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Morse–Smale cascades on 3-manifolds

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Abstract. This is a survey of recent (from 2000) results obtained by the authors in collaboration with Russian and foreign colleagues. The major theme of our investigations involves Morse–Smale cascades on orientable 3-manifolds and includes a complete topological classification of them, a determination of the interconnection between their dynamics and the topology of the ambient manifold, a criterion for embeddability in a topological flow, and necessary and sufficient conditions for such cascades to have an energy function.

Bibliography: 76 titles.

Keywords: Morse–Smale diffeomorphism, topological classification, embedding in a flow, energy function.

Contents

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Introduction

In 1937 Andronov and Pontryagin [2] introduced the concept of a *rough system* of differential equations defined in a bounded part of the plane: a system that retains its qualitative properties under small changes in the right-hand side. They proved that the flow generated by such a system is characterized by the following properties:

1) the set of fixed points and periodic orbits is finite, all its elements are hyperbolic;
2) there are no separatrices from a saddle to a saddle;
3) all ω- and α-limit sets are contained in the union of the fixed points and the periodic orbits (limit cycles).

The above properties are also known to characterize rough flows on a two-dimensional sphere. The principal difficulty in passing from a two-dimensional sphere to orientable surfaces of positive genus is the possibility that there may exist new types of motion (non-closed recurrent trajectories). That there are no such trajectories for rough flows without equilibrium states on a two-dimensional torus follows from Maier's 1939 paper [42]. In 1959 Peixoto [56] introduced the concept of *structural stability* of flows to generalize the concept of roughness. We recall that a flow f^t is *structurally stable* if, for any sufficiently close flow g^t, there exists a homeomorphism h sending trajectories of the system g^t to trajectories of the system f^t. The original definition of a rough flow involved the additional requirement that the homeomorphism h be C^0-close to the identity map. The concepts of 'roughness' and 'structural stability' are now known to be equivalent, though this fact is highly non-trivial. In [57] and [58] Peixoto proved that the above conditions 1)–3) are necessary and sufficient for the structural stability of a flow on an orientable closed (compact and without boundary) surface and showed that such flows are dense in the space of all C^1-flows.

An immediate generalization of properties of rough flows on orientable surfaces leads to *Morse–Smale* systems (continuous and discrete). The non-wandering set of such a system consists of finitely many fixed points and periodic orbits, each of which is hyperbolic, and the stable and unstable manifolds W^s_p and W^u_q intersect transversally[1] for any distinct non-wandering points p, q.

Morse–Smale systems are named after Smale's 1960 paper [68], where he first introduced flows with the above properties and proved that they satisfy inequalities similar to the Morse inequalities. That Morse–Smale systems are structurally stable was later shown by Smale and Palis [52], [55]. However, already in 1961 Smale [70]

[1] Two smooth submanifolds X_1 and X_2 of an n-manifold X are said to *intersect transversally* (*be in general position*) if either $X_1 \cap X_2 = \varnothing$ or $T_xX_1 + T_xX_2 = T_xX$ for any point $x \in (X_1 \cap X_2)$ (here T_xA denotes the tangent space to a manifold A at a point x).

proved that such systems do not exhaust the class of all rough systems, constructing for this purpose a structurally stable diffeomorphism on the two-dimensional sphere $\mathbb{S}^2$ with infinitely many periodic points. This diffeomorphism is now known as the 'Smale horseshoe'. Nevertheless, these systems have great value both in applications (because they adequately describe any regular stable processes) and in studying the topology of the phase space (because of the deep interrelation between the dynamics of these systems and the ambient manifold).

The key problem in the study of dynamical systems is the determination of the set of complete topological invariants: properties of a system that uniquely determine the decomposition of the phase space into trajectories up to topological equivalence (conjugacy). We recall that two flows f^t and f'^t (two diffeomorphisms f and f') on an n-manifold M^n are said to be *topologically equivalent* (*topologically conjugate*) if there exists a homeomorphism $h\colon M^n \to M^n$ that carries trajectories of f^t to trajectories of f'^t (that is, $f'h = hf$). The topological classification of dynamical systems occupies a special place in the qualitative theory of differential equations. Besides the immediate use of the topological invariants obtained, very valuable information can be derived from the discovery of fundamentally new dynamical phenomena. Thus far this problem has a rich history.

The equivalence class of Morse–Smale flows on a circle is uniquely determined by the number of its fixed points. For cascades on a circle, Maier [42] found in 1939 a complete topological invariant consisting of a triple of numbers: the number of periodic orbits, their periods, and the so-called ordinal number. In 1955 Leontovich and Maier [40] introduced a compete topological invariant — the scheme of a flow — for flows with finitely many singular trajectories on a two-dimensional sphere. This scheme contained a description of singular trajectories (equilibrium states, periodic orbits, separatrices of saddle equilibrium states) and their relative positions. In 1971 Peixoto [59] formalized the notion of a Leontovich–Maier scheme and proved that for a Morse–Smale system on an arbitrary surface a complete topological invariant is given by the isomorphism class of a directed graph associated with it whose vertices are in a one-to-one correspondence with the equilibrium states and closed trajectories and whose edges correspond to the connected components of the invariant manifolds of the equilibrium states and closed trajectories, where the isomorphisms preserve specially chosen subgraphs.[2]

Morse–Smale flows (cascades) on manifolds of dimension $n \geqslant 3$ ($n \geqslant 2$) feature a new type of motion compared with lower-dimensional systems, because of possible (*heteroclinic*) intersections of the invariant manifolds of distinct saddle points. Afraĭmovič (Afraimovich) and Šil'nikov (Shil'nikov) [1] proved that the restriction of Morse–Smale flows to the closure of the set of heteroclinic trajectories is conjugate to a suspension over a topological Markov chain. Nevertheless, an invariant similar to the Peixoto graph proved to be sufficient for describing a complete topological invariant for a broad subclass of such systems, and in particular for Morse–Smale diffeomorphisms on surfaces with finitely many heteroclinic orbits (Bezdenezhnykh and Grines [6], [7], 1985; Grines [25], 1993),[3] for flows with finitely many singular trajectories on 3-manifolds (Umanskii, [73], 1990), for flows on the sphere $\mathbb{S}^n$,

[2]In [51] Oshemkov and Sharko pointed out a certain inaccuracy concerning the Peixoto invariant due to the fact that an isomorphism of graphs does not distinguish between types of decompositions into trajectories for a domain bounded by two periodic orbits.

$n \geqslant 3$, without closed orbits (Piljugin [60], 1978), and for diffeomorphisms with saddle points of Morse index 1 on closed orientable manifolds M^n, $n > 3$ (Grines, Gurevich, Medvedev [26], [27], 2007–2008).

Topological classification of even the simplest Morse–Smale diffeomorphisms on 3-manifolds does not fit into the concept of singling out a skeleton consisting of stable and unstable manifolds of periodic orbits. The reason for this lies primarily in the possible 'wild' behaviour of separatrices of saddle points. More specifically, even though the closure of a separatrix may differ from a separatrix by only one point, it may fail to be even a topological submanifold. Pixton [61] (1977) was the first to construct a diffeomorphism with wild separatrices — for this he employed the Artin–Fox curve [3] to realize the invariant manifolds of a saddle fixed point (see Fig. 5). In 2000 Bonatti and Grines [9] investigated the class of diffeomorphisms on a three-dimensional sphere (diffeomorphisms in the Pixton class $\mathscr{P}$) that have non-wandering set consisting of four fixed points: two sinks, a source, and a saddle. They showed that the Pixton class contains a countable set of pairwise topologically non-conjugate diffeomorphisms. Furthermore, the topological conjugacy class of a diffeomorphism $f \in \mathscr{P}$ is uniquely determined by the embedding type of a one-dimensional separatrix in the basin of a sink, which is described by a new topological invariant: a smooth embedding of the circle $\mathbb{S}^1$ (the orbit space of a one-dimensional separatrix) in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ (the space of wandering orbits in the basin of a sink).

The appearance of new topological invariants gives rise to a natural research problem, the investigation of bifurcations that facilitate passing from one class of topologically conjugate diffeomorphisms to another. For the bifurcations appearing here a distinctive feature is that the structure of the non-wandering set is not changed, whereas the qualitative change of a diffeomorphism takes place solely because of a change in the embedding type of separatrices of saddle points. In [12] the present authors collaborated with Bonatti and Medvedev to prove that a passage from one topological conjugacy class to another in the set of Pixton diffeomorphisms can be realized using a sequence of two bifurcations of saddle–node type. We note that this result provides a solution in the Pixton class of the problem posed by Palis and Pugh in [54] on finding a smooth arc with some 'good' properties (for example, with finitely many bifurcations) to connect two structurally stable dynamical systems (two flows or two diffeomorphisms). Recall that two C^r-diffeomorphisms ($r \geqslant 0$) $f, f' \colon X \to X$ are said to be C^r-isotopic if there exists a C^r-homotopy $F \colon X \times [0,1] \to X$ between f and f' such that the map $f_t \colon X \to X$ defined by $f_t(x) = F(x,t)$ is a C^r-diffeomorphism for each $t \in [0,1]$. Here the family $\{f_t, t \in [0,1]\}$ of C^r-diffeomorphisms is called a C^r-arc connecting f and f'. Newhouse and Peixoto proved in [50] that any two Morse–Smale flows on a closed manifold can be connected by an arc with finitely many bifurcations. However, for

[3] One should also point out that Langevin [39] proposed a different approach to finding topological invariants for such diffeomorphisms. Though no classification results were given in [39], the ideas there nevertheless turned out to be very fruitful and have been put to use in the classification of diffeomorphisms, as is demonstrated, in particular, in the present survey. A classification of Morse–Smale diffeomorphisms on surfaces with infinitely many heteroclinic orbits which required the machinery of topological Markov chains follows from the paper [15] by Bonatti and Langevin, where necessary and sufficient conditions for topological conjugacy of Smale diffeomorphisms (C^1-structurally stable diffeomorphisms) on surfaces were established.

discrete dynamical systems the situation is different. For example, it follows from work of Matsumoto [41] and Blanchard [8] that any closed orientable surface admits isotopic Morse–Smale diffeomorphisms that cannot be connected by such an arc.

Another difference between Morse–Smale diffeomorphisms in dimension 3 from their surface analogues lies in the variety of heteroclinic intersections: a connected component of such an intersection may be not only a point as in the two-dimensional case, but also a curve, compact or non-compact (see Figs. 2, 3). The problem of a topological classification of Morse–Smale cascades on 3-manifolds either without heteroclinic points (gradient-like cascades) or without heteroclinic curves was solved in a series of papers from 2000 to 2006 by Bonatti, Grines, Medvedev, Pécou, and Pochinka [9], [11], [13], [14]. A complete topological classification of the class $\mathrm{MS}(M^3)$ of orientation-preserving Morse–Smale diffeomorphisms on a closed orientable 3-manifold M^3 was announced by Pochinka in [62], [63], and a complete proof was given in [64]. In what follows, $\mathrm{MS}(M^n)$, $n \geqslant 1$, will denote the set of orientation-preserving Morse–Smale diffeomorphisms on a closed orientable n-manifold M^n.

Since Morse–Smale systems exist on any compact manifold, the problem of the interrelation between the dynamics of such systems and the topological structure of the ambient manifolds is interesting and important. The first step in this direction was made by Smale [68], who employed the Morse inequalities to establish a connection between the Betti numbers of the ambient manifold and the number and index of equilibrium states and closed trajectories of a Morse–Smale flow on it. Franks [24] gave analogues of the Morse inequalities for Morse–Smale flows without equilibrium states. We note that the set of Morse–Smale flows without rest points consists of periodic trajectories. For such flows on manifolds of dimension $n \geqslant 4$, Asimov [4] constructed a special round-handle decomposition of the ambient manifold and proved that if a manifold admits a round-handle decomposition, then on the manifold there exists a Morse–Smale flow without rest points. The topological structure of a three-dimensional manifold admitting a Morse–Smale flow without rest points was investigated by Morgan [49], who showed that the ambient manifold is either a Seifert manifold or a special union of Seifert spaces and 'thick' tori $\mathbb{T}^2 \times [0,1]$.

Progress in finding relations between the topology of a 3-manifold and the dynamics of a Morse–Smale cascade defined on it has been made by Bonatti, Grines, Medvedev, Pécou, and Zhuzhoma. For example, a complete topological classification of the phase spaces of Morse–Smale cascades without heteroclinic curves on 3-manifolds was obtained in [10], and in [33] it was shown that the ambient 3-manifold of a gradient-like diffeomorphism with tamely embedded frames of one-dimensional separatrices of saddle points admits a Heegaard splitting whose genus is uniquely determined by the periodic data of the diffeomorphism.

Another conceptual direction in the qualitative theory is connected with the 'fundamental theorem of dynamical systems' established by Conley [19] in 1978. According to this theorem, any continuous dynamical system (flow or cascade) has a continuous Lyapunov function (that is, a function which is decreasing along trajectories of the system outside the chain recurrent set and is constant on the chain components. From several points of view, information about the existence of an energy function for a smooth dynamical system is of greater interest, that is, a smooth Lyapunov function whose set of critical points coincides with the chain recurrent set of the system.

The most complete results in this direction were obtained for Morse–Smale systems for which the chain recurrent set coincides with the set of fixed points and periodic orbits. In 1961 Smale [69] proved that a *gradient-like flow* (a Morse–Smale flow without closed trajectories) always has an energy function which is a Morse function. In 1968 Meyer [44] generalized this result and constructed an energy function which is a Morse–Bott function for an arbitrary Morse–Smale flow. Here we recall that a point $p\in M^n$ is called a *critical point* of a C^r-smooth ($r\geqslant 2$) function $\psi\colon M^n\to\mathbb{R}$ if $\dfrac{\partial\psi}{\partial x_1}(p)=\cdots=\dfrac{\partial\psi}{\partial x_n}(p)=0$ ($\operatorname{grad}\psi(p)=0$) in some local coordinates $x_1,\dots,x_n$ ($x_j(p)=0$ for all $j=1,\dots,n$). A critical point p is *non-degenerate* if the Hessian matrix of second derivatives $\dfrac{\partial^2\psi}{\partial x_i\,\partial x_j}(p)$ is non-singular; otherwise p is *degenerate*. A function $\psi\colon M^n\to\mathbb{R}$ is a *Morse function* if all its critical points are non-degenerate, and ψ is a *Morse–Bott function* if, at any critical point, the Hessian is non-singular in the direction normal to the critical level set.

In 1977 Pixton [61] proved that a Morse–Smale diffeomorphism on a surface has an energy function that is a Morse function. Moreover, he constructed on a 3-sphere a diffeomorphism without an energy function and showed that this phenomenon is related to wild embeddings of separatrices of saddle points. In [31], [32] the present authors collaborated with Laudenbach to investigate conditions for the existence of an energy function for Morse–Smale cascades on 3-manifolds. From these investigations it became clear that many Morse–Smale cascades on 3-manifolds fail to have an energy function. This means that for such a system any Lyapunov function which is also a Morse function has additional critical points (distinct from the periodic points of the diffeomorphism). And this leads us to the concept of a Lyapunov function with a minimum number of critical points, which was called a *quasi-energy function* in [30]. There a subclass of cascades was singled out in the Pixton class as those which fail to have an energy function but for which a quasi-energy function can be constructed. However, the problem of constructing a quasi-energy function for arbitrary Morse–Smale cascades is still a matter for the future.

The existence of an energy function for a Morse–Smale cascade imposes certain restrictions on its dynamics. It does not guarantee, however, that a diffeomorphism admitting an energy function can be imbedded in some flow, even if there are no heteroclinic intersections. Recall that a diffeomorphism f is *embeddable in a C^m-flow* ($m\geqslant 0$) if it is the time-one shift along trajectories of some C^m-flow X^t ($f=X^1$). The problem of embeddability a diffeomorphism in a flow is classical; a detailed survey of results in this area can be found in [74]. In particular, from the papers [52], [55], in which Morse–Smale diffeomorphisms are shown to be structurally stable, it follows that for any manifold M^n there exists a non-empty open (in $\operatorname{Diff}^1(M^n)$) set of Morse–Smale diffeomorphisms that embed in a topological flow (a C^0-flow). We note that, according to [16], the set of C^2-diffeomorphisms that embed in a C^1-smooth flow is nowhere dense in the space of Morse–Smale diffeomorphisms. In [52] Palis found some necessary conditions for a diffeomorphism $f\in\operatorname{MS}(M^n)$ to embed in a topological flow, and for $n=2$ he showed that these conditions are also sufficient and posed the problem of extending this result to higher dimensions. In [28] and [29] the present authors collaborated with Gurevich and Medvedev to solve the Palis problem in dimension $n=3$.

The present survey is structured as follows.

In § 1 we include properties of Morse–Smale diffeomorphisms necessary for an understanding of their dynamics. Namely, here we shall be concerned with asymptotic behaviour, the topology of the embedding, and the structure of the space of orbits that lie on separatrices of periodic points.

Section 2 deals with the Pixton class, which provides a framework for understanding the results to be presented. In § 2.1 we present an approach to the topological classification of diffeomorphisms in the Pixton class. By extending this approach we arrive at a complete topological classification of arbitrary Morse–Smale diffeomorphisms on 3-manifolds. In § 2.2 we construct a simple arc connecting two topologically non-conjugate cascades in the Pixton class.

Section 3 presents a complete topological classification (including a realization) of diffeomorphisms in the set $\mathrm{MS}(M^3)$. More specifically, we prove that for a diffeomorphism $f\in\mathrm{MS}(M^3)$ the equivalence class of its scheme S_f is a complete topological invariant, and the scheme contains information about the periodic data and the topology of the embedding of the two-dimensional invariant manifolds of saddle points of f in the phase space. Moreover, properties of this scheme are employed to single out a set $\mathscr{S}$ of abstract schemes for each $S\in\mathscr{S}$ of which a diffeomorphism $f_S\in\mathrm{MS}(M^3)$ is constructed such that the schemes S_{f_S} and S are equivalent.

In § 4 we derive relations between the number $g_f=(r_f-l_f+2)/2$ and the topology of a manifold M^3, where r_f and is the number of saddle periodic points, and l_f is the number of node (source or sink) periodic points of a diffeomorphism $f\in\mathrm{MS}(M^3)$. A topological classification is obtained for three-dimensional closed orientable manifolds that admit Morse–Smale diffeomorphisms without heteroclinic curves. These manifolds are shown to be either a three-dimensional sphere in the case $g_f=0$, or a connected sum[4] of g_f copies of $\mathbb{S}^2\times\mathbb{S}^1$. The case when f is gradient-like and has tamely embedded frames of one-dimensional separatrices exhibits another relation between the number g_f and the topology of M^3. In this case the ambient manifold M^3 admits a Heegaard splitting[5] of genus g_f.

