ON ISOPERIMETRIC SETS OF RADIALLY SYMMETRIC MEASURES

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Abstract. We study the isoperimetric problem for the radially symmetric measures. Applying the spherical symmetrization procedure and variational arguments we reduce this problem to a one-dimensional ODE of the second order. Solving numerically this ODE we get an empirical description of isoperimetric regions of the planar radially symmetric exponential power laws. We also prove some isoperimetric inequalities for the log-convex measures. It is shown, in particular, that the symmetric balls of large size are isoperimetric sets for strictly log-convex and radially symmetric measures. In addition, we establish some comparison results for general log-convex measures.

1. Introduction

Let μ be a Borel measure on $\mathbb{R}^d$ (or a Riemannian manifold) and A be a Borel set. We consider its surface measure $\mu^+(\partial A)$

$$\mu^+(\partial A) = \underline{\lim}_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

where A^ε is the ε-neighborhood of A. Recall that a set A is called isoperimetric if it has the minimal surface measure among of all the sets with the same measure $\mu(A)$.

The isoperimetric function $\mathcal{I}_\mu(t)$ of μ is defined by

$$\mathcal{I}_\mu(t) = \inf_A \{\mu^+(\partial A) : \mu(A) = t\}.$$

In what follows we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. For the Lebesgue measure we also use the common notations λ and dx. If μ has a continuous density ρ, then the surface measure μ^+ has the following representation: $\mu^+ = \rho \cdot \mathcal{H}^{d-1}$. We denote by κ_d the constant appearing in the Euclidean isoperimetric inequality λ: $\lambda^{1-\frac{1}{d}}(A) \le \kappa_d \mathcal{H}^{d-1}(\partial A)$.

In this paper we study the isoperimetric sets of the radially symmetric measures, i.e. measures with densities of the type

$$\mu = \rho(r)\ dx = e^{-v(r)}\ dx.$$

Only a small number of spaces with an exact solution to the isoperimetric problem is known so far. The most well known examples are

1) Euclidean space with the Lebesgue measure (solutions are the balls)
2) Spheres S^{d-1} and hyperbolic spaces H^{d-1} (solutions are the metric balls)

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3) Gaussian measure $\gamma = \frac{1}{(\sqrt{2\pi})^d} e^{-\frac{|x|^2}{2}} dx$ (solutions are the half-spaces).

Some other examples can be found in [30]. See also recent developments in [31], [24], [12], [17], [16], [11].

Whereas S^{d-1} and H^{d-1} are the model spaces in geometry, the Gaussian measures are the most important model measures in probability theory. The solutions to the isoperimetric problem for the Gaussian measures have been obtained by Sudakov and Tsirel'son [33] (see also Borell [8]). The proof given in [33] used the solution to the isoperimetric problem on the sphere. Ehrhard [18] found later another proof based on the Steiner symmetrization for Gaussian measures (see [30] for generalizations to product measures).

Some exact solutions to the isoperimetric problem are known in the one-dimensional case. For instance, the half-lines are the isoperimetric sets for the probability log-concave one-dimensional distributions (see [7]). This result was generalized in [31].

Another interesting result has been obtained by Borell in [9]. He has shown that the balls about the origin

$$B_R = \{|x| \le R\}$$

are solutions to the isoperimetric problem for the non-probability measure

$$\mu = e^{r^2}\ dx.$$

Several extensions of this result can be found in [31], [24]. It was conjectured in [31] that the balls about the origin are solutions to the isoperimetric problem for $\mu = \rho(r)\ dx$ provided $\log \rho(r)$ is smooth and convex. We deal below with a slightly changed version of this conjecture. Namely, we are interested in radially symmetric measures with increasing convex $r \to \log \rho(r)$ (e.g. $\mu = e^{r^\alpha}$, $\alpha \ge 1$).

To our knowledge, no any other non-trivial exact solutions to the isoperimetric problem coming from the probability theory are known. For instance, it was proved in [31] that there exist isoperimetric regions for log-concave radially symmetric distributions which are neither balls nor halfplanes, but no precise example was given.

The paper is organized as follows. In Section 3 we prove a symmetrization result for rotationally invariant measures. This result is not new. During the preparation of the manuscript we learned from Frank Morgan about a recent symmetrization result for warped products of manifolds in [28] (Proposition 3, Proposition 5). See also remarks in Section 3.2 of [30] and Section 9.4 of [19]. In this paper we provide alternative arguments which are close to the classical proof that the Steiner symmetrization does not increase the surface measure.

By variational arguments we show that every stationary set for the measure $\mu = e^{-v(r)}\ dx$ on the plane which has Ox as the revolution axis and a real analytic boundary is either a ball or has the form

$$A = \{(r,\theta) : -f(r) < \theta < f(r)\},\ \ f(r) \in [0,\pi], \tag{1}$$

where $f(r)$ is a solution to

$$\Big[\frac{r^2 f'}{\sqrt{1+r^2(f')^2}}\Big]' - v'(r)\Big[\frac{r^2 f'}{\sqrt{1+r^2(f')^2}}\Big] = c \cdot r \tag{2}$$

for some constant c.

We analyze this equation for several precise examples. It turns out that apart from special cases (Lebesgue measure) only small part of the solutions to this ODE can describe an isoperimetric set. It looks in general impossible to determine analytically the constant c and the initial conditions for (2) such that the corresponding solution describes an isoperimetric region. Nevertheless, performing numerical computations it is possible to find *empirically* the desired parameters, since most of the solutions to (2) are either non-smooth or non-closed curves.

We are especially interested in is the exponential power law

$$\rho(r) = C_\alpha e^{-r^\alpha}$$

on the plane. We justify by numerical computations that for the super-Gaussian laws $\alpha > 2$ the isoperimetric regions are non-compact and can be obtained by a separation of the plane in two pieces by an axially symmetric convex curve. Unlike this, the isoperimetric regions for the exponential law $\rho = C_1 e^{-r}$ are compact convex axially symmetric sets (which are not the balls) and their complements. For some values $\alpha \in (1,2)$ there exist isoperimetric regions of both types.

In the last section we analyze a non-probabilistic case: $\mu = e^V \ dx$, where V is a convex potential. The interest in this type of measures is partially motivated by problems coming from the differential geometry. The measures of this type are natural "flat" analogs of the negatively curved spaces. In fact, both types of spaces enjoy very similar isoperimetric inequalities. Note that the famous Cartan-Hadamard conjecture on a comparison isoperimetric inequality for the manifolds with negative sectional curvatures is still an open problem.

We prove some results related to the cited conjecture from [31]. We show, in particular, that the large balls are the isoperimetric sets for $\mu = e^V \ dx$ under assumption that $V = r^\alpha, \alpha > 1$ (more generally, V is convex, radially symmetric and superlinear). Applying mass transportational arguments we prove that every log-convex radially symmetric measure $\mu = e^V \ dx$ satisfies

$$\mu^+(\partial A) \ge \frac{1}{\sqrt{1+\pi^2}} \mu^+(\partial B)$$

for every Borel set A and a ball B about the origin satisfying $\mu(A) = \mu(B)$.

We also prove some comparison theorems for log-convex measures of general type. We show, in particular, that $\mu = e^V dx$ with a convex non-negative V enjoys the Euclidean isoperimetric inequality. Finally, we prove some results for the products of the one-dimensional (non-probability!) log-convex measures. The case of probability product measures has been studied in [7], [4], [6]. We establish a log-convex (one-dimensional) analog of a Caffarelli's contraction theorem for the optimal transportation of the uniformly log-concave measures. More precise, we show that every one-dimensional log-convex measure $\mu = e^V dx$ satisfying

$$V'' e^{-2V} \ge 1,$$

V is even and $V(0) = 0$ is a 1-Lipschitz image of the model measure $\nu = \frac{dx}{\cos x}$. In particular, this implies the following comparison result:

$$\mathcal{I}_\mu(t) \ge \mathcal{I}_\nu(t) = e^{t/2} + e^{-t/2}.$$

Finally, we estimate the isoperimetric profile for a large class of the log-convex product measures.

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2. Existence, regularity and geometric properties of isoperimetric sets

It is known that under broad assumptions the isoperimetric regions do exist for measures with a finite total volume. Some results on existence for measures with an infinite total volume can be found in [31].

We will widely use the fact that the isoperimetric low-dimensional surfaces are regular. A classical result on regularity of the isoperimetric sets was obtained by Almgren [1]. We use the following refinement obtained by F. Morgan [26]. The original formulation is given in terms of a Riemannian metric, but the result still holds for the trivial Riemannian metric and a potential with the same regularity.

Let A be an open set with smooth boundary ∂A and $\{\phi_t\}, \phi_0 = \mathrm{Id}$ be any smooth family of diffeomorphisms satisfying $\mu(A_t) = \mu(A)$, where $A_t = \phi_t(A)$. We call A stationary if

$$\frac{d}{dt}\mu^+(\partial A_t)|_{t=0} = 0.$$

We call A stable if

$$\frac{d^2}{dt^2}\mu^+(\partial A_t)|_{t=0} \geq 0.$$

Clearly, isoperimetric sets must be stationary and stable.

Theorem 2.1. *For $d \leq 7$ let S be an isoperimetric hypersurface for $\mu = e^{-v}\, dx$. Assume that v is $C^{k-1,\alpha}$, $k \geq 1, 0 < \alpha < 1$ and Lipschitz. Then S is locally a $C^{k,\alpha}$ manifold. If v is real analytic, then S is real analytic.*

For $d > 7$ the statement holds up to a closed set of singularities with Hausdorff dimension less than or equal to $d - 7$.

In addition, we will use more special facts about log-concave radially symmetric distributions proved in [31].

1) The balls about the origin are not isoperimetric (even stable) for strictly log-concave radially symmetric distributions (Theorem 3.10).
2) The isoperimetric sets for strictly log-concave distributions have connected boundaries (Corollary 3.9).

3. Spherical symmetrization

In this section we deal with the radially symmetric measures. We start with the case $d = 2$. Denote by (r, θ) the standard polar coordinate system. Assume that $\mu = \rho(r)\, dx$ is supported on B_R, $R \in (0, \infty]$ and ρ is smooth and positive on B_R.

Assumption: A is an open set with Lipschitz boundary ∂A (i.e. ∂A is a finite union of graphs of Lipschitz functions).

We remark that according to Theorem 2.1 the isoperimetric hypersurfaces satisfy this assumption.

Definition 3.1. We say that a set A^* is obtained from $A \subset R^2$ by the circular symmetrization with respect to the x-axis, if for every $r \geq 0$ the set $\partial B_r \cap A^*$ has the same length as $\partial B_r \cap A$ and, in addition, $\partial B_r \cap A^*$ has the form $\{-f(r) < \theta < f(r)\}$ for some $f \in [0, \pi]$. If $\partial B_r \subset A$, we require that $\partial B_r \cap A^* = \partial B_r$.

Remark 3.2. By the Fubini theorem A and A^* have the same μ-measure. In addition, the circular symmetrization can be defined with respect to any ray starting from the origin.

