element* of a preference relation. There are two versions of what is to be considered as a maximal element.

Definition 1. An alternative i is a *weak maximal element* of a relation ρ or weak ρ-maximal in A if $\forall j\ j\rho i \Rightarrow i\rho j$.[2]

Definition 2. An alternative i is a *strong maximal element* of a relation ρ or strong ρ-maximal in A if $\forall j\neq i\ (j, i)\notin\rho$.

The set of strong maximal elements is always a subset of the set of weak maximal elements. If ρ is *asymmetric* (i.e. $\forall(i, j)\ (i, j)\in\rho \Rightarrow (j, i)\notin\rho$), these sets

[1] Throughout the paper plain lowercase letters without indices denote alternatives or numbers, plain capital letters without indices - sets of alternatives, Greek letters - relations. Vectors are denoted by bold small letters, vector components - by plain small letters with one index. Matrices are denoted by bold capital letters, matrix elements - by plain small letters with two indices.
[2] Here and below $\forall j$ stands for $\forall j\in A$.

coincide. If ρ is also acyclic they are always nonempty. If ρ is transitive then the set of its weak maximal elements is always nonempty as well. This is not true for the set of strong maximal elements. In this paper, a term “maximal element” is used instead of “weak maximal element”.

Let *MAX*(ρ) denote the set of all alternatives that are ρ-maximal in *A*. If **R** is the matrix representing ρ, then $i \notin MAX(\rho) \Leftrightarrow \exists j$: $j \neq i$ & $r_{ij}=0$ & $r_{ji}=1$. Let $\mathbf{Q}=\overline{\mathbf{R}+\overline{\mathbf{R}}^{tr}}$, then ($\exists j$: $j\neq i$ & $r_{ij}=0$ & $r_{ji}=1$) ⇔ ($\exists j$: $q_{ij}=1$). Then $i\in MAX(\rho) \Leftrightarrow q_{ij}=0$ for all j, $1\leq j\leq n$. Let $\mathbf{v}=\mathbf{Q}\cdot\mathbf{a}$, then $v_i=\sum_{k=1}^{n} q_{ik}\cdot a_k=0$ iff $q_{ij}=0$ for all j, $1\leq j\leq n$, then $v_i=0$ iff $i\in MAX(\rho)$. Therefore $\overline{\mathbf{v}}=\overline{\mathbf{Q}\cdot\mathbf{a}}=\overline{(\mathbf{R}+\overline{\mathbf{R}}^{tr})\cdot\mathbf{a}}=\mathbf{max}(\rho)$ is the characteristic vector for the set *MAX*(ρ).

The matrix expression for **max**(ρ) looks simpler when ρ is asymmetric or *complete* (i.e. $\forall(i,j)$ $(i,j)\in\rho \vee (j,i)\in\rho$).

If ρ is asymmetric then $\forall j\neq i$ $r_{ji}=1 \Rightarrow r_{ij}=0$, and $r_{ii}=0$ for all $i\in A$. Then $i\notin MAX(\rho) \Leftrightarrow \exists j$: $j\neq i$ & $r_{ji}=1$. Let $\mathbf{Q}=\mathbf{R}^{tr}$, then ($\exists j$: $j\neq i$ & $r_{ji}=1$) ⇔ ($\exists j$: $q_{ij}=1$). Consequently, $i\in MAX(\rho) \Leftrightarrow q_{ij}=0$ for all j, $1\leq j\leq n$, therefore $\mathbf{max}(\rho)=\overline{\mathbf{Q}\cdot\mathbf{a}}=\overline{\mathbf{R}^{tr}\cdot\mathbf{a}}$. It follows also from Definition 2 that this formula gives us the matrix-vector representation of the set of strong maximal elements of any ρ.

If ρ is complete then $\forall j\neq i$ $r_{ij}=0 \Rightarrow r_{ji}=1$. Consequently, $i\notin MAX(\rho) \Leftrightarrow \exists j$: $j\neq i$ & $r_{ij}=0$. Let $\mathbf{Q}=\overline{\mathbf{R}+\mathbf{E}}$, then ($\exists j$: $j\neq i$ & $r_{ij}=0$) ⇔ ($\exists j$: $q_{ij}=1$). Then $i\in MAX(\rho) \Leftrightarrow q_{ij}=0$ for all j, $1\leq j\leq n$, therefore $\mathbf{max}(\rho)=\overline{\mathbf{Q}\cdot\mathbf{a}}=\overline{(\overline{\mathbf{R}+\mathbf{E}})\cdot\mathbf{a}}$.

Let us formulate this result as

Lemma 1.

1) If **R** is the matrix representing a relation ρ, then the characteristic vector **max**(ρ) for the set of ρ-maximal elements *MAX*(ρ) is $\mathbf{max}(\rho)=\overline{(\overline{\mathbf{R}+\overline{\mathbf{R}}^{tr}})\cdot\mathbf{a}}$;

2) if ρ is asymmetric then $\mathbf{max}(\rho)=\overline{\mathbf{R}^{tr}\cdot\mathbf{a}}$;

3) ∀ρ $\mathbf{smax}(\rho)=\overline{\mathbf{R}^{tr}\cdot\mathbf{a}}$ is the vector of the set of strong ρ-maximal elements;

4) if ρ is complete then $\mathbf{max}(\rho)=\overline{(\overline{\mathbf{R}+\mathbf{E}})\cdot\mathbf{a}}$.

Definition 3. A binary relation π is called the *asymmetric part* of a binary relation ρ if $\forall(i, j)\ (i, j)\in\pi \Leftrightarrow ((i, j)\in\rho \vee (j, i)\notin\rho)$.

Corollary of Lemma 1. If π is the asymmetric part of ρ then $MAX(\rho)=MAX(\pi)$.

Proof of the Corollary. Let **P** be the matrix representing π. It follows from $\overline{a \vee b} = \bar{a} \wedge \bar{b}$ that $\mathbf{P}=\overline{(\mathbf{R}^{tr} + \overline{\mathbf{R}})}$. Since π is asymmetric, by Lemma 1(2) $\mathbf{max}(\pi)=\overline{\mathbf{P}^{tr}(\mathbf{R})\cdot\mathbf{a}}$. Since $\overline{(\mathbf{R}^{tr})}=(\overline{\mathbf{R}})^{tr}$, we obtain $\mathbf{P}^{tr}=(\overline{(\mathbf{R}^{tr} + \overline{\mathbf{R}})})^{tr}=\overline{(\mathbf{R}^{tr} + \overline{\mathbf{R}})^{tr}}=\overline{(\mathbf{R} + \overline{\mathbf{R}}^{tr})}$. Then $\mathbf{max}(\pi)=\overline{\mathbf{P}^{tr}(\mathbf{R})\cdot\mathbf{a}}=\overline{\overline{(\mathbf{R} + \overline{\mathbf{R}}^{tr})}\cdot\mathbf{a}}=\mathbf{max}(\rho) \Leftrightarrow MAX(\rho)=MAX(\pi)$. □

An ordered pair (i, j) such that $i\rho j$ is also called a *ρ-step*. A *path* from i to j is an ordered sequence of steps starting at i and ending at j, such that the second alternative in each step coincides with the first alternative of the next step. If all steps in a path belong to the same relation ρ, we call it *ρ-path*, i.e. a ρ-path is an ordered sequence of alternatives $i, j_1, \ldots, j_{k-1}, j$, such that $i\rho j_1, j_1\rho j_2, \ldots, j_{k-2}\rho j_{k-1}, j_{k-1}\rho j$. The number of steps in a path is path's *length*. An alternative j is called *reachable in k steps* from i if there is a path of length k from i to j. A ρ-path from i to j is called a *minimal ρ-path* if $i\neq j$ and there is no ρ-path from i to j which is shorter. Also minimal ρ-paths must not be cycles.

Let $\kappa(\rho)$ denote the *transitive closure* of ρ: $(i, j)\in\kappa(\rho)$ if j is reachable from i via ρ. By definition, $\kappa(\rho)$ is reflexive. Let $\kappa_k(\rho)$ denote the *k-transitive closure* of ρ. The k-transitive closure is an abridged version of the transitive closure: $(i, j)\in\kappa_k(\rho) \Leftrightarrow i=j$ or j is reachable from i in no more than k steps via ρ. If d is the maximum of lengths of all minimal ρ-paths in A (i.e. if d is the diameter of the digraph, which represents ρ) then $\kappa_k(\rho)=\kappa(\rho) \Leftrightarrow k\geq d$. The value $d=d(\rho)$ will be called the *ρ-diameter* of A.

Relations μ, τ and υ

Now let us consider the framework of the social choice problem. A group of agents have to choose alternatives from the set A. The number of agents is greater than one. Each agent has preferences over alternatives from A. The preferences of the group are represented by *majority relation* μ, which is a binary relation on A, $\mu \subset A\times A$, constructed thus: $(i, j)\in\mu$ if and only if an alternative i is strongly preferred to an alternative j by majority of all agents. By assumption, majority is defined so that μ is asymmetric. The analysis presented here is mostly independent from the definition of what is majority as long as the respective majority relation is asymmetric. Decisions may be made under any version of majority: simple majority, absolute majority, qualified majority or unanimity – statements of all lemmas and theorem of the paper and almost all other statements, except for those that rely on McGarvey's theorem [19], will hold. If neither $(i, j)\in\mu$, nor $(j, i)\in\mu$ holds, and if $i\neq j$ then (i, j) is called a *tie*. A set of ties τ is a *symmetric* binary relation on A, $(i, j)\in\tau \Leftrightarrow (j, i)\in\tau$. We presume that τ is *irreflexive*: $\forall i\ (i, i)\notin\tau$. Let υ denote the relation, which is the union of μ, τ and ε, $\upsilon=\mu\cup\tau\cup\varepsilon$. It is complete and *reflexive* ($\forall i\ (i, i)\in\upsilon$). Relations μ, τ and υ can be interpreted as group's *strong social preference* (μ), *social indifference* (τ) and *weak social preference* (υ) relations.

A binary relation is called a *tournament* if it is asymmetric and complete. If simple majority rule is used, if the number of voters is odd and no individual voter is indifferent between any two alternatives, then μ is a tournament and τ is empty.