Section 5 gives results involving the existence of an energy function for a Morse–Smale cascade on a 3-manifold. For example, in § 5.1 we construct a quasi-energy function for a subset of the class of Pixton diffeomorphisms. In § 5.2 we introduce for a gradient-like diffeomorphism the concept of a self-indexing energy function and prove that a criterion for its existence is related to the presence of special Heegaard

[4]Let X_1 and X_2 be two compact n-manifolds and let $D_1\subset X_1$ and $D_2\subset X_2$ be subspaces homeomorphic to $\mathbb{D}^n$, with $h_1\colon\mathbb{D}^n\to D_1$ and $h_2\colon\mathbb{D}^n\to D_2$ the corresponding homeomorphisms. Let $g\colon\partial D_1\to\partial D_2$ be a homeomorphism such that the map $h_2^{-1}gh_1\big|_{\partial\mathbb{D}^n}\colon\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$ reverses the orientation. The space $X_1\sharp X_2=(X_1\setminus\operatorname{int}D_1)\cup_g(X_2\setminus\operatorname{int}D_2)$ is called a *connected sum* of X_1 and X_2.

[5]A three-dimensional closed manifold is called a *handlebody of genus* $g\geqslant 0$ if it is obtained from a 3-ball by an orientation-reversing identification of g pairs of pairwise disjoint 2-disks on the boundary of the ball. The boundary of a handlebody is a closed surface of genus g. A *Heegaard splitting* of genus $g\geqslant 0$ for a manifold M^3 is a representation of M^3 as a gluing together of two handlebodies of genus g by means of some diffeomorphism of their boundaries; the common boundary of these bodies is called a *Heegaard surface* in the manifold M^3.

surfaces of genus g_f. In § 5.3 we introduce for an arbitrary Morse–Smale diffeomorphism of a three-dimensional manifold the concept of a dynamically ordered Morse–Lyapunov function, whose properties are closely connected with the dynamics of the diffeomorphism. Necessary and sufficient conditions for the existence of an energy function with these properties are shown to be governed by the embedding type of the invariant manifolds of saddle orbits of the diffeomorphism.

In Section 6 Morse–Smale diffeomorphisms on 3-manifolds are used to demonstrate fundamentally new obstructions to embedding in a flow as compared to their analogues on surfaces. It is proved that a diffeomorphism $f\in \mathrm{MS}(M^3)$ embeds in a topological flow if and only if its scheme S_f is trivial.

1. Basic properties

Definition 1.1. A diffeomorphism $f\colon M^n\to M^n$ on a smooth closed (compact, without boundary) connected orientable n-manifold M^n ($n\geqslant 1$) is called a *Morse–Smale diffeomorphism* if

1) the non-wandering set Ω_f is finite and hyperbolic;

2) the stable and unstable manifolds W^s_p and W^u_q intersect transversally for any periodic points p, q.

In this section we present the properties of Morse–Smale diffeomorphisms needed to understand their dynamics. Some of the results below were announced and proved in the survey [71] and the papers [52], [55]; for detailed proofs see the monograph [34]. All facts are given for the class $\mathrm{MS}(M^n)$ of orientation-preserving Morse–Smale diffeomorphisms $f\colon M^n\to M^n$ on an orientable manifold M^n.

1.1. Dynamics. Let $f\in \mathrm{MS}(M^n)$. By Definition 1.1 the non-wandering set Ω_f of f consists of finitely many periodic points ($\Omega_f=\mathrm{Per}_f$). The hyperbolic structure of Ω_f means that at each periodic point $p\in\Omega_f$ of period m_p the eigenvalues of the Jacobian matrix $\left(\dfrac{\partial f^{m_p}}{\partial x}\right)\bigg|_p$ do not have unit modulus. It follows that at any periodic point p there are invariant manifolds, the *stable* manifold W^s_p and the *unstable* manifold W^u_p, which are defined in topological terms as follows:

$$W^s_p=\Bigl\{x\in M^n\colon \lim_{n\to+\infty} d(f^{nm_p}(x),p)=0\Bigr\},$$
$$W^u_p=\Bigl\{x\in M^n\colon \lim_{n\to+\infty} d(f^{-nm_p}(x),p)=0\Bigr\},$$

where d is the metric on M^n. Moreover, $\dim W^s_p=n-q_p$ and $\dim W^u_p=q_p$, where q_p is the number of eigenvalues of the Jacobian matrix $\left(\dfrac{\partial f^{m_p}}{\partial x}\right)\bigg|_p$ with modulus greater than 1 and is called the *Morse index*. In what follows, for any subset P of Ω_f we denote by W^u_P (respectively, W^s_P) the union of all the unstable (stable) manifolds of all the points in P. A connected component ℓ^s_p (ℓ^u_p) of the set $W^s_p\setminus p$ ($W^u_p\setminus p$) is called a *stable* (*unstable*) *separatrix* of a point p. The number ν_p which equals $+1$ or -1 depending on whether the map $f^{m_p}\big|_{W^u_p}$ preserves or reverses the orientation is called the *orientation type* of a point p. The triple $(m_p,q_p,\nu_p)=(m_{\mathscr O_p},q_{\mathscr O_p},\nu_{\mathscr O_p})$ is called the *periodic data* of a point p or an orbit $\mathscr O_p$.

A periodic point p is called a *saddle* point if $0 < q_p < n$, otherwise p is called a *node*; further, p is called a *sink* (respectively, *source*) if $q_p = 0$ ($q_p = n$). Since a diffeomorphism $f \in \mathrm{MS}(M^n)$ preserves orientation, the orientation type of a node point is $+1$, whereas for a saddle point both cases $+1$ and -1 are possible. Given $q \in \{0, \dots, n\}$, we denote by Ω_q the set of periodic points with Morse index q, by k_f the total number of periodic orbits, and by k_q the number of periodic orbits with Morse index not exceeding q.

Qualitative properties (from the viewpoint of topological conjugacy) of Morse–Smale diffeomorphisms are largely determined by the embedding, position, and asymptotic behaviour of the invariant manifolds of periodic points, as described in the following assertion.

Statement 1.1. *Let* $f \in \mathrm{MS}(M^n)$. *Then*:

1) $M^n = \bigcup_{p \in \Omega_f} W^u_p$;

2) W^u_p *is a smooth submanifold*[6] *of* M^n *that is homeomorphic to* $\mathbb{R}^{q_p}$ *for any periodic point* $p \in \Omega_f$;

3) $\mathrm{cl}(\ell^u_p) \setminus (\ell^u_p \cup p) = \bigcup_{r \in \Omega_f \colon \ell^u_p \cap W^s_r \neq \varnothing} W^u_r$ *for any unstable separatrix* ℓ^u_p *of any periodic point* $p \in \Omega_f$.

Since the stable manifold of a periodic point of a diffeomorphism f is the unstable manifold of the same point regarded as a periodic point of the diffeomorphism f^{-1}, all the assertions for unstable manifolds also hold (with corresponding modifications) for stable manifolds.

In view of the theorem on local topological classification of a hyperbolic fixed point of a diffeomorphism (see Theorem 5.5 in [53]), the map f^{m_p} at p is locally conjugate to the linear diffeomorphism $a_{q_p,\nu_p} \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$a_{q_p,\nu_p}(x_1, \dots, x_n) = \left(\nu_p \cdot 2x_1, 2x_2, \dots, 2x_{q_p}, \nu_p \,\frac{x_{q_p+1}}{2}, \frac{x_{q_p+2}}{2}, \dots, \frac{x_n}{2}\right).$$

We call $a_{q,\nu} \colon \mathbb{R}^n \to \mathbb{R}^n$ a *canonical diffeomorphism*. Further, we let $a^u_{q,\nu}$ and $a^s_{q,\nu}$ denote the restrictions of $a_{q,\nu}$ to $Ox_1 \dots x_q$ and $Ox_{q+1} \dots x_n$, and we call them the *canonical expansion* and the *canonical contraction*, respectively. By virtue of assertion 2) in Statement 1.1, $W^u_{\mathscr{O}_p}$ is a smooth submanifold of M^n. Hence, the map $f\big|_{W^u_{\mathscr{O}_p}} \colon W^u_{\mathscr{O}_p} \to W^u_{\mathscr{O}_p}$ is a diffeomorphism. These facts enable us to give a global topological classification of the maps $f\big|_{W^u_{\mathscr{O}_p}}$, in the following form.

[6] A subset A of a C^r-manifold X ($r \geqslant 0$) is called a C^r*-submanifold* if, for some integer $0 \leqslant k \leqslant n$, each point of A lies in a chart (U, ψ) of X such that $\psi(U \cap A) = \mathbb{R}^k$ or $\psi(U \cap A) = \mathbb{R}^k_+$, where $\mathbb{R}^k = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_{k+1} = \dots = x_n = 0\}$ and $\mathbb{R}^k_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^k \colon x_k \geqslant 0\}$. Furthermore, A becomes a C^r-manifold with the charts $\bigl\{\bigl(U \cap A, \psi\big|_{U \cap A}\bigr)\bigr\}$. A C^0-submanifold is also called a *topological submanifold*.

It is a classical fact in topology that a subset A of a C^r-manifold X with $r \geqslant 1$ is a C^r-submanifold if and only if it is the image of a C^r-embedding; that is, there exist a C^r-manifold B and a regular C^r-map $g \colon B \to X$ (the rank of the Jacobian matrix of g at any point equals the dimension of the manifold B) which maps B homeomorphically to the subspace $A = g(B)$ with the topology induced from X. Such a map g is called a C^r*-embedding*.

Statement 1.2. *Let $f \in \mathrm{MS}(M^n)$. Then for any periodic point $p \in \Omega_f$ the diffeomorphism $f^{m_p}\big|_{W^u_p} : W^u_p \to W^u_p$ is topologically conjugate to the canonical expansion $a^u_{q_p,\nu_p} : \mathbb{R}^{q_p} \to \mathbb{R}^{q_p}$.*

In the case when a periodic point of a diffeomorphism $f \in \mathrm{MS}(M^n)$ is a saddle point, information can be acquired not only from the embedding in the ambient space of its invariant manifolds, but also from the embedding of an f-invariant neighbourhood of its orbit.

For $q \in \{1, \dots, n-1\}$ and $t \in (0, 1]$ let $\mathscr{N}^t_q = \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1^2 + \dots + x_q^2)(x_{q+1}^2 + \dots + x_n^2) < t\}$ and let $\mathscr{N}_q = \mathscr{N}^1_q$. We note that the set $\mathscr{N}^t_q$ is invariant under the canonical diffeomorphism $a_{q,\nu}$, which has a unique saddle fixed point at the origin O, with unstable and stable manifolds $W^u_O = Ox_1 \dots x_q$ and $W^s_O = Ox_{q+1} \dots x_n$.

Definition 1.2. Let $f \in \mathrm{MS}(M^n)$. A neighbourhood N_σ of a saddle point $\sigma \in \Omega_f$ is said to be *linearizing* if there exists a homeomorphism $\mu_\sigma : N_\sigma \to \mathscr{N}_{q_\sigma}$ conjugating the diffeomorphism $f^{m_\sigma}\big|_{N_\sigma}$ with the canonical diffeomorphism $a_{q_\sigma,\nu_\sigma}\big|_{\mathscr{N}_{q_\sigma}}$.

The neighbourhood $N_{\mathscr{O}_\sigma} = \bigcup_{k=0}^{m_\sigma - 1} f^k(N_\sigma)$ equipped with the map $\mu_{\mathscr{O}_\sigma}$ composed of the homeomorphisms $\mu_\sigma f^{-k} : f^k(N_\sigma) \to \mathscr{N}_{q_\sigma}$, $k = 0, \dots, m_\sigma - 1$, will be called a *linearizing neighbourhood* of the orbit $\mathscr{O}_\sigma$, and the homeomorphism $\mu_{\mathscr{O}_\sigma}$ will be called a *linearizing homeomorphism.*

Statement 1.3. *Any saddle point* (*orbit*) *of a diffeomorphism $f \in \mathrm{MS}(M^n)$ has a linearizing neighbourhood.*

In the neighbourhood $\mathscr{N}_q$ we define a pair of transversal foliations $\mathscr{F}^u_q$, $\mathscr{F}^s_q$ as follows:

$$\mathscr{F}^u_q = \bigcup_{(c_{q+1},\dots,c_n) \in Ox_{q+1}\dots x_n} \{(x_1, \dots, x_n) \in \mathscr{N}_q : (x_{q+1}, \dots, x_n) = (c_{q+1}, c_n)\},$$

$$\mathscr{F}^s_q = \bigcup_{(c_1,\dots,c_q) \in Ox_1\dots x_q} \{(x_1, \dots, x_n) \in \mathscr{N}_q : (x_1, \dots, x_q) = (c_1, \dots, c_q)\}.$$

We note that the canonical diffeomorphism $a_{q,\nu}$ maps leaves of the foliation $\mathscr{F}^u_q$ ($\mathscr{F}^s_q$) into leaves of the same foliation. By virtue of Statement 1.3, for any saddle point σ of a diffeomorphism $f \in \mathrm{MS}(M^3)$ the foliations $\mathscr{F}^u_{q_\sigma}$ and $\mathscr{F}^s_{q_\sigma}$ induce, via the linearizing homeomorphism, the f-invariant foliations $F^u_{\mathscr{O}_\sigma}$ and $F^s_{\mathscr{O}_\sigma}$ on the linearizing neighbourhood $N_{\mathscr{O}_\sigma}$; these foliations are said to be *linearizing* (see Fig. 1).

In view of assertion 1) in Statement 1.1, the invariant manifolds of periodic points of a diffeomorphism $f \in \mathrm{MS}(M^n)$ are submanifolds of M^n. Nevertheless, the closure of an invariant manifold of a saddle point may have a complicated topological structure. This phenomenon may be of a dynamical or of a purely topological nature. The first case here corresponds to the situation when a separatrix of a saddle point is involved in heteroclinic intersections.

Definition 1.3. If σ_1 and σ_2 are different periodic saddle points of a diffeomorphism $f \in \mathrm{MS}(M^n)$ for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is said to be *heteroclinic*. Furthermore:

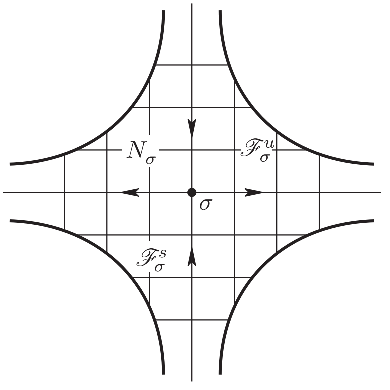


Figure 1. Linearizing foliations in a linearizing neighbourhood.

• if $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$ (respectively, $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 1$), then a connected component of the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic manifold* (a *heteroclinic curve*) (see Fig. 2);

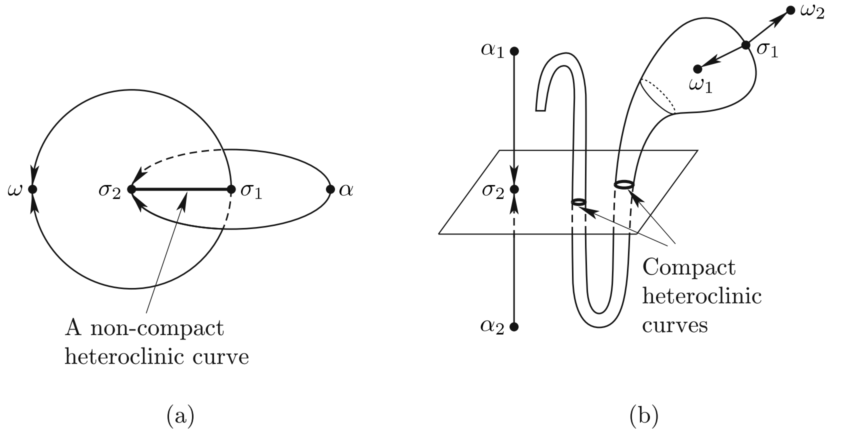


Figure 2. Heteroclinic curves.

• if $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 0$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is countable, any point of it is called a *heteroclinic point*, and the orbit of a heteroclinic point is called a *heteroclinic orbit* (see Fig. 3).

Definition 1.4. A diffeomorphism $f \in \mathrm{MS}(M^n)$ is said to be *gradient-like* if the condition $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ with distinct points $\sigma_1, \sigma_2 \in \Omega_f$ implies that $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$.

The transversality condition of the invariant manifolds readily implies that a diffeomorphism $f \in \mathrm{MS}(M^n)$ is gradient-like if and only if it has no heteroclinic points.

By assertion 3) in Statement 1.1, the closure of a separatrix of a saddle point involved in a heteroclinic intersection does not have the structure of a topological

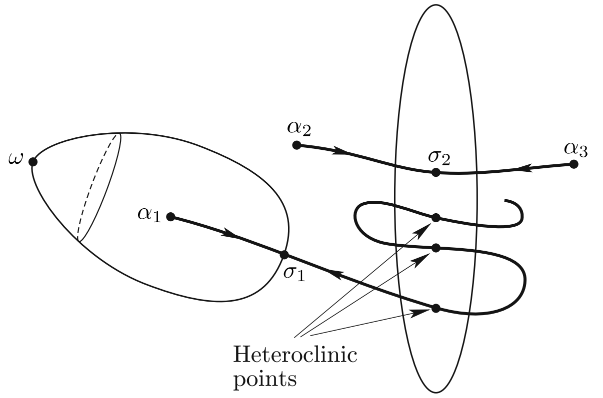


Figure 3. Heteroclinic points.

manifold; otherwise, being the one-point compactification of a Euclidean space, it is a topologically embedded manifold. Recall that a C^0-map $g\colon B \to X$ is a *topological embedding* of a topological manifold B in a manifold X if it maps B homeomorphically onto the subspace $g(B)$ with topology induced from X. The image $A = g(B)$ is called a *topologically embedded manifold*. We note that in general a topologically embedded manifold is not a topological submanifold. If A is a submanifold, then it is said to be *tame* or *tamely embedded*; otherwise it is said to be *wild* or *wildly embedded*, and the points at which the conditions for being a topological submanifold are not satisfied are called *wild points* or *points of wild embedding*.

Statement 1.4. *Let $f \in \mathrm{MS}(M^n)$ and let σ be a saddle point of f such that an unstable separatrix ℓ^u_σ is not involved in heteroclinic intersections. Then*

$$\mathrm{cl}(\ell^u_\sigma) \setminus (\ell^u_\sigma \cup \sigma) = \{\omega\},$$

where ω is a sink periodic point. Furthermore, if $q_\sigma = 1$, then $\mathrm{cl}(\ell^u_\sigma)$ is an arc topologically embedded in M^n, and if $q_\sigma \geqslant 2$, then $\mathrm{cl}(\ell^u_\sigma)$ is the sphere $\mathbb{S}^{q_\sigma}$ topologically embedded in M^n.

By assertion 2) in Statement 1.1, the set $\ell^u_\sigma \cup \sigma$ is a smooth submanifold of M^n. However, it may well be that the manifold $\mathrm{cl}(\ell^u_\sigma)$ is wild at ω, in which case the separatrix ℓ^u_σ is said to be *wild*; otherwise, it is *tame*.

For $n = 2$ the results of Moisa [48] guarantee that any compact arc, and hence any separatrix without heteroclinic points, is tamely embedded in M^2. An example (not related to the dynamics) of a compact wild arc in $\mathbb{S}^3$ with one wild point was given by Artin and Fox [3] in 1948. In 1977 Pixton [63] realized this arc by separatrices of a saddle point of a Morse–Smale diffeomorphism on a 3-sphere (see Fig. 4). The next result provides a criterion for a tame embedding of separatrices of saddle points of a diffeomorphism $f \in \mathrm{MS}(M^3)$ in the ambient 3-manifold.