Proposition 3.3. *The circular symmetrization does not increase the surface measure*

$$\mu^+(\partial A^*) \le \mu^+(\partial A).$$

Assume that A is connected, $\mu^+(\partial A^*) = \mu^+(\partial A)$, and $\mathrm{card}(\partial A \cap \partial B_r) < \infty$ *for every $r > 0$. Then $\mu(A^* \setminus U(A)) = 0$ for some rotation $U(r,\theta) = re^{i(\theta+\theta_0)}$.*

Proof. Without loss of generality we deal with a compact A with $\mathcal{H}^{d-1}(\partial A) < \infty$. It is known (see Theorem 3.42 in [2]) that there exists a sequence A_n of smooth sets such that $I_{A_n} \to I_A$ almost everywhere (in the Lebesgue measure sense) and $\mu^+(\partial A_n) \to \mu^+(\partial A)$. Thus, to prove the first part of the Proposition, it is sufficient to deal with sets with smooth boundaries. We can even assume that every ∂A_n is a level set of a polynomial function P_n restricted to a compact subset (A_n obtained in the proof of Theorem 3.42 are level sets of smooth functions, we only apply the Weierstrass polynimila approximation theorem). It is also possible to require that $\mathrm{card}(A_n \cap \partial B_r) < \infty$. Thus, without loss of generality it is sufficient to consider the case when ∂A consists of finite number $n(r)$ of ordered Lipschitz curves $r \to r(\cos f_i(r), \sin f_i(r))$, $r \to r(\cos g_i(r), \sin g_i(r))$ such that

$$f_1 \le g_1 < f_2 \le g_2 < \cdots < f_n \le g_n$$

and

$$A = \{(r,\theta) : f_i(r) < \theta < g_i(r)\}$$

(we suppose that $f_i = g_i$ just for a finite number of r). The further proof mimics the classical proof that Steiner's symmetrization does not increase the perimeter. Indeed, for every curve $r \to r(\cos\varphi(r), \sin\varphi(r))$ one has

$$ds^2 = (1 + r^2(\varphi')^2)dr^2.$$

Hence

$$\mu^+(\partial A) = \int_0^\infty \Bigl(\sum_{i=1}^{n(r)} \sqrt{(1 + r^2(f_i')^2)} + \sqrt{(1 + r^2(g_i')^2}\Bigr)\rho(r)\ I_r\ dr,$$

$$\mu^+(\partial A^*) = 2\int_0^\infty \sqrt{1 + r^2\Bigl(\sum_{i=1}^{n(r)} \frac{f_i' - g_i'}{2}\Bigr)^2}\rho(r)\ I_r\ dr,$$

where $I_r = \{n(r) \ne 0\}$ (all the r such that $\partial B_r \cap \partial A \ne \emptyset$). Note that the function $\sqrt{1+x^2}$ is convex, hence

$$\sum_{i=1}^{n}\Bigl(\sqrt{1+a_i^2} + \sqrt{1+b_i^2}\Bigr) \ge 2\sqrt{\Bigl(1 + \frac{1}{4}\sum_{i=1}^{n}\bigl(a_i - b_i\bigr)^2\Bigr)}$$

and we get the desired inequality.

Now assume that $\mu^+(\partial A) = \mu^+(\partial A^*)$, ∂A is Lipschitz and $\mathrm{card}(\partial A \cap \partial B_r) < \infty$ for every $r > 0$. Then the above formulae hold. Clearly, $\mu^+(\partial A) = \mu^+(\partial A^*)$ is possible only if $n = 1$, hence $A = \{(r,\theta) : f(r) < \theta < g(r)\}$. Moreover, $f' = -g'$ for $r \in [r_1, r_2]$. Hence $f + g$ is constant on $[r_1, r_2]$. The equality is possible only if $n = 1$ on some interval $r_1 < r < r_2$ (maybe unbounded) and $n = 0$ outside. In this case A^* is obtained from A by a rotation on the constant angle $\frac{f+g}{2}$ (up to a zero measure set). □

Corollary 3.4. *Let A be an isoperimetric set for a planar radially symmetric density. Assume, in addition, that A is connected, open, has an analytic boundary, and $\mathcal{H}^1(\partial B_r \cap \partial A) = 0$ for every $r > 0$. Then A is stable under the circular symmetrization with respect to some ray.*

Proof. It is sufficient to note that the analyticity of ∂A and $\mathcal{H}^1(\partial B_r \cap \partial A) = 0$ imply that $\mathrm{card}(\partial B_r \cap \partial A) < \infty$ and apply the previous Proposition. □

Analogously to the circular symmetrization let us introduce the spherical symmetrization on $\mathbb{R}^d$.

Definition 3.5. We say that a set $A^* \subset \mathbb{R}^d$ is obtained from $A \subset \mathbb{R}^d$ by the spherical symmetrization with respect to the ray $R_a = \{ta : t \geq 0\}$ associated to a vector $a \in \mathbb{R}^d$, $a \neq 0$, if for every $r \geq 0$ the set the $\partial B_r \cap A^*$ has the same Hausdorff $\mathcal{H}^{d-1}$-measure as $\partial B_r \cap A$ and, in addition, $\partial B_r \cap A^*$ is a spherical cap centered at $R_a \cap \partial B_r$.

We don't prove here that the spherical symmetrization does not increase the surface measure (see [28]). Nevertheless, we show that every isoperimetric set satisfying some additional technical assumptions is stable under a spherical symmetrization. To this end let us introduce an intermediate operation.

Definition 3.6. Let $d \geq 3$ and A be a Borel set. The set $A^*_{x_1,x_2}$ is determined by the following requirements. Fix coordinates $x_3, \cdots, x_d$. The intersection of every circle

$$C_R = \{(x_1, x_2) : x_1^2 + x_2^2 = R^2\}$$

with $A^*_{x_1,x_2}$ is an open arc $l_R \subset C_R$ satisfying:

1) l_R has the same length as $l_R \cap A$ ($l_R = C_R$ if $C_R \subset A$)
2) the center $M = (x_1, x_2)$ of l_R is uniquely determined by the requirement $x_1 \geq 0, x_2 = 0$.

In what follows we associate to any arbitrary vector $a = (a^1, \cdots, a^d)$ the following matrix Q_a:

$$(Q_a)_{ij} = a^i \cdot a^j.$$

The following lemma follows from the convexity of the function $x \to \det^{1/2}(I + Q_x)$.

Lemma 3.7. *Let M be a symmetric positive matrix. For a number of vectors $v_1, \cdots, v_{2n}$ the following inequality holds*

$$\sum_{k=1}^{2n} \det{}^{1/2}(M + Q_{v_k}) \geq 2 \cdot \det{}^{1/2}(M + Q_v),$$

where $v = \frac{\sum_{k=1}^{n} v_k - \sum_{k=n+1}^{2n} v_k}{2}$. In addition, an equality holds if and only if $n = 1$ and $v_1 = -v_2$.

Proposition 3.8. *Let μ be a radially symmetric measure. Then $\mu^+(\partial A^*_{x_1 x_2}) \leq \mu^+(\partial A)$.*

Assume that every isoperimetric set of μ has analytic boundary. Let A be isoperimetric, connected, and $\mathcal{H}^1(C \cap \partial A) = 0$ for every circle

$$C = \{x_1^2 + x_i^2 = R^2; x_j = a_j\},\ j \in \{2, \cdots, d\} \setminus \{i\}$$

with fixed $i \in \{2, \cdots, d\}$, $R > 0$, $a_j \in \mathbb{R}$. Then there exists a ray $R_a = \{t \cdot a, t \geq 0\}$, $a \in R^d \setminus \{0\}$ such that every nonempty intersection of A with any ball $B_r = \{x : |x| \leq r\}$ is a spherical cap (up to zero measure) with the center at $R_a \cap B_r$.

Proof. Let us show the first part. Without loss of generality we assume that A is compact. Consider the symmetrized set $A^*_{x_1x_2}$. Clearly, $A^*_{x_1x_2}$ has the same μ-measure as A. Let us show that $A^*_{x_1x_2}$ has a smaller surface measure. Arguing in the same way as in the previous Proposition we can assume that ∂A consists of finite number of smooth surfaces S_k and intersection of every S_k with every circle $C = \{x_1^2 + x_2^2 = R^2\}$ (other x_i are fixed) consists of finite number of points. Let us parametrize every surface S_k in the following way:

$$F_k : (r, \tilde{x}) \to \big(\sqrt{r^2 - \tilde{x}^2} \cos\theta_i(r, \tilde{x}), \sqrt{r^2 - \tilde{x}^2} \sin\theta_i(r, \tilde{x}), \tilde{x}\big),$$

where r is the distance from F_k to the origin and θ_k is the angle between the O_{x_1}-axis and the projection of F_k onto $O_{x_1x_2}$-plane, and $\tilde{x} = (x_3, \cdots, x_d)$. In addition,

$$A \cap C = \cup_{k=1}^n \{\theta_k < \theta < \theta_{n+k}\}.$$

The first fundamental form G_k of S_k has the following representation in $(r, \tilde{x})$-coordinates:

$$G_k = M + Q_k,$$

where

$$M_{rr} = \frac{r^2}{r^2 - \tilde{x}^2}, \ M_{rx_i} = -\frac{rx_i}{r^2 - \tilde{x}^2}, \ M_{x_ix_j} = \delta_{ij} + \frac{x_ix_j}{r^2 - \tilde{x}^2}$$

and

$$Q_k = (r^2 - \tilde{x}^2) Q_{\nabla\theta_k},$$

where $\nabla\theta_k = (\partial_r\theta_k, \partial_{x_3}\theta_k, \cdots, \partial_{x_d}\theta_k)$. Hence $\mu^+(\partial A)$ is equal to

$$\sum_{k=1}^{2n} \int_0^\infty \int_{B_r} \det{}^{1/2}\big(M + (r^2 - \tilde{x}^2) Q_{\nabla\theta_k}\big) I(r, \tilde{x}) \ d\tilde{x} \ \rho(r) \ dr.$$

Here $I(r, \tilde{x})$ is the set of $(r, \tilde{x})$ such that C has a non-empty intersection with A. Clearly,

$$\mu^+(\partial A^*_{x_1x_2}) = 2 \int_0^\infty \int_{B_r} \det{}^{1/2}\big(M + (r^2 - \tilde{x}^2) Q_{\nabla\theta}\big) I(r, \tilde{x}) \ d\tilde{x} \ \rho(r) \ dr,$$

where

$$\tilde{\theta} = \frac{\sum_{k=1}^n \theta_k - \sum_{k=n+1}^{2n} \theta_k}{2}.$$

The desired inequality follows from Lemma 3.7.

Let us prove the second part. Take an isoperimetric set A satisfying the assumptions. Note that the above formulae hold for A and, in addition, $\mu(\partial A) = \mu(\partial A^*)$. This is possible if and only if $n = 1$ and $\nabla(\theta_2 + \theta_1) = 0$ on $r_1 \leq r \leq r_2$ and $n = 0$ for other values of r. But this means that $A^*_{x_1x_2}$ is obtained from A by a rotation. Applying consequently x_1x_i-symmetrizations to A, we obtain a set $\tilde{A}$. Since every x_1x_i-symmetrization does not increase the surface measure, the set $\tilde{A}$ is obtained by a rotation of A (up to measure zero). In addition, $\tilde{A}$ is symmetric with respect to any hyperplane $\pi_i = \{x_i = 0\}$, $i > 1$ and $\partial\tilde{A}$ is connected.

Now let us show that $\tilde{A}$ is symmetric with respect to any hyperplane π passing through the origin and containing x_1-axis. Indeed, since μ and $\tilde{A}$ are symmetric, the hyperplane π divides $\tilde{A}$ in two pieces $A^+ \cup A^-$ with the same measure. Clearly, the Hsiang symmetrization $A^+ \cup s_\pi(A^+)$ (s_π is the reflection with respect to π) is an isoperimetric set. Since the isoperimetric sets have smooth boundaries, it is possible if and only if $\partial \tilde{A}$ intersects π orthogonally. Hence $A \cap \partial B_r$ is a spherical cap with the center at the x_1-axis for every $r > 0$. □

4. Stationary circular symmetric sets

In this section we study the stationary sets of a radially symmetric measure $\mu = e^{-v(r)}\, dx$ on the plane.