Let $\mathbf{M}=[m_{ij}]$, $\mathbf{T}=[t_{ij}]$ and $\mathbf{U}=[u_{ij}]$ denote the matrices representing μ, τ and υ, respectively. It is evident that $\mathbf{U}=\mathbf{M}+\mathbf{T}+\mathbf{E}=\overline{\mathbf{M}^{\mathrm{tr}}}$, $\mathbf{M}+\mathbf{M}^{\mathrm{tr}}+\mathbf{E}=\overline{\mathbf{T}}$.

The *lower section* of an alternative i is a set $L(i)=\{j\in A\colon i\mu j\}$. Correspondingly, the *upper section* of i is a set $D(i)=\{j\in A\colon j\mu i\}$. The *horizon* of i is a set $H(i)=\{j\in A\colon i\tau j\}$. Let $\mathbf{l}(i)$, $\mathbf{d}(i)$ and $\mathbf{h}(i)$ denote characteristic vectors of $L(i)$, $D(i)$ and $H(i)$, respectively. They are calculated by the following formulae.

$$(1)\qquad \mathbf{l}(i)=\mathbf{M}^{\mathrm{tr}}\cdot\mathbf{e}(i),\ \mathbf{d}(i)=\mathbf{M}\cdot\mathbf{e}(i),\ \mathbf{h}(i)=\mathbf{T}\cdot\mathbf{e}(i)$$

The proof is obvious.

3. Representations of various tournament solutions in the general case

A decision is defined as a choice of a subset of A. It is presumed that social decisions should be based on social preferences represented by μ. Therefore collective decision-making is modeled by *social choice functions S* that map the set of all possible majority relations {μ} into a set of subsets of A: $\{\mu\}\rightarrow 2^A$. McGarvey [19] proved that if majority is simple then the set of all possible majority relations on A equals the set of all asymmetric relations on A.

The Condorcet winner and the core

The first candidate for optimal social choice function is the *core Cr* - a set of all μ-maximal alternatives in A, $Cr=MAX(\mu)$, $i\in Cr \Leftrightarrow D(i)=\varnothing$. Since μ is an asymmetric part of υ, by Lemma 1 and its Corollary $Cr=MAX(\mu)=MAX(\upsilon)$ and

$$\mathbf{cr}=\mathbf{max}(\mu)=\mathbf{max}(\upsilon)=\overline{\mathbf{M}^{\mathrm{tr}}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}}\cdot\mathbf{a}}=\overline{\overline{(\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}.$$

If $\mathbf{R}=\mathbf{X}+\mathbf{Y}$, then $r_{ij}=1 \Leftrightarrow x_{ij}=1 \vee y_{ij}=1$. Let $\mathbf{R}=\mathbf{M}+\mathbf{E}$. If $\forall j\neq i\ r_{ij}=1$ then i is a *Condorcet winner* - an alternative dominating any other alternative via the majority relation (there can be at most one such alternative). Therefore $\mathbf{cw}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$ is the characteristic vector of the set CW containing only the Condorcet winner cw, $CW=\{cw\}$.

The main problem with the set of Condorcet winners and the core is that they are almost always empty, which makes these social choice functions impractical. But they provide us with an important normative principle – the outcome of an optimal social choice function should always coincide with the set of Condorcet winners, when the Condorcet winner exists (*Condorcet consistency*). Finally the social choice should be based on social preferences only. That means that a social choice function must satisfy *neutrality* - it should be invariant with respect to the automorphism group of the digraph corresponding to the given relation (if a permutation of alternatives' numbers does not change the relation then alternatives chosen after the permutation should be labeled with the same numbers as ones chosen before).

Definition 3. A social choice function S: $\{\mu\}\rightarrow 2^A$ is called a (*tournament*) *solution* if

1) it is always nonempty, $\forall\mu$ $S(\mu)\neq\emptyset$;

2) it satisfies Condorcet consistency, $\forall\mu$ $CW\neq\emptyset \Rightarrow S(\mu)=CW$;

3) it is neutral.

To distinguish solutions thus defined from other types of solutions the term "tournament solution" is used. It is applied irrespectively of whether a solution considered was defined for the set of all asymmetric relations or just for tournaments. If two social choice functions coincide when their domains are restricted to the set of tournaments, they may be considered as different *versions* of the same tournament solution.

We limit our account only to several such solutions, and refer a reader to an extensive account of tournament solutions made by Laslier [20] for information on other concepts proposed in the literature.

The uncovered set

The first tournament solution we consider is the uncovered set UC [1,2]. We substitute a subrelation α, $\alpha\subseteq\upsilon$, called the *covering relation* for the weak social preference relation υ and choose strong α-maximal elements. The covering relation has its normative justification (provided below), therefore a choice from the uncovered set can be regarded as optimal.

The versions of the covering relation are presented in Table 1. These versions will be denoted α^{Nn}, where N is a Roman numeral denoting a row and n is a small letter denoting a column. An alternative i *covers* an alternative j (version Nn), $i\alpha^{\mathrm{Nn}}j$, if the condition in the cell (N, n) holds.

Table 1. The versions of the covering relation

N\n	a	b	c
I	$i\mu j$ & $L(j)\subset L(i)\cup H(i)$	$(i\mu j \vee L(i)\cap D(j)\neq\emptyset)$ & $L(j)\subseteq L(i)\cup H(i)$	$L(j)\subseteq L(i)\cup H(i)$
II	$i\mu j$ & $L(j)\subset L(i)$	$L(j)\subset L(i)$	$L(j)\subseteq L(i)$
III	$i\mu j$ & $D(i)\subset D(j)$	$D(i)\subset D(j)$	$D(i)\subseteq D(j)$

IV	$i\mu j \,\&\, ((L(j)\subset L(i) \,\&\, D(i)\subseteq D(j)) \vee (L(j)\subseteq L(i) \,\&\, D(i)\subset D(j)))$	$(L(j)\subset L(i) \,\&\, D(i)\subseteq D(j)) \vee (L(j)\subseteq L(i) \,\&\, D(i)\subset D(j))$	$L(j)\subseteq L(i) \,\&\, D(i)\subseteq D$
V	$H(j)\cup L(j)\subset L(i)$	$(H(j)\setminus\{i\})\cup L(j)\subseteq L(i) \,\&\, H(i)\cup H(j)\neq\{i, j\}$	$(H(j)\setminus\{i\})\cup L(j)\subseteq L(i$

Since relations α^{Na} and α^{Nb} are asymmetric, all their maximal elements are strong. Since α^{Nb} is an asymmetric part of α^{Nc}, maximal elements of α^{Nb} are weak maximal elements of α^{Nc}. Let us define an *uncovered alternative* as a strong maximal element of α^{Nn} and the *uncovered set* UC^{Nn} as the set of such alternatives.

Let us consider the following example.

Example 1. $A=\{a, b, c, d, e\}$ and $\mu=\{(a, b), (b, c), (c, d), (c, e), (d, e), (e, b)\}$.

In Example 1 $L(a)=\{b\}$, $L(b)=\{c\}$, $L(c)=\{d, e\}$, $L(d)=\{e\}$, $L(e)=\{b\}$, $D(a)=\varnothing$, $D(b)=\{a\}$, $D(c)=\{b\}$, $D(d)=\{c\}$, $D(e)=\{c, d\}$, $H(a)=\{c, d, e\}$, $H(b)=\{d\}$, $H(c)=\{a\}$, $H(d)=\{a, b\}$, $H(e)=\{a\}$, therefore $\alpha^{\mathrm{Ia}}=\{(a, b), (c, d), (d, e)\}$, $\alpha^{\mathrm{Ib}}=\{(a, b), (a, c), (c, d), (d, e)\}$, $\alpha^{\mathrm{Ic}}=\{(a, b), (a, c), (a, d), (a, e), (c, d), (d, a), (d, e), (e, a)\}$, $\alpha^{\mathrm{IIa}}=\alpha^{\mathrm{IIb}}=\{(c, d)\}$, $\alpha^{\mathrm{IIc}}=\{(a, e), (c, d), (e, a)\}$, $\alpha^{\mathrm{IIIa}}=\{(a, b), (d, e)\}$, $\alpha^{\mathrm{IIIb}}=\alpha^{\mathrm{IIIc}}=\{(a, b), (a, c), (a, d), (a, e), (d, e)\}$, $\alpha^{\mathrm{IVa}}=\alpha^{\mathrm{Va}}=\varnothing$, $\alpha^{\mathrm{IVb}}=\alpha^{\mathrm{IVc}}=\alpha^{\mathrm{Vb}}=\alpha^{\mathrm{Vc}}=\{(a, e)\}$, consequently $UC^{\mathrm{Ia}}=\{a, c\}$; $UC^{\mathrm{Ib}}=\{a\}$; $UC^{\mathrm{Ic}}=\varnothing$; $UC^{\mathrm{IIa}}=UC^{\mathrm{IIb}}=\{a, b, c, e\}$; $UC^{\mathrm{IIc}}=\{b, c\}$; $UC^{\mathrm{IIIa}}=\{a, c, d\}$; $UC^{\mathrm{IIIb}}=UC^{\mathrm{IIIc}}=\{a\}$; $UC^{\mathrm{IVa}}=UC^{\mathrm{Va}}=A$; $UC^{\mathrm{IVb}}=UC^{\mathrm{IVc}}=UC^{\mathrm{Vb}}=UC^{\mathrm{Vc}}=\{a, b, c, d\}$.

The version α^{IIIb} of the covering relation might be called the oldest one. It was proposed by Fishburn [1] and is related to the game-theoretic concept of majorization proposed by Gillies [21]. Its normative rationale is the following: if i is better than j and if all alternatives, which are better than i, are also better than j, then it is suboptimal to choose j [1]. Similar justifications can be constructed for other versions. Independently of Fishburn, Miller [2] proposed α^{IIc} for tournaments and proved that (in tournaments) it is equivalent to α^{IIa}, α^{IIb} and α^{IIIb}. Also he suggested α^{IVc} for the general case (when $\tau\neq\varnothing$). After Miller's work, Richelson [3] proposed α^{IVb}. Bordes [4] considered uncovered sets based on versions α^{IIa}, α^{IIb}, α^{IIIb} and α^{IVb} as different choice functions in the case $\tau\neq\varnothing$ and proposed α^{IIIa} and α^{IVa} to complete the picture. He also proposed $L(j)\subset L(i)$ & $D(i)\subset D(j)$ as a version of covering relation but (then) wrongly attributed it to Miller. Independently of Bordes, the version α^{IVa} was suggested by McKelvey [5]. Duggan [6] proposed α^{Ia} and α^{Va}. Here again to complete the picture we propose the versions α^{Ib}, α^{Ic}, α^{IIIc}, α^{Vb}, α^{Vc} of the covering relation and UC^{IIc} as a choice function different (in the general case) from UC^{IIb} and other versions of the uncovered set.