Statement 1.5. *Let $f \in \mathrm{MS}(M^3)$, let ω be a sink point, and let ℓ^u_σ be a one-dimensional (respectively, two-dimensional) separatrix of a saddle σ such that $\ell^u_\sigma \subset W^s_\omega$.*

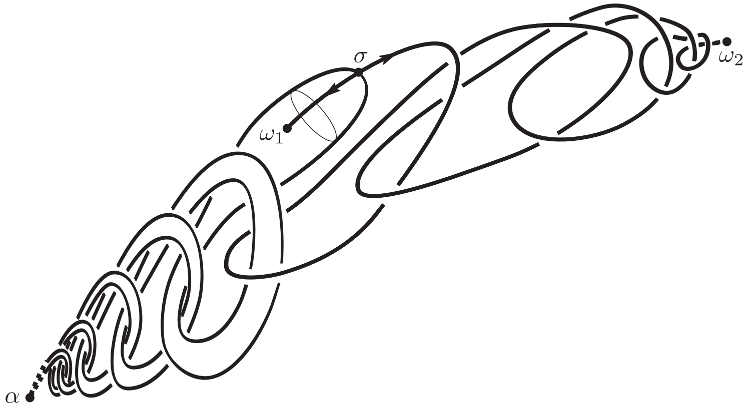


Figure 4. Pixton's example.

Then ℓ_σ^u is tamely embedded in M^3 if and only if there exists a smooth 3*-ball $B_\omega \subset W_\omega^s$ containing ω in its interior and such that ℓ_σ^u intersects ∂B_ω in a single point* (*in a single circle*)*.*

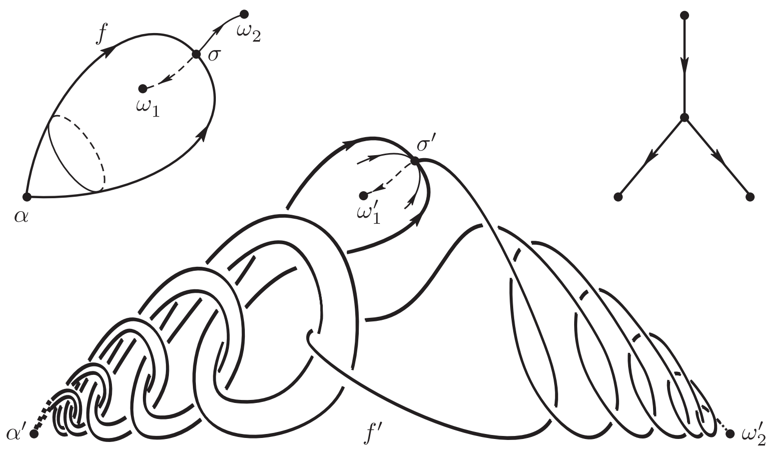


Figure 5. Pixton diffeomorphisms that are not topologically conjugate.

Since a conjugating homeomorphism must take invariant manifolds of periodic points of one diffeomorphism into analogous manifolds of another, it becomes apparent that there exist at least two topologically non-conjugate diffeomorphisms with isomorphic graphs — this involves a diffeomorphism f for which all the separatrices are tame and a diffeomorphism f' having some wild separatrices (see Fig. 5). Consequently, the graph is not a complete topological invariant, and thus we need mechanisms capable of tracing not only the asymptotic behaviour of separatrices, but also the topology of their embedding. For this purpose we must digress briefly into the theory of group actions on manifolds. The main source of the information in the following subsection is the remarkable book [37].

1.2. Orbit spaces. Let $g\colon X\to X$ be a diffeomorphism of a manifold X and let n_X be the number of connected components of X. We denote by X/g the space of g-orbits on X, and by $p_g\colon X\to X/g$, the natural projection. To simplify matters, we shall assume that the space X/g is connected. One says that g acts *discontinuously* on X if, for any compact set $K\subset X$, the set of $k\in\mathbb{Z}$ for which $g^k(K)\cap K\neq\varnothing$ is finite. In this case the projection p_g is a covering that induces a homomorphism $\eta_g\colon\pi_1(X/g)\to\mathbb{Z}$ as follows. Let $p_g^{-1}(\widehat{x})$ be the inverse image of a point $\widehat{x}\in X/g$ under p_g. We note that $p_g^{-1}(\widehat{x})$ is the orbit of any point $x\in p_g^{-1}(\widehat{x})$. Let $\widehat{c}$ be a loop in X/g such that $\widehat{c}(0)=\widehat{c}(1)=\widehat{x}$. By the monodromy theorem, there exists a loop c in X based at x ($c(0)=x$) which is a lift of the path $\hat{c}$. Moreover, there exists an element $k\in n_X\mathbb{Z}$ such that $c(1)=g^k(x)$. Here $n_X\mathbb{Z}$ denotes the set of integers which are multiples of n_X. Let $\eta_g\colon\pi_1(X/g)\to n_X\mathbb{Z}$ be a map carrying $[\widehat{c}]$ to k. By definition, a *fundamental domain for the action of g on X* is a closed set $D_g\subset X$ for which there exists a set $\widetilde{D}_g$ with the following properties:

1) $\mathrm{cl}(\widetilde{D}_g)=D_g$;

2) $g^k(\widetilde{D}_g)\cap\widetilde{D}_g=\varnothing$ for all $k\in(\mathbb{Z}\setminus\{0\})$;

3) $\bigcup_{k\in\mathbb{Z}}g^k(\widetilde{D}_g)=X$.

Statement 1.6. *Let the diffeomorphism g act discontinuously on an n-manifold X. Then*:

1) *the natural projection $p_g\colon X\to X/g$ is a covering*;

2) *the orbit space X/g is a smooth n-manifold*;

3) *for a fundamental domain D_g of the action of g on X, the orbit spaces D_g/g and X/g are homeomorphic*;

4) *the map $\eta_{X/g}\colon\pi_1(X/g)\to n_X\mathbb{Z}$ is an epimorphism.*

As an illustration of this result, consider the orbit space $\widehat{\mathscr{W}}^u_{q,\nu}=(\mathbb{R}^q\setminus O)/a^u_{q,\nu}$ of the action of the canonical expansion $a^u_{q,\nu}$ on $\mathbb{R}^q\setminus O$ with $q\in\{1,\dots,n\}$ and $\nu\in\{+1,-1\}$. This action is discontinuous, and $\{(x_1,\dots,x_q)\in\mathbb{R}^q\colon 1\leqslant x_1^2+\dots+x_q^2\leqslant 4\}$ is a fundamental domain of it (see Fig. 6). This leads us to the following list of spaces:

1) the space $\widehat{\mathscr{W}}^u_{1,-1}$ is homeomorphic to the circle $\mathbb{S}^1$;

2) the space $\widehat{\mathscr{W}}^u_{1,+1}$ is homeomorphic to the product of circles $\mathbb{S}^1\times\mathbb{S}^0$;

3) the space $\widehat{\mathscr{W}}^u_{2,-1}$ is homeomorphic to a Klein bottle;

4) the space $\widehat{\mathscr{W}}^u_{2,+1}$ is homeomorphic to the two-dimensional torus $\mathbb{T}^2$;

5) the space $\widehat{\mathscr{W}}^u_{q,-1}$, $q\geqslant 3$, is homeomorphic to a generalized Klein bottle (the topological space obtained from $\mathbb{S}^{q-1}\times[0,1]$ by identifying its boundaries by means of the map $g\colon\mathbb{S}^{q-1}\times\{0\}\to\mathbb{S}^{q-1}\times\{1\}$ given by the formula $g(x_1,x_2,\dots,x_q,0)=(-x_1,x_2,\dots,x_q,1)$);

6) the space $\widehat{\mathscr{W}}^u_{q,+1}$ is homeomorphic to $\mathbb{S}^{q-1}\times\mathbb{S}^1$.

The next result partially explains how the problem of topological classification of diffeomorphisms reduces to manipulations with topological objects.

Statement 1.7. *Assume that the diffeomorphisms g and g' act discontinuously on the manifolds X and X', respectively, and that the spaces X/g and X'/g' are connected. Then*:

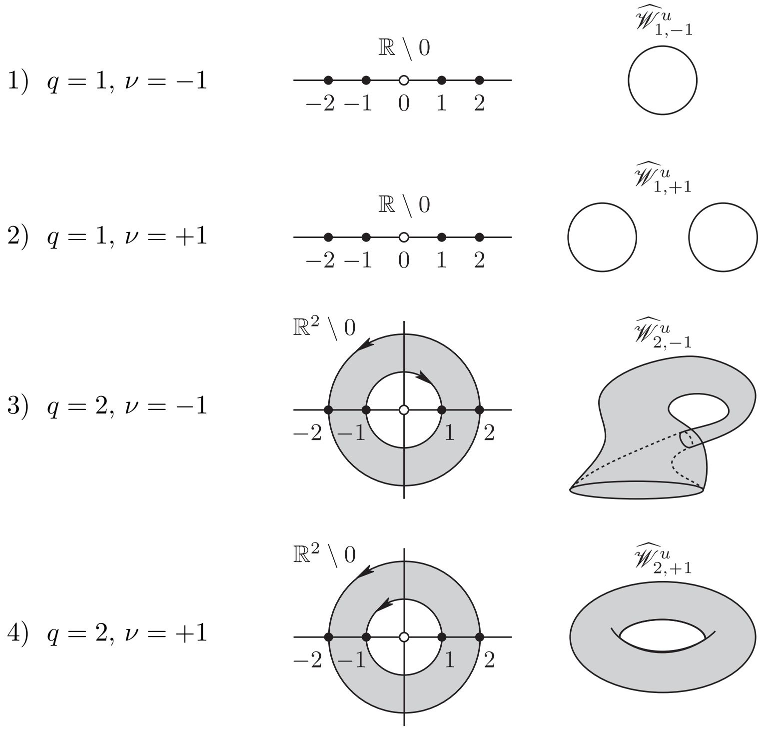


Figure 6. Orbit spaces of the canonical expansion.

1) *if* $h\colon X \to X$ *is a homeomorphism (diffeomorphism) such that* $hg = g'h$, *then the map* $\widehat{h}\colon X/g \to X'/g'$ *given by* $\widehat{h} = p_{g'}hp_g^{-1}$ *is a homeomorphism and* $\eta_g = \eta_{g'}\widehat{h}_*$, *where* $\widehat{h}_*$ *is the isomorphism induced by the homeomorphism* $\widehat{h}$;

2) *if* $\widehat{h}\colon X/g \to X'/g'$ *is a homeomorphism (diffeomorphism) such that* $\eta_g = \eta_{g'}\widehat{h}_*$, *then for any points* $x \in X$ *and* $x' \in p_{g'}^{-1}(\widehat{h}(p_g(\widehat{x})))$ *there exists a unique homeomorphism* $h\colon X \to X'$ *that is a lift of* $\widehat{h}$ *and is such that* $hg = g'h$ *and* $h(x) = x'$.

Using Statements 1.2 and 1.7, we arrive at the following fact.

Statement 1.8. *Let* p *be a periodic point with Morse index* $q_p \geqslant 1$ *for a diffeomorphism* $f \in \mathrm{MS}(M^n)$. *Then the orbit space* $\widehat{W}^u_{\mathscr{O}_p} = (W^u_{\mathscr{O}_p} \setminus \mathscr{O}_p)/f$ *is a smooth* q_p*-manifold that is homeomorphic to* $\widehat{\mathscr{W}}^u_{q_p,\nu_p}$.

2. The Pixton class

Let $\mathscr{P}$ denote the class of Morse–Smale diffeomorphisms $f \in \mathrm{MS}(M^3)$ with non-wandering set made up of a source fixed point α, a saddle fixed point σ, and sink fixed points ω_1 and ω_2. Considering that Pixton's example lies in this class, we call $\mathscr{P}$ the *Pixton class*.

2.1. Complete topological invariant. In this subsection we shall be concerned with an approach to topological classification of Pixton diffeomorphisms; by generalizing this approach we shall obtain a complete topological classification of arbitrary Morse–Smale diffeomorphisms on 3-manifolds (see § 3).

2.1.1. *Necessary and sufficient conditions for topological conjugacy.* We set $V_f = W^u_{\alpha_f} \setminus \alpha_f$ and $\widehat{V}_f = V_f/f$ and denote by $p_f \colon V_f \to \widehat{V}_f$ the natural projection. By Statement 1.8, the quotient space $\widehat{V}_f$ is homeomorphic to the product $\mathbb{S}^2 \times \mathbb{S}^1$, and by Statement 1.6 the projection $p_f \colon V_f \to \widehat{V}_f$ is a covering that induces an epimorphism $\eta_f \colon \pi_1(\widehat{V}_f) \to \mathbb{Z}$. In view of Statement 1.8 the space $\widehat{L}^s_f = p_f(W^s_{\sigma_f} \setminus \sigma_f)$ is a two-dimensional torus smoothly embedded in the manifold $\widehat{V}_f$. Moreover, the torus $\widehat{L}^s_f$ is *homotopically non-trivial*, that is, $i_{\widehat{L}^s_f *}(\pi_1(\widehat{L}^s_f)) \neq 0$, where $i_{\widehat{L}^s_f *} \colon \widehat{L}^s_f \to \widehat{V}_f$ is the inclusion map. Furthermore, from the definition of η_f it follows that $\eta_f(i_{\widehat{L}^s_f *}(\pi_1(\widehat{L}^s_f))) = \mathbb{Z}$ (see Fig. 7).

Definition 2.1. The tuple $S_f = (\widehat{V}_f, \eta_f, \widehat{L}^s_f)$ is called the *scheme* of the diffeomorphism $f \in \mathscr{P}$.

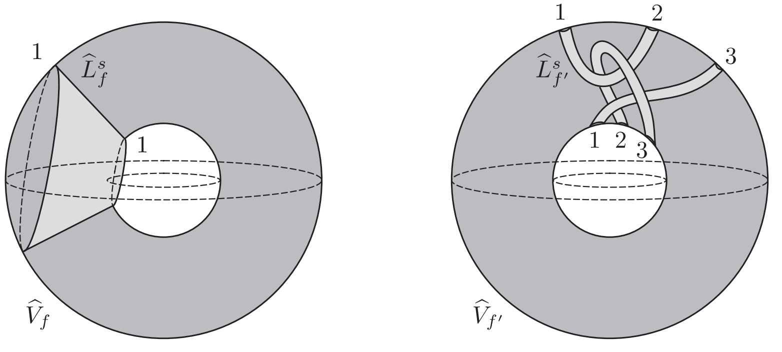


Figure 7. Schemes of Pixton diffeomorphisms.

In Fig. 7 we show the geometric components of the schemes S_f and $S_{f'}$ of the diffeomorphisms f and f' with phase portraits shown in Fig. 5. More precisely, Figure 7 illustrates fundamental domains for the action of the diffeomorphisms f and f' on V_f and $V_{f'}$, respectively. Each fundamental domain is a three-dimensional annulus, from which the orbit spaces $\widehat{V}_f$ and $\widehat{V}_{f'}$ are obtained by identifying the boundary spheres of the annulus by means of the corresponding diffeomorphism f or f'. Also, the orbit spaces $\widehat{L}^s_f$ and $\widehat{L}^s_{f'}$ are obtained from cylinders by identifying the circles with the same numbers.

From Statement 1.7 it follows that if diffeomorphisms $f, f' \in \mathscr{P}$ are topologically conjugate, then their schemes are equivalent in the sense of the following definition.

Definition 2.2. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in \mathscr{P}$ are said to be *equivalent* if there exists a homeomorphism $\widehat{\varphi} \colon \widehat{V}_f \to \widehat{V}_{f'}$ such that:

1) $\eta_f = \eta_{f'} \widehat{\varphi}_*$;
2) $\widehat{\varphi}(\widehat{L}^s_f) = \widehat{L}^s_{f'}$.

More precisely, the equivalence class of a scheme is a complete topological invariant for a Pixton diffeomorphism.

Theorem 2.1. *Diffeomorphisms $f, f' \in \mathscr{P}$ are topologically conjugate if and only if their schemes S_f, $S_{f'}$ are equivalent.*

The idea of the proof is as follows. In view of assertion 1) in Statement 1.1, the ambient manifold can be represented as the union $M^3 = V_f \cup W^u_\sigma \cup \Omega_f$. By Statement 1.7 the existence of a homeomorphism $\widehat{\varphi}\colon \widehat{V}_f \to \widehat{V}_{f'}$ that realizes the equivalence of the schemes S_f and $S_{f'}$ implies the existence of a homeomorphism $\varphi\colon V_f \to V_{f'}$ that conjugates the diffeomorphisms $f\big|_{V_f}$ and $f'\big|_{V_{f'}}$ and is such that $\varphi(W^s_\sigma \setminus \sigma) = W^s_{\sigma'} \setminus \sigma'$. In general, this homeomorphism cannot be extended to the set W^u_σ. However, φ can be modified in the linearizing neighbourhood N_σ so that it takes leaves of the linearizing foliation F^s_σ into leaves of the linearizing foliation $F^s_{\sigma'}$, and hence it is uniquely extendable to the required conjugating homeomorphism.

This modification is implemented in the orbit spaces $\widehat{V}_f$ and $\widehat{V}_{f'}$ into which the linearizing neighbourhoods N_σ and $N_{\sigma'}$ are projected as tubular neighbourhoods $N(\widehat{L}^s_f)$ and $N(\widehat{L}^s_{f'})$ of the tori $\widehat{L}^s_f$ and $\widehat{L}^s_{f'}$, the latter neighbourhoods being fibred by the two-dimensional leaves of $\widehat{F}^s_\sigma$ and $\widehat{F}^s_{\sigma'}$ that are the projections by p_f and $p_{f'}$ of the two-dimensional linearizing leaves of F^s_σ and $F^s_{\sigma'}$, respectively (see Fig. 8). The foliation F^s_σ (respectively, $F^s_{\sigma'}$) has a unique compact leaf $\widehat{L}^s_f$ ($\widehat{L}^s_{f'}$), and its holonomy group[7] $\operatorname{Hol}(\widehat{L}^s_f, \widehat{x})$ ($\operatorname{Hol}(\widehat{L}^s_{f'}, \widehat{x}'))$ is an infinite cyclic group. The generator of this group is a germ of the expansion of this interval with one fixed point. Since all such expansions are topologically conjugate in a neighbourhood of the fixed point, the holonomy groups $\operatorname{Hol}(\widehat{L}^s_f, \widehat{x})$ and $\operatorname{Hol}(\widehat{L}^s_{f'}, \widehat{x}')$ are conjugate. Moreover, the conjugating homeomorphism can be taken to agree with $\widehat{\varphi}$ on $\widehat{L}^s_f$. Hence, there exist neighbourhoods $U(\widehat{L}^s_f)$ and $U(\widehat{L}^s_{f'})$ of the tori $\widehat{L}^s_f$ and $\widehat{L}^s_{f'}$ and a homeomorphism $\widehat{\varphi}_0\colon U(\widehat{L}^s_f) \to U(\widehat{L}^s_{f'})$ coinciding with $\widehat{\varphi}$ on $\widehat{L}^s_f$ that carries leaves of the foliations $\widehat{F}^s_\sigma\big|_{U(\widehat{L}^s_f)}$ into leaves of the foliations $\widehat{F}^s_{\sigma'}\big|_{U(\widehat{L}^s_{f'})}$ (see, for example, [17], Theorem 2 of Chap. IV).