Lemma 4.1. *Assume that for some smooth f one has:*

$$A = \{(r,\theta) : -f(r) < \theta < f(r)\}, \ \ f(r) \in [0,\pi] \tag{3}$$

and A is a stationary set. Then f satisfies

$$\dot{u} - \dot{v}u = cr, \tag{4}$$

where c is a constant and

$$u = \frac{r^2 \dot{f}}{\sqrt{1 + r^2(\dot{f})^2}}. \tag{5}$$

Proof. One has

$$\mu(A) = 2\int_0^\infty r f(r)\ \rho(r) dr,$$

$$\mu^+(\partial A) = 2\int_0^\infty \sqrt{1 + r^2 (f')^2}\ \rho(r) dr.$$

Let us compute a variation of $\mu^+(\partial A)$ under the constraint $\mu(A) = C$. Consider an infinitesimal variation $f + \varepsilon\varphi$ of f by a smooth compactly supported function φ. Since we keep $\mu(A)$ constant, we assume that $\int \varphi r \rho\ dr = 0$. One obtains

$$\int_0^\infty \frac{r^2 f' \varphi'}{\sqrt{1 + r^2 (f')^2}}\ \rho(r) dr = 0.$$

Integrating by parts one gets

$$\int_0^\infty \Bigl(\bigl[\frac{r^2 f'}{\sqrt{1 + r^2 (f')^2}}\bigr]' - v'(r) \bigl[\frac{r^2 f'}{\sqrt{1 + r^2 (f')^2}}\bigr] \Bigr)\ \varphi\ \rho(r)\ dr = 0.$$

Taking into account that this holds for every smooth φ with $\int \varphi r \rho\ dr = 0$, one gets that $f(r)$ satisfies

$$\bigl[\frac{r^2 f'}{\sqrt{1 + r^2 (f')^2}}\bigr]' - v'(r) \bigl[\frac{r^2 f'}{\sqrt{1 + r^2 (f')^2}}\bigr] = c \cdot r$$

for some constant c. The proof is complete. □

Remark 4.2. Clearly, Lemma 4.1 can be generalized to higher dimensions. Let $d = 3$ and ∂A is parametrized in the following way:

$$\begin{cases} x = r \sin f(r) \cos\varphi \\ y = r \sin f(r) \sin\varphi \\ z = r \cos f(r). \end{cases}$$

Then

$$\mu(A) = 4\pi \int_0^\infty r^2 (1 - \cos f) \rho(r) dr,$$

$$\mu^+(\partial A) = 2\pi \int_0^\infty r \sin f \sqrt{1 + r^2 (f')^2}\ \rho(r) dr.$$

Arguing as above, we obtain

$$\Bigl(\frac{r^3 f'}{\sqrt{1+(rf')^2}} \Bigr)' - v' \frac{r^3 f'}{\sqrt{1+(rf')^2}} = \frac{r}{\sqrt{1+(rf')^2}} \mathrm{ctg} f - c r^2.$$

Remark 4.3. We remark that equations (4)-(5) follows also from a result of [31]: an isoperimetric surface S with density e^{-V} has a constant generalized mean curvature

$$(d-1)H - \langle n, \nabla V \rangle,$$

where H is the Euclidean mean curvature of S and n is the normal vector of S.

Note, however, that the spheres about the origin always have constant generalized mean curvature for every radially symmetric density. The ball B_{r_0} corresponds to the singular function $f(r) = 2\pi \chi_{[0,r_0]}$ and r is tangent to u at r_0. In addition, the halfspace $H_v = \{x : \langle x, v \rangle \ge 0\}$ through the origin gives another example of a surface of a constant generalized mean curvature. This set corresponds to the constant solution $f = \frac{\pi}{2}$.

Example 4.4. Consider the standard Gaussian measure ($v = \frac{r^2}{2}$). Among all the solutions to (4) take the ones growing not faster than a linear function. These are the constants. For $u = c$ solve (5). The solution

$$f = \arccos\Bigl(\frac{r_0}{r}\Bigr), \quad r \ge r_0$$

defines an isoperimetric surface (a halfspace).

Lemma 4.5. *Assume that v is real analytic, A is a connected isoperimetric set and ∂A is connected. Then A is either a ball or a circularly symmetric set (with respect to some ray).*

If A is circularly symmetric with respect to the x-axis, then

$$\partial A \cap \{y \ge 0\} \subset \{r e^{if(r)},\ r \in [r_0, r_1]\},$$

where f is a solution to (4)-(5) and $[r_0, r_1]$ is the maximal interval of the existence of the solution to (4)-(5).

Proof. According to the regularity results (see Section 2), ∂A is real analytic. If $\mathcal{H}^1(\partial A \cap \partial B_r) > 0$, then ∂A contains an arc and by the uniqueness of analytic continuation A is a ball. If $\mathcal{H}^1(\partial A \cap \partial B_r) = 0$, then A is symmetric with respect to some circular symmetrization by Corollary 3.4.

Assume that A is circularly symmetric with respect to the x-axis. The boundary $\partial B_R \cap \partial A$ contains exactly two points for every $R > 0$ from an open interval (a, b) and (3) holds in a neighborhood of R with f solving (4)-(5). Let $[r_0, r_1]$ be the interval of existence to (4)-(5). Assume that $l : r \to e^{if(r)}$, $r \in [r_0, r_1]$ does not cover the intersection of ∂A with the halfplane $\{y \ge 0\}$. Clearly, in this case $l([r_0, r_1])$ is compact and $f'(r_1) = \infty$. There exists a unique analytic continuation l_δ of this curve for $r \in [r_1, r_1 + \delta)$ for some $\delta > 0$. This continuation does satisfy (4)-(5) as well (note that by the uniqueness of the continuation l_δ can not be an arc). Since v is radially symmetric, by the uniqueness of the solution to an ODE with given initial data, the curve l_δ coincides with the reflection of $l((r_1 - \delta', r_1])$ with respect to the line $\theta = f(r_1)$. But this clearly contradicts to the fact that $A \cap \partial B_r$ is an arc for every $r > 0$. Hence $\partial A \subset \{r e^{if(r)},\ r \in [r_0, r_1]\}$. $\square$

Let us give some examples when (4)-(5) is explicitly solvable.

Example 4.6. Consider the Lebesgue measure ($v = 0$). Let us find the solution to (4)-(5) such that $u(r_0) = r_0$, $u(r_1) = r_1$. One easily obtains

$$u = \frac{r^2 + r_0 r_1}{r_0 + r_1}, \ \ r_0 + r_1 \neq 0.$$

The solutions are the balls having the segment $[r_0, r_1]$ as the intersection with the x-axis. The formula makes sense also for negative values of r_0, r_1. Note that the case $r_0 + r_1 = 0$ corresponds to a constant u, hence to the balls about the origin. In addition, infinite values of r_0, r_1 correspond to the half-spaces (which are also stationary).

Example 4.7. **Stationary symmetric sets for** $\mu = \frac{dx}{r}$.

In this case the isoperimetric sets do not exist (see [12]). We show by solving explicitly (4)-(5) that the only stationary sets with smooth connected boundaries are the balls about the origin and halfplanes passing through the origin.

Indeed, consider a stationary set which is not a ball and not a halfspace ($u \neq 0$). Solving $\dot{u} - \frac{1}{r}u = ar$ we get $u = ar^2 + \lambda r$. Note that the solution to (5) exists for r satisfying $|u(r)| \le r$, thus

$$|\lambda + ar| \le 1.$$

By the symmetry arguments it is enough to consider the case $\lambda \ge 0$. Assume first that $\lambda > 0$. Note that for $\lambda \le 1$ one has

$$f(r) = \int_0^r \frac{as + \lambda}{s\sqrt{1 - (\lambda + as)^2}}\, ds.$$

But this integral diverges. Hence, it makes only sense to consider the case

$$\lambda > 1, \ a < 0.$$

One obtains

$$f(r) = \int_{\frac{1-\lambda}{a}}^r \frac{as + \lambda}{s\sqrt{1 - (\lambda + as)^2}}\, ds, \quad \frac{\lambda - 1}{-a} \le r \le \frac{1 + \lambda}{-a}.$$

Computing this integral we get

$$a \int_{\frac{1-\lambda}{a}}^r \frac{1}{\sqrt{1 - (\lambda + as)^2}}\, ds = \arcsin(\lambda + ar) - \frac{\pi}{2},$$

$$\lambda \int_{\frac{1-\lambda}{a}}^r \frac{ds}{s\sqrt{1 - (\lambda + as)^2}} = \lambda \int_{\frac{a}{1-\lambda}}^{\frac{1}{r}} \frac{dt}{\sqrt{t^2 - (t\lambda + a)^2}}.$$

The latter integral is equal to

$$\frac{\lambda}{\sqrt{\lambda^2 - 1}} \Big(\frac{\pi}{2} + \arcsin\Big(\frac{\lambda^2 + \lambda ar - 1}{ar} \Big) \Big).$$

Finally, we obtain

$$f(r) = \arcsin(\lambda + ar) - \frac{\pi}{2} + \frac{\lambda}{\sqrt{\lambda^2 - 1}} \Big(\frac{\pi}{2} + \arcsin\Big(\frac{\lambda^2 + \lambda ar - 1}{ar} \Big) \Big).$$

One easily verify that this curve is non-closed. Indeed, if $r = r_1 = -\frac{1+\lambda}{a}$, one has

$$f(r_1) = \pi \Big(1 - \frac{\lambda}{\sqrt{\lambda^2 - 1}} \Big)$$

and this is neither 0 nor π.

Solving (4)-(5) for $\lambda = 0$ and taking into account that the solution should be smooth for $r \neq 0$ we obtain a family of circles containing the origin

$$x^2 + y^2 = \frac{x}{a}.$$

It is easy to check that these circles have infinite length.

Remark 4.8. Some very interesting results on isoperimetric sets for measures $\mu = r^p\, dx$ have been recently obtained in [16].

5. Computations of isoperimetric sets for exponential power laws

In this section we compute numerically the isoperimetric sets of the measures

$$\mu_\alpha = C_\alpha e^{-r^\alpha}\, dx,\ \ \alpha \geq 1.$$

The isoperimetric estimates for these kind of laws have been obtained in [20].

Note that by the results mentioned in Section 2:

1) The isoperimetric sets do exist and have at least $C^{1,\varepsilon}$ boundary for some $\varepsilon > 0$. Note that the origin is the only point where the potential is not analytic. Thus, the isoperimetric curve is analytic at any other point.
2) The boundaries of isoperimetric sets are connected.
3) Balls about the origin are NOT isoperimetric for $\alpha > 1$.

Solving equation (4) we get

$$u = ae^{r^\alpha} \int_r^\infty se^{-s^\alpha}\, ds + \lambda e^{r^\alpha}. \tag{6}$$

For $\lambda = 0$, u has a growth of the order $r^{2-\alpha}$. In any other case u has a growth of the order e^{r^α}.