In a tournament all versions of the uncovered set coincide and are equal to a set of maximal elements of 2-transitive closure of μ, $UC=MAX(\kappa_2(\mu))$, which is yet one more normative justification for this solution. Another interpretation of this equality is given in Section 4.

Nonemptiness of UC^{Na} and UC^{Nb}, N=II÷IV, is guaranteed by transitivity of α^{Na} and α^{Nb}, N=II÷IV. Nonemptiness of UC^{Va} follows from nonemptiness of UC^{IVa} and from inclusion $\alpha^{IVa}\supseteq\alpha^{Va}$. Relation α^{Vc} can not have cycles of length greater then two, therefore α^{Vb}, which is an asymmetric part of α^{Vc}, is acyclic and UC^{Vb} is always nonempty as well. Thus versions UC^{Na} and UC^{Nb}, N=II÷V are true tournament solutions.

The relations α^{Ia} and α^{Ib} may have cycles, and for any q-majority rule (except unanimity) it is possible to find a relation μ such that $UC^{Ia}=UC^{Ib}=\varnothing$, e.g. one may construct a μ-cycle of sufficiently great length.

Example 2. $A=\{a, b, c\}$, $\mu=\{(a, c), (b, c)\}$.

Example 2 can be constructed for any q-majority rule. Here $\tau=\{(a, b), (b, a)\}$, $\upsilon=\{(a, c), (a, b), (b, a), (b, c)\}$, $\alpha^{Na}=\alpha^{Nb}=\mu$, $\alpha^{Nc}=\upsilon$, as a result $UC^{Na}=UC^{Nb}=\{a, b\}$, $UC^{Nc}=\varnothing$, N=I÷V. Thus social choice functions UC^{Nc} are not true solutions for all types of majority rule, and UC^{In} are not true solutions for all types of majority rule except unanimity. Nevertheless, it is important to know how they are calculated, since these versions of the uncovered set can be used for calculation of several other tournament solutions, such as the union of minimal weakly externally stable sets and classes of k-stable sets.

Let us construct the matrices representing α^{Nn}. An alternative i belongs to versions UC^{Na} or UC^{Nc} if there exists a path of a certain type from i to any other alternative. The conditions for $i \in UC^{Nn}$ are listed in Table 2.

Table 2. The versions of the uncovered alternatives

N\n	a	c
I	$\forall j \neq i\ i\mu j \vee i\tau j \vee \exists k$: $i\mu k\ \&\ k\mu j$	$\forall j \neq i\ i\mu j \vee \exists k$: $i\mu k\ \&\ k\mu j$
II	$\forall j \neq i\ i\mu j \vee i\tau j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j)$	$\forall j \neq i\ i\mu j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j)$
III	$\forall j \neq i\ i\mu j \vee i\tau j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\tau k\ \&\ k\mu j)$	$\forall j \neq i\ i\mu j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\tau k\ \&\ k\mu j)$
IV	$\forall j \neq i\ i\mu j \vee i\tau j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j) \vee (i\tau k\ \&\ k\mu j)$	$\forall j \neq i\ i\mu j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j)$ $(i\tau k\ \&\ k\mu j)$
V	$\forall j \neq i\ i\mu j \vee i\tau j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j) \vee (i\tau k\ \&\ k\mu j) \vee (i\tau k\ \&\ k\tau j)$	$\forall j \neq i\ i\mu j \vee \exists k$: $(i\mu k\ \&\ k\mu j) \vee (i\mu k\ \&\ k\tau j)$ $(i\tau k\ \&\ k\mu j) \vee (i\tau k\ \&\ k\tau j)$

Calculating the uncovered set in a tournament, Banks [22] considered a product $\mathbf{R}=\mathbf{M}\cdot\mathbf{M}$ and pointed out that an element $r_{ij}=\sum_{k=1}^{n} m_{ik} \cdot m_{kj}$ is not equal to zero iff ($\exists k$: $i\mu k$ & $k\mu j$). That is $r_{ij} \neq 0$ iff there is a two-step μ-path from i to j. Since we presume that all vectors and matrices are Boolean ones, $r_{ij}=1$ iff there is a two-step μ-path from i to j and $r_{ij}=0$ otherwise. Respectively, if $\mathbf{R}=\mathbf{M}\cdot\mathbf{T}$ then $r_{ij}=1$ iff there is a two-step path from i to j, where the first step is a μ-step $i\mu k$, and the second step is a τ-step $k\tau j$, $r_{ij}=1 \Leftrightarrow$ ($\exists k$: $i\mu k$ & $k\tau j$), otherwise $r_{ij}=0$. Analogously, if $\mathbf{R}=\mathbf{T}\cdot\mathbf{M}$ then $r_{ij}=1 \Leftrightarrow$ ($\exists k$: $i\tau k$ & $k\mu j$), and if $\mathbf{R}=\mathbf{T}\cdot\mathbf{T}$ then $r_{ij}=1 \Leftrightarrow$ ($\exists k$: $i\tau k$ & $k\tau j$).

Let $\mathbf{Q}=\mathbf{M}^2+\mathbf{M}+\mathbf{T}+\mathbf{E}$. If $q_{ij}=1$ then either $i=j$, or $i\tau j$, or $i\mu j$, or $\exists k$: $i\mu k$ & $k\mu j$ hold. Consequently, if $q_{ij}=1$, then i is not covered by j according to the version α^{Ia}. If $\exists j$: $q_{ij}=0$, then neither $i=j$, nor $i\tau j$, nor $i\mu j$ holds, hence $j\mu i$. Also $\sum_{k=1}^{n} m_{ik} \cdot m_{kj}=0$ $\Rightarrow$ $(m_{kj}=1 \Rightarrow m_{ik}=0) \Rightarrow (k\mu j \Rightarrow (k\mu i \vee k\tau i))$. Therefore, if $q_{ij}=0$, then i is covered by j according to α^{Ia}. Then ($q_{ij}=1 \Rightarrow (j, i) \notin \alpha^{Ia}$ and $q_{ij}=0 \Rightarrow (j, i) \in \alpha^{Ia}$) $\Leftrightarrow$ $\overline{\mathbf{Q}}^{tr}=\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}^{tr}=\mathbf{R}$ is the matrix representation of α^{Ia}. Similar considerations produce matrices representing all α^{Na} and α^{Nc} (see Table 3)

Table 3. The matrix representation of the versions α^{Na} and α^{Nc} of the covering relation

N\n	a	c

I	$\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})^{tr}}$	$\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{tr}}$
II	$\overline{(\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})^{tr}}$	$\overline{(\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{tr}}$
III	$\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})^{tr}}$	$\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{tr}}$
IV	$\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})^{tr}}$	$\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{tr}}$
V	$\overline{(\mathbf{T}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})^{tr}}$	$(\mathbf{T}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+$

By Lemma 1 (Statement 3) $\mathbf{uc}^{Nn}=\mathbf{max}(\alpha^{Nn})=\overline{\mathbf{R}^{tr}\cdot\mathbf{a}}$ and we obtain the following formulae for the characteristic vectors $\mathbf{uc}^{Na}$ and $\mathbf{uc}^{Nc}$ of the uncovered sets UC^{Na} and UC^{Nc} (Table 4).

Table 4. Characteristic vectors of the uncovered sets UC^{Na} and UC^{Nc}

N\n	a	c
I	$\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}$	$\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$
II	$\overline{\overline{(\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}$	$\overline{\overline{(\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$
III	$\overline{\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}$	$\overline{\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$
IV	$\overline{\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}$	$\overline{\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$
V	$\overline{\overline{(\mathbf{T}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}$	$(\mathbf{T}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+$

A relation α^{Nb} is an asymmetric part of the relation α^{Nc}, therefore by Lemma 1 (Statement 1) and its Corollary expressions for characteristic vectors $\mathbf{uc}^{Nb}$ are given by the expression $\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{tr})}\cdot\mathbf{a}}$, where $\mathbf{R}$ is the matrix representing α^{Nc} (see Table 3). They are too long to be written in full here.

Finally, it should be noted that in terms of **M** and **U** the expressions for $\mathbf{uc}^{Nn}$ are simpler. These formulae are given in Section 6.

The uncaptured set

The concept of the uncaptured set *UCp* was proposed by Duggan [7]. The majority relation μ lacks maximal elements due to intransitivity, whereas transitivity of a relation guarantees nonemptiness of the set of its maximal elements. The choice from the *uncaptured set* is defined and justified as the choice of maximal elements of all maximal transitive subrelations of μ.

Duggan [7] proved that the uncaptured set coincides with the set of maximal elements of a certain subrelation of μ, which he called the *capturing relation*, $\beta\subseteq\mu$, $UCp=MAX(\beta)$. By definition an alternative i is *captured* by an alternative j, $(j, i)\in\beta$ if none of the following propositions holds: 1) $(j, i)\notin\mu$; 2) $\exists k$: $(i\mu k \,\&\, k\mu j)\vee(i\mu k \,\&\, k\tau j)\vee(i\tau k \,\&\, k\mu j)$; 3) $\exists k, l$: $(i\mu k \,\&\, k\mu l \,\&\, l\mu j)\vee(i\mu k \,\&\, k\tau l \,\&\, l\mu j)$. Therefore $i\in UCp$ if any alternative dominating i is either 1) reachable from i in two steps, at least one of which is a μ-step, or 2) reachable from i in three steps, the first and the last of which are μ-steps.

It is evident that in tournaments the uncaptured set is the set of maximal elements of 3-transitive closure of μ, $UCp=MAX(\kappa_3(\mu))$, which is another justification for this solution.

Let $\mathbf{Q}=\mathbf{M}\cdot\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E}$. If $q_{ij}=1$ then either $i=j$, or $i\tau j$, or $i\mu j$, or $\exists k$: $(i\mu k \,\&\, k\mu j)\vee(i\mu k \,\&\, k\tau j)\vee(i\tau k \,\&\, k\mu j)$, or $\exists k, l$: $(i\mu k \,\&\, k\mu l \,\&\, l\mu j)\vee(i\mu k \,\&\, k\tau l \,\&\, l\mu j)$ hold. Consequently, if $q_{ij}=1$ then i is not captured by j, $(j, i)\notin\beta$, and $(j, i)\in\beta$ if $q_{ij}=0$. Therefore

$$\mathbf{R}=\overline{\mathbf{Q}}^{\mathrm{tr}}=\overline{(\mathbf{M}\cdot\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}\cdot\mathbf{M}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}^{\mathrm{tr}}$$

is the matrix representation of the capturing relation β.