Consider the map $\widehat{\phi}$ on $U(\widehat{L}^s_f)$ defined by $\widehat{\phi} = \widehat{\varphi}^{-1}\widehat{\varphi}_0$. We identify the neighbourhood $U(\widehat{L}^s_f)$ with the set $\mathbb{T}^2 \times [-1,1]$, and proceed to construct an isotopy $\widehat{\phi}_t\colon \mathbb{T}^2 \times [-1,1] \to \mathbb{T}^2 \times [-1,1]$ so as to have $\widehat{\phi}_0 = \widehat{\phi}$, $\widehat{\phi}_1 = \mathrm{id}\big|_{\mathbb{T}^2\times[-1,1]}$, and

[7] Let $\mathscr{F}$ be a smooth foliation of codimension m on an n-manifold and let F be a compact leaf of this foliation. Let Σ be a smooth local m-dimensional secant of the leaves of $\mathscr{F}$ that passes through a point $x \in F$. Then to any closed loop $c \subset F$ based at x there corresponds a diffeomorphism $\psi_c\colon \Sigma \to \Sigma$ with fixed point x that maps a point $y \in \Sigma$ on a leaf of $\mathscr{F}$ into the first-return point of this leaf to the secant Σ along the loop c. If $c' \in [c] \in \pi_1(F, x)$, then the maps $\psi_{c'}$ and ψ_c coincide in some neighbourhood of x, that is, they lie in the same *germ* of diffeomorphisms of Σ at x. Thus, the map $c \to \psi_c$ induces a homomorphism $\Phi\colon \pi_1(F, x) \to G(\Sigma, x)$ from the fundamental group of the leaf F at x into the group of germs of diffeomorphisms of Σ at the point $\widehat{x}$. The group $\operatorname{Hol}(F, x) = \Phi(\pi_1(F, x))$ is called the *holonomy group* of F at x. Holonomies F and F' are said to be *conjugate* if there exist secants Σ and Σ' transversal to F and F' at points $x \in F$ and $x' \in F'$ together with a homeomorphism $h\colon F \cup \Sigma \to F' \cup \Sigma'$ such that $\psi_{h(c)} = h\psi_c h^{-1}$ for any $[c] \in \pi_1(F, x)$ near the point x'.

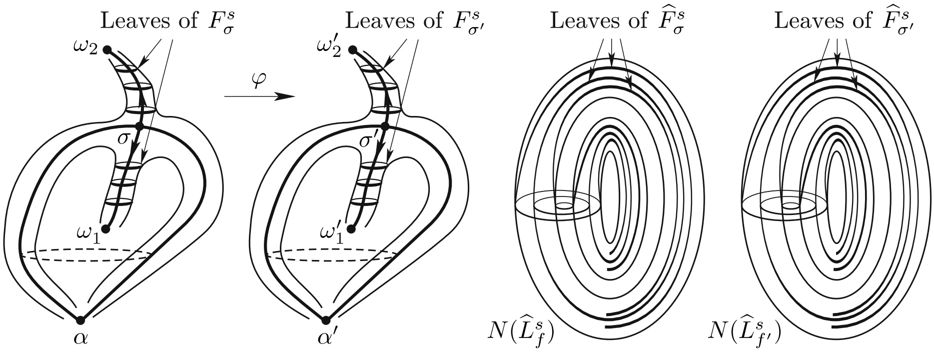


Figure 8. Construction of a conjugating homeomorphism.

$\widehat{\phi}_t(\mathbb{T}^2\times\{0\})=\mathbb{T}^2\times\{0\}$ for $t\in[0,1]$. To this end we define the maps $h_{+,t}\colon \mathbb{T}^2\times[t,1]\to\mathbb{T}^2\times[0,1]$ and $h_{-,t}\colon \mathbb{T}^2\times[-1,-t]\to\mathbb{T}^2\times[-1,0]$, $t\in[0,1)$, as follows:

$$h_{+,t}(x,s)=\left(x,\frac{s-t}{1-t}\right),\qquad h_{-,t}(x,s)=\left(x,\frac{s+t}{-1+t}\right).$$

Further, we set

$$\widehat{\phi}_t(x,s)=\begin{cases} h^{-1}_{-,t}\widehat{\phi}h_{-,t}(x,s), & s\in[-1,-t],\\ (x,s), & |s|\leqslant t,\\ h^{-1}_{+,t}\widehat{\phi}h_{+,t}(x,s), & s\in[t,1],\end{cases}$$

and continuously extend the family $\widehat{\phi}_t$, $t\in[0,1)$, by the map $\widehat{\phi}_1(x,s)=(x,s)$. Then there exists a homeomorphism $\widehat{\Phi}\colon U(\widehat{L}^s_f)\to U(\widehat{L}^s_f)$ which is the identity map on the boundary $\partial U(\widehat{L}^s_f)$ and agrees with $\widehat{\phi}$ in some neighbourhood of $\widehat{L}^s_f$ (see, for example, Corollary 3.14 in [10]). It follows that the homeomorphism $\widehat{\varphi}\widehat{\Phi}$ is the required modification of the homeomorphism $\widehat{\varphi}$.

2.1.2. *Realization.* To describe the idea behind the realization of Pixton diffeomorphisms, we represent the manifold $\mathbb{S}^2\times\mathbb{S}^1$ as the orbit space $(\mathbb{R}^3\setminus O)/a^u_{3,+1}$ for the action of the canonical expansion on $\mathbb{R}^3\setminus O$. By Statement 1.6, the natural projection $p_{\mathbb{S}^2\times\mathbb{S}^1}\colon\mathbb{R}^3\setminus O\to\mathbb{S}^2\times\mathbb{S}^1$ of this action is a covering that induces an epimorphism $\eta_{\mathbb{S}^2\times\mathbb{S}^1}\colon\pi_1(\mathbb{S}^2\times\mathbb{S}^1)\to\mathbb{Z}$. From Statements 1.2 and 1.7 it follows that the scheme of any diffeomorphism $f\in\mathscr{P}$ is equivalent (in the sense of Definition 2.2) to the tuple $S=(\mathbb{S}^2\times\mathbb{S}^1,\eta_{\mathbb{S}^2\times\mathbb{S}^1},\widehat{L}^s)$, where $\widehat{L}^s\subset\mathbb{S}^2\times\mathbb{S}^1$ is a two-dimensional torus such that $\eta(i_{\widehat{L}^s*}(\pi_1(\widehat{L}^s)))=\mathbb{Z}$. Such a tuple S is called an *abstract scheme*. The following realization theorem holds.

Theorem 2.2. *For any abstract scheme $S=(\mathbb{S}^2\times\mathbb{S}^1,\eta_{\mathbb{S}^2\times\mathbb{S}^1},\widehat{L}^s)$ there exists a diffeomorphism $f_S\in\mathscr{P}$ whose scheme is equivalent to S.*

The possibility of realizing an abstract scheme $S=(\mathbb{S}^2\times\mathbb{S}^1,\eta_{\mathbb{S}^2\times\mathbb{S}^1},\widehat{L}^s)$ by a Pixton diffeomorphism is based on the following observation. We take a tubular neighbourhood $N(\widehat{L}^s)\subset\mathbb{S}^2\times\mathbb{S}^1$ of the torus $\widehat{L}^s$, and define $L^s=p^{-1}_{\mathbb{S}^2\times\mathbb{S}^1}(\widehat{L}^s)$

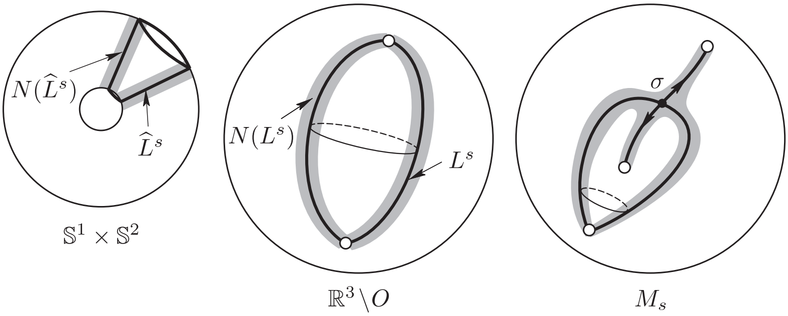


Figure 9. Realization of an abstract scheme.

and $N(L^s) = p^{-1}_{\mathbb{S}^2\times\mathbb{S}^1}(N(\widehat{L}^s))$. Since the neighbourhood $N(\widehat{L}^s)$ is homeomorphic to the orbit space $(\mathscr{N}_1 \setminus W^u_O)/a_{1,+1}$, it follows by Statement 1.7 that there exists a diffeomorphism $\nu_s\colon N(L^s) \to \mathscr{N}_1 \setminus W^u_O$ which conjugates the diffeomorphisms $a^u_{3,+1}\big|_{N(L^s)}$ and $a_{1,+1}\big|_{\mathscr{N}_1\setminus W^u_O}$. This fact enables one to 'glue the neighbourhood of the linear saddle' in the manifold $\mathbb{R}^3 \setminus O$ (see Fig. 9). The latter means that on the manifold $M_s = (\mathbb{R}^3 \setminus O) \cup_{\nu_s} \mathscr{N}_1$ there exists a diffeomorphism $f_{M_s}\colon M_s \to M_s$ whose non-wandering set consists of a single hyperbolic saddle point σ with Morse index 1, and its restriction to the manifold $R_u = M_s \setminus W^u_\sigma$ is topologically conjugate to the canonical expansion $a^u_{3,+1}$.

The manifold $R_s = M_s \setminus W^s_\sigma$ is obtained from the manifold $\mathbb{R}^3 \setminus O$ by removing the set $N(L^s)$ and attaching the set $\mathscr{N}_1 \setminus W^s_O$. Then the orbit space R_s/f_s is obtained from the original manifold $\mathbb{S}^2 \times \mathbb{S}^1$ by a so-called *surgery along the torus* $\widehat{L}^s$. This surgery consists in removing the tubular neighbourhood $N(\widehat{L}^s)$ (homeomorphic to the orbit space $\widehat{\mathscr{N}^s_{1,+1}} = (\mathscr{N}_1 \setminus W^u_O)/a_{1,+1}$) and gluing two solid tori (homeomorphic to the orbit space $\widehat{\mathscr{N}^u_{1,+1}} = (\mathscr{N}_1 \setminus W^s_O)/a_{1,+1}$) onto the boundary of the resulting manifold in such a way that the meridian of a solid torus (the boundary of a two-dimensional disk in the solid torus that is not contractible on the boundary torus) is identified with a homotopically trivial curve in $\mathbb{S}^2 \times \mathbb{S}^1$ (see Fig. 10). Since the torus $\widehat{L}^s$ is homotopically non-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, it bounds a solid torus in $\mathbb{S}^2 \times \mathbb{S}^1$ (see, for example, Theorem 4 in [9]). Further, the manifold obtained from the two solid tori (glued along the boundaries by a diffeomorphism taking a meridian into a meridian) is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (see, for example, Proposition 7.1 in [23]), and hence the manifold $(\mathbb{S}^2 \times \mathbb{S}^1)_{\widehat{L}^s}$ obtained by the surgery on $\mathbb{S}^2 \times \mathbb{S}^1$ along $\widehat{L}^s$ consists of two copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Then Statement 1.7 implies that the manifold $R_s = M_s \setminus W^s_\sigma$ consists of two connected components, and the restriction of the diffeomorphism f_{M_s} to each of them is topologically conjugate to the canonical contraction $a^s_{0,+1}$.

We compactify the manifold M_s by adding three points, and we extend the diffeomorphism f_{M_s} by continuity by adding three hyperbolic nodes as fixed points: one source and two sinks. The resulting diffeomorphism f_S is therefore in the Pixton class, and its scheme is equivalent to S.

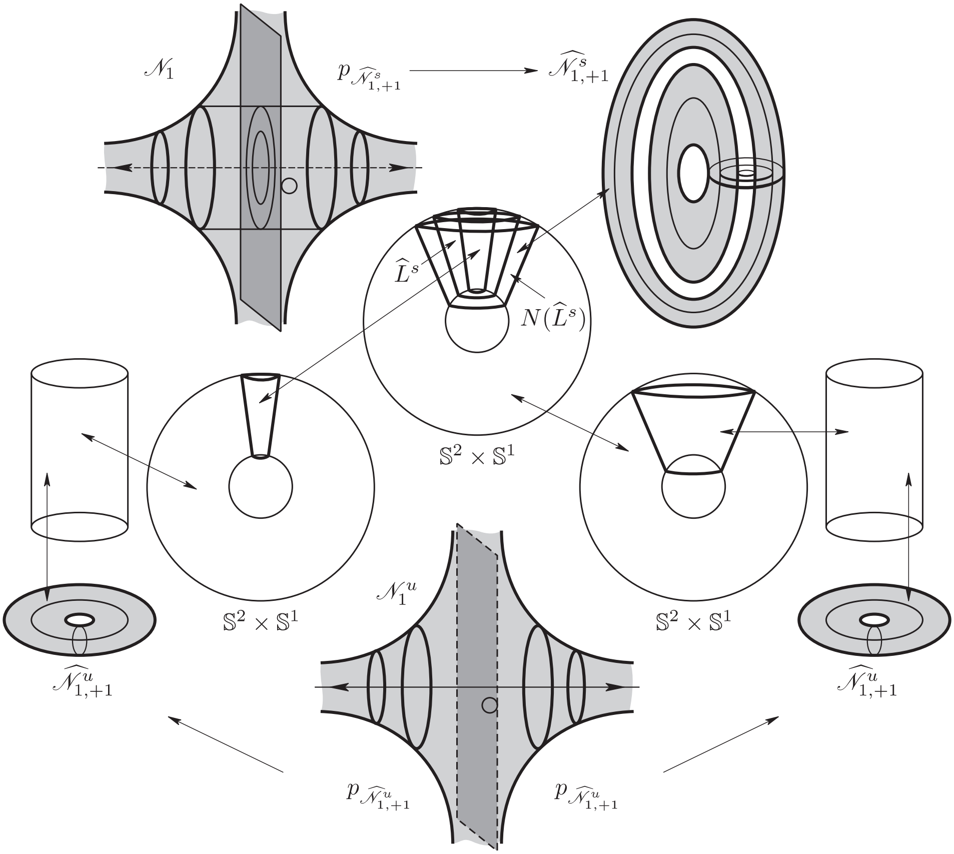


Figure 10. Surgery along a torus.

2.1.3. *Topology of the ambient manifold.*

Theorem 2.3. *For any diffeomorphism $f \in \mathscr{P}$, the ambient manifold is diffeomorphic to the 3-sphere $\mathbb{S}^3$.*

The proof of Theorem 2.3 is based on the following remarkable result describing the topological structure of a neighbourhood of a sphere with one wild point.

Lemma 2.1. *Let $\eta\colon \mathbb{S}^2 \to M^3$ be a topological embedding of the 2-sphere. Assume that η is smooth everywhere except at one point, and let $\Sigma = \eta(\mathbb{S}^2)$. Then any neighbourhood V of the sphere Σ contains a neighbourhood K diffeomorphic to $\mathbb{S}^2 \times [0,1]$.*

The proof of Lemma 2.1 amounts to constructing a smooth 3-ball $B \subset M^3$ containing a wild point x_0 in its interior and such that $\partial B \cap \Sigma$ consists of a single circle. Then the required neighbourhood K is obtained from B by adding a tubular neighbourhood of the two-dimensional disk $D = \Sigma \setminus \operatorname{int} B$. Figure 11 shows how to construct such a ball B for the Artin–Fox wild sphere. We note that the construction depends essentially on the fact that the sphere Σ has at most one wild

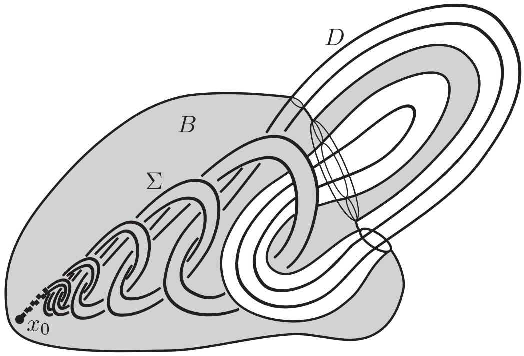


Figure 11. Construction of the ball B for the Artin–Fox sphere.

point. This is not accidental, because Lemma 2.1 fails in case there are two or more wild points. An example of a two-dimensional sphere in $\mathbb{R}^3$ with two wild points is depicted in Fig. 12.

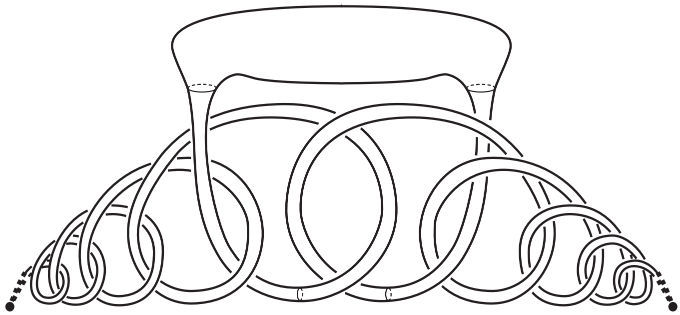


Figure 12. Two-dimensional sphere in $\mathbb{R}^3$ with two wild points.

The scheme of the proof of Theorem 2.3 is now easily seen. According to Statements 1.1 and 1.4, the set $\Sigma = W^s_\sigma \cup \alpha$ is a topologically embedded sphere which is smooth everywhere except at α. Since by assertion 1) in Statement 1.1 the ambient manifold M^3 can be represented as $M^3 = \Sigma \cup W^s_{\omega_1} \cup W^u_{\omega_2}$, it follows that the neighbourhood K, as required in Lemma 2.1, has boundary spheres S_1 and S_2 in the basins $W^s_{\omega_1}$ and $W^u_{\omega_2}$, respectively. Since each basin is homeomorphic to $\mathbb{R}^3$, the set $M^3 \setminus \operatorname{int} K$ consists of smooth 3-balls $B_1 \subset W^s_{\omega_1}$ and $B_2 \subset W^u_{\omega_2}$ such that $\partial B_1 = S_1$ and $\partial B_2 = S_2$. Thus, M^3 is the result of gluing together two smooth 3-balls B_1 and $B_2 \cup K$ along the boundary, and hence is diffeomorphic to $\mathbb{S}^3$ (see, for example, [23]).

2.2. Bifurcations that change the embedding type of separatrices. We let $J(\mathbb{S}^3)$ denote the set of orientation-preserving 'North Pole–South Pole' diffeomorphisms, that is, the diffeomorphisms whose non-wandering sets consist of exactly two hyperbolic points: a source and a sink). The main result of the present subsection is as follows.