Equation (5) is equivalent to the equation

$$\dot{f} = \frac{u}{r\sqrt{r^2 - u^2}}.$$

Let $l = re^{if}$ be a curve solving (4)-(5). Without loss of generality we may assume that $a \geq 0$, because for the opposite values of a and λ one obtains the curve which is symmetric to l with respect to the y-reflection. Note that $|u(r)| < r$ for any r in the interval of existence for l. For $\alpha = 1$ and $\lambda = 0$ one has $u(r) = a(1 + r)$. For $\alpha = 2$ and $\lambda = 0$ one has $u(r) = a$. Thus for $\alpha = 1$, $\lambda = 0$ and $0 \leq a < 1$ the interval of existence of l is $(r_0, +\infty)$ for some $r_0 > 0$ and is empty for $a \geq 1$. For $\alpha = 2$, $\lambda = 0$ the interval of existence of l is $(r_0, +\infty)$ for some $r_0 > 0$. We find empirically that for $\lambda = 0$ and $\alpha \geq 1$ the equation $u(r) = r$ has no more than one solution. This implies that the interval of existence of l is infinite. If $\lambda \neq 0$ then the growth rate of $u(r)$ is higher than 1 and the existence interval of l is either empty or finite.

Let us describe different types of behavior of l.

1) The curve is non-compact. This corresponds to the case when $\lambda = 0$. Indeed, otherwise equation $r^2 - u^2 \geq 0$ is satisfied on a compact interval and l is compact.

Assume that the curve does not touch the origin and r_0 is the smallest value of r such that B_r and l has a non-empty intersection. The angle of rotation of the curve when r changes from r_0 to r is equal to

$$f(r) = \int_{r_0}^{r} \frac{u}{s\sqrt{s^2 - u^2}}\, ds. \tag{7}$$

The full rotation of l is equal to $\Delta_f = \int_{r_0}^{\infty} \frac{u}{s\sqrt{s^2-u^2}}\, ds$. In the case of compact curve the full rotation is equal to $\Delta_f = \int_{r_0}^{r_1} \frac{u}{s\sqrt{s^2-u^2}}\, ds$, where $[r_0, r_1]$ is the largest existence interval for l.

The curves can be self-intersecting or non self-intersecting.

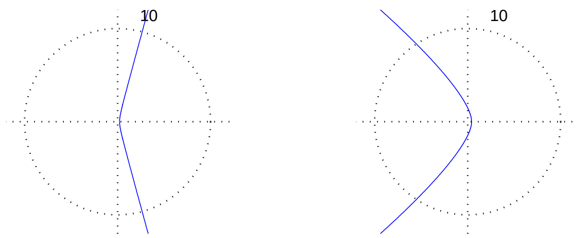


Figure 1: Non-compact and non self-intersecting curves. Smooth non-compact solutions to (4)-(5) in the super-Gaussian case ($\alpha = 3$, $a = 0.5$) and sub-Gaussian case ($\alpha = 1.3$, $a = 0.5$).

If the full rotation exceeds π then two branches of l have an intersection. Obviously, in this case the curve can not be a boundary of an isoperimetric set. Clearly, since isoperimetric sets have smooth boundary, the part of a curve between r_0 and the point of intersection r_1 can be isoperimetric only if $\dot{f}(r_1) = \infty$.

It was realized by numerical computations that for $\alpha \geq 2$ the full rotation is less then π and l is not self-intersecting. If $\alpha = 1$ then for all a the curve l is self-intersecting. It is easy to verify that for $\alpha = 1$ the full rotation is infinite and for $\alpha > 1$ the full rotation is finite.

2) The curve intersects itself non-smoothly. This happens only for $1 \leq \alpha < 2$. Clearly, in this case the curves are not isoperimetric.

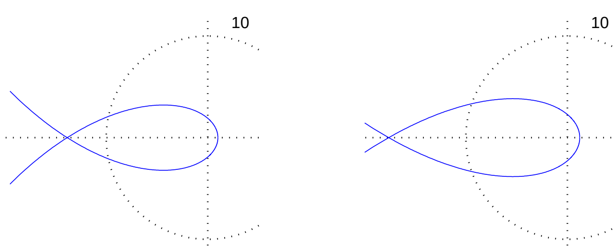


Figure 2: Self-intersecting curves. Solutions to (4)-(5) in the case $\alpha = 1$, $a = 0.5$ and $\alpha = 1.1$, $a = 0.7$.

3) The curve is compact but not closed ($f'_r = \infty$) for $|f| < \pi$. This type of behavior occurs for any $\alpha \geq 1$. These curves can not be isoperimetric since they have no analytic circular symmetric continuation (see Lemma 4.5).

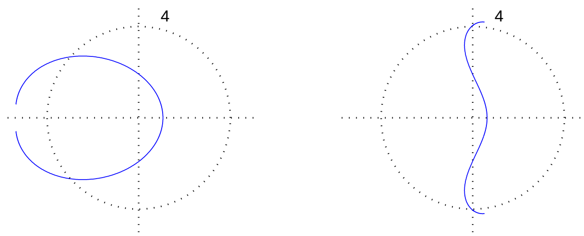


Figure 3: Non-closed solutions to (4)-(5) in the case $\alpha = 1$ ($a = 0.5, \lambda = 0.01$ and $a = 0.5, \lambda = -0.1$).

Clearly, in this case $0 < \Delta_f < \pi$.

4) The curve is closed and smooth ($f'_r = \infty$ if $f = \pm\pi$). These types of curves do appear for any $1 \leq \alpha < 2$.

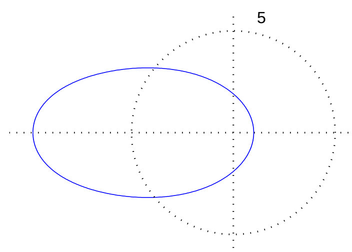


Figure 4: Closed smooth solution to (4)-(5) in the case $\alpha = 1$, $a = 0.5$.

5) Special case: curve starting from the origin. The main difference to previous cases: for smooth curves one has $f(0) = \pi/2$ (unlike $f(r_0) = 0$ for $r_0 > 0$).

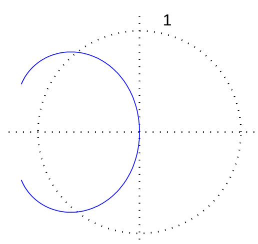


Figure 5: Curve starting from the origin $\alpha = 1$.

We compute numerically the isoperimetric sets for the super-Gaussian ($\alpha > 2$), sub-Gaussian ($1 < \alpha < 2$), and exponential $\alpha = 1$ distributions. We stress that the results below are partially justified by numerical computations.

1) Super-Gaussian case, $\alpha > 2$**.**

In this case the balls around the origin are not isoperimetric. For every a and $\lambda = 0$ the solutions to (4)-(5) are non-compact and non self-intersecting (case 1)). In this case

(8) $$u = ae^{r^\alpha} \int_r^\infty se^{-s^\alpha}\, ds$$

It was verified numerically that all the compact curves ($\lambda \neq 0$) solving (4)-(5) correspond to the case 3) but not to 2) or 4). The same happens to curves starting from the origin. Hence, the compact curves are not isoperimetric.

Conclusion: For any given value of measure there exists a unique (up to a rotation) open isoperimetric set A and a unique parameter a such that $\partial A = \{re^{if(r)}\}$ with u given by (8) and f given by (7). The set A is one of the sets obtained by dividing the plane by the curve l in two pieces.

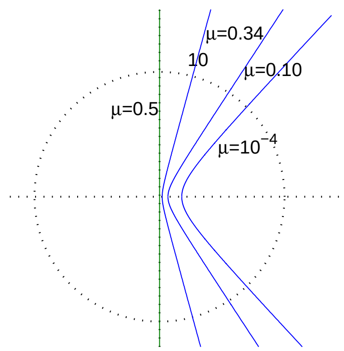


Figure 6: Experimental computation of the isoperimetric sets for $\alpha = 3$. The number m is equal to the value of μ of the corresponding **convex** region. The green line (A is a halfspace) corresponds to the case $\mu(A) = 1/2$.

2) Exponential case, $\alpha = 1$.

In this case the balls about the origin can not be excluded as eventual isoperimetric sets.

One has

$$u = a(r+1) + \lambda e^r. \tag{9}$$

It is easy to see that case 1) is not possible because (7) diverges as r goes to infinity.

In case 5) it can be verified numerically that the full rotation of the curves is less than $\pi/2$ (in this case θ starts from $\pi/2$) (see Figure 5.).

Now fix the parameter a and change λ. For positive values of λ the full rotation of the curve depends monotonically on λ. The same holds for $-a < \lambda < 0$. It turns out that for negative values of λ the curves are either non-closed or intersect themselves non-smoothly. This observation allows to describe the isoperimetric sets. Indeed, there exists a unique positive value of λ such that both ends of the curve meet smoothly. In addition, there exists $\lambda < 0$ such that both ends of the curve meet smoothly but in this case the curve is self-intersecting.

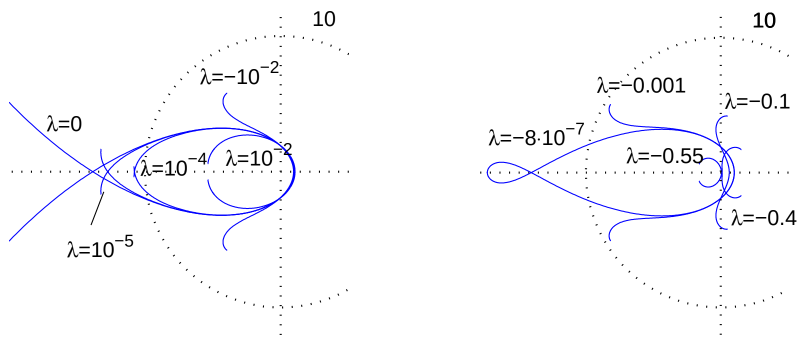


Figure 7: Solutions to (4)-(5) in the case $\alpha = 1$, $a = 0.5$ and different values of λ. The smooth curve appears for $\lambda \approx 10^{-4}$.

Thus we conclude that *every a can have only one λ corresponding to a smooth curve.*

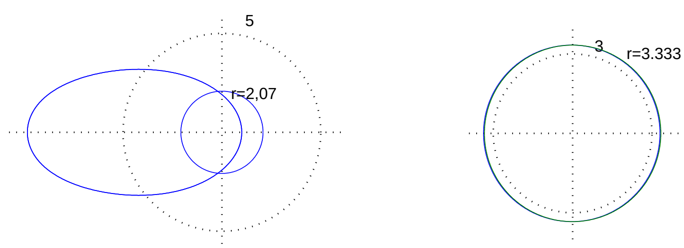


Figure 8: Isoperimetric sets in the case $\alpha = 1$, $a = 0.5$ and $a = 0.7$ and the corresponding balls of the same measure (they almost coincide on the picture).

We managed to find a family of smooth stationary curves solving (4)-(5). They do not touch the origin. According to Lemma 4.5 they have only the circles around the origin as competitors. The computations show: circles are always worse!

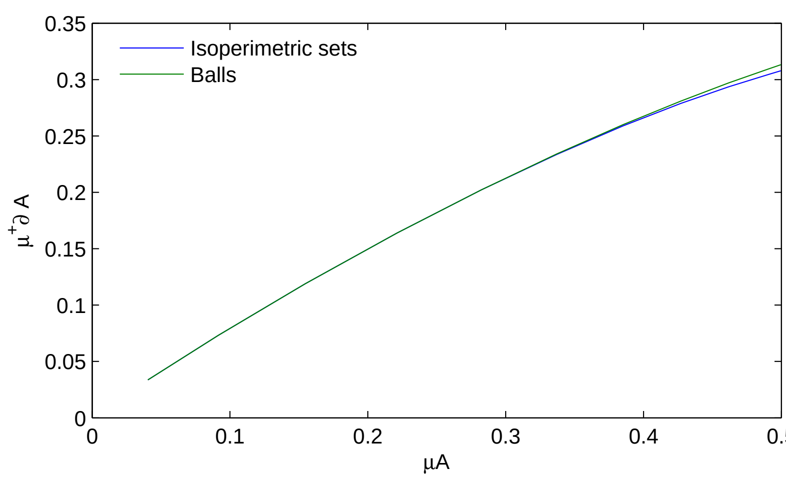


Figure 9: Isoperimetric function and dependence of the surface measure from the measure for the balls.