Since the capturing relation is asymmetric, by Lemma 1 (Statement 2) we obtain the following formula for the characteristic vector **ucp** of the uncaptured set UCp:

$$\mathbf{ucp}=\mathbf{max}(\beta)=\overline{\mathbf{R}^{\mathrm{tr}}\cdot\mathbf{a}}=$$
$$\overline{\overline{(\mathbf{M}\cdot\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}\cdot\mathbf{M}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}.$$

The weak and strong top cycles and the untrapped set

A nonempty set B, $B\subseteq A$, is called a *dominant set* if each alternative in B dominates each alternative outside B, $i\in B \Leftrightarrow (\forall j\notin B\ i\mu j)$ [11, 15, 16]. A dominant set will be called a *weak top cycle* set (denoted WTC) if it is minimal, i.e. if none of its proper subsets is a dominant set [1, 14, 15, 23].

Dominant sets must not be confused with *dominating* and (*)-*dominating* sets [29]. The latter concepts are defined for simple, non-directed graphs, which are used as representations of various networks. They are based on a notion of proximity and bear similarity to externally stable sets discussed in the next paragraph.

A nonempty set B, $B \subseteq A$, is called an *undominated set* if no alternative outside B dominates some alternative in B, $i \in B \Leftrightarrow (\forall j \notin B \Rightarrow (j, i) \notin \mu)$ [11]. An undominated set is called a *strong top cycle* set if it is maximal, i.e. if none of its proper subsets is an undominated set [12]. If such a set is not unique, then the solution is defined as the union of these sets [13]. It is denoted *STC*.

In Example 1 the undominated sets are $\{a\}$ and A and the only dominant set is A, therefore $STC=\{a\}$ and $WTC=A$.

In Example 2 undominated sets are $\{a\}$, $\{b\}$, $\{a, b\}$ and A, dominant sets are {a, b} and A. As a result, minimal undominated sets are $\{a\}$ and $\{b\}$, the minimal dominant set is $\{a, b\}$, consequently $STS=WTS=\{a, b\}$.

Solutions *STC* and *WTC* have multiple justifications. Concepts of dominant and undominated sets may be considered as generalizations of such notions as the Condorcet winner and the undominated alternative. Therefore *WTC* and *STC* are the substitutes for the Condorcet winner and the core, respectively. Alternatives from *STC* or *WTS* may be considered as the best choices because they belong to the subsets of A, which are the best ones (i.e. ones that do not contain subsets better than themselves) among a number of certain subsets endowed with some good property related to comparisons of alternatives.

STC also possesses an important property of *absorption*. If proposals are put to vote one by one, then any superset of an undominated set is absorbing in a sense that no voting trajectory leads outside of it. An absorbing set is absolutely stable. If at a certain round of voting an alternative from such a set is made a decision, then from this moment on it can be changed only for an alternative from the same set. Thus *STC* is a union of minimal absorbing sets.

The following normative justification will help us to construct matrix-vector representations for *STC* and *WTC*. Since transitivity guarantees the existence of maximal elements, Deb [24] proposed to use transitive closures of majority relations instead of relations themselves. He proved that sets of maximal elements of transitive closures of μ and υ coincide with *STC* and *WTC*: $STC=MAX(\kappa(\mu))$, $WTC=MAX(\kappa(\upsilon))$. In Example 2 $\kappa(\mu)=\mu$, $\kappa(\upsilon)=\upsilon=\{(a, b), (a, c), (b, a), (b, c)\}$, therefore $MAX(\kappa(\mu))=MAX(\kappa(\upsilon))=\{a, b\}$.

Let us consider matrices $\mathbf{M}_{(k)}=\sum_{i=1}^{k}\mathbf{M}^{i}+\mathbf{E}$ and $\mathbf{U}_{(k)}=\sum_{i=1}^{k}\mathbf{U}^{i}$; $m_{(k)ij}=1$ if there is a μ-path from i to j, $j\neq i$, of length no greater than k, also $m_{(k)ii}=1$ for all i. Consequently $\mathbf{M}_{(k)}$ represents the k-transitive closure of μ - $\kappa_k(\mu)$. Respectively, $\mathbf{U}_{(k)}$ represents $\kappa_k(\upsilon)$. By definition $\kappa_k(\mu)\neq\kappa_{d(\mu)}(\mu)$ if $k<d(\mu)$ and $\kappa_{d(\mu)}(\mu)=\kappa_k(\mu)=\kappa(\mu)$ for any $k\geq d(\mu)$. Analogously, $\kappa_k(\upsilon)\neq\kappa_{d(\upsilon)}(\upsilon)$ if $k<d(\upsilon)$ and $\kappa_{d(\upsilon)}(\upsilon)=\kappa_k(\upsilon)=\kappa(\upsilon)$ for any $k\geq d(\upsilon)$. Consequently, $\mathbf{M}_{(d(\mu))}$ and $\mathbf{U}_{(d(\upsilon))}$ are the representations of $\kappa(\mu)$ and $\kappa(\upsilon)$, respectively. Let us note that $\mathbf{M}_{(d(\mu))}\neq\mathbf{M}_{(d(\mu)-1)}$ & $\mathbf{M}_{(d(\mu))}=\mathbf{M}_{(d(\mu)+1)}$ and $\mathbf{U}_{(d(\upsilon))}\neq\mathbf{U}_{(d(\upsilon)-1)}$ & $\mathbf{U}_{(d(\upsilon))}=\mathbf{U}_{(d(\upsilon)+1)}$ hold.

Since $STC=MAX(\kappa(\mu))$, $\rho=\kappa(\mu)$, $\mathbf{R}=\mathbf{M}_{(d)}$, by Lemma 1(1) the characteristic vector of the strong top cycle set STC is

$$\mathbf{stc}=\overline{(\mathbf{M}_{(d)}+\overline{\mathbf{M}}^{tr}_{(d)})\cdot\mathbf{a}}.$$

A μ-diameter $d=d(\mu)$, is determined by a condition $\mathbf{M}_{(d)}\neq\mathbf{M}_{(d-1)}$ & $\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)}$.

Since $WTC=MAX(\kappa(\upsilon))$, $\rho=\kappa(\mu)$, $\mathbf{R}=\mathbf{U}_{(d)}$. Completeness of υ implies completeness of $\kappa_k(\upsilon)$ and $\kappa(\upsilon)$. Therefore by Lemma 1 (Statement 4) the characteristic vector of the weak top cycle set WTC is

$$\mathbf{wtc}=\overline{\overline{(\mathbf{U}_{(d)}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}}_{(d)}\cdot\mathbf{a}}.$$

A υ-diameter $d=d(\upsilon)$ is determined by a condition $\mathbf{U}_{(d)}\neq\mathbf{U}_{(d-1)}$ & $\mathbf{U}_{(d)}=\mathbf{U}_{(d+1)}$.

Nonemptiness of the set of maximal elements is also guaranteed by acyclicity. The *untrapped set UT* was defined by Duggan [7] as the set of maximal elements of all maximal acyclic subrelations of μ. Duggan [7] proved that the untrapped set coincides with the set of maximal elements of a certain subrelation of μ, which he called the *trapping relation* $\gamma\subseteq\mu$, $UT=MAX(\gamma)$.

An alternative *i traps* an alternative *j* if $i\mu j$ and i is not reachable from j via μ, $(i,j)\in\gamma \Leftrightarrow (i,j)\in\mu$ & $(j,i)\notin\kappa(\mu)$ [7].

In Example 1 $\gamma=\{(a, b)\}$ and $UT=\{a, c, d, e\}$. In Example 2 $\gamma=\mu$, $UT=MAX(\mu)=\{a, b\}$.

Let $\mathbf{Q}=\mathbf{M}_{(d(\mu))}+\mathbf{T}$. If $q_{ij}=1$ then either $i\tau j$ or $i\kappa(\mu)j$ holds. Consequently, if $q_{ij}=1$ then i is not trapped by j, $(j, i)\notin\gamma$, and $(j, i)\in\gamma$ if $q_{ij}=0$. Therefore if $d=d(\mu)$

then $\mathbf{R}=\overline{\mathbf{Q}}^{\mathrm{tr}}=\overline{(\mathbf{M}_{(\mathrm{d})}+\mathbf{T})}^{\mathrm{tr}}$ represents γ. Since trapping is asymmetric, by Lemma 1 (Statement 2) we obtain the following formula for the characteristic vector **ut** of the untrapped set *UT*:

$$\mathbf{ut}=\mathbf{max}(\gamma)=\overline{\mathbf{R}^{\mathrm{tr}}\cdot\mathbf{a}}=\overline{\overline{\mathbf{Q}}\cdot\mathbf{a}}=\overline{\overline{(\mathbf{M}_{(\mathrm{d})}+\mathbf{T})}\cdot\mathbf{a}}.$$

A diameter $d=d(\mu)$ again is determined by a condition $\mathbf{M}_{(d)}\neq\mathbf{M}_{(d\text{-}1)}$ & $\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)}$ or, alternatively, $\mathbf{M}^{d}\neq\mathbf{M}_{(d\text{-}1)}$ & $\mathbf{M}^{d+1}=\mathbf{M}_{(d)}$.

If μ is a tournament, solutions *STC*, *UT* and *WTC* coincide and are called the top cycle set *TC*.

The minimal weakly stable, minimal externally stable and minimal weakly externally stable sets

To the best of our knowledge, the idea of *weak stability* appears for the first time in [25]. The concept of the minimal weakly stable set as a social choice rule was introduced by Aleskerov and Kurbanov [9]. A nonempty set *B*, $B\subseteq A$, is called a *weakly stable set* if it has the following property: if *i* belongs to *B*, then for any *j* outside *B*, which dominates *i*, there is *k* in *B*, which dominates *j*. In terms of upper and lower sections *B* is weakly stable if $\forall j\notin B\ B\cap L(j)\neq\varnothing \Rightarrow B\cap D(j)\neq\varnothing$. A weakly stable set is called a *minimal weakly stable set* if none of its proper subsets is a weakly stable set. If such set is not unique, then the solution is defined as the union of these sets.