Theorem 2.4. *For any diffeomorphisms $f, f' \in \mathscr{P}$ there exists a smooth arc $\{f_t \in \mathrm{Diff}(\mathbb{S}^3)\}$ such that*:

1) $f_0 = f$, $f_1 = f'$;

2) $f_t \in \mathscr{P}$ *for all* $t \in [0, 1/3) \cup (2/3, 1]$;

3) $f_t \in J(\mathbb{S}^3)$ *for all* $t \in (1/3, 2/3)$;

4) *the non-wandering set of the diffeomorphism* $f_{i/3}$, $i = 1, 2$, *consists of two hyperbolic fixed points — a source and a sink — and one non-hyperbolic saddle–node fixed point.*

The proof of Theorem 2.4 is divided into two parts:

(I) construction of a smooth arc with one saddle–node bifurcation point between an arbitrary Pixton diffeomorphism and some North Pole–South Pole diffeomorphism;

(II) construction of a smooth isotopy between any North Pole–South Pole diffeomorphisms which consists of diffeomorphisms of the same type.

(I) The key to the first part of the proof is provided by a basic property of Pixton diffeomorphisms, which we now give in the following proposition.

Proposition 2.1. *For any diffeomorphism $f \in \mathscr{P}$, at least one of the one-dimensional separatrices is tame.*

This result enables us to put the tame separatrix on a coordinate axis in the local coordinates of the corresponding sink, and then to realize on it the standard confluence of the saddle with the sink, thereby solving problem (I).

Passage to the orbit space will also be useful in proving Proposition 2.1.

Given a diffeomorphism $f \in \mathscr{P}$, we let γ_1 and γ_2 denote the unstable separatrices of a point σ_f. By Statement 1.4, the closure $\mathrm{cl}(\gamma_i)$ ($i = 1, 2$) of a one-dimensional unstable separatrix of the point σ is homeomorphic to a simple compact arc and consists of this separatrix and two points: σ and the sink. Assume for definiteness that the point ω_i lies on the arc $\mathrm{cl}(\gamma_i)$. For $i = 1, 2$ we set $V_i = W^s_{\omega_i} \setminus \omega_i$ and $\widehat{V}_i = V_i/f$. According to results in § 1.2, the natural projection $p_i \colon V_i \to \widehat{V}_i$ is a covering that induces an epimorphism $\eta_i \colon \pi_1(\widehat{V}_i) \to \mathbb{Z}$. Here the manifold $\widehat{V}_i$ is homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$, and the set $\widehat{\gamma}_i = p_i(\gamma_i)$ is a *knot* (a homeomorphic image of a circle) in $\widehat{V}_i$ such that $\eta_i(i_{\widehat{\gamma}_i *}(\pi_1(\widehat{\gamma}_i))) = \mathbb{Z}$.

The following criterion for tameness of the one-dimensional separatrix γ_i, $i = 1, 2$, is the key to the proof of Proposition 2.1.

Lemma 2.2. *The separatrix γ_i, $i = 1, 2$, is tame if and only if there exists a tubular neighbourhood $N(\widehat{\gamma}_i)$ of the knot $\widehat{\gamma}_i$ in $\widehat{V}_i$ such that the manifold $\widehat{V}_i \setminus N(\widehat{\gamma}_i)$ is homeomorphic to a solid torus.*

The sufficiency of the condition in Lemma 2.2 depends on the fact that the existence of the neighbourhood $N(\widehat{\gamma}_i)$ implies the existence of a homeomorphism $\widehat{\varphi}_i \colon \widehat{V}_i \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $\widehat{\varphi}_i(\widehat{\gamma}_i) = \{x_i\} \times \mathbb{S}^1$ for some $x_i \in \mathbb{S}^2$. In view of Statement 1.7 this implies the existence of a homeomorphism $\varphi_i \colon W^s_{\omega_i} \to \mathbb{R}^3$ for which $\varphi_i(\gamma_i) = \mathbb{R}_+$.

The scheme for proving necessity of the condition in Lemma 2.2 is as follows. If the separatrix γ_i is tame, then by the definition of tameness there exists a smooth

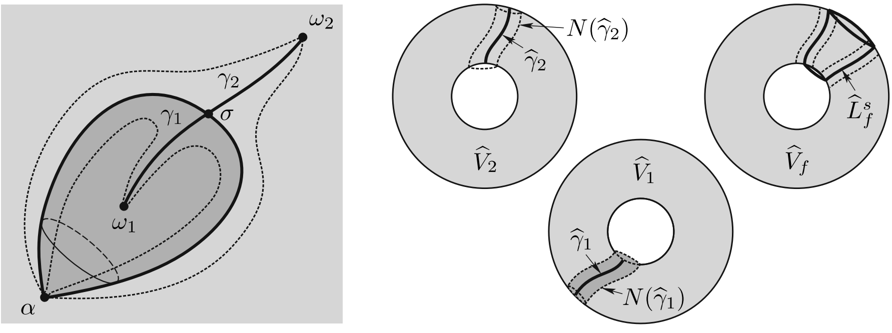


Figure 13. Duality of quotient spaces.

3-ball $B_i \subset W^s_{\omega_i}$ containing ω_i such that γ_i intersects ∂B_i in a single point. By standard topological methods it can be deformed into a smooth 3-ball (again denoted by B_i) with the additional property that $f(B_i) \subset \operatorname{int} B_i$.

We set $K_i = B_i \setminus \operatorname{int} f(B_i)$. By construction, the three-dimensional annulus K_i is a fundamental domain for the action of f on V_i, and hence K_i/f is homeomorphic to $\widehat{V}_i$ according to Statement 1.6. We choose a tubular neighbourhood $N_i \subset K_i$ of the arc $l_i = \gamma_i \cap K_i$ so that the image $f(d_i)$ of the two-dimensional disk $d_i = N_i \cap \partial B_i$ coincides with the intersection $N_i \cap \partial f(B_i)$. By construction, the set $G_i = K_i \setminus N_i$ is bounded by a two-dimensional sphere composed of the disks $\delta_i = \partial B_i \setminus d_i$, $f(\delta_i)$ and the two-dimensional annulus $\partial N_i \cap \operatorname{int} G_i$. It follows that G_i is a three-dimensional ball. Furthermore, the tubular neighbourhood $N(\widehat{\gamma}_i)$ is obtained from N_i by identifying the disks d_i and $f(d_i)$, and its complement is obtained from G_i by identifying the disks δ_i and $f(\delta_i)$ by means of the diffeomorphism f, and hence is a solid torus.

Thus, the proof of Proposition 2.1 amounts to checking that the complement of at least one of the tubular neighbourhoods $N(\widehat{\gamma}_1)$, $N(\widehat{\gamma}_2)$ is a solid torus. The latter fact follows from the duality of the quotient spaces $(\widehat{V}_1 \setminus N(\widehat{\gamma}_1)) \cup (\widehat{V}_2 \setminus N(\widehat{\gamma}_2)) = \widehat{V}_f \setminus N(\widehat{L}^s_f)$ (see Fig. 13) and the fact that a homotopically non-trivial torus bounds a solid torus in $\mathbb{S}^2 \times \mathbb{S}^1$.

(II) That there is a smooth arc connecting any two orientation-preserving diffeomorphisms, and hence any two North Pole–South Pole diffeomorphisms, on $\mathbb{S}^3$) is a classical result due to Cerf [18]. We assert that this arc can be chosen to be composed solely of North Pole–South Pole diffeomorphisms. The analogous result fails to hold in dimension six (see Theorem 4.3.5 in [34]), because of the existence of distinct smooth structures on a seven-dimensional sphere (as proved by Milnor [45]).

This problem reduces to the construction of a smooth arc $\{l_t \in J(\mathbb{S}^3),\ t \in [0,1]\}$, connecting any diffeomorphism $f \in J(\mathbb{S}^3)$ with the *canonical North Pole–South Pole diffeomorphism* $g\colon \mathbb{S}^3 \to \mathbb{S}^3$ given by

$$g(x_1,x_2,x_3,x_4) = \left(\frac{4x_1}{5-3x_4}, \frac{4x_1}{5-3x_4}, \frac{4x_3}{5-3x_4}, \frac{5x_4-3}{5-3x_4}\right).$$

It is readily verified that $a^s_{3,+1} = \vartheta g \vartheta^{-1}$, where $\vartheta\colon \mathbb{S}^3 \to \mathbb{R}^3$ is the *stereographic projection* $\vartheta(x_1, x_2, x_3, x_4) = \left(\dfrac{x_1}{1-x_4}, \dfrac{x_2}{1-x_4}, \dfrac{x_3}{1-x_4}\right)$. By construction, the arc $\{l_t \in J(\mathbb{S}^3)\}$ consists of the following parts:

1. a diffeomorphism $l_t \in J(\mathbb{S}^3)$ for $t \in [0, 1/4]$, where $l_0 = f$ and $l_{1/4}$ is a C^2-diffeomorphism in $J(\mathbb{S}^3)$;
2. a C^2-diffeomorphism $l_t \in J(\mathbb{S}^3)$ for $t \in [1/4, 1/2]$, where $l_{1/2}$ is a C^2-diffeomorphism in the class $NS(\mathbb{S}^3) \subset J(\mathbb{S}^3)$ of diffeomorphisms with source at $N(0,0,0,1)$ and sink at $S(0,0,0,-1)$;
3. a diffeomorphism $l_t \in NS(\mathbb{S}^3)$ for $t \in [1/2, 3/4]$, where $l_{3/4}$ is a diffeomorphism in the class $E_g(\mathbb{S}^3) \subset NS(\mathbb{S}^3)$ of diffeomorphisms h such that there exist neighbourhoods $V_h(N)$, $V_h(S)$ of the points N, S with $h\big|_{V_h(N)\cup V_h(S)} = g\big|_{V_h(N)\cup V_h(S)}$;
4. a diffeomorphism $l_t \in E_g(\mathbb{S}^3)$ for $t \in [3/4, 1]$ such that $l_1 = g$.

The construction of the first part of the arc $\{l_t\}$ is based on the structural stability of f and on the density of C^2-diffeomorphisms in the space of all diffeomorphisms.

To construct the second part of $\{l_t\}$, we denote by α and ω the source and sink for the diffeomorphism $l_{1/4}$. Let D_α and D_ω (D_S and D_N) be disjoint 3-disks containing α and ω (S and N). Then there exists a C^2-smooth arc $\{H_t \in \mathrm{Diff}(\mathbb{S}^3)\}$ with the following properties: $H_0 = \mathrm{id}$, $H_1(D_N) = D_\alpha$, $H_1(D_S) = D_\omega$, $H_1(N) = \alpha$, and $H_1(S) = \omega$ (see, for example, Theorem 3.2 in [35]). It follows that $H_t^{-1} l_{1/4} H_t$ is an isotopy connecting $l_{1/4}$ with $l_{1/2} = H_1^{-1} l_{1/4} H_1$, and after reparametrization this produces the required arc.

The construction of the third part is based on Belitskii's theorem (see, for example, [66]) on the smooth conjugacy of a C^2-diffeomorphism to its linear part in a neighbourhood of a hyperbolic node fixed point, and on the fact that there exists a smooth arc connecting two linear contractions (expansions) on $\mathbb{R}^3$ (see, for example, Proposition 5.4 in [53]).

In the construction of the last part of the arc, we again pass to the space of wandering orbits. To do this we note that for any diffeomorphism $h \in E_g(\mathbb{S}^3)$ its wandering set is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$, and there exists an $r^+_h \in \mathbb{R}$ ($r^-_h \in \mathbb{R}$) such that h agrees with g on $\mathbb{S}^2 \times [r^+_h, +\infty)$ ($\mathbb{S}^2 \times (-\infty, r^-_h]$). Further, there exists a diffeomorphism $\psi_h\colon \mathbb{S}^3 \to \mathbb{S}^3$ which conjugates the diffeomorphisms h and g and agrees with the identity map on $D^+_h \cup D^-_h$, where $D^+_h = \mathbb{S}^2 \times [r^+_h, +\infty) \cup S$ and $D^-_h = \mathbb{S}^2 \times (-\infty, r^-_h] \cup N$. In view of Statement 1.7 the diffeomorphism ψ_h induces, via the covering $p_{\mathbb{S}^2\times\mathbb{S}^1}\colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{S}^1$, a diffeomorphism $\widehat{\psi}_h\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ which acts as the identity on the fundamental group. In view of Proposition 0.4 in [9], one of the diffeomorphisms $\widehat{\psi}_h$ and $\widehat{v}\widehat{\psi}_h$ is isotopic to the identity map, where the diffeomorphism $\widehat{v}$ is defined as follows. For any $\Theta \in \mathbb{R}$ we let $R_\Theta\colon \mathbb{S}^2 \to \mathbb{S}^2$ denote the map of rotation through the angle Θ about the axis passing through the points $(0,0,1)$ and $(0,0,-1)$. For $\lambda \in \mathbb{R}$ let $\tilde{v}_\lambda\colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$ be the diffeomorphism such that $\tilde{v}_\lambda(s, r) = (R_{2\pi(r-\lambda)}, r)$ on $K_\lambda = \mathbb{S}^2 \times [\lambda, \lambda+1)$ and it agrees with the identity map outside K_λ. Then $\widehat{v} = p_{\mathbb{S}^2\times\mathbb{S}^1} \widetilde{v}_\lambda (p_{\mathbb{S}^2\times\mathbb{S}^1}\big|_{K_\lambda})^{-1}\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ is the required diffeomorphism.

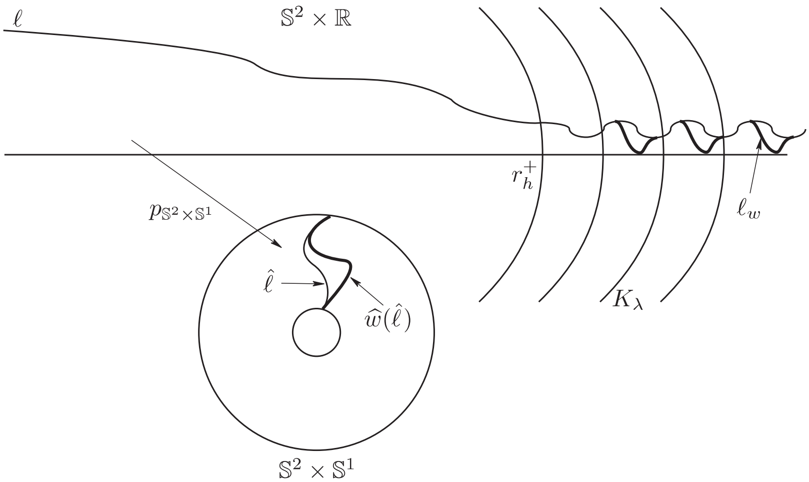


Figure 14. Illustration for part (I).

Thus, the diffeomorphism $\widehat{\psi}_{l_{3/4}}$ may be regarded as isotopic to the identity map (otherwise, $l_{3/4}$ can be connected with the diffeomorphism $\nu_1 l_{3/4}$ by the smooth arc $\{\nu_t l_{3/4}\} \subset E_g(\mathbb{S}^3)$, where $\nu_t(S) = S$, $\nu_t(N) = N$, and ν_t is defined on $\mathbb{S}^2 \times \mathbb{R}$ by

$$\nu_t(s,r) = \begin{cases} (s,r), & (s,r) \in \mathbb{S}^2 \times (-\infty, \lambda], \\ (R_{2\pi(r-\lambda)t}, r), & (s,r) \in K_\lambda, \\ (R_{2\pi t}, r), & (s,r) \in \mathbb{S}^2 \times [\lambda+1, +\infty), \end{cases}$$

for $t \in [0,1]$ and $\lambda > r^+_{l_{3/4}}$). By the fragmentation lemma (see [5]), there are diffeomorphisms $\widehat{w}_1, \dots, \widehat{w}_q \colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ that are smoothly isotopic to the identity, satisfy $\widehat{\psi}_{l_{3/4}} = \widehat{w}_1 \cdots \widehat{w}_q$, and are such that for any $i = 1, \dots, q$ there exists a support U_i of the isotopy $\{\widehat{w}_{i,t}\}$ between the identity and $\widehat{w}_i$ which has the following properties: there exists a $\lambda_i \in [r^+_{l_{3/4}} + 2(i-1), r^+_{l_{3/4}} + 2i)$ such that some connected component of the set $p^{-1}_{\mathbb{S}^2\times\mathbb{S}^1}(U_i)$ is a subset of K_{λ_i}. Let $w_{i,t} \colon \mathbb{S}^3 \to \mathbb{S}^3$ be the diffeomorphism which agrees with $\left(p_{\mathbb{S}^2\times\mathbb{S}^1}\big|_{K_{\lambda_i}}\right)^{-1} \widehat{w}^{-1}_{i,t} p_{\mathbb{S}^2\times\mathbb{S}^1}$ on K_{λ_i} and with the identity map outside K_{λ_i}. Let $\mu_t = w^{-1}_{q,t} \cdots w^{-1}_{1,t} \psi_{l_{3/4}} \colon \mathbb{S}^3 \to \mathbb{S}^3$. Then after a reparametrization $\{\mu_t g \mu_t^{-1}\}$ is the required arc.

Exactly the same idea of passing to the quotient space is used in part (I) of the proof when putting tame separatrices on coordinate axis. Figure 14 depicts an h-invariant arc ℓ for some diffeomorphism $h \in E_g(\mathbb{S}^3)$ and shows its projection $\widehat{\ell}$ to the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. Then we apply to the knot $\widehat{\ell}$ a diffeomorphism $\widehat{w} \colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ that is smoothly isotopic to the identity and such that the support U of the isotopy $\{\widehat{w}_t\}$ between the identity map and $\widehat{w}$ has the following property: there exists a $\lambda \in [r_h^+, r_h^+ + 2)$ such that some connected component of the set $p^{-1}_{\mathbb{S}^2\times\mathbb{S}^1}(U)$ is a subset of K_λ. Further, the figure shows an arc ℓ_w that coincides with ℓ on the set D_h^- and is invariant under the diffeomorphism wh, where $w \colon \mathbb{S}^3 \to \mathbb{S}^3$ is the

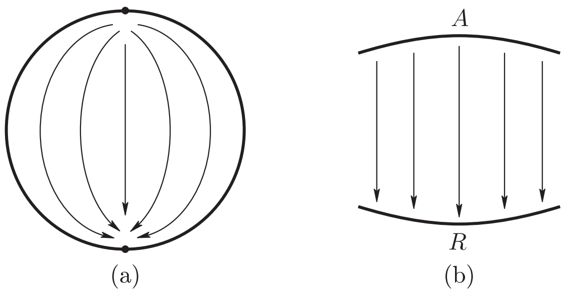


Figure 15. A 'source–sink' diffeomorphism (a) and its generalization (b).

diffeomorphism that coincides with $\left(p_{\mathbb{S}^2\times\mathbb{S}^1}\big|_{K_\lambda}\right)^{-1}\widehat{w}^{-1}p_{\mathbb{S}^2\times\mathbb{S}^1}$ on K_λ and with the identity map outside K_λ.