Conclusion: For any given value of measure there exists a unique (up to a rotation) isoperimetric curve $l = \partial A$ defined by a unique couple of parameters a, λ such that the corresponding isoperimetric set is either the compact convex set inside of l or its complement. In this case $\partial A = \{re^{if(r)}\}$, u is given by (9), and f given by (7).

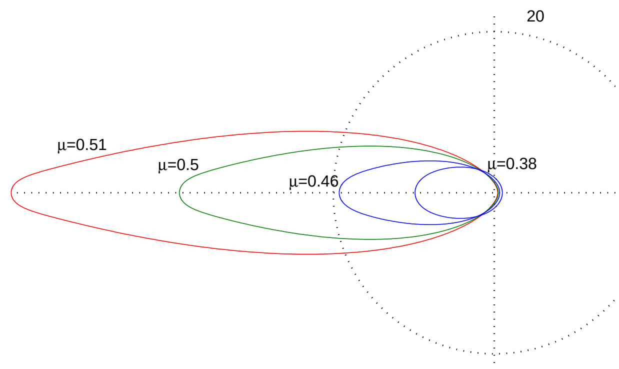


Figure 10: Experimental computation of the isoperimetric sets for $\alpha = 1$. The number m is equal to the value of μ of the corresponding **non-convex** region. The green line corresponds to the case $\mu(A) = 1/2$. The red line ($m = 0.51$) corresponds to a stationary smooth curve which is not isoperimetric (the best choice can be found among of the family of blue curves with $m = 0.49$). For small values of m the curves are asymptotically round in shape.

3) Sub-Gaussian case, $1 < \alpha < 2$.

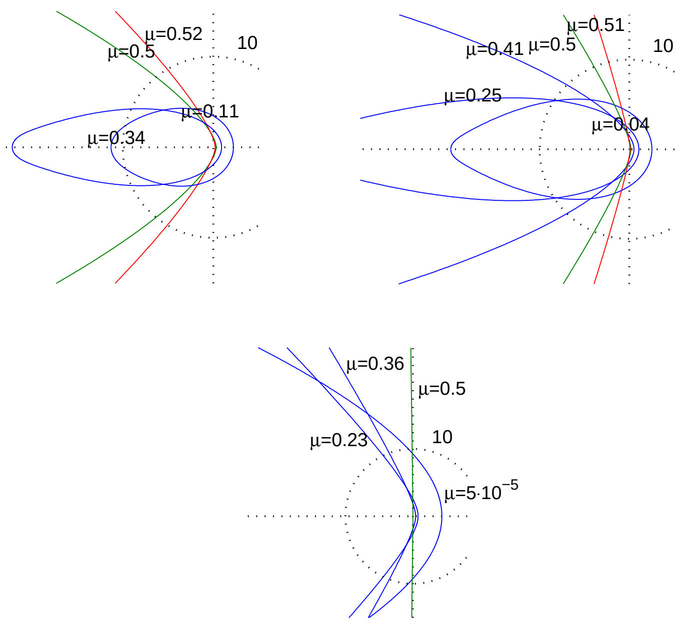
In this case there exist compact as well non-compact smooth stationary curves. However, some of them are not isoperimetric (similarly to the red curve from the Figure 10). It was verified numerically that there exist critical values $1 < \alpha_0 < \alpha_1 \leq 2$ such that the isoperimetric sets are compact for $1 < \alpha < \alpha_0$ ($\alpha_0 \approx 1.08$) and non-compact for $\alpha_1 < \alpha$. Obviously, $\alpha_1 \leq 2$. Some heuristic arguments demonstrate that α_1 can be equal to 1.5. Indeed, we know that $u \sim ar^{2-\alpha}$. Applying formula for Δ_f with $u = ar^{2-\alpha}$ we get that the full rotation $\Delta_f < \pi$ for $\alpha > 1.5$.

Conclusion:

1) there exists $a_0 = a_0(\alpha)$ such that for $a = a_0$ the corresponding curve given by (6) is the boundary of an isoperimetric set of measure 0.5;
2) for $a < a_0$ the curve given by (6) is not isoperimetric;
3) for $a > a_0$ the curve is isoperimetric and the measure of the part of the plane that does not contain the origin is less than 0.5;
4) there exists $a_1 = a_1(\alpha)$ such that for every $a < a_1$ the curve given by (6) is non-compact (this corresponds to the case $\lambda = 0$) and the full rotation is less than π (see Figure 1);
5) for every $a > a_1$ there exists a unique λ such that the isoperimetric curve is compact, closed and smooth (analogously to the case $\alpha = 1$);

6) for $1 < \alpha < \alpha_0$ one has $a_0 > a_1$. Thus for $\alpha < \alpha_0$ all the isoperimetric curves are compact (exponential type);
7) for $\alpha_1 > \alpha > \alpha_0$ the critical value a_0 is less than a_1. Thus for $\alpha_1 > \alpha > \alpha_0$ there exist compact isoperimetric curves as well as non-compact isoperimetric curves.
8) for $\alpha > \alpha_1$ and for every $a > 0$ the full rotation of corresponding curve given by (6) is less then π. Thus every isoperimetric curve is non-compact (super-Gaussian type).

Figures 11-13: Experimental computation of the isoperimetric sets for $\alpha = 1.1; 1.2; 1.5$. The number m is equal to the value of μ of the corresponding **non-convex** region. The green line corresponds to the case $\mu(A) = 1/2$. The red line corresponds to a stationary smooth curve which is not isoperimetric.



6. Log-convex measures

In this section we investigate the so called log-convex measures, i.e. measures of the form $\mu = e^V\ dx$, where V is convex. The measures of this type can be considered as natural analogs of the negatively curved spaces in geometry.

Recall that the hyperbolic space H^{d-1} serves as a model space for the negatively curved spaces. Solutions to the isoperimetric problem are given by the metric balls. The hyperbolic plane H^2 enjoys the following isoperimetric inequality:

$$4\pi\nu(A) - K\nu^2(A) \le [\nu^+(\partial A)]^2, \tag{10}$$

where ν is the Riemannian volume on H^2, $\nu^+(\partial A)$ is the length of the boundary ∂A, $\nu(A) < \infty$ and $K < 0$ is the constant Gauss curvature.

Analyzing (10) and Borell's result [9] one can conclude that a natural isoperimetric inequality for a log-convex measure has the form

$$\mu(A)^{1-\frac{1}{d}} + \mu(A)\psi(\mu(A)) \le C(d)\mu^+(\partial A) \tag{11}$$

for some increasing ψ. Here the first term in the left-hand side is "responsible" for small values of $\mu(A)$ and the second one for large one's.

The following conjecture was suggested in [31].

Conjecture 6.1. *Let $\mu = e^{v(r)}dx$ be a radially symmetric measure with a convex smooth potential v on R^d, $d \ge 2$. Then the isoperimetric regions for μ are the balls about the origin $B_R = \{x : |x| \le R\}$.*

Here we prove some particular cases of this conjecture and some related results.

6.1. **Divergence theorem and radially symmetric measures.** We start with an elementary lemma based on the divergence theorem. This lemma allows to describe asymptotically the isoperimetric function for radially symmetric measures. We deal below with open sets with Lipschitz boundaries.

Lemma 6.2. *For $\mu = e^V dx$ with a sufficiently regular V one has*

$$\mu^+(\partial A) \ge \int_A \Bigl(\mathrm{div}\frac{\nabla V}{|\nabla V|} + |\nabla V|\Bigr)d\mu.$$

In particular, if $V = v(r)$, then

$$\mu^+(\partial A) \ge \int_A \Bigl(\frac{d-1}{r} + v'(r)\Bigr)d\mu.$$

If V has convex sublevel sets, then

$$\mu^+(\partial A) \ge \int_A |\nabla V| d\mu.$$

Proof. The result follows from the trivial inequality

$$\mu^+(\partial A) \ge \int_{\partial A} \Bigl\langle n_A, \frac{\nabla V}{|\nabla V|}\Bigr\rangle e^V d\mathcal{H}^{d-1}$$

and integration by parts. □

Example 6.3. Measures $\nu_2 = e^r dx$ and $\nu_1 = e^{\sum_{i=1}^d |x_i|}dx$ satisfy the following Cheeger-type inequalities:

$$\nu_2^+(\partial A) \ge \nu_2(A),$$

$$\nu_1^+(\partial A) \ge \sqrt{d}\ \nu_1(A).$$

Let us apply this result in the radially symmetric case.

Lemma 6.4. *Let μ be any Borel positive measure on $\mathbb{R}^d$ and $F : \mathbb{R}^d \to \mathbb{R}^+$ be any function such that the corresponding distribution function $\mu_F(t) = \mu(C_t)$, where $C_t := \{x : F(x) \le t\}$, is continuous and strictly increasing from 0 to $\mu(\mathbb{R}^d)$. Then for every Borel set A with finite measure one has*

$$\int_A F d\mu \ge \int_{C_t} F d\mu,$$

where t is chosen in such a way that $\mu(A) = \mu(C_t)$.

Proof. First we note that the existence of t satisfying $\mu(A) = \mu(C_t)$ follows from the assumptions. Next

$$\begin{aligned}\int_A F(x)d\mu &= \int_{A\cap C_t} F(x)d\mu + \int_{A\cap C_t^c} F(x)d\mu \\ &\geq \int_{A\cap C_t} F(x)d\mu + \int_{A\cap C_t^c} t d\mu \\ &= \int_{A\cap C_t} F(x)d\mu + t\mu(A^c \cap C_t) \geq \int_{C_t} F(x)d\mu.\end{aligned}$$

In the proof we use that $\mu(A^c \cap C_t) = \mu(A \cap C_t^c)$ (this is because $\mu(A) = \mu(C_t)$). □

Remark 6.5. Applying Lemma 6.4 and Lemma 6.2 to a radially symmetric measure $\mu = \exp v(r)\ dx$ and the function $F = v'(r)$ with increasing v' one obtains the following estimate of the isoperimetric function:

$$\mathcal{I}_\mu(t) \geq \int_{B_{r(t)}} v'(r)\ d\mu,$$

where $r(t)$ satisfies $\mu(B_{r(t)}) = t$. Note that the term $\frac{d-1}{r}$ is negligible for any increasing v' and large values of t. Thus the obtained estimate is asymptotically sharp for large sets. See also Proposition 6.7 and Theorem 6.10

Example 6.6. Let $\mu = e^{r^\alpha} dx$, $\alpha > 1$. For large values of $\mu(A)$ (say $\mu(A) \geq 1$) one has

$$\mu^+(\partial A) \geq C\mu(A) \log^{1-\frac{1}{\alpha}} \mu(A).$$

Indeed, apply Lemma 6.4 to $F = r^{\alpha-1}$. We get $\mu^+(\partial A) \geq \alpha \int_A r^{\alpha-1}\ d\mu \geq \alpha \int_{B_{r(\mu(A))}} r^{\alpha-1}\ d\mu$. For large r one has

$$\mu(B_r) \sim c_1 r^d e^{r^\alpha}, \quad \int r^{\alpha-1}\ d\mu \sim c_2 r^{d+\alpha-1} e^{r^\alpha}$$

and we easily get the desired asymptotics.