When μ is a tournament the notion of weak stability coincides with von Neumann-Morgenstern's concept of *external stability*: *B* is externally stable if $\forall j\notin B\ B\cap D(j)\neq\varnothing$, that is *B* is a externally stable set if there is one-step μ-path from some alternative in *B* to any alternative outside *B* [26]. If one assumes (in contradiction with our assumption of μ's asymmetry) that μ is symmetric, then definition of an externally stable set turns into a definition of a dominating set [29] mentioned in the previous paragraph, since symmetric binary relations are represented by simple (non-directed) graphs.

Weak stability is a generalization of external stability, since the latter implies the former but not vice versa. Thus it is possible to define the second version of this solution: a union of *minimal externally stable sets MES*. A definition of a minimal externally stable set was given by Wuffl, Feld and Owen

[8], but they consider the framework of spatial voting games and do not use *MES* as a solution concept. The union of minimal externally stable sets was proposed as a social choice rule in general (non-spatial) setting and as a version of the union of minimal weakly sets by Subochev [10]. It is evident that the union of minimal dominating sets will always coincide with the universal set A.

To see the difference between *MWS* and *MES* let us consider Example 1. It is not difficult to check that the only minimal weakly stable sets are $\{a\}$ and $\{c, e\}$, therefore $MWS=\{a, c, e\}$. At the same time the only minimal externally stable sets are $\{a, c\}$ and $\{a, b, d\}$, consequently $MES=\{a, b, c, d\}$.

Like *STC* and *WTC* the concepts of *MWS* and *MES* may be justified as choices from the narrowest subsets of *A* that possess some good property related to binary comparisons of alternatives. The idea of the minimal externally stable set may also be viewed as a generalization of the concept of *WTC*: the former is obtained from the latter through the substitution of ∃ for ∀ in the definition of the dominant set.

Another interpretation of *MWS* and *MES* is presented in Section 4, where generalizations of *UC* and *MWS* are discussed.

To calculate *MES* we will use the following theorem: $i \in MES \Leftrightarrow i \in UC^{\mathrm{IIIa}} \vee \exists j\text{: } j \in L(i) \;\&\; j \in UC^{\mathrm{IIIa}}$ [10]. That is an alternative belongs to *MES* iff it ether belongs to UC^{IIIa}, or belongs to an upper section of some alternative from UC^{IIIa}. For instance, in Example 1 *b* is covered (version IIIa) by *a* but dominates *c*, which is uncovered (version IIIa), therefore according to the theorem *b* must belong to *MES*, and indeed it does, since *b* belongs to the minimal externally stable set $\{a, b, d\}$.

Consequently, *MES* is a union of UC^{IIIa} and upper sections of all alternatives from UC^{IIIa}. By the formula (1) for the characteristic vector $\mathbf{d}(UC^{\mathrm{IIIa}})$ of the union of upper sections of all alternatives from UC^{IIIa} we obtain

$$\mathbf{d}(UC^{\mathrm{IIIa}}) = \sum_{i \in UC^{\mathrm{IIIa}}} \mathbf{d}(i) = \sum_{i \in UC^{\mathrm{IIIa}}} \mathbf{M} \cdot \mathbf{e}(i) = \mathbf{M} \cdot \sum_{i \in UC^{\mathrm{IIIa}}} \mathbf{e}(i) = \mathbf{M} \cdot \mathbf{uc}^{\mathrm{IIIa}}.$$

Thus $\mathbf{mes}=\mathbf{uc}^{\mathrm{IIIa}}+\mathbf{d}(UC^{\mathrm{IIIa}})=\mathbf{uc}^{\mathrm{IIIa}}+\mathbf{M}\cdot\mathbf{uc}^{\mathrm{IIIa}}=(\mathbf{M}+\mathbf{E})\cdot\mathbf{uc}^{\mathrm{IIIa}}$ and finally

$$\mathbf{mes}=(\mathbf{M}+\mathbf{E})\cdot \overline{\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{T}+\mathbf{E})}\cdot\mathbf{a}}.$$

Unfortunately, we could not get similar representation for *MWS*.

Let us also propose a new (third) version of the minimal weakly stable set. This version is important for calculation of rankings based on the idea of relative stability of alternatives and sets (described in Section 4). A set B will be called a *weakly externally stable set* if $\forall i \notin B \; B \cap (D(i) \cup H(i)) \neq \varnothing$. That is B is a weakly externally stable set if there is a one-step υ-path from some alternative in B to any alternative outside B, $\forall i \notin B \;\; \exists j: j \in B \;\&\; j \upsilon i$. Correspondingly, B is not a weakly externally stable set if $\exists i$: $B \subseteq L(i)$, i.e. if there is an alternative dominating all alternatives from B via μ. The weak external stability, like external stability, is monotonous, i.e. if $B \subseteq C$ then weak external stability of B implies weak external stability of C.

A new social choice function is defined as the union of all *minimal weakly externally stable sets* in A and denoted *MWES*. In a tournament solutions *MWS*, *MES* and *MWES* coincide and are denoted as *MWS*.

A criterion to determine whether an alternative belongs to *MWES* is given by the following

Lemma 2. An alternative i belongs to a minimal weakly externally stable set *MWES* iff i is uncovered according to α^{IIa} or if some alternative from the lower section of i or from the horizon of i is not covered (version α^{IIc}) by any alternative from the upper section of i,

$$i \in MWES \Leftrightarrow i \in UC^{\mathrm{IIa}} \vee \exists j: j \in L(i) \cup H(i) \;\&$$
$$\& \; (\forall k \in D(i) \; j \mu k \vee \exists l: (j \mu l \;\&\; l \mu k) \vee (j \mu l \;\&\; l \tau k))).$$

The proof of Lemma 2 is given in Appendix.

Lemma 2 allows us to find the matrix representation of *MWES*. Let $\mathbf{R}=\overline{\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E}}=\overline{\mathbf{M}\cdot\mathbf{U}+\mathbf{E}}$. Then $r_{ij}=0 \Leftrightarrow (j, i) \notin \alpha^{\mathrm{IIc}}$. Let $\mathbf{b}$ and $\mathbf{c}$ be characteristic vectors of sets B and C, $B \subseteq A$, $C \subseteq A$, respectively. Let $\mathbf{v}=\mathbf{R}\cdot\mathbf{b}$, then $v_i=1 \Leftrightarrow \exists j \in B$: $(j, i) \in \alpha^{\mathrm{IIc}}$. Consequently, $\overline{\mathrm{v}}_{\mathrm{i}}=1 \Leftrightarrow \forall j \in B \; (j, i) \notin \alpha^{\mathrm{IIc}}$. Then $(\mathbf{c}\cdot\overline{\mathbf{v}})=\sum_{k=1}^{n} \mathrm{c}_k \cdot \overline{\mathrm{v}}_k=1$ iff there is at least one alternative in C not covered (version α^{IIc}) by any alternative from B. Now let $B=D(i)$, $C=L(i) \cup H(i)$ and $f_i=(\mathbf{c}\cdot\overline{\mathbf{v}})$. Then $f_i=1$ iff there is some alternative from the lower section of i or from the horizon of i not covered (version α^{IIc}) by any alternative from the upper section of i, i.e. $\mathbf{f}$ is a characteristic vector of precisely those alternatives that satisfy the aforestated condition. Let **sch**(**R**) denote a vector made of diagonal elements of **R**, **v**=**sch**(**R**) $\Leftrightarrow$ $v_i=r_{ii}$ for all i, $1\le i\le n$. According to the formulae (1)

$$b_j=d(i)_j=(\mathbf{M}\cdot\mathbf{e}(i))_j=m_{ji};\ c_k=l(i)_k+h(i)_k=(\mathbf{M}^{tr}\cdot\mathbf{e}(i))_k+(\mathbf{T}\cdot\mathbf{e}(i))_k=m_{ik}+t_{ki}=m_{ik}+t_{ik}.$$

As a result $f_i=\sum_{j,k=1}^{n}(m_{ik}+t_{ik})\cdot\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}}_{kj}\cdot m_{ji}$,

that is $\mathbf{f}=\mathbf{sch}((\mathbf{M}+\mathbf{T})\cdot\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}}\cdot\mathbf{M})$.

Let **mwes** denote a characteristic vector of *MWES*. Then by Lemma 2 $\mathbf{mwes}=\mathbf{uc}^{IIa}+\mathbf{f}$. Therefore $\mathbf{mwes}=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}}\cdot\mathbf{a}+\mathbf{sch}((\mathbf{M}+\mathbf{T})\cdot\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}}\cdot\mathbf{M})$.

4. Classes of *k*-stable alternatives and *k*-stable sets

The concept of weak stability has the following rationale. An alternative can be interpreted as a particular state of affairs, which will emerge as a result of an act of choice. An undominated alternative is therefore an *absolutely* stable state: if it is a status quo, there will be no majority in favor of any change. Though not a stable state in an absolute sense, a dominated alternative might be *relatively* stable: such a state can be changed by majority vote for something else, but if this happens it is always possible to restore this state through a series of binary comparisons. The idea of relative stability of alternatives, which are understood as states, underlies the concept of sets of *k*-stable alternatives introduced by Aleskerov and Subochev [17]; see also [10]. Such solutions as the uncovered set, the uncaptured set and the top cycle are particular versions of these sets.

But it is also possible to interpret sets of alternatives as states or *macrostates*. We say that an alternative x as a state of affairs (a *microstate*) realizes a macrostate X, $X\subseteq A$, when $x\in X$. For instance, the fact, that a certain person (alternative) is occupying an elected office, might be treated as a microstate, and the fact, that a certain political party has this person as their member and exerts control over the office through him/her, might be treated as a macrostate. Components of the *STC* are therefore *absolutely* stable macrostates: it is not possible to change a status quo, when it belongs to a minimal undominated set, by proposing any alternative outside this set. The concepts of weakly stable and *k*-stable sets [10, 18] represent the idea of a *relatively* stable macrostate: a representative of a party might be voted out of the office and changed for an opposition figure, but the party is always able to restore their control in just one round (weakly stable set) or at most k rounds (k-stable set) of voting, respectively, by proposing other candidates (given the possibility to make such proposals). Similar interpretation is given by Wuffl, Feld and Owen [8] to justify the solution they proposed (Finagle's point), which is based on the concept of external stability and thus is related to *MWS* and *MES*.