3. Topological classification

In this section we give a complete topological classification of diffeomorphisms in the class $\mathrm{MS}(M^3)$. Our classification is conceptually a development of methods presented in § 2.1. The basic idea is to represent the dynamics of a diffeomorphism $f\in\mathrm{MS}(M^3)$ in a 'source–sink' form, where by 'source' and 'sink' we understand invariant closed sets, one of which, say A, is an attractor, and the other, say R, is a repeller.[8] The set $V=M^n\setminus(A\cup R)$ consists of wandering points that move under the action of the diffeomorphism from the source to the sink (see Fig. 15). In choosing a pair A, R that is suitable for a topological invariant, particular emphasis will be placed on two things: first, the space $\widehat{V}=V/f$ of wandering orbits, together with the projections of separatrices of saddle points embedded in it, should be canonically describable, and second, the conjugating homeomorphism on V should be extendable to a homeomorphism to the attractor and the repeller. For Pixton diffeomorphisms the set $\operatorname{cl}W^u_\sigma$ is such an attractor, the source α is a repeller, and the set V coincides with $W^u_\alpha\setminus\alpha$. In the general situation such a choice is described in the next subsection.

3.1. Necessary and sufficient conditions for topological conjugacy. Assume that $f\in\mathrm{MS}(M^3)$. We represent the dynamics of f in a 'source–sink' form as follows. Let $A_f=W^u_{\Omega_0\cup\Omega_1}$, $R_f=W^s_{\Omega_2\cup\Omega_3}$, and $V_f=M^3\setminus(A_f\cup R_f)$. In this case the set A_f (respectively, R_f) is a connected attractor (repeller) of the diffeomorphism f with topological dimension not exceeding 1, and the set V_f is a connected 3-manifold. Furthermore, the quotient space $\widehat{V}_f=V_f/f$ is a closed connected orientable 3-manifold on which the natural projection $p_f\colon V_f\to\widehat{V}_f$ induces an epimorphism $\eta_f\colon\pi_1(\widehat{V}_f)\to\mathbb{Z}$. We also set $\widehat{L}^s_f=p_f(W^s_{\Omega_1}\setminus A_f)$ and $\widehat{L}^u_f=p_f(W^u_{\Omega_2}\setminus R_f)$.

[8]Recall that a compact subset A of M^n is an *attractor for a diffeomorphism* $f\colon M^n\to M^n$ if there exists a neighbourhood U of A such that $f(U)\subset\operatorname{int}U$ and $A=\bigcap_{n\in\mathbb{N}}f^n(U)$. This neighbourhood U is said to be *trapping*. A set $R\subset M^n$ is a *repeller* for f if R is an attractor for f^{-1}.

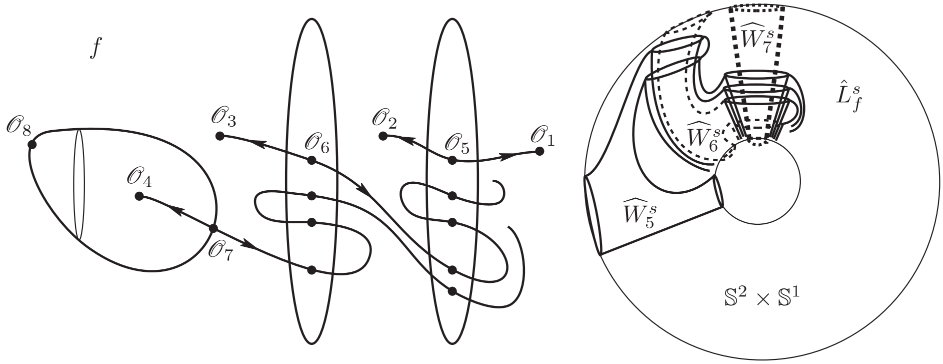


Figure 16. The scheme of a diffeomorphism $f \in \mathrm{MS}(\mathbb{S}^3)$.

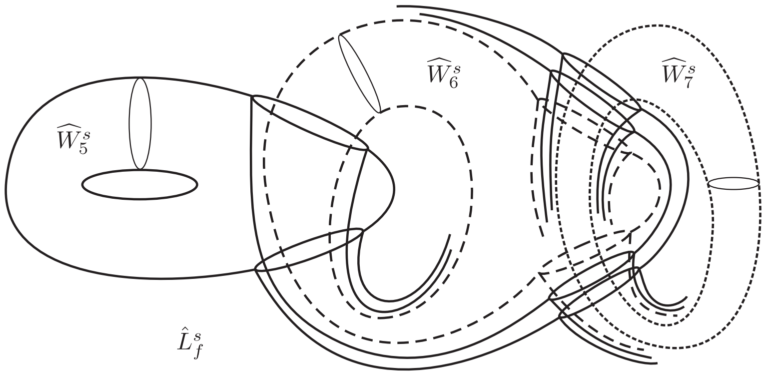


Figure 17. Union of projections to $\widehat{V}_f$ of two-dimensional separatrices.

Definition 3.1. The tuple $S_f = (\widehat{V}_f, \eta_f, \widehat{L}^s_f, \widehat{L}^u_f)$ is called the *scheme* of a diffeomorphism $f \in \mathrm{MS}(M^3)$.

Each arcwise connected component of the set $\widehat{L}^s_f$ ($\widehat{L}^u_f$) is the projection under p_f of the stable (unstable) two-dimensional manifold of a saddle orbit and is homeomorphic to a two-dimensional torus or a Klein bottle with an empty, finite, or countable set of punctured points, the number of which equals the cardinality of the set of heteroclinic orbits on this manifold. Figure 16 depicts the phase portrait and the scheme of a diffeomorphism $f \in \mathrm{MS}(\mathbb{S}^3)$ (in the form of a fundamental domain whose boundaries are not identified). It is also assumed that the nonwandering set of f is fixed, the periodic orbits are numbered as in the figure, and $\widehat{W}^s_j = p_f(W^s_{\mathscr{O}_j})$, $j = 5, 6, 7$. The union $\widehat{L}^s_f = \widehat{W}^s_5 \cup \widehat{W}^s_6 \cup \widehat{W}^s_7$ is shown separately in Fig. 17, where $\widehat{W}^s_7$ is a non-punctured torus, $\widehat{W}^s_6$ is a torus with finitely many punctured points, and $\widehat{W}^s_5$ is a torus with a countable number of punctured points.

Note that in the example in Fig. 16 the set $\widehat{L}^u_f$ is empty, and the set $\widehat{L}^s_f$ consists of one connected component. In the general situation each of the sets $\widehat{L}^s_f$ and $\widehat{L}^u_f$

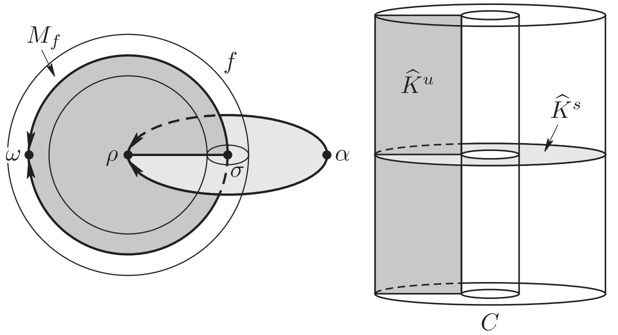


Figure 18. The scheme of a diffeomorphism $f \in \mathrm{MS}(\mathbb{S}^3)$.

has finitely many connected components, like those shown in Fig. 17. Moreover, the sets $\widehat{L}^s_f$ and $\widehat{L}^u_f$ intersect transversally (the intersection is non-empty in general). The simplest case of such an intersection is depicted in Fig. 18 with the example of the scheme of a diffeomorphism $f \in \mathrm{MS}(\mathbb{S}^3)$ whose wandering set contains a non-compact heteroclinic curve. For this diffeomorphism the trapping neighbourhood M_f of the attractor $A_f = \mathrm{cl}\, W^u_\sigma$ is a solid torus, the fundamental domain $M_f \setminus \mathrm{int}\, M_f$ for the action of f on V_f is homeomorphic to $\mathbb{T}^2 \times [-1,1]$, and $\widehat{V}_f$ is a three-dimensional torus obtained from the manifold C in Fig. 18 by identifying the upper and lower annuli, as well as the outer and inner annuli. As a result of this identification, the annulus $\widehat{K}^s$ goes over into the torus $\widehat{L}^s_f$, and the rectangle $\widehat{K}^u$ into the torus $\widehat{L}^u_f$.

In view of Statement 1.7 a necessary condition for topological conjugacy of diffeomorphisms $f, f' \in \mathrm{MS}(M^3)$ is that their schemes be equivalent in the sense of the following definition.

Definition 3.2. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in \mathrm{MS}(M^3)$ are said to be *equivalent* if there exists a homeomorphism $\widehat{\varphi}\colon \widehat{V}_f \to \widehat{V}_{f'}$ with the following properties:

1) $\eta_f = \eta_{f'}\widehat{\varphi}_*$;

2) $\widehat{\varphi}(\widehat{L}^s_f) = \widehat{L}^s_{f'}$ and $\widehat{\varphi}(\widehat{L}^u_f) = \widehat{L}^u_{f'}$.

Theorem 3.1. *Morse–Smale diffeomorphisms $f, f' \in \mathrm{MS}(M^3)$ are topologically conjugate if and only if that their schemes are equivalent.*

The underlying idea of the proof of sufficiency in Theorem 3.1 is similar to that used for the Pixton class and depends on the fact that the homeomorphism $\varphi\colon V_f \to V_f$ which is the lift of the homeomorphism $\widehat{\varphi}$ and is not in general extendable to the set $A_f \cup R_f$, can be modified on the union of the linearizing neighbourhoods N_σ in such a way that it takes two-dimensional linearizing foliations of f into analogous foliations of f'. The difference is that for a diffeomorphism $f \in \mathrm{MS}(M^3)$, in contrast to Pixton diffeomorphisms, the linearizing neighbourhoods of different saddle points may intersect, and this necessitates matching the linearizing foliations on such intersections. Thus, the fundamental technique here involves constructing a compatible system of neighbourhoods, which is the topic of the next subsection.

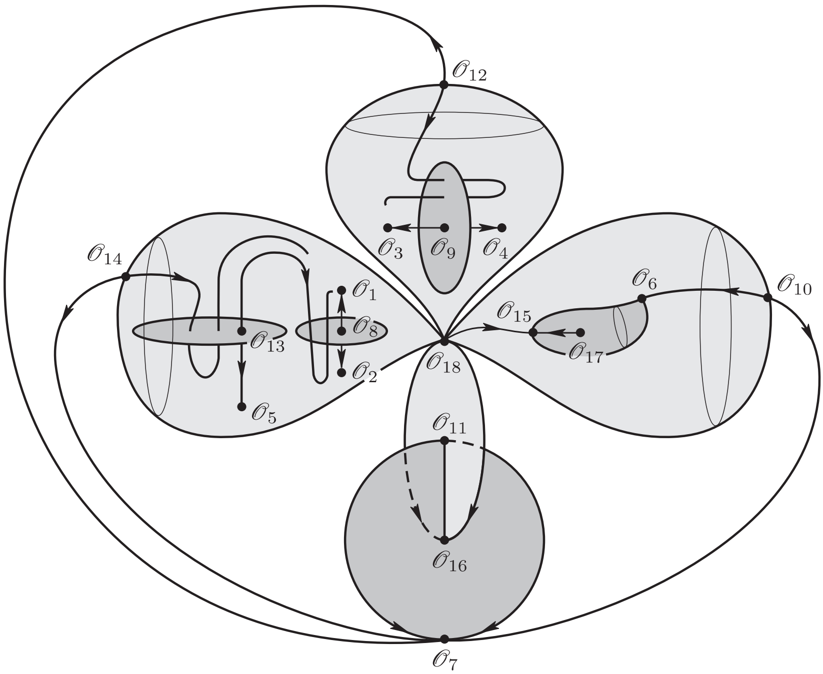


Figure 19. The phase portrait of a Morse–Smale diffeomorphism $f\colon \mathbb{S}^3 \to \mathbb{S}^3$ with dynamically ordered periodic orbits.

3.2. Dynamical order. Characteristic manifolds and spaces. Compatible family of neighbourhoods. Following Smale [71], we consider the relation $\prec$ on the set of periodic orbits:

$$\mathcal{O}_p \prec \mathcal{O}_r \quad \Leftrightarrow \quad W^s_{\mathcal{O}_p} \cap W^u_{\mathcal{O}_r} \neq \varnothing.$$

The relation $\prec$ is a partial order, because the number of periodic orbits is finite, and their invariant manifolds intersect transversally. Moreover, this relation can be completed to form an order relation, for example, as follows.

Definition 3.3. An ordering $\mathcal{O}_1, \dots, \mathcal{O}_{k_f}$ of the periodic orbits of a diffeomorphism $f \in \mathrm{MS}(M^n)$ is said to be *dynamical* if it satisfies the following conditions:

1) if $\mathcal{O}_i \prec \mathcal{O}_j$, then $i \leqslant j$;
2) if $q_{\mathcal{O}_i} < q_{\mathcal{O}_j}$, then $i < j$.

Figure 19 shows the phase portrait of a Morse–Smale diffeomorphism $f\colon \mathbb{S}^3 \to \mathbb{S}^3$ with dynamically ordered periodic orbits in the case when the non-wandering set Ω_f consists of fixed points. It is readily verified that a dynamical ordering of the periodic orbits exists for any diffeomorphism $f \in \mathrm{MS}(M^n)$. To check this it suffices to observe that the condition $\mathcal{O}_i \prec \mathcal{O}_j$ implies that $q_{\mathcal{O}_i} \leqslant q_{\mathcal{O}_j}$. Indeed, since the intersection $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j}$ is transversal, the condition $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \varnothing$ implies that $\dim W^s_{\mathcal{O}_i} + \dim W^u_{\mathcal{O}_j} - n \geqslant 0$. Hence, $n - q_{\mathcal{O}_i} + q_{\mathcal{O}_j} - n \geqslant 0$, and therefore $q_{\mathcal{O}_i} \leqslant q_{\mathcal{O}_j}$.

If the orbits of a diffeomorphism $f \in \mathrm{MS}(M^n)$ are dynamically ordered, then for any periodic orbit $\mathcal{O}_i$ we set $m_i = m_{\mathcal{O}_i}$, $q_i = q_{\mathcal{O}_i}$, $\nu_i = \nu_{\mathcal{O}_i}$, $W^s_i = W^s_{\mathcal{O}_i}$, and $W_i^u = W_{\mathscr{O}_i}^u$. To construct a compatible family of neighbourhoods, we employ a sequence of representations of the dynamics of f in a 'source–sink' form; this sequence is connected with a dynamical ordering of the orbits. Namely, for $i = 1,\dots,k_f-1$ we let

$$A_i=\bigcup_{j=1}^{i} W_j^u, \qquad R_i=\bigcup_{j=i+1}^{k_f} W_j^s, \qquad V_i=M^n\setminus(A_i\cup R_i).$$

Then the set A_i (respectively, R_i) is an attractor (repeller) of the diffeomorphism f, and $V_i = W^s_{A_i\cap\Omega_f}\setminus A_i = W^u_{R_i\cap\Omega_f}\setminus R_i$. Let $\widehat{V}_i = V_i/f$ and denote by $p_i\colon V_i\to\widehat{V}_i$ the natural projection. We call V_i the *characteristic manifold* and its orbit space $\widehat{V}_i$ the *characteristic space*. These concepts will be used below in an essential way in solving the realization problem for $\mathrm{MS}(M^3)$-diffeomorphisms, and also in constructing global Lyapunov functions for them.

By virtue of Statement 1.3, for any saddle orbit $\mathscr{O}_i$, $i=k_0+1,\dots,k_2$, there exists a linearizing neighbourhood $N_{\mathscr{O}_i}$ with a pair of linearizing foliations $F^s_{\mathscr{O}_i}$, $F^u_{\mathscr{O}_i}$. Given any saddle orbit $\mathscr{O}_i$, we set $N_i=N_{\mathscr{O}_i}$, $F_i^u=F^u_{\mathscr{O}_i}$, and $F_i^s=F^s_{\mathscr{O}_i}$. Also, for any point $x\in N_i$ we let $F^u_{i,x}$ ($F^s_{i,x}$) denote the unique leaf of F_i^u (F_i^s) passing through x.

Definition 3.4. Let $f\in\mathrm{MS}(M^3)$. A family N_f of linearizing neighbourhoods $N_{k_0+1},\dots,N_{k_2}$ of all orbits of f is said to be *compatible* and the linearizing foliations in these neighbourhoods are said to be *compatible* if the following conditions are satisfied:

1) if $W^s_{i_1}\cap W^u_{i_2}=\varnothing$ for $i_1<i_2$, then $N_{i_1}\cap N_{i_2}=\varnothing$;

2) if $W^s_{i_1}\cap W^u_{i_2}\neq\varnothing$ and $q_{i_1}=q_{i_2}$, then $(F^s_{i_2,x}\cap N_{i_1})\subset F^s_{i_1,x}$ and $(F^u_{i_1,x}\cap N_{i_2})\subset F^u_{i_2,x}$ for $x\in(N_{i_1}\cap N_{i_2})$;

3) if the set $H=W^s_{\Omega_1}\cap W^u_{\Omega_2}$ is non-empty, then there exists an f-invariant neighbourhood $N(H)\subset M^3$ of H equipped with an f-invariant foliation G consisting of two-dimensional disks that are transversal to H and such that $(F^s_{i_1,x}\cap G_x\cap N_{i_2})\subset F^s_{i_2,x}$ and $(F^u_{i_2,x}\cap G_x\cap N_{i_1})\subset F^u_{i_1,x}$ for any point $x\in(N_{i_1}\cap N_{i_2}\cap N(H))$ with $q_{i_1}<q_{i_2}$, where G_x is the leaf of G passing through x.

A compatible system of neighbourhoods is a modification of an admissible system of tubular families, as constructed in [52], [55] and having the properties 1) and 2). The condition 3) is a technical supplement to the definition given by Palis and Smale and is used in a very essential way in constructing a conjugating homeomorphism in the proof of Theorem 3.1 and in singling out the set of abstract schemes in § 3.3. Figure 20 depicts a foliated neighbourhood of a point A lying on a heteroclinic curve in $W^s_{i_1}\cap W^u_{i_2}$ for Morse–Smale diffeomorphisms on 3-manifolds. The lower part of the figure shows the phase portraits of diffeomorphisms with heteroclinic curves on $\mathbb{S}^3$.

Theorem 3.2. *Any diffeomorphism $f\in\mathrm{MS}(M^3)$ has a compatible system of neighbourhoods.*

The proof of this theorem amounts to the successive construction of compatible foliations according to the following plan.