Analogously, for $0 < \tau \leq d$

$$\mu_\tau = \frac{dx}{(1-r^2)^{1+\tau}},\ x \in B_1,$$

satisfies

$$\mu_\tau^+(\partial A) \geq C\mu_\tau(A)^{1+\frac{1}{\tau}}.$$

For $\tau = 0$ one has

$$\mu_0^+(\partial A) \geq e^{C\mu_0(A)}.$$

Moreover, applying some refinements of the arguments from above, we show that **for the strictly log-convex radially symmetric measures the large balls are isoperimetric sets**.

Proposition 6.7. *Let $\mu = e^{v(r)}\ dx$ be a radially symmetric measure on $\mathbb{R}^d$ with increasing v. Assume that there exists a smooth function $f : [0,\infty) \to \mathbb{R}$, satisfying the following assumptions:*

1) $|f| \leq 1$
2) $f(r_0) = 1$
3) *function $F(r) = f'(r) + f(r)\Big(v'(r) + \frac{d-1}{r}\Big)$ is increasing on $[0,\infty)$.*

Then among the sets satisfying $\mu(A) = \mu(B_{r_0})$ the ball B_{r_0} has the minimal surface measure μ^+.

Proof. Set: $\omega(x) = f(r) \cdot \frac{x}{r}$. Take a set A with $\mu(A) = \mu(B_{r_0})$. Without loss of generality we assume that ∂A is smooth and denote by n_A the outer unit normal to ∂A. Applying integration-by-parts we get

$$\mu^+(\partial A) \geq \int_{\partial A} \langle \omega, n_A \rangle e^{v(r)} \, dx = \int_A \operatorname{div}\big(\omega \cdot e^{v(r)}\big) \, dx =$$
$$= \int_A \Big[f'(r) + f(r)\Big(v'(r) + \frac{d-1}{r}\Big)\Big] e^{v(r)} \, dx = \int_A F(r) e^{v(r)} \, dx.$$

Using that F is increasing in r, we get by Lemma 6.4 that $\int_A F(r) e^{v(r)} \, dx \geq \int_{B_{r_0}} F(r) e^{v(r)} \, dx$. Thus

$$\mu^+(\partial A) \geq \int_{B_{r_0}} F(r) e^{v(r)} \, dx.$$

It remains to note that for $A = B_{r_0}$ we have equalities in all the computations above. Hence

$$\mu^+(\partial A) \geq \mu^+(\partial B_{r_0}).$$

□

Corollary 6.8. *Assume that $v'' \geq 1$. Let $r_0 \geq \sqrt{d+2}$. Then among all the sets satisfying $\mu(A) = \mu(B_{r_0})$ the ball B_{r_0} has the minimal surface measure.*

Proof. Without loss of generality we assume that $v(0) = 0$. Set:

$$f(r) = \begin{cases} \frac{3}{2\sqrt{d+2}}\big(r - \frac{r^3}{3(d+2)}\big), & r \leq \sqrt{d+2} \\ 1 & r > \sqrt{d+2}. \end{cases}$$

Assume first that $v = r^2/2$. Note that f is continuously differentiable and increasing. One has

$$F(r) = \begin{cases} \frac{3}{2\sqrt{d+2}}\big(d + \frac{2r^2}{3} - \frac{r^4}{3(d+2)}\big), & r \leq \sqrt{d+2} \\ r + \frac{d-1}{r} & r > \sqrt{d+2}, \end{cases}$$

$$F'(r) = \begin{cases} \frac{2r}{\sqrt{d+2}}\big(1 - \frac{r^2}{d+2}\big), & r \leq \sqrt{d+2} \\ 1 - \frac{d-1}{r^2} & r > \sqrt{d+2}. \end{cases}$$

Clearly, f and F satisfy assumptions of the Proposition 6.7 for every $r_0 \geq \sqrt{d+2}$.

It is easy to check that for v satisfying $v'' > 1$ (hence $v' \geq r$) one has $F'(r) \geq \frac{2r}{\sqrt{d+2}}\big(1 - \frac{r^2}{d+2}\big)$ for $r \leq \sqrt{d+2}$ and $F'(r) \geq 1 - \frac{d-1}{r^2}$ for $r \geq \sqrt{d+2}$. The proof is complete. □

Remark 6.9. One can construct more examples using Proposition 6.7. It is applicable under assumption of certain strict convexity of v. It was pointed out to the author by Frank Morgan that the arguments of Proposition 6.7 imply the following results:

1) If $v' = r^a$ with $a > 0$, then the balls B_r are minimizers if $r > r_0$, where r_0 satisfies $r_0 = \sqrt[a+1]{\frac{d+2}{a}}$.

2) In $\mathbb{R}^d$ with the Riemannian metric

$$dr^2 + (e^{u(r)})^2 d\theta^2$$

and density $e^{v(r)}$ (with respect to the Riemannian volume) satisfying

$$(d-1)u'' + v'' + \frac{d-1}{r^2} \geq 1$$

the balls B_r with $r \geq \sqrt{d+2}$ are minimizers.

6.2. **An estimate of the isoperimetric function for radially symmetric log-convex measures.** The main aim of this subsection is the following theorem.

Theorem 6.10. *Let $\mu = e^{v(r)}dx$ be a measure with a convex increasing potential v. Then for every Borel set A the following inequality holds*

$$\mu^+(\partial A) \geq \frac{1}{\sqrt{1+\pi^2}}\ \mu^+(\partial B_r),$$

provided $\mu(A) = \mu(B_r)$.

Roughly speaking, this means that the balls about the origin define the isoperimetric profile up to some universal constant. In the proof we apply the mass transportation techniques.

The idea of applying the mass transportation to isoperimetric inequalities belongs to M. Gromov. In particular, he applied the triangular mappings (Knothe mappings) to obtain the classical isoperimetric inequality. Unfortunately, it seems hard to obtain sharp constants for isoperimetric inequalities by using only the mass transportation arguments in more general situations. Nevertheless, they can be used for proving the isoperimetric inequalities with the best rate. Gromov arguments for the Euclidean isoperimetric problem are nowadays well-known and can be found in many papers and books. Let us give another interesting example.

Example 6.11. The following isoperimetric inequality holds in the hyperbolic space H_d:

$$\nu^+(\partial A) \geq \max\Bigl[\frac{1}{\kappa_d}\nu^{1-\frac{1}{d}}(A), (d-1)\nu(A)\Bigr].$$

Consider the hyperbolic space $H_d = \mathbb{R}^{d-1} \times R^+$,

$$g = \frac{dy_1^2 + \cdots dy_d^2}{y_d^2} = g_0(dy_1^2 + \cdots dy_d^2).$$

Then $\nu = \frac{dy_1\cdots dy_d}{y_d^d} I_{\{y_d>0\}}$. Consider a bounded Borel set $A \subset \{y_d > \varepsilon\}$ with $\varepsilon > 0$. Let T be the optimal Euclidean transportation map pushing forward $\nu|_A$ to the Lebesgue measure restricted to some Euclidean ball $B_r \subset \mathbb{R}^d$ with a center to be chosen later. By the change of variables formula $g_0^{d/2} = \det D^2W$ on A, hence $0 = \log\det\Bigl(\frac{1}{\sqrt{g_0}}D^2W\Bigr) \leq \frac{\mathrm{Tr} D^2W}{\sqrt{g_0}} - d$. Integrating by parts we obtain

$$d\nu(A) \leq \int_A \frac{\mathrm{Tr}D^2W}{\sqrt{g_0}} g_0^{d/2}\, d\lambda = \int_{\partial A} \frac{\langle n_A, \nabla W\rangle}{g_0^{1/2}} g_0^{d/2}\, d\mathcal{H}^{d-1} - \int_A \langle \nabla W, \nabla g_0^{\frac{d-1}{2}}\rangle\, d\lambda.$$

Noting that $|\nabla_M f|^2_M = \frac{1}{g_0}\Bigl(\frac{\partial f}{\partial x_i}\Bigr)^2$ and estimating $\frac{\langle n_A, T\rangle}{g_0^{1/2}}$ by $|T||n_A|_M$ we get $d\nu(A) \leq \sup|T|\nu^+(\partial A) + (d-1)\int_A \frac{\langle T, e_d\rangle}{y_d^d}\, d\lambda$. Applying the change of variables one gets

$$d\nu(A) \le \sup |T| \nu^{+}(\partial A) + (d-1) \int_{B_r} y_d \, d\lambda.$$

Choosing the center of B_r at the point $(0, -tr)$ with $t \ge 0$ we get $\sup |T| \le (t+1)r$. In addition, using $\int_{B_r} (y + rt) \, d\lambda = 0$, one obtains

$$d\nu(A) \le (t+1) r \nu^{+}(\partial A) - (d-1) \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})} r^{d+1} t.$$

Taking into account that $\nu(A) = \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})} r^d$ and choosing $t = 0, t = +\infty$, one obtains the claim.

Definition 6.12. Let μ, ν be a couple of probability measures. We say that T is a radial mass transportation of μ to ν if $\nu \circ T^{-1} = \mu$ and it has the form

$$T(x) = g(r) \cdot N(x)$$

with $r = |x|$ and $|N(x)| = 1$. In particular, $T(\partial B_r) \subset \partial B_{g(r)}$.

There are different ways to transport μ to ν by a radial transportation mapping. Consider the decomposition $\mathbb{R}^d = [0, \infty) \times S^{d-1}$. We use below the following construction. Let ν_r, μ_r be the one-dimensional images of ν, μ under the mapping $x \to |x| = r$. Let $g(r)$ be the increasing function pushing forward ν_r to μ_r. For every fixed r we denote by $\nu^r(\theta)$, $\mu^r(\theta)$ the corresponding conditional measures on S^{d-1} obtained by disintegration of ν, μ. Let

$$T_r^{S^{d-1}} = T_r^{S^{d-1}}(\theta) : S^{d-1} \to S^{d-1}$$

be the optimal transportation mapping pushing forward $\nu^r(\theta)$ to $\mu^r(\theta)$ and minimizing the squared Riemannian distance on S^{d-1}.