Let μ be a tournament. An alternative i is called *generally stable* if every other alternative in A is reachable from i via μ [17]. Every alternative in A is reachable from i iff it is in the top cycle set *TC* [23], thus all alternatives in *TC* and only they are generally stable. The choice from *TC* is therefore justified as the choice of generally stable alternatives. Let $s(i, j)$ denote a minimal length function, which is equal to the length of a minimal μ-path from i to j.

Let $s(i, i)=0$ and $s(i, j)=\infty$ if j is not reachable from i via μ. Then i is generally stable iff $\max_{j\in A} s(i, j)<\infty$.

Generally stable alternatives are not equally stable – they may be more stable or less stable than the others. An alternative i is called *k-stable* if $\max_{j\in A} s(i, j)=k<\infty$ [17]. An alternative with smaller k is therefore considered as more stable one. Let $SP_{(k)}$ denote a class of k-stable alternatives in A. We define a solution $P_{(k)}$ as the set of those generally stable alternatives, from which it is possible to reach any given alternative in A in no more than k μ-steps, $P_{(k)}=SP_{(1)}+SP_{(2)}+\ldots+SP_{(k)}$. Solution $P_{(k)}$ is a generalization of the uncovered and the uncaptured sets since this definition and $\tau=\varnothing$ imply $P_{(k)}=MAX(\kappa_k(\mu))$, therefore $UC=P_{(2)}$ and $UCp=P_{(3)}$.

The ideas of the generally stable and k-stable sets replicate the concepts of the generally stable and k-stable alternatives. A set B, $B\subseteq A$, is called a *k-stable set* if for any alternative j outside B, $j\in A\backslash B$, there exists a μ-path of length s: $s\leq k$ to j from some alternative i from B, $i\in B$, but at the same time there is at least one alternative j outside B, $j\in A\backslash B$, such that it is not reachable in less than k μ-steps from any i: $i\in B$ [10, 18]. A k-stable set will be called a *minimal k-stable set* if none of its proper subsets is a k-stable set. It follows from this definition that a weakly (externally) stable set is a 1-stable set.

$SS_{(k)}$ denotes a class of those alternatives, which belong to some minimal k-stable set, but do not belong to any minimal stable set with the degree of stability less than k. We define a solution $S_{(k)}$ as a union of those minimal generally stable sets, from which it is possible to reach any alternative outside a set in no more than k steps: $S_{(k)}=SS_{(1)}+SS_{(2)}+\ldots+SS_{(k)}$. Solution $S_{(k)}$ generalizes the concept of the minimal weakly (externally) stable set, $S_{(1)}=MWS$.

Let $A=\{a, b, c, d, e\}$ and $\mu=\{(a, c), (a, d), (a, e), (b, a), (b, d), (b, e), (c, b), (c, e), (d, c), (e, d)\}$. Then $SP_{(1)}=\varnothing$, $SP_{(2)}=\{a, b, c\}$, $SP_{(3)}=\{d\}$, $SP_{(4)}=\{e\}$, $P_{(2)}=UC=\{a, b, c\}$, $P_{(3)}=UCp=\{a, b, c, d\}$, $SS_{(1)}=MWS=\{a, b, c, d\}$, $SS_{(2)}=\varnothing$, $SS_{(3)}=\varnothing$, $SS_{(4)}=\{e\}$, $S_{(1)}=S_{(2)}=S_{(3)}=\{a, b, c, d\}$, $S_{(4)}=A$.

It is possible to use the filtration by classes $SP_{(k)}$ and $SS_{(k)}$ as new methods of ranking alternatives in a tournament (described in [10, 18]).

Let $\mathbf{p}_{(k)}$ denote the characteristic vector of $P_{(k)}$. Since μ is complete, all $\kappa_k(\mu)$ are complete as well. By Lemma 1(4) $\mathbf{p}_{(k)}=\overline{\overline{(\mathbf{M}_{(k)}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}}$. Since all matrices are Boolean ones, the following equation holds $\sum_{i=1}^{k}\mathbf{M}^{i}+\mathbf{E}=(\mathbf{M}+\mathbf{E})^{k}$.

Consequently $\mathbf{M}_{(k)}=\sum_{i=1}^{k}\mathbf{M}^{i}+\mathbf{E}=\mathbf{U}^{k}$ and $\mathbf{p}_{(k)}=\overline{\overline{\mathbf{U}^{k}}\cdot\mathbf{a}}$.

Let $\mathbf{sp}_{(k)}$ denote the characteristic vector of $SP_{(k)}$. By definition of $P_{(k)}$ $i\in SP_{(k)} \Leftrightarrow i\in P_{(k)}$ & $i\notin P_{(k-1)}$. Therefore $sp_{(k)i}=p_{(k)i}\cdot\overline{p_{(k-1)i}}=\overline{\overline{p_{(k)i}}+p_{(k-1)i}}$, that is

$$\mathbf{sp}_{(k)}=\overline{\overline{\mathbf{p}_{(k)}}+\mathbf{p}_{(k-1)}}=\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}+\overline{\overline{\overline{\mathbf{M}}_{(k-1)}\cdot\mathbf{a}}}}=\overline{\overline{\mathbf{U}^{k}}\cdot\mathbf{a}+\overline{\overline{\overline{\mathbf{U}^{k-1}}\cdot\mathbf{a}}}}.$$

If $k=1$ then $\mathbf{M}_{(k-1)}=\mathbf{E}$. $\overline{\overline{\overline{\mathbf{E}}}\cdot\mathbf{a}}=\mathbf{o} \Rightarrow \mathbf{sp}_{(1)}=\mathbf{p}_{(1)}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\mathbf{cw}$ - the Condorcet winner.

If $k=2$ then $\mathbf{M}_{(k-1)}=\mathbf{M}+\mathbf{E}$. $\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\mathbf{cw}$. If there is the Condorcet winner, then $\overline{(\mathbf{M}^2+\mathbf{M}+\mathbf{E})\cdot\mathbf{a}}=\overline{\mathbf{cw}}$. $\mathbf{cw}+\overline{\mathbf{cw}}=\mathbf{a} \Rightarrow \mathbf{sp}_{(2)}=\overline{\overline{\mathbf{cw}}+\mathbf{cw}}=\overline{\mathbf{a}} \Leftrightarrow SP_{(2)}$ is empty. If there is no Condorcet winner then $\mathbf{sp}_{(2)}=\mathbf{p}_{(2)}=\overline{\overline{(\mathbf{M}^2+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\mathbf{uc}$, which corresponds to $SP_{(2)}=UC$ when $CW=\varnothing$.

It is also evident that $\mathbf{p}_{(3)}=\overline{\overline{(\mathbf{M}^3+\mathbf{M}^2+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\mathbf{ucp}$.

A is finite therefore $\exists m=\max\limits_{i\in WTC}(\max\limits_{j\in A} s(i, j))$. Then $P_{(m)}=TC$, $\mathbf{tc}=\mathbf{p}_{(m)}$ and $SP_{(k)}=\varnothing$ for all k: $k>m$. By definition of d, which is the μ-diameter of A, $m=\max\limits_{i\in WTC}(\max\limits_{j\in A} s(i, j))\leq\max\limits_{i\in A}(\max\limits_{j\in A} s(i, j))=d$, so there is no need to multiply $\mathbf{U}$ till the value of d is determined - it is enough to find m and stop. It was shown in [10] (see also [18]) that if $SP_{(1)}=CW=\varnothing$ then $SP_{(k)}\neq\varnothing$ for all k: $2\leq k\leq m$ and $SP_{(k)}=\varnothing$ for all k: $k>m$. Therefore the value of m can be determined from the condition $\mathbf{p}_{(m-1)}\neq\mathbf{p}_{(m)}$ & $\mathbf{p}_{(m)}=\mathbf{p}_{(m+1)}$.

A relation μ is asymmetric, but if there is no Condorcet winner, all k-transitive closures $\kappa_k(\mu)$, $k\geq 2$, possess the symmetric part, since all $\kappa_k(\mu)$ are complete and $|MAX(\kappa_2(\mu))|=|P_{(2)}|=|UC|\geq 3$ for any A: $|A|\geq 4$ [2]. Let $\kappa_k(\mu)$ be denoted as $\upsilon_{(k)}$, $\upsilon_{(k)}=\kappa_k(\mu)$. Let $\mu_{(k)}$ and $\tau_{(k)}$ denote asymmetric and symmetric parts of $\kappa_k(\mu)$, respectively. By definition $\upsilon_{(1)}=\upsilon$, $\mu_{(1)}=\mu$, $\tau_{(1)}=\tau\cup\varepsilon$.

Let us consider $\upsilon_{(k)}$ and $\mu_{(k)}$ as versions of relations υ and μ. If a set B, $B\subseteq A$, is a k-stable set it follows from the definition of a k-stable set that any alternative j outside B will be reachable from some alternative i from B in one $\upsilon_{(k)}$-step, i.e. $\forall j\notin B\ \exists i$: $i\in B$ & $i\upsilon_{(k)}j$. If the degree of stability of B is greater than k, then $\exists j$: $(j\notin B$ & $\forall i\in B\ (i, j)\notin\upsilon_{(k)})$. Consequently, if B is a minimal k-stable set with respect to μ, it must be a minimal weakly externally stable set with respect to $\upsilon_{(k)}$. Conversely, if B is a minimal weakly externally stable set with respect to $\upsilon_{(k)}$, then it must be a minimal stable set with degree of stability no less than k with respect to μ.