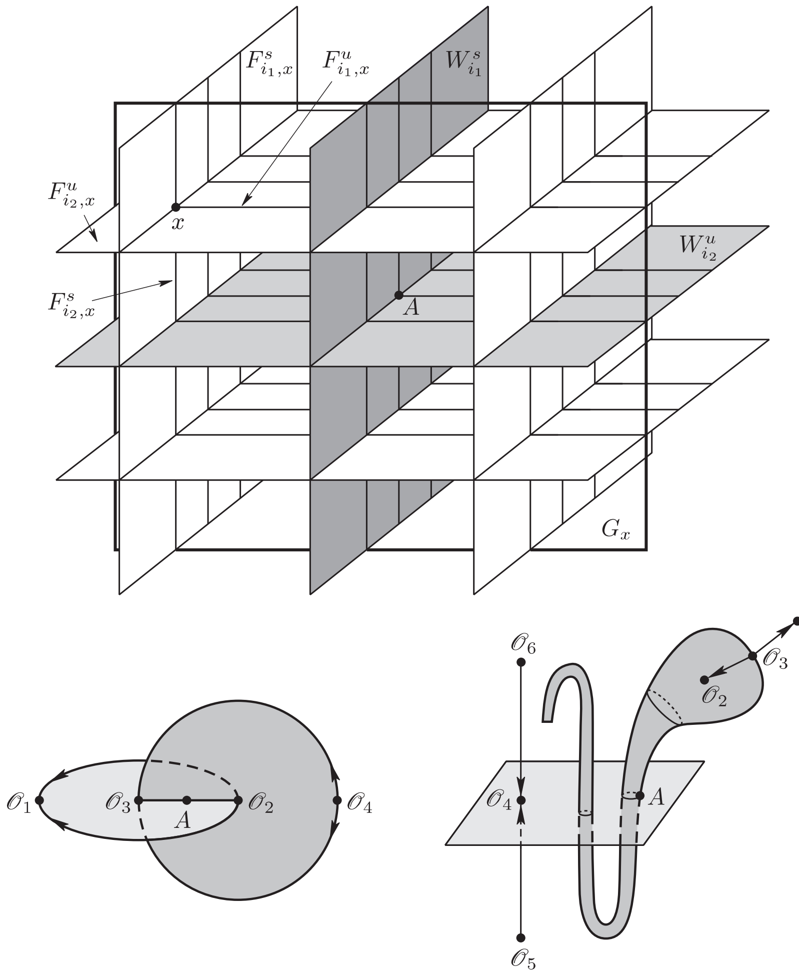


Figure 20. Foliated neighbourhood of a point on a heteroclinic curve.

1. Using induction on $i = k_0 + 1, \dots, k_1$ and passing to the orbit space $\widehat{V}_i$, we construct a two-dimensional foliation F^s_i with the property 2) in Definition 3.4. For $i = k_0 + 1$, F^s_i is the inverse image of the foliation of the tubular neighbourhood $\widehat{N}^u_i$ of the manifold $p_i(W^u_i)$ into two-dimensional disks. For $i > k_0 + 1$, the foliation in the tubular neighbourhood $\widehat{N}^u_i$ is modified so that any connected component of the intersection $p_i(F^s_i) \cap \widehat{N}^u_i$, $k_0 + 1, \dots, i - 1$, lies on a disk of the foliation.

2. A similar argument involving passage to the diffeomorphism f^{-1} produces a two-dimensional foliation F^u_i with the property 2) in Definition 3.4 for $i = k_1 + 1, \dots, k_2$.

3. In the space $\widehat{V}_f$ we consider the projection $\widehat{H}$ of heteroclinic curves which coincides with the intersection $\widehat{L}^s_f \cap \widehat{L}^u_f$. By construction, $\widehat{H}$ is a compact set, and in some neighbourhood $N(\widehat{H})$ of $\widehat{H}$ there exists a foliation $\widehat{G}$ consisting of

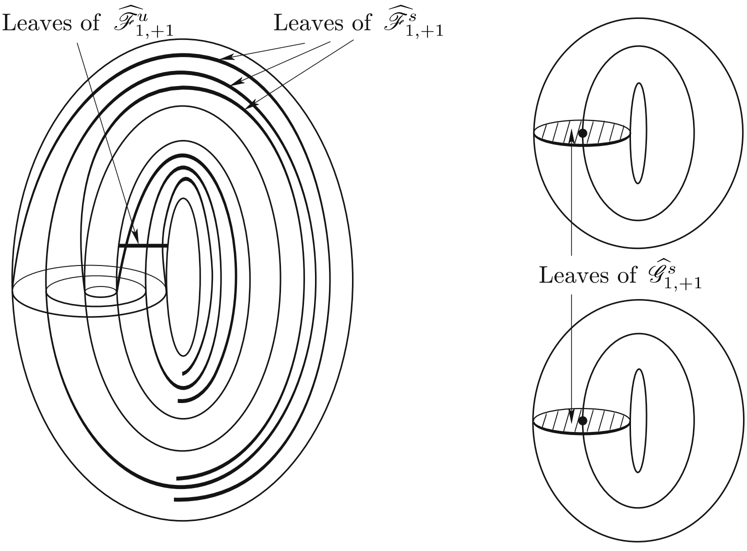


Figure 21. Leaves of the foliations $\widehat{\mathscr{F}}^s_{1,+1}$, $\widehat{\mathscr{F}}^u_{1,+1}$, $\widehat{\mathscr{G}}^s_{1,+1}$.

two-dimensional disks that are transversal both to $\widehat{H}$ and to the two-dimensional foliations constructed in steps 1 and 2. The foliation G is the inverse image of the foliation $\widehat{G}$.

4. Using induction on $i = k_1+1,\dots,k_2$ and passing to the orbit space $\widehat{V}_{i-1}$, we construct a one-dimensional foliation F^s_i with the properties 2) and 3) in Definition 3.4. This foliation is the inverse image of the foliation of the tubular neighbourhood $\widehat{N}^u_i$ of the manifold $p_{i-1}(W^u_i)$ into one-dimensional disks, modified so that any connected component of the intersection $p_{i-1}(G\cap F^s_j)\cap\widehat{N}^u_i$ for $j = k_0+1,\dots,k_1$ and of the intersection $p_{i-1}(F^s_j)\cap\widehat{N}^u_i$ for $j = k_1+1,\dots,i-1$ lies on a disk of the foliation.

5. A similar argument involving passage to the diffeomorphism f^{-1} gives a one-dimensional foliation F^u_i with the properties 2), 3) in Definition 3.4 for $i = k_1+1,\dots,k_2$.

3.3. Realization using the abstract scheme. The solution of the realization problem is based primarily on the possibility of a canonical description of the sets $\widehat{L}^s_f$ and $\widehat{L}^u_f$ in the scheme S_f of a diffeomorphism $f\in \mathrm{MS}(M^3)$.

Recall that $a_{1,\nu}\colon \mathbb{R}^3\to\mathbb{R}^3$ is the canonical diffeomorphism $a_{1,\nu}(x_1,x_2,x_3) = (\nu\cdot 2x_1, \nu\cdot x_2/2, x_3/2)$, and $a^s_{1,\nu} = a_{1,\nu}\big|_{W^s_O}$ is the canonical contraction. The orbit space $\widehat{\mathscr{W}}^s_{1,\nu} = (W^s_O\setminus O)/a^s_{1,\nu}$ of the canonical contraction is a two-dimensional torus for $\nu=+1$ or a Klein bottle for $\nu=-1$. Further, the set $\mathscr{N}_1 = \{(x_1,x_2,x_3)\in\mathbb{R}^3\colon x_1^2(x_2^2+x_3^2)<1\}$ is $a_{1,\nu}$-invariant, and $\widehat{\mathscr{N}}^s_{1,\nu} = (\mathscr{N}_1{}^s)/a_{1,\nu}$ is a tubular neighbourhood of the surface $\widehat{\mathscr{W}}^s_{1,\nu}$, where $\mathscr{N}_1{}^s = \mathscr{N}_1\setminus W^u_O$. The natural projection $p_{\widehat{\mathscr{N}}^s_{1,\nu}}\colon \mathscr{N}_1{}^s\to\widehat{\mathscr{N}}^s_{1,\nu}$ is a covering and it induces an epimorphism $\eta_{\widehat{\mathscr{N}}^s_{1,\nu}}\colon \pi_1(\widehat{\mathscr{N}}^s_{1,\nu})\to\mathbb{Z}$.

We let $\widehat{\mathscr{F}}^s_{1,\nu}$ and $\widehat{\mathscr{F}}^u_{1,\nu}$ denote the pair of transversal foliations on $\widehat{\mathscr{N}}^s_{1,\nu}$ whose leaves are the projections under $p_{\widehat{\mathscr{N}}^s_{1,\nu}}$ of the leaves of the foliations $\mathscr{F}^s_1$ and $\mathscr{F}^u_1$ (see the left-hand part of Fig. 21). Considering an at most countable set $X \subset \widehat{\mathscr{W}}^s_{1,\nu}$, we denote by Z the union of all leaves of $\widehat{\mathscr{F}}^u_{1,\nu}$ that pass through points of X. Let

$$\widehat{\mathscr{W}}^s_{1,\nu,X} = \widehat{\mathscr{W}}^s_{1,\nu} \setminus X, \qquad \widehat{\mathscr{N}}^s_{1,\nu,X} = \widehat{\mathscr{N}}^s_{1,\nu} \setminus Z,$$
$$\widehat{\mathscr{F}}^s_{1,\nu,X} = \widehat{\mathscr{F}}^s_{1,\nu} \setminus Z, \qquad \widehat{\mathscr{F}}^u_{1,\nu,X} = \widehat{\mathscr{F}}^u_{1,\nu} \setminus Z.$$

Definition 3.5. Let $\widehat{V}$ be a closed 3-manifold equipped with a map η consisting of epimorphisms to $\mathbb{Z}$ from the fundamental groups of all the connected components of $\widehat{V}$. A compact set $\widehat{L}^s \subset \widehat{V}$ will be said to be an *s-lamination* if it consists of a finite number n_s of arcwise connected components $\widehat{W}^s_1, \dots, \widehat{W}^s_{n_s}$ such that any component is a smooth submanifold, the component $\widehat{W}^s_1$ is a closed surface, $(\operatorname{cl} \widehat{W}^s_i \setminus \widehat{W}^s_i) \subset \bigcup_{j=1}^{i-1} \widehat{W}^s_j$ for $i > 1$, and moreover, for any $i = 1, \dots, n_s$ there exist a tubular neighbourhood $N(\widehat{W}^s_i)$ of the surface $\widehat{W}^s_i$, numbers $m^s_i \in \mathbb{N}$ and $\nu^s_i \in \{-1, +1\}$, a set $X^s_i \subset \widehat{\mathscr{W}}^s_{1,\nu^s_i}$, and a homeomorphism $\widehat{\mu}^s_i \colon N(\widehat{W}^s_i) \to \widehat{\mathscr{N}}^s_{1,\nu^s_i,X^s_i}$ with the following properties:

1) $\widehat{\mu}^s_i(\widehat{W}^s_i) = \widehat{\mathscr{W}}^s_{1,\nu^s_i,X^s_i}$;

2) $\eta([c]) = m^s_i \cdot \eta_{\widehat{\mathscr{N}}^s_{1,\nu^s_i}}(\widehat{\mu}^s_i([c]))$ for any closed curve $c \subset N(\widehat{W}^s_i)$;

3) for any $j < i$ and any leaf $\mathscr{D}$ of the foliation $\widehat{\mathscr{F}}^s_{1,\nu^s_i,X^s_i}$, the set $\widehat{\mu}^s_j(N(\widehat{W}^s_j) \cap (\widehat{\mu}^s_i)^{-1}(\mathscr{D}))$ is either empty or is a subset of a leaf of the foliation $\widehat{\mathscr{F}}^s_{1,\nu^s_j,X^s_j}$.

We note that an s-lamination is a lamination in the sense of the classical definition.[9] The following proposition is a direct corollary of Statement 1.1 and Theorem 3.2.

Proposition 3.1. *The set $\widehat{L}^s_f$ in the scheme S_f of any diffeomorphism $f \in \mathrm{MS}(M^3)$ is an s-lamination.*

Using the canonical description of an s-lamination, we can introduce a surgery operation on the manifold $\widehat{V}$ along an s-lamination $\widehat{L}^s$. This operation generalizes the surgery along a torus and exhibits a subtle property of the embedding of the lamination $\widehat{L}^s_f$ in the manifold $\widehat{V}_f$. To this end, we consider the canonical expansion $a^u_{1,\nu} = a_{1,\nu}\big|_{W^u_O}$. Its orbit space $\widehat{\mathscr{W}}^u_{1,\nu} = (W^u_O \setminus O)/a^u_{1,\nu}$ is a pair of knots for $\nu = +1$ or a single knot for $\nu = -1$. The set $\widehat{\mathscr{N}}^u_{1,\nu} = (\mathscr{N}^u_1)/a_{1,\nu}$ is a tubular neighbourhood of the manifold $\widehat{\mathscr{W}}^u_{1,\nu}$, where $\mathscr{N}^u_1 = \mathscr{N}_1 \setminus W^s_O$. The natural projection $p_{\widehat{\mathscr{N}}^u_{1,\nu}} \colon \mathscr{N}^u_1 \to \widehat{\mathscr{N}}^u_{1,\nu}$ is a covering that induces a map $\eta_{\widehat{\mathscr{N}}^u_{1,\nu}}$ made up of non-trivial homomorphisms

[9] Given an n-dimensional manifold X ($n \geqslant 2$) and a subset $Y \subset X$, assume that Y is a union $\bigcup_{j\in J} L_j$ of disjoint m-dimensional ($1 \leqslant m \leqslant n-1$) connected manifolds L_j (leaves). The family $\mathscr{L} = \{L_j, j \in J\}$ is called an m-dimensional lamination with support $Y = \operatorname{supp} \mathscr{L}$ if, for any point $x \in Y$, there exist a neighbourhood $U_x \subset X$ and a homeomorphism $\psi \colon U_x \to \mathbb{R}^n$ such that any connected component of the intersection $U_x \cap L_j$ (provided it is non-empty) is mapped by ψ into an m-dimensional hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{m+1} = c_{m+1}, \dots, x_n = c_n\}$.

into $\mathbb{Z}$ on the fundamental group of each connected component of the manifold $\widehat{\mathscr{N}}^u_{1,\nu}$. Let $\widehat{\mathscr{G}}^s_{1,\nu}$ denote the foliation of $\widehat{\mathscr{N}}^s_{1,\nu}$ whose leaves are the projections by $p_{\widehat{\mathscr{N}}^u_{1,\nu}}$ of the leaves of the foliation $\mathscr{F}^s_1$ (see Fig. 21). We consider the diffeomorphism $\zeta_{1,\nu}\colon \widehat{\mathscr{N}}^s_{1,\nu}\setminus\widehat{\mathscr{W}}^s_{1,\nu}\to\widehat{\mathscr{N}}^u_{1,\nu}\setminus\widehat{\mathscr{W}}^u_{1,\nu}$ defined by $\zeta_{1,\nu}=p_{\widehat{\mathscr{N}}^u_{1,\nu}}(p_{\widehat{\mathscr{N}}^s_{1,\nu}}\big|_{\widehat{\mathscr{N}}^s_{1,\nu}\setminus\widehat{\mathscr{W}}^s_{1,\nu}})^{-1}$.

Let $\widehat{L}^s=\bigcup_{i=1}^{n_s}\widehat{W}^s_i$ be an s-lamination on the manifold $\widehat{V}$. Since $\widehat{W}^s_1$ is a closed surface, the homeomorphism μ^s_1 can be assumed to be a diffeomorphism. The surgery of $\widehat{V}$ along $\widehat{W}^s_1$ by means of the diffeomorphism $\zeta_{\widehat{W}^s_1}=\zeta_{1,\nu}\mu^s_1\big|_{N(\widehat{W}^s_1)\setminus\widehat{W}^s_1}$ will be called *surgery along the first surface of the s-lamination*. For $i=1,\dots,n_s-1$ we set $\check{W}^s_i=p_{\widehat{W}^s_1}(\widehat{W}^s_{i+1}\cup(\widehat{\mathscr{W}}^u_{1,\nu}\cap G^s_i))$, where G^s_i is the union of the leaves $\mathscr{D}$ of $\widehat{\mathscr{G}}^s_{1,\nu}$ such that $p_{\widehat{W}^s_1}(\widehat{L}^s\setminus\widehat{W}^s_1)\cap p_{\widehat{W}^s_1}(\mathscr{D})\neq\varnothing$. Let $\check{L}^s=\bigcup_{i=1}^{n_s-1}\check{W}^s_i$. By construction, the set $\check{L}^s$ is again an s-lamination on the manifold $\widehat{V}_{\widehat{W}^s_1}$, and will be called the *derivative of the s-lamination* $\widehat{L}^s$. The manifold $\widehat{V}_{\widehat{L}^s}$ will be said to be obtained from the manifold $\widehat{V}$ by *surgery along the s-lamination* $\widehat{L}^s$ if it is obtained from $\widehat{V}$ by successive surgeries along the first surfaces of the derivatives of the laminations. Arguing just as in the case of surgeries along a torus, we obtain the following result.

Proposition 3.2. *For any diffeomorphism $f\in\mathrm{MS}(M^3)$, any connected component of the manifold $\widehat{V}_{\widehat{L}^s}$ is diffeomorphic to $\mathbb{S}^2\times\mathbb{S}^1$.*

Similarly, we define a u-lamination on the manifold $\widehat{V}$ and consider the manifold $\widehat{V}_{\widehat{L}^u}$ obtained by surgery of $\widehat{V}$ along the u-lamination $\widehat{L}^u$. Moreover, the statements analogous to Propositions 3.1 and 3.2 can be proved. It turns out that the necessary conditions in these propositions provide a sufficient condition for singling out the set $\mathscr{S}$ of abstract schemes.

Definition 3.6. A tuple $S=(\widehat{V},\eta,\widehat{L}^s,\widehat{L}^u)$ is called an *abstract scheme* if:

1) $\widehat{V}$ is a closed 3-manifold whose fundamental group admits an epimorphism $\eta\colon\pi_1(\widehat{V})\to\mathbb{Z}$;

2) $\widehat{L}^s$ and $\widehat{L}^u$ are transversal s- and u-laminations on $\widehat{V}$, respectively;

3) any connected component of the manifold obtained by surgery of $\widehat{V}$ along the s-lamination $\widehat{L}^s$ (the u-lamination $\widehat{L}^u$) is diffeomorphic to $\mathbb{S}^2\times\mathbb{S}^1$.

Theorem 3.3. *For any abstract scheme $S\in\mathscr{S}$ there exists a diffeomorphism $f\in\mathrm{MS}(M^3)$ whose scheme is equivalent to S.*

The existence of an epimorphism η on $\widehat{V}$ leads to the existence, first, of a smooth connected non-compact 3-manifold V without boundary that covers the space $\widehat{V}$, and second, of a diffeomorphism $f_V\colon V\to V$ which is a positive generator of the covering transformation group[10] $G(V,\ p\colon V\to\widehat{V},\ \widehat{V})$. The rest of the proof of Theorem 3.3 proceeds as for the realization of Pixton diffeomorphisms: we successively glue n_s hyperbolic saddle orbits with Morse index 1 and n_u hyperbolic saddle orbits with Morse index 1 into the manifold V. Using property 3) of the abstract scheme, we can compactify the resulting manifold by adding finitely many

[10] By definition, the *covering transformation group* $G(\bar{X},p,X)$ of a covering $p\colon\bar{X}\to X$ is the group of all homeomorphisms $\bar{h}\colon\bar{X}\to\bar{X}$ for which $p\bar{h}=p$.

hyperbolic sink and source orbits, the number of which is equal to the number of connected components of the manifolds $\widehat{V}_{\widehat{L}^s}$ and $\widehat{V}_{\widehat{L}^u}$, respectively.