Then

$$T = g(r) \cdot T_r^{S^{d-1}}(x/r)$$

is the desired mapping. Recall that $T_r^{S^{d-1}}$ has the form

$$T_r^{S^{d-1}} = \exp(\nabla_{S^{d-1}} \varphi)$$

for some $\frac{1}{2}d^2$-convex potential φ. Here $\nabla_{S^{d-1}}$ is the spherical gradient on S^{d-1}, exp is the exponential mapping on S^{d-1} and, in addition,

$$|\nabla_{S^{d-1}} \varphi(x)| = d(x, T_r^{S^{d-1}}(x)). \tag{12}$$

For a fixed x consider a unit vector v such that $v \perp x$. One has

$$\partial_r T = g_r \cdot T_r^{S^{d-1}} + g \cdot \frac{\partial T_r^{S^{d-1}}}{\partial r},$$

$$\partial_v T = g \cdot \frac{\partial T_r^{S^{d-1}}}{\partial v}.$$

We use below a computation obtained in [13] (pp. 48, 96) (see also [14]). Consider a mapping $\tilde{T} = \exp(\nabla_{S^{d-1}} \varphi)$. Choose for a fixed $x \in S^{d-1}$ an orthonormal basis in the tangent space to S^{d-1} such that the first vector is equal to $\frac{\nabla_{S^{d-1}} \varphi}{\theta}$, where $\theta = |\nabla_{S^{d-1}} \varphi|$. In this basis the Jacobian matrix looks like

$$D\tilde{T} = \begin{pmatrix} 1 + a & b^t \\ \frac{\sin\theta}{\theta} b & \cos\theta \cdot I + \frac{\sin\theta}{\theta} \cdot D \end{pmatrix},$$

where

$$\text{Hess } \varphi = \begin{pmatrix} a & b^t \\ b & D \end{pmatrix}.$$

The d^2-convexity condition takes the form $H \ge 0$, where

$$H = \begin{pmatrix} 1+a & b^t \\ b & \frac{\theta}{\tan\theta} I + D \end{pmatrix}.$$

Taking into account that $B_r = rS^{d-1}$ and $\partial_r T_r, \partial_v T_r$ are orthogonal to T_r one gets (13)

$$\det DT = g_r \Bigl(\frac{g}{r}\Bigr)^{d-1} \det DT_r = g_r \Bigl(\frac{g}{r}\Bigr)^{d-1} \det \begin{pmatrix} 1+a & b^t \\ \frac{\sin\theta}{\theta} b & \cos\theta \cdot I + \frac{\sin\theta}{\theta} \cdot D \end{pmatrix}$$

Proof. (*Theorem 6.10*). Now fix a Borel set A and take T pushing forward $\mu|_A$ to $\mu|_{B_{r_0}}$. Srt:

$$\mu = \rho(r)\ dx$$

and apply (13)

$$\rho(g(r)) g_r \Bigl(\frac{g}{r}\Bigr)^{d-1} \det DT_r = \rho(r). \tag{14}$$

Note that the change of variables formula requires some additional justification. According to the results of Section 3 we can first symmetrize A (see Proposition 3.8 or [28]) and deal from the very beginning only with sets with a revolution axis. In this case (13) clearly holds. Indeed, every T_r is smooth because it is an optimal mapping sending a spherical cap onto S^{d-1}.

Take the logarithm of the both sides of (14). We apply inequality $\ln\det M \le \mathrm{T}rM - n$ which holds for every symmetric positive $n \times n$-matrix M. It is easy to check, that the following inequality holds :

$$\ln\rho - \ln\rho(g) \le \frac{g}{r}\Bigl(1 + a + (d-2)\cos\theta + \frac{\sin\theta}{\theta}\mathrm{Tr}D\Bigr) + g_r - d.$$

Using the estimates $\frac{\sin\theta}{\theta} \le 1$, $\frac{\theta}{\tan\theta} \le 1$ and positivity of H one obtains

$$\begin{aligned} &1 + a + (d-2)\cos\theta + \frac{\sin\theta}{\theta}\mathrm{Tr}D = \\ &= 1 + a + \frac{\sin\theta}{\theta}\Bigl(\mathrm{Tr}D + \frac{\theta}{\tan\theta}(d-2)\Bigr) \le 1 + a + \Bigl(\mathrm{Tr}D + \frac{\theta}{\tan\theta}(d-2)\Bigr) \\ &\le a + d - 1 + \mathrm{Tr}D = \Delta_{S^{d-1}}\varphi + d - 1. \end{aligned}$$

Let us integrate over A with respect to $\mu = \rho\ dx$:

$$\int_A \rho\log\rho\ dx - \int_A \rho\log\rho(g)\ dx \le \int_A \Bigl(\frac{g}{r}(\Delta_{S^{d-1}}\varphi + d - 1) + g_r\Bigr)\rho\ dx - d\mu(A).$$

Applying integration by parts we get

$$\begin{aligned} \int_A g_r\rho\ dx &= \int_A \langle \nabla g, \frac{x}{r}\rangle \rho\ dx \\ &= \int_{\partial A} g\langle n_A, \frac{x}{r}\rangle\ \rho\ d\mathcal{H}^{d-1} - (d-1)\int_A \frac{g\rho}{r}\ dx - \int_A g\rho_r\ dx. \end{aligned}$$

Hence

$$\int_A \log\frac{\rho}{\rho(g)}\rho\ dx + d\mu(A) + \int_A g\rho_r\ dx \le \int_{\partial A} g\langle n_A, \frac{x}{r}\rangle\ \rho\ d\mathcal{H}^{d-1} + \int_A \frac{g}{r}\rho\cdot\Delta_{S^{d-1}}\varphi\ dx.$$

By the coarea formula

$$\int_A \frac{g}{r}\rho \cdot \Delta_{S^{d-1}}\varphi \; dx = \int_0^\infty \Big[\frac{1}{r}\int_{\partial B_r \cap A} \Delta_{S^{d-1}}\varphi \; d\mathcal{H}^{d-1}\Big] \; g\rho \; dr.$$

Integrating by parts on $\partial B_r = rS^{d-1}$ we get that for every smooth ξ

$$\int_{\partial B_r} \Delta_{S^{d-1}}\varphi \; \xi \; d\mathcal{H}^{d-1} = -\int_{\partial B_r} \langle \nabla_{S^{d-1}}\varphi, \nabla_{S^{d-1}}\xi\rangle \; d\mathcal{H}^{d-1}.$$

Note that $\frac{1}{r}\nabla_{S^{d-1}}\xi$ is nothing else but the projection $Pr_{TS^{d-1}}\nabla\xi$ of the $\nabla\xi$ onto the tangent space to ∂B_r. Hence

$$\begin{aligned}\int \frac{g}{r}\rho\xi \cdot \Delta_{S^{d-1}}\varphi \; dx &= -\int_0^\infty \Big[\int_{\partial B_r} \langle Pr_{TS^{d-1}}\nabla\xi, \nabla_{S^{d-1}}\varphi\rangle \; d\mathcal{H}^{d-1}\Big] \; g\rho \; dr \\ &= -\int \langle Pr_{TS^{d-1}}\nabla\xi, \nabla_{S^{d-1}}\varphi\rangle \; g\rho \; dx.\end{aligned}$$

Approximating I_A by smooth functions we get

$$\int_A \frac{g}{r}\rho \cdot \Delta_{S^{d-1}}\varphi \; dx = \int_{\partial A} \langle Pr_{TS^{d-1}} n_A, \nabla_{S^{d-1}}\varphi\rangle g\rho \; d\mathcal{H}^{d-1}.$$

Hence

$$\begin{aligned}&\int_A \rho \log \frac{\rho}{\rho(g)} \; dx + d\mu(A) + \int_A g\rho_r \; dx \le \\ &\le \int_{\partial A} g\big\langle n_A, \frac{x}{r}\big\rangle \; \rho \; d\mathcal{H}^{d-1} + \int_{\partial A} \langle Pr_{TS^{d-1}} n_A, \nabla_{S^{d-1}}\varphi\rangle g\rho \; d\mathcal{H}^{d-1}.\end{aligned}$$

Since $\exp(\nabla_{S^{d-1}}\varphi)$ takes values in the unit sphere, one has $|\nabla_{S^{d-1}}\varphi| \le \pi$ (see (12)) and the right-side does not exceed

$$\int_{\partial A} g\sqrt{1+\pi^2}\rho \; d\mathcal{H}^1 \le r_0\sqrt{1+\pi^2}\mu^+(\partial A).$$

Note that $g(r) \le r$. Hence

$$\int_A \log \frac{\rho}{\rho(g)} \rho \; dx \ge 0$$

and

$$\int_A g\rho_r \; dx = \int_A g v_r \; d\mu = \int_{B_{r_0}} r v_r(g^{-1}) \; d\mu \ge \int_{B_{r_0}} r v_r \; d\mu.$$

Finally, we obtain

$$d\mu(B_{r_0}) + \int_{B_{r_0}} r v_r \; d\mu \le r_0\sqrt{1+\pi^2}\mu^+(\partial A).$$

The divergence theorem implies that the left-hand side is equal to $r_0\mu^+(\partial B_{r_0})$. Hence

$$\mu^+(\partial B_{r_0}) \le \sqrt{1+\pi^2}\mu^+(\partial A).$$

$\square$

6.3. **Product measures. A comparison theorem.** We start this subsection with a comparison result. The comparison theorems are very important tools for studying the isoperimetric estimates. The most well-known example is the Levy-Gromov's isoperimetric inequality for the Ricci positive manifolds. Its probabilistic version is given by the Bakry-Ledoux comparison theorem [3] (see also [27]).

Theorem (Bakry-Ledoux): Assume that

$$\mu = e^{-V}\, dx,$$

is a probability measure with V satisfying

$$D^2 V \ge c \cdot \mathrm{Id}, \quad c > 0$$

and γ_c is the Gaussian measure with the covariance operator $c \cdot \mathrm{Id}$. Then

$$\mathcal{I}_{\mu} \ge \mathcal{I}_{\gamma_c}.$$

The Bakry-Ledoux theorem is an immediate corollary of the following result:

Theorem (Caffarelli): For every probability measure $\mu = e^{-V}\, dx$ with $D^2 V \ge I$ the optimal transportation mapping $T = \nabla \varphi$ with convex φ which pushes forward the standard Gaussian measure γ onto μ is 1-Lipschitz (see [10], Theorem 11 and recent developments in [5], [23], [34], [21]).

Note that the spaces from these examples are positively curved (i.e. with a positive Bakry-Emery tensor). Concerning the negatively curved spaces, it is still an open problem, whether the Cartan-Hadamard conjecture holds in general case.

Cartan-Hadamard conjecture: Let M be a complete, smooth, simply connected Riemannian manifold with sectional curvatures bounded from above by a constant nonpositive value c. The isoperimetric function $\mathcal{I}_M$ satisfies $\mathcal{I}_M \ge \mathcal{I}_c$, where $\mathcal{I}_c$ is the isoperimetric function of the model space with the constant sectional curvature c.

The conjecture is known to be true for a long time for $d = 2$ (see, for instance, [35]). Other known cases: $d = 3$ (B. Kleiner, [22]), $d = 4$, $c = 0$ (C. Croke, [15]). Some new proofs and recent developments can by found in M. Ritoré [29], F. Schulze [32].

In this paper we prove a comparison result for the products of log-convex measures. It turns out that a natural model measure for the one-dimensional log-convex distributions has the following form:

$$\nu_A = \frac{dx}{\cos Ax}, \ -\frac{\pi}{2A} < x < \frac{\pi}{2A}.$$

Its potential V satisfies

$$V'' e^{-2V} = A^2.$$

By a result from [31] (Corollary 4.12) the isoperimetric sets for strictly log-concave even measures on the line are symmetric intervals containing the origin.

Using this result it is easy to compute the isoperimetric profile of ν_A:

$$\mathcal{I}_{\nu_A}(t) = e^{At/2} + e^{-At/2}.$$

It turns out that in the log-convex case the following quantity is a natural measure of convexity of the potential:

$$W^{''} e^{-2W}$$

(unlike $W^{''}$ in the probabilistic case).

We establish here the following analog of the Caffarelli result.

Proposition 6.13. *Let $\mu = e^W dx$ be a measure with even convex potential W. Assume that*

$$W'' e^{-2W} \ge A^2,$$

and $W(0) = 0$. Then μ is the image of ν_A under a 1-Lipschitz increasing mapping.

Proof. Without loss of generality one can assume that W is smooth and $W'' e^{-2W} > A^2$. Let φ be a convex potential such that $T = \varphi'$ sends μ to ν_A. In addition, we require that T is antisymmetric. Clearly, φ' satisfies

$$e^W = \frac{\varphi''}{\cos A\varphi'}.$$

Assume that x_0 is a local maximum point for φ''. Then at this point

$$\varphi^{(3)}(x_0) = 0 \quad \varphi^{(4)}(x_0) \le 0.$$

Differentiating the change of variables formula at x_0 twice we get

$$W'' = \frac{\varphi^{(4)}}{\varphi''} - \Bigl(\frac{\varphi^{(3)}}{\varphi''}\Bigr)^2 + \frac{A^2}{\cos^2 A\varphi'}(\varphi'')^2 + A\frac{\sin A\varphi'}{\cos A\varphi'}\varphi'''.$$

Consequently one has at x_0

$$W'' \le \frac{A^2}{\cos^2 A\varphi'}(\varphi'')^2 = A^2 e^{2W}.$$

But this contradicts to the main assumption.