Let $\mathbf{ss}_{(k)}$ and $\mathbf{s}_{(k)}$ denote characteristic vectors of classes of k-stable sets $SS_{(k)}$ and solutions $S_{(k)}$, respectively. Let $MWES(\upsilon_{(k)})$ and $\mathbf{mwes}(\upsilon_{(k)})$ denote the union of minimal weakly externally stable sets and its characteristic vector with respect to $\upsilon_{(k)}$. Then $i\in SS_{(k)} \Rightarrow i\in MWES(\upsilon_{(k)})$ and $i\in MWES(\upsilon_{(k)}) \Rightarrow i\in SS_{(x)}$, x: $x\leq k$, that is $SS_{(k)}\subseteq MWES(\upsilon_{(k)})$ and $MWES(\upsilon_{(k)})\subseteq S_{(k)}$. Consequently $\mathbf{s}_{(k)}=\mathbf{mwes}(\upsilon_{(k)})+\mathbf{s}_{(k-1)}$. By definition $SS_{(1)}$ is the union of externally stable sets (with respect to μ), thus we obtain the following inductive formulae for calculation of $\mathbf{s}_{(k)}$:

$$\mathbf{s}_{(1)}=\mathbf{ss}_{(1)}=\mathbf{mes}=(\mathbf{M}+\mathbf{E})\cdot\mathbf{p}_{(2)}=\mathbf{U}\cdot\overline{\overline{\mathbf{U}}^2\cdot\mathbf{a}},$$

$$\mathbf{s}_{(k)}=\mathbf{s}_{(k-1)}+\mathbf{mwes}(\upsilon_{(k)})=\mathbf{s}_{(k-1)}+\overline{\overline{(\widetilde{\mathbf{M}}\cdot\widetilde{\mathbf{U}}+\widetilde{\mathbf{U}})}\cdot\mathbf{a}}+\mathbf{sch}((\widetilde{\mathbf{M}}+\widetilde{\mathbf{T}})\cdot\overline{\overline{(\widetilde{\mathbf{M}}\cdot\widetilde{\mathbf{U}}+\mathbf{E})}\cdot\widetilde{\mathbf{M}}}),$$

where $\widetilde{\mathbf{U}}=\mathbf{M}_{(k)}$, $\widetilde{\mathbf{M}}=\overline{(\mathbf{M}^{\mathrm{tr}}_{(k)}+\overline{\mathbf{M}}_{(k)})}$.

Since $P_{(k)}\subseteq S_{(k)}\subseteq P_{(k+2)}\subseteq TC$ [10] iterations will stop at some k between m-2 and m, when $\mathbf{s}_{(k)}$ becomes equal to $\mathbf{tc}=\mathbf{p}_{(m)}$. Finally $i\in SS_{(k)} \Leftrightarrow i\in S_{(k)}$ & $i\notin S_{(k-1)}$. Therefore $ss_{(k)i}=s_{(k)i}\cdot\overline{s_{(k-1)i}}=\overline{\overline{s_{(k)i}}+s_{(k-1)i}}$, that is $\mathbf{ss}_{(k)}=\overline{\overline{\mathbf{s}}_{(k)}+\mathbf{s}_{(k-1)}}$.

5. Weak uncovered set and Levchenkov sets

With a slight modification of our approach we may also obtain algorithms for calculations of two tournament solutions resulted from weakening of the notion of covering relation. Below as in the previous Section we consider only such μ that are tournaments.

Weak covering à la Laffond and Lainé

It is said that an alternative *x* *weakly covers y*, if *x*μ*y* and *card*{*z*: *y*μ*z* & *z*μ*x*}≤1, that is there is at most one 2-step μ-path from *y* to *x*. The *weak uncovered set WUC* is a strong top cycle with respect to the relation of weak covering [27]. This solution is a refinement of the uncovered set, *WUC*⊆*UC*.

To construct the matrix representing the weak covering relation one needs not only to know if there is a 2-step μ-path from *y* to *x*, but how many are there such paths. If elements of **M** are considered to be plain ordinary numbers 0 and 1 then the entry z_{ij} of its square $\mathbf{Z}=\mathbf{M}^2$ is the number of 2-step μ-paths from *i* to *j*. Thus to calculate *WUC* one may construct a Boolean matrix $\widetilde{\mathbf{M}}$ representing the weak covering relation from the ordinary matrices **M** and $\mathbf{Z}=\mathbf{M}^2$: for $i\neq j$ $\widetilde{\mathrm{m}}_{\mathrm{ij}}=1 \Leftrightarrow$ ($m_{ij}=1$ ∨ $z_{ij}\geq 2$); $\widetilde{\mathrm{m}}_{\mathrm{ii}}=0$ for all *i*. Then $\mathbf{wuc}=\overline{\overline{(\widetilde{\mathbf{M}}_{(\mathrm{d})}+\overline{\widetilde{\mathbf{M}}}{}^{\mathrm{tr}}_{(\mathrm{d})})\cdot\mathbf{a}}}$, where $\widetilde{\mathbf{M}}_{(k)}=\sum_{\mathrm{i}=1}^{\mathrm{k}}\widetilde{\mathbf{M}}^{\mathrm{i}}+\mathbf{E}$. The value of *d* is determined by a condition $\widetilde{\mathbf{M}}_{(d)}\neq\widetilde{\mathbf{M}}_{(d-1)}$ & $\widetilde{\mathbf{M}}_{(d)}=\widetilde{\mathbf{M}}_{(d+1)}$.

Weak covering à la Levchenkov

It is said that *z* is a *partner* of *x* against *y* if *x*μ*z* and *z*μ*y* holds. Let *q* be a natural number. It is defined that *x q-surpasses y* if *y* has no partner against *x* and either *x*μ*y* or *x* has at least *q* partners against *y* [28]. It is evident that *x* *q*-surpasses *y* if either *x* covers *y*, or *y*μ*x* & (∀*z* *y*μ*z* ⇒ *x*μ*z*) & *card*{*z*: *x*μ*z* & *z*μ*y*}≥*q* holds. A *Levchenkov set* of order *q* $LS_{(q)}$ is a set of maximal elements of a relation "*q*-surpasses". These solutions are refinements of the uncovered set,

$$LS_{(1)} \subseteq LS_{(2)} \subseteq \ldots \subseteq LS_{(q)} \subseteq \ldots \subseteq UC.$$

A Boolean matrix $\mathbf{R}_{(q)}$ representing the relation "q-surpasses" is constructed from $\mathbf{M}$ and $\mathbf{Z}=\mathbf{M}^2$ in the following way: for $i \neq j$ $r_{(q)ij}=1 \Leftrightarrow z_{ji}=0$ & $(m_{ij}=1 \vee z_{ij} \geq q)$; $r_{(q)ij}=0$ for all i. Since the relation "q-surpasses" is asymmetric, by Lemma 1 (Statement 2) for the characteristic vector $\mathbf{ls}_{(q)}$ of Levchenkov set of order q we obtain $\mathbf{ls}_{(q)}=\overline{\mathbf{R}_{(q)}^{\mathrm{tr}} \cdot \mathbf{a}}$.

For more information on these solutions see [20].

6. Conclusion

We have constructed an instrument for finding optimal social choice sets - a calculation technique based on matrix representations of corresponding preference relations. To avoid the problem of integer overflows we suggested the use of Boolean matrices instead of plain ordinary ones.

It turns out that the calculations needed to obtain any solution considered are of polynomial complexity. This opens the way for construction of algorithms with minimal complexity. This problem will be addresses in further publications.

Since any binary relation can be represented by a directed graph, calculation of choice functions, which are based on a majority preference relation, can be formulated as a graph-theoretical problem. On the other hand, simple non-directed graphs, which represent symmetric binary relations, are extensively used as models of various networks, for instance, social networks [30, 31, 32]. Therefore, a natural direction for further research would be to compare tournament solutions and algorithms presented in this paper with social network solution concepts, e.g., the dominating set (which has been briefly mentioned in Section 3).

It is also worth noting here that the constructed matrix-vector representation allowed us to introduce several new versions of some solutions, namely, versions of the uncovered set and minimal weakly externally stable set.

The following Theorem summarizes the results of this paper. We use matrices $\mathbf{M}$ and $\mathbf{U}$ instead of $\mathbf{M}$ and $\mathbf{T}$, since it makes all expressions simpler.

Theorem. Let **cw**, **cr**, $\mathbf{uc}^{\mathrm{Nn}}$, **mes**, **mwes**, **ucp**, **stc**, **ut** and **wtc**, respectively, denote characteristic vectors of the following solutions: the Condorcet winner *CW*, the core Cr, fifteen versions of the uncovered set UC^{Nn}, the union of minimal externally stable sets MES, the union of minimal weakly externally stable sets $MWES$, the uncaptured set UCp, the strong top cycle set STC, the untrapped set UT, the weak top cycle set WTC.

Let $\mathbf{sp}_{(k)}$, $\mathbf{ss}_{(k)}$, $\mathbf{p}_{(k)}$ and $\mathbf{s}_{(k)}$ denote characteristic vectors for classes of k-stable alternatives $SP_{(k)}$, classes of k-stable sets $SP_{(k)}$, and solutions $P_{(k)}=SP_{(1)}+SP_{(2)}+\ldots+SP_{(k)}$ and $S_{(k)}=SS_{(1)}+SS_{(2)}+\ldots+SS_{(k)}$, respectively. Let $\mathbf{a}$ denote the characteristic vector of the set A, ε denote the relation of identity represented by $\mathbf{E}=[\delta_{ij}]$, $d=d(\rho)$ denote a ρ-diameter of A. Let $\mathbf{M}$, $\mathbf{T}$, $\mathbf{U}$ denote Boolean matrices representing relations μ, τ and $\upsilon=\mu\cup\tau\cup\varepsilon$ on A. Finally, let $\mathbf{M}_{(k)}=\sum\limits_{i=1}^{k}\mathbf{M}^{i}+\mathbf{E}$ and $\mathbf{U}_{(k)}=\sum\limits_{i=1}^{k}\mathbf{U}^{i}$.