Hence φ'' has no local maximum. Note that φ is even. This implies that that 0 is the global minimum of φ''. Hence

$$\varphi'' \ge \varphi''(0) = 1.$$

Clearly, T^{-1} is the desired mapping. □

The corollary below can be seen as an elementary "flat" version of the Cartan-Hadamard-type comparison results or as a log-convex version of the Bakry-Ledoux comparison theorem (see also 6.16).

Corollary 6.14. *Let μ be a product measure*

$$\mu = \prod_{i=1}^{d} e^{W_i} \ dx_i$$

with

$$W_i^{''} e^{-2W_i} \ge A^2, \ \ W_i \text{ is even and } W_i(0) = 0.$$

Then

$$\mathcal{I}_\mu \ge \mathcal{I}_{\mu_A},$$

where $\mu_A = \prod_{i=1}^d \frac{dx_i}{\cos Ax_i}$ is the measure on $[-\frac{\pi}{2A}, \frac{\pi}{2A}]^d$.

It is not clear, whether this result can be generalized to the multi-dimensional case. More generally, it is not clear, which measure of convexity V is responsible for the isoperimetric properties (in the probabilistic case this is the Hessian of the potential). Surprisingly, in certain situation some lower bounds on

$$\det D^2 V \cdot e^{-V}$$

turn out to be sufficient for some isoperimetric estimates.

We denote by κ_d the constant appearing in the Euclidean isoperimetric inequality $\lambda(A)^{1-\frac{1}{d}} \le \kappa_d \mathcal{H}^{d-1}(\partial A)$.

Proposition 6.15. *Let $V \ge 0$ be convex and, in addition,*

$$e^{-V} \det D^2 V \ge K^d \tag{15}$$

for some $K \ge 0$. Then for some constant $C(d)$ the following inequality holds

$$\mu(A)^{1-\frac{1}{d}} + KC(d)\ \mu(A)^{1+\frac{1}{d}} \le \kappa_d \mu^+(\partial A). \tag{16}$$

Proof. Let ∇W be the optimal transportation pushing forward $\mu|_A$ to $\lambda|_{B_r}$. By the change of variables formula (see [25])

$$e^V = \det D^2 W_a$$

on I_A, where $D^2 W_a$ is the second Alexandrov derivative of W (recall that $D^2 W \ge D^2_a W$, where $D^2 W$ is the distributional derivative). Taking the logarithm of the both sides and applying the standard estimate one gets

$$0 \le V \le \Delta W_a - d.$$

Integrating over A one gets

$$d\mu(A) \le \int_A \Delta W\, d\mu = \int_{\partial A} \langle n_A, \nabla W \rangle e^V\ d\mathcal{H}^{d-1} - \int_A \langle \nabla W, \nabla V \rangle e^V\, d\lambda.$$

Hence

$$d\mu(A) + \int_A \langle \nabla W, \nabla V \rangle e^V\, d\lambda \le r\mu^+(\partial A). \tag{17}$$

By the change of variables $\int_A \langle \nabla W, \nabla V \rangle e^V\, d\lambda = \int_{B_r} \langle x, \nabla V \circ \nabla \Phi \rangle\, d\lambda$, where $\Phi = W^*$ is the corresponding convex conjugated function. Note that $\nabla \Phi$ is the optimal transport of $\lambda|_{B_r}$ onto $\mu|_A$. Taking into account that $x = \nabla \frac{|x|^2 - r^2}{2}$ and integrating by parts one gets

$$\begin{aligned}\int_{B_r} \langle x, \nabla V \circ \nabla \Phi \rangle\, d\lambda &\ge \frac{1}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \mathrm{Tr}\Bigl[D^2 V\bigl(\nabla\Phi\bigr) \cdot D^2_a \Phi\Bigr]\, d\lambda \\ &= \frac{1}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \mathrm{Tr}\Bigl[(D^2_a\Phi)^{\frac{1}{2}} \cdot D^2 V\bigl(\nabla\Phi\bigr) \cdot (D^2_a\Phi)^{\frac{1}{2}}\Bigr]\, d\lambda.\end{aligned}$$

Since $\nabla\Phi$ pushes forward $\lambda|_{B_r}$ to $\mu|_A$, by the change of variables

$$e^{V(\nabla\Phi)} \det D^2_a \Phi = 1.$$

Note that $(D^2_a\Phi)^{\frac{1}{2}} \cdot D^2 V\bigl(\nabla\Phi\bigr) \cdot (D^2_a\Phi)^{\frac{1}{2}}$ is a symmetric matrix. It is nonnegative, since V and Φ are convex. Hence

$$\begin{aligned}&\frac{1}{d}\mathrm{Tr}\Bigl[(D^2_a\Phi)^{\frac{1}{2}} \cdot D^2 V\bigl(\nabla\Phi\bigr) \cdot (D^2_a\Phi)^{\frac{1}{2}}\Bigr] \ge \Bigl(\det D^2_a\Phi \cdot \det D^2 V\bigl(\nabla\Phi\bigr)\Bigr)^{\frac{1}{d}} = \\ &= \Bigl(e^{-V(\nabla\Phi)} \cdot \det D^2 V\bigl(\nabla\Phi\bigr)\Bigr)^{\frac{1}{d}} \ge K.\end{aligned}$$

Finally we obtain that for some constant C depending only on d

$$d\mu(A) + \frac{dKC}{2} r^{d+2} \le r\mu^+(\partial A).$$

The desired result follows from the relation $r = \Bigl(\frac{\mu(A)}{\lambda(B_1)}\Bigr)^{\frac{1}{d}} = d\kappa_d \mu^{\frac{1}{d}}(A)$. □

Corollary 6.16. *If $V \ge 0$ and convex, then $\mu = e^V\ dx$ satisfies the Euclidean isoperimetric inequality.*

Example 6.17. Note that condition (15) is tensorizable. The following product measure on $[-\pi/2, \pi/2]^d$ satisfies inequality (16) :

$$\mu = \prod_{i=1}^{d} \frac{dx_i}{\cos^2 x_i}.$$

We finish this section with another isoperimetric estimate for log-convex product measures.

Theorem 6.18. *Consider a log-convex measure*

$$\mu = \prod_{i=1}^{d} e^{V_i(x_i)}\ dx_i$$

such that every V_i is convex, even and $V_i(0) = 0$. Assume that there exists a concave increasing function $G : [0,\infty) \to [0,\infty)$ satisfying

$$e^{-V_i} G'(V_i') V_i^{''} \ge 1.$$

Then for some constants $c_1(d), c_2(d)$ the following inequality holds

$$\mu(A)^{1-\frac{1}{d}} + c_1(d)\ \mu(A) \cdot G^{-1}\bigl(c_2 \mu^{1/d}(A)\bigr) \le \kappa_d \mu^+(\partial A).$$

Proof. According to the general result on Steiner symmetrization for product measures (see [30]), the Steiner symmetrization with respect to any axis does not increase the surface measure of the set. Since the family of symmetric intervals are isoperimetric sets for every one-dimensional measure $e^{V_i}\ dx_i$, we can assume from the very beginning that A is symmetric with respect to every mapping $x \to (\pm x_1, \cdots, \pm x_i, \cdots, \pm x_n)$. Let ∇W be the optimal transportation pushing forward $\mu|_A$ to $\lambda|_{B_r}$. In the same way as in the previous proposition we prove

$$d\mu(A) + \int_A \langle \nabla W, \nabla V \rangle e^V\ d\lambda \le r\mu^+(\partial A)$$

and

$$\int_A \langle \nabla W, \nabla V \rangle e^V\ d\lambda = \int_{B_r} \langle x, \nabla V \circ \nabla \Phi \rangle\, d\lambda.$$

By the symmetry reasons the functions x_i and $V_i'(\nabla\Phi) = V_i'(\Phi_{x_i})$ have the same sign. Hence by the Jensen inequality (which is applicable because G^{-1} is convex)

$$\int_{B_r} \langle x, \nabla V \circ \nabla \Phi \rangle\, d\lambda = \sum_{i=1}^{d} \int_{B_r} |x_i||V_i'(\Phi_{x_i})|\, d\lambda \ge$$

$$\ge \sum_{i=1}^{d} \int_{B_r} |x_i|\ d\lambda \cdot G^{-1}\Bigl(\frac{\int_{B_r} |x_i| G(V_i'(|\Phi_{x_i}|))\ d\lambda}{\int_{B_r} |x_i|\ d\lambda}\Bigr).$$

Denote $S = (G(V_i'))$. One has

$$\sum_i \int_{B_r} |x_i| G(V_i'(|\Phi_{x_i}|))\, d\lambda = \int_{B_r} \langle x, S \circ \nabla\Phi \rangle\ d\lambda.$$

The latter is larger than

$$\int_{B_r} \langle x, \nabla S \circ \nabla \Phi \rangle\, d\lambda \ge \frac{1}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \mathrm{Tr}\Bigl[D^2 S\bigl(\nabla\Phi\bigr) \cdot D_a^2 \Phi\Bigr]\, d\lambda$$

$$= \frac{1}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \mathrm{Tr}\Bigl[(D_a^2\Phi)^{\frac{1}{2}} \cdot D^2 S\bigl(\nabla\Phi\bigr) \cdot (D_a^2\Phi)^{\frac{1}{2}}\Bigr]\, d\lambda.$$

Note that DS is diagonal and nonnegative. Applying the arithmetic-geometric inequality for the trace and determinant we get

$$\int_{B_r} \langle x, \nabla S \circ \nabla \Phi \rangle \, d\lambda \geq \frac{d}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \bigl[\det(D^2_a \Phi) \cdot \det D^2 S\bigl(\nabla \Phi\bigr)\bigr]^{1/d} \, d\lambda.$$

The latter is equal to

$$\frac{d}{2} \int_{B_r} \bigl(r^2 - |x|^2\bigr) \Bigl[e^{-V(\nabla \Phi)} \det D^2 S\bigl(\nabla \Phi\bigr)\Bigr]^{1/d} \, d\lambda.$$

Note that

$$e^{-V(\nabla \Phi)} \det D^2 S\bigl(\nabla \Phi\bigr) = \Bigl[\prod_{i=1}^{d} e^{-V_i} G'(V'_i) V''_i\Bigr] \circ \nabla \Phi \geq 1.$$

Hence for some $C(d)$

$$\int_{B_r} \langle x, \nabla S \circ \nabla \Phi \rangle \, d\lambda \geq C(d) r^{d+2}.$$

Thus

$$\int_{B_r} \langle x, \nabla V \circ \nabla \Phi \rangle \, d\lambda \geq C_1(d) r^{d+1} G^{-1}(C_2(d) r).$$

The result follows from the relation $\mu(A) = \lambda(B_r)$. □

Corollary 6.19. *The measure*

$$\mu = \prod_{i=1}^{d} \frac{dx_i}{\cos x_i}.$$

on $[-\pi/2, \pi/2]^d$ *satisfies*

$$\mu(A)^{1-\frac{1}{d}} + C_1 e^{C_2 \mu^{1/d}(A)} \leq \kappa_d \mu^+(\partial A)$$

(one can take $G = \ln(x + \sqrt{1+x^2})$*).*

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