Then

1) $\mathbf{cw}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{cr}=\overline{\overline{\mathbf{U}}\cdot\mathbf{a}}=\overline{\mathbf{M}^{\mathrm{tr}}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{Ia}}=\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{Ib}}=\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{\mathrm{tr}})}\cdot\mathbf{a}}$, where $\mathbf{R}=\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{\mathrm{tr}}}$,

$\mathbf{uc}^{\mathrm{Ic}}=\overline{\overline{(\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IIa}}=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IIb}}=\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{\mathrm{tr}})}\cdot\mathbf{a}}$, where $\mathbf{R}=\overline{(\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{\mathrm{tr}}}$,

$\mathbf{uc}^{\mathrm{IIc}}=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IIIa}}=\overline{\overline{(\mathbf{U}\cdot\mathbf{M}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IIIb}}=\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{\mathrm{tr}})}\cdot\mathbf{a}}$, where $\mathbf{R}=\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{\mathrm{tr}}}$,

$\mathbf{uc}^{\mathrm{IIIc}}=\overline{\overline{(\mathbf{U}\cdot\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IVa}}=\overline{\overline{(\mathbf{U}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{IVb}}=\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{\mathrm{tr}})}\cdot\mathbf{a}}$, where $\mathbf{R}=\overline{(\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{\mathrm{tr}}}$,

$\mathbf{uc}^{\mathrm{IVc}}=\overline{\overline{(\mathbf{U}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{uc}^{\mathrm{Va}}=\overline{\overline{\mathbf{U}\cdot\mathbf{U}}\cdot\mathbf{a}}$,

$\mathbf{uc}^{Vb}=\overline{\overline{(\mathbf{R}+\overline{\mathbf{R}}^{tr})}\cdot\mathbf{a}}$, where $\mathbf{R}=\overline{(\mathbf{T}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{T}+\mathbf{M}\cdot\mathbf{M}+\mathbf{M}+\mathbf{E})^{tr}}$,

$\mathbf{uc}^{Vc}=\overline{\overline{(\mathbf{T}\cdot\mathbf{T}+\mathbf{U}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}\cdot\mathbf{a}}$,

$\mathbf{ucp}=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}\cdot\mathbf{M}+\mathbf{U}\cdot\mathbf{M}+\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{mes}=(\mathbf{M}+\mathbf{E})\cdot\overline{\overline{(\mathbf{U}\cdot\mathbf{M}+\mathbf{U})}\cdot\mathbf{a}}$,

$\mathbf{mwes}=\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{U})}\cdot\mathbf{a}}+\mathbf{sch}((\mathbf{M}+\mathbf{T})\cdot\overline{\overline{(\mathbf{M}\cdot\mathbf{U}+\mathbf{E})}\cdot\mathbf{M}})$,

$\mathbf{stc}=\overline{\overline{(\mathbf{M}_{(d)}+\overline{\mathbf{M}}_{(d)}^{tr})}\cdot\mathbf{a}}$, $d=d(\mu)$: $(\mathbf{M}_{(d)}\neq\mathbf{M}_{(d-1)})$ & $(\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)})$,

$\mathbf{ut}=\overline{\overline{(\mathbf{M}_{(d)}+\mathbf{T})}\cdot\mathbf{a}}$, $d=d(\mu)$: $(\mathbf{M}_{(d)}\neq\mathbf{M}_{(d-1)})$ & $(\mathbf{M}_{(d)}=\mathbf{M}_{(d+1)})$,

$\mathbf{wtc}=\overline{\overline{\mathbf{U}}_{(d)}\cdot\mathbf{a}}$, $d=d(\upsilon)$: $(\mathbf{U}_{(d)}\neq\mathbf{U}_{(d-1)})$ & $(\mathbf{U}_{(d)}=\mathbf{U}_{(d+1)})$.

2) If μ is a tournament, then $\mathbf{U}=\mathbf{M}+\mathbf{E}$, $\mathbf{M}_{(k)}=\mathbf{U}_{(k)}=\mathbf{U}^{k}$ and

$\mathbf{p}_{(k)}=\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^{k}}\cdot\mathbf{a}}$,

$\mathbf{sp}_{(k)}=\overline{\overline{\mathbf{p}_{(k)}}+\mathbf{p}_{(k-1)}}=\overline{\overline{\overline{\mathbf{M}}_{(k)}\cdot\mathbf{a}+\overline{\overline{\mathbf{M}}_{(k-1)}\cdot\mathbf{a}}}}=\overline{\overline{\mathbf{U}^{k}}\cdot\mathbf{a}+\overline{\overline{\overline{\mathbf{U}^{k-1}}}\cdot\mathbf{a}}}$,

$\mathbf{cw}=\mathbf{p}_{(1)}=\mathbf{sp}_{(1)}=\overline{\overline{(\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}}\cdot\mathbf{a}}$,

$\mathbf{uc}=\mathbf{p}_{(2)}=\overline{\overline{(\mathbf{M}^{2}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^{2}}\cdot\mathbf{a}}$,

$\mathbf{ucp}=\mathbf{p}_{(3)}=\overline{\overline{(\mathbf{M}^{3}+\mathbf{M}^{2}+\mathbf{M}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^{3}}\cdot\mathbf{a}}$,

$\mathbf{stc}=\mathbf{ut}=\mathbf{wtc}=\mathbf{p}_{(m)}=\overline{\overline{(\mathbf{M}_{(m)}+\mathbf{E})}\cdot\mathbf{a}}=\overline{\overline{\mathbf{U}^{m}}\cdot\mathbf{a}}$, m: $\mathbf{p}_{(m-1)}\neq\mathbf{p}_{(m)}$ & $\mathbf{p}_{(m)}=\mathbf{p}_{(m+1)}$,

$\mathbf{s}_{(1)}=\mathbf{ss}_{(1)}=\mathbf{mes}=(\mathbf{M}+\mathbf{E})\cdot\mathbf{p}_{(2)}=\mathbf{U}\cdot\overline{\overline{\mathbf{U}^{2}}\cdot\mathbf{a}}$,

$\mathbf{s}_{(k)}=\mathbf{s}_{(k-1)}+\mathbf{mwes}(\upsilon_{(k)})=\mathbf{s}_{(k-1)}+\overline{\overline{(\widetilde{\mathbf{M}}\cdot\widetilde{\mathbf{U}}+\widetilde{\mathbf{U}})}\cdot\mathbf{a}}+$

$+\mathbf{sch}((\widetilde{\mathbf{M}}+\widetilde{\mathbf{T}})\cdot\overline{\overline{(\widetilde{\mathbf{M}}\cdot\widetilde{\mathbf{U}}+\mathbf{E})}\cdot\widetilde{\mathbf{M}}})$,

$\mathbf{ss}_{(k)}=\overline{\overline{\mathbf{s}}_{(k)}+\mathbf{s}_{(k-1)}}$,

where $\widetilde{\mathbf{U}}=\mathbf{M}_{(k)}$, $\widetilde{\mathbf{M}}=\overline{(\mathbf{M}_{(k)}^{tr}+\overline{\mathbf{M}}_{(k)})}$.

Appendix

Proof of Lemma 2. Suppose $i \in B$, B is a minimal weakly externally stable set. Then $B \setminus \{i\}$ is not weakly externally stable $\Leftrightarrow \exists j \notin B \setminus \{i\}$: $(B \setminus \{i\}) \subseteq \mathrm{L}(j)$. At the same time if $j \notin B$ then $B \cap (D(j) \cup H(j)) \neq \varnothing$. Consequently either $j=i$ or $i \in D(j) \cup H(j)$ holds, that is $i \in D(j) \cup H(j) \cup \{j\}$. A condition $i \in D(j) \cup H(j) \cup \{j\}$ is equivalent to $j \in L(i) \cup H(i) \cup \{i\}$.

A condition $(B \setminus \{i\}) \subseteq L(j)$ is equivalent to $B \subseteq L(j) \cup \{i\}$. Since by assumption B is a weakly externally stable set and $B \subseteq L(j) \cup \{i\}$ then $L(j) \cup \{i\}$ must be a weakly externally stable set as well (due to monotonicity of weak external stability). Consequently, if B is a minimal weakly externally stable set and $i \in B$ then it is necessary that $\exists j$: $j \in L(i) \cup H(i) \cup \{i\}$ & $L(j) \cup \{i\}$ is a weakly externally stable set.

Let us prove that this condition is sufficient for the existence of a minimal weakly externally stable set B such that $i \in B$. Suppose $\exists j$: $j \in L(i) \cup H(i) \cup \{i\}$ & $L(j) \cup \{i\}$ is a weakly externally stable set. If $L(j) \cup \{i\}$ is minimal then $B = L(j) \cup \{i\}$. If it is not the case then $\exists C$: $C \subset L(j) \cup \{i\}$ and C is a minimal weakly externally stable set. By definition $L(j)$ is not a weakly externally stable set. Since C is weakly externally stable, C is not a subset of $L(j)$ (monotonicity of weak external stability). But $C \subset L(i) \cup \{i\}$, therefore $i \in C$ and $B = C$.

Thus, i belongs to a minimal weakly externally stable set iff $\exists j$: $j \in L(i) \cup H(i) \cup \{i\}$ and $L(j) \cup \{i\}$ is a weakly externally stable set.

$L(i) \cup \{i\}$ is not a weakly externally stable set $\Leftrightarrow \exists k$: $(L(i) \cup \{i\}) \subseteq L(k) \Leftrightarrow k \mu i$ & $L(i) \subseteq L(k) \Leftrightarrow i \notin UC^{\mathrm{IIa}}$. Therefore, $L(i) \cup \{i\}$ is a weakly stable set iff $i \in UC^{\mathrm{IIa}}$.

Suppose $\exists j$: 1) $j \in L(i) \cup H(i)$ & 2) $L(j) \cup \{i\}$ is not a weakly externally stable set. Then (2) $\Leftrightarrow \exists k$: $(L(j) \cup \{i\}) \subseteq L(k)$. Then $(L(j) \cup \{i\}) \subseteq L(k) \Leftrightarrow L(j) \subseteq L(k)$ & $\{i\} \subseteq L(k)$. Then $\{i\} \subseteq L(k) \Leftrightarrow k \in D(i)$. $j \in L(i) \cup H(i)$ & $k \in D(i)) \Rightarrow k \neq j$. Then ($k \neq j$ & $L(j) \subseteq L(k)) \Leftrightarrow k \alpha^{\mathrm{IIc}} j$. Consequently, $\exists j$: 1) $j \in L(i) \cup H(i)$ & 2) $L(j) \cup \{i\}$ is not a weakly externally stable set $\Leftrightarrow \exists j, k$: 1) $j \in L(i) \cup H(i)$ & 2) $k \in D(i)$ & 3) $k \alpha^{\mathrm{IIc}} j$. Therefore a set $L(j) \cup \{i\}$: $j \in L(i) \cup H(i)$ is weakly externally stable iff j is not covered by any alternative from the upper section of i according to α^{IIc}.

Therefore $\exists j$: 1) $j \in L(i) \cup H(i) \cup \{i\}$ and 2) $L(j) \cup \{i\}$ is a weakly externally stable set $\Leftrightarrow$ either $i \in UC^{IIa}$, or $\exists j$: 1) $j \in L(i) \cup H(i)$ & 2) j is not covered by any alternative from the upper section of i according to α^{IIc}.

As a result, i belongs to the set *MWES* iff either i is uncovered according to the version α^{IIa} of the covering relation, or some alternative from the lower section of i or from the horizon of i is not covered by any alternative from the upper section of i according to the version α^{IIc} of the covering relation. □

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