4. Interrelation between dynamics and topology of the ambient manifold

In this section we present some relationships between the topology of the ambient manifold M^3 and the number of saddle and node periodic points of a diffeomorphism $f \in \mathrm{MS}(M^3)$.

4.1. Classification of the 3-manifolds admitting Morse–Smale diffeomorphisms without heteroclinic curves. Let $\mathrm{MS}_*(M^3)$ denote the class of Morse–Smale diffeomorphisms without heteroclinic curves on 3-manifolds. The following is a classification theorem for phase spaces of diffeomorphisms in this class.

Theorem 4.1. *Let $f \in \mathrm{MS}_*(M^3)$ be a diffeomorphism for which Ω_f consists of r_f saddle points and l_f node points. Then $g_f = (r_f - l_f + 2)/2$ is a non-negative integer, and the following assertions hold*:

1) *if $g_f = 0$, then M^3 is a 3-sphere*;

2) *if $g_f > 0$, then M^3 is a connected sum of g_f copies of $\mathbb{S}^2 \times \mathbb{S}^1$.*

Conversely, for any non-negative integers r, l, g such that $g = (r - l + 2)/2$ is a non-negative integer, there exists a diffeomorphism $f \in \mathrm{MS}_*(M^3)$ with the following properties:

a) *if $g = 0$, then M^3 is a 3-sphere, and if $g > 0$, then M^3 is a connected sum of g copies of $\mathbb{S}^2 \times \mathbb{S}^1$*;

b) *the non-wandering set of a diffeomorphism f consists of r saddle points and l node points.*

Theorem 4.1 has the following immediate corollary.

Corollary 4.1. *If the ambient 3-manifold of a Morse–Smale diffeomorphism f is different from both $\mathbb{S}^3$ and a connected sum of finitely many copies of $\mathbb{S}^2 \times \mathbb{S}^1$, then the wandering set of f must contain heteroclinic curves.*

We describe the idea of the proof of Theorem 4.1.

The direct assertion. It suffices to deal with diffeomorphisms in the subclass $\mathrm{MS}_{**}(M^3) \subset \mathrm{MS}_*(M^3)$ of diffeomorphisms f for which the non-wandering set consists solely of fixed points such that among them there is at least one saddle point with Morse index 2 and all the separatrices of the saddle points are invariant under f (if $r_f = 0$, then $f \in \mathrm{MS}_*(M^3)$ is a 'source–sink' diffeomorphism and M^3 is homeomorphic to $\mathbb{S}^3$ (see, for example, Theorem 2.2.1 in [34]), and hence $l_f = 2$ and $g_f = 0$, so that the theorem holds; if $r_f \neq 0$, then there exists a non-zero integer q such that $f^q \in \mathrm{MS}_{**}(M^3)$). The proof is by induction on the number $r_f > 0$ of saddle points of $f \in \mathrm{MS}_{**}(M^3)$. So we consider the case $r_f > 0$ and assume that the required assertion has been proved for $r_{f'} < r_f$.

Since a diffeomorphism $f \in \mathrm{MS}_{**}(M^3)$ has no heteroclinic curves, it follows that separatrices of saddle points with distinct Morse indices do not intersect. Hence, because the non-wandering set Ω_f is finite, there is at least one saddle point p_0 in $\Omega_2 \neq \varnothing$ whose two-dimensional unstable separatrix is not involved in heteroclinic intersections. By assertion 3) in Statement 1.1, there exists a sink $\omega \in \Omega_f$ such that

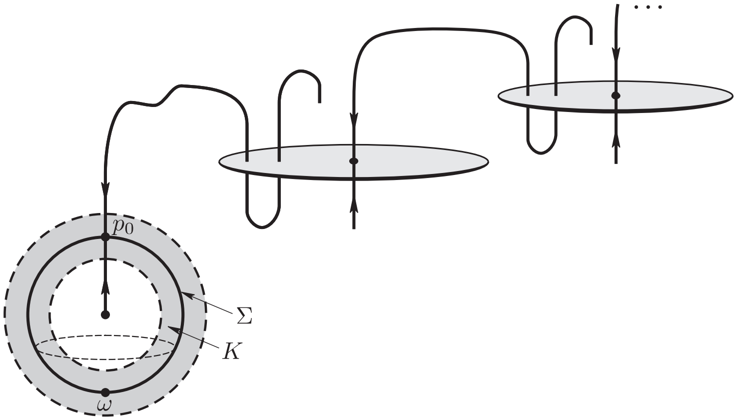


Figure 22. Illustration for the proof of the direct assertion in Theorem 4.1.

the separatrix $W^u_{p_0} \setminus p_0$ is contained in W^s_ω. Let $\Sigma = W^u_{p_0} \cup \omega$. By Statement 1.4, Σ is a sphere topologically embedded in M^3 and is smooth everywhere except possibly at the single point ω. By Lemma 2.1, there exists a neighbourhood K of Σ which is diffeomorphic to $\mathbb{S}^2 \times [0,1]$ (see Fig. 22). Then Σ is an attractor, and we can assume without loss of generality that $f(K) \subset \operatorname{int} K$.

Removing the domain $\operatorname{int} K$ from the manifold M^3, we obtain a compact manifold with two boundary components S_1 and S_2. Let M_1 be the compact manifold without boundary that is obtained from $M^3 \setminus \operatorname{int} K$ by attaching two closed 3-balls B_1 and B_2 to its boundary. One can easily construct a Morse–Smale diffeomorphism $f_1 \colon M_1 \to M_1$ such that f_1 agrees with f on $M^3 \setminus f^{-1}(K)$, has exactly two attracting fixed points $\omega_1 \in B_1$ and $\omega_2 \in B_2$, and has no other periodic points in $B_1 \cup B_2$. Then f_1 has the same number of fixed points as f, the number of its saddle fixed points is $r_{f_1} = r_f - 1$, and the number of sinks and sources is $l_{f_1} = l_f + 1$. We consider two cases: a) $M^3 \setminus K$ is disconnected, and b) $M^3 \setminus K$ is connected.

In case a) the manifold M_1 is a disjoint union of two manifolds $\widetilde{M}_1$ and $\check{M}_1$, and M^3 is a connected sum $\widetilde{M}_1 \# \check{M}_1$. Let $\tilde{f}_1$ and $\check{f}_1$ denote the restrictions of f_1 to the respective manifolds $\widetilde{M}_1$ and $\check{M}_1$. We have $r_{\tilde{f}_1} < r_f$ and $r_{\check{f}_1} < r_f$, so the induction hypothesis ensures that $\widetilde{M}_1$ and $\check{M}_1$ are connected sums of $g_{\tilde{f}_1} = (r_{\tilde{f}_1} - l_{\tilde{f}_1} + 2)/2$ and $g_{\check{f}_1} = (r_{\check{f}_1} - l_{\check{f}_1} + 2)/2$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$, respectively (by a manifold consisting of 0 copies of $\mathbb{S}^2 \times \mathbb{S}^1$ we mean $\mathbb{S}^3$). As a corollary, M^3 is a connected sum of

$$\frac{r_{\tilde{f}_1} - l_{\tilde{f}_1} + 2}{2} + \frac{r_{\check{f}_1} - l_{\check{f}_1} + 2}{2} = \frac{r_{f_1} - l_{f_1} + 4}{2} = \frac{r_f - l_f + 2}{2}$$

copies of $\mathbb{S}^2 \times \mathbb{S}^1$. This proves the theorem in case a).

In case b) the manifold M_1 is connected, and hence $M^3 = M_1 \# M_*$, where M_* is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ (see, for example, [43]). This is illustrated in Fig. 23, which schematically shows the manifold M^3 containing a sphere Σ not separating it, with a tubular neighbourhood K bounded by the spheres S_1 and S_2. Then there exists a solid cylinder H intersecting each sphere S_1 and S_2 precisely in a single

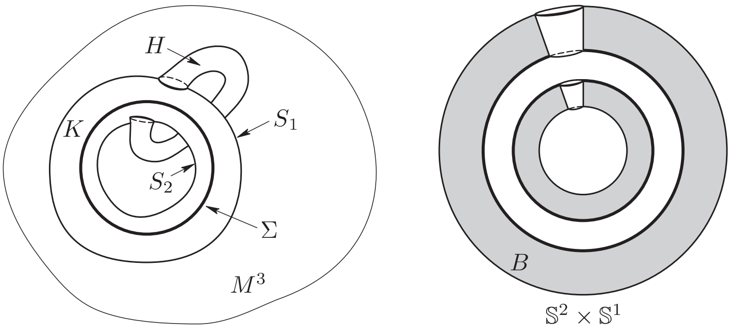


Figure 23. Illustration for case b).

two-dimensional disk. In the manifold M_1 the balls B_1 and B_2 are attached to the spheres S_1 and S_2, and the union $B_1 \cup H \cup B_2$ is a 3-ball in M_1. Another 3-ball B (shaded) is contained in $\mathbb{S}^2 \times \mathbb{S}^1$. Also, the manifold $\mathbb{S}^2 \times \mathbb{S}^1 \setminus B$ is homeomorphic to $K \cup H$. Then the connected sum $M_1 \# (\mathbb{S}^2 \times \mathbb{S}^1)$ obtained using these 3-balls is homeomorphic to M^3.

As before, we denote by r_{f_1} the number of saddles and by l_{f_1} the number of sinks and sources of the diffeomorphism f_1. We have $r_{f_1} = r_f - 1$, so by the induction hypothesis M_1 is either $\mathbb{S}^3$ if $(r_{f_1} - l_{f_1} + 2)/2 = 0$ or a connected sum of $(r_{f_1} - l_{f_1} + 2)/2$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Consequently, M^3 is a connected sum of $(r_{f_1} - l_{f_1} + 2)/2 + 1 = (r_f - l_f + 2)/2$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$, and thus the conclusion of the theorem also holds in this case.

The converse assertion. In this part of the proof of Theorem 4.1, for any non-negative integers r, l, g such that $g = (r - l + 2)/2$ is a non-negative integer, we construct a gradient-like vector field X on M^3 with the following properties:

a) if $g = 0$, then M^3 is a 3-sphere, and if $g > 0$, then M^3 is a connected sum of g copies of $\mathbb{S}^2 \times \mathbb{S}^1$;

b) the non-wandering set of the flow X consists of r saddle points and l node equilibrium states.

In this case the required diffeomorphism will be the time-one map of the flow generated by the resulting vector field.

a) In case M^3 is a sphere, we have $l = r + 2$. On the 3-ball B let X_0 denote a vector field that is transversal to S, directed outwards, has a unique source in the interior, and does not have closed trajectories. Also, on the compact 3-ball B let X_1 be a Morse–Smale vector field that is transversal to the boundary $S = \partial B$ and has precisely one sink, r sources, r saddles with two-dimensional unstable manifolds, and no closed trajectories. Gluing together two copies of B along the boundaries, one copy with the field X_0 and the other with the field X_1, we obtain the 3-sphere $\mathbb{S}^3$ equipped with a Morse–Smale vector field without heteroclinic intersections and closed trajectories and having precisely l sources and sinks and r saddles (see Fig. 24, where the flow X is constructed for $r = 4$).

b) In the case when M^3 is a connected sum of $g > 0$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$, M^3 is obtained by gluing together two copies of handlebodies B_g of genus g by means of a diffeomorphism of its boundary $S_g = \partial B_g$ that is isotopic to the identity map

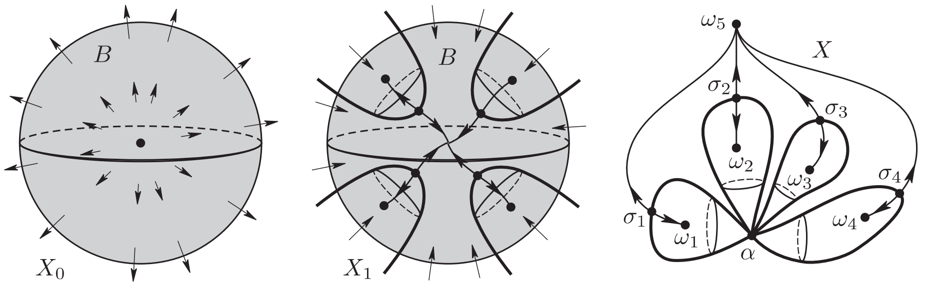


Figure 24. Construction of the vector field X on the sphere $\mathbb{S}^3$.

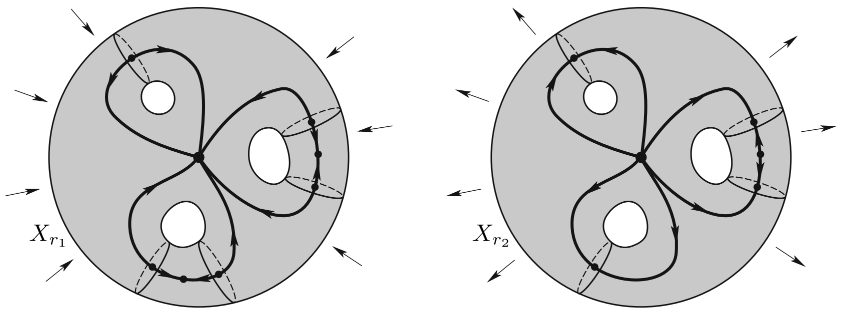


Figure 25. Construction of the vector field X on a connected sum of $g > 0$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

(see, for example, [23]). Let r be written as the sum $r = r_1 + r_2$, where $r_j \geqslant g$, $j = 1, 2$. For $j = 1, 2$ we construct the vector fields X_{r_1} and X_{r_2} on B_g as in Fig. 25 (in which X_{r_1} and X_{r_2} are shown for $g = 3$, $r_1 = 5$, $r_2 = 4$).

Gluing together two copies of the handlebody B_g along the boundaries, one copy with the field X_{r_1} and the other with the field X_{r_2}, we obtain a connected sum of g copies of $\mathbb{S}^2 \times \mathbb{S}^1$ equipped with a Morse–Smale vector field without heteroclinic intersections and closed trajectories and having precisely l sources and sinks and r saddles.

4.2. Heegaard splitting of the ambient 3-manifold of a gradient-like diffeomorphism. Let $\mathrm{MS}_0(M^n)$ denote the subclass of gradient-like diffeomorphisms in $\mathrm{MS}(M^n)$. By Statement 1.4, the closure $\mathrm{cl}\,\ell$ of any one-dimensional unstable separatrix ℓ of a saddle point σ of a diffeomorphism $f \in \mathrm{MS}_0(M^n)$ is homeomorphic to the closed interval consisting of this separatrix and two points: σ and some sink ω. Let L_ω be the union of the unstable one-dimensional separatrices of saddle points that contain ω in their closures. Since W^s_ω is homeomorphic to $\mathbb{R}^n$ (see assertion 2) in Statement 1.1) and the set $F_\omega = L_\omega \cup \omega$ is a union of simple arcs with a unique common point ω, we call F_ω a *frame of one-dimensional unstable separatrices*, in analogy with a frame of arcs in $\mathbb{R}^n$, and we make the following definition.

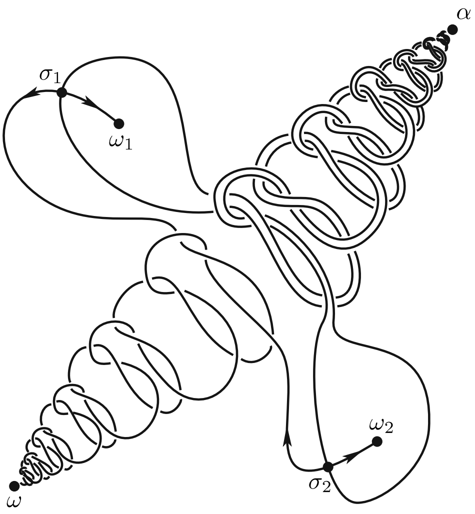


Figure 26. Gradient-like diffeomorphism on $\mathbb{S}^3$ with mildly wild frame of separatrices.

Definition 4.1. A frame F_ω of separatrices is said to be *tame* if there exists a homeomorphism $\psi_\omega\colon W^s_\omega \to \mathbb{R}^n$ such that $\psi_\omega(F_\omega)$ is a frame of rays in $\mathbb{R}^n$. Otherwise, the frame of separatrices is said to be *wild*.

If α is a source for a diffeomorphism f, then we define a tame (wild) frame F_α of one-dimensional stable separatrices similarly. Figure 26 shows the phase portrait of a diffeomorphism on $\mathbb{S}^3$. Debrunner and Fox proved in [20] that the frame of separatrices F_ω is wild in this case, and they referred to such frames of $n > 1$ arcs as mildly wild, because removing any arc from the frame makes it tame.

The main result in this subsection is the following.

Theorem 4.2. *If all the frames of one-dimensional separatrices of a diffeomorphism $f \in \mathrm{MS}_0(M^3)$ are tame, then the ambient manifold M^3 admits a Heegaard splitting of genus g_f.*

The idea behind the proof is as follows. Since the one-dimensional frames of separatrices of f are tame, for the attractor $A_f = W^u_{\Omega_0\cup\Omega_1}$ we can construct a connected trapping neighbourhood M_{A_f} which is a smooth handlebody of some genus g_{A_f} and is composed of sets B and C, where B is a union of $|\Omega_0|$ (here $|X|$ is the cardinality of a set X) three-dimensional balls containing sinks, and C is a union of $|\Omega_1|$ three-dimensional balls containing saddles of index 1. Also, any connected component of C intersects B in precisely two two-dimensional disks belonging to the intersection $\partial B \cap \partial C$, and $M_{A_f} \setminus A_f$ is diffeomorphic to the manifold $S_{g_{A_f}} \times (0, 1]$, where $S_{g_{A_f}}$ is an orientable surface of genus g_{A_f} (see Fig. 27).

By construction, among the connected components of C there are g_{A_f} balls whose removal from M_{A_f} results in a connected manifold. Further removal of the remaining $|\Omega_1|-g_{A_f}$ balls gives the manifold B. It follows that $1+|\Omega_1|-g_{A_f} = |\Omega_0|$, and therefore $g_{A_f} = 1 + |\Omega_1| - |\Omega_0|$.

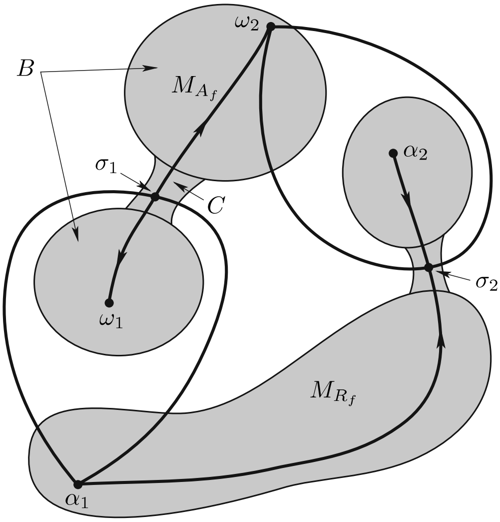


Figure 27. Trapping neighbourhoods of a one-dimensional attractor and a one-dimensional repeller.

Similarly, by passing to the diffeomorphism f^{-1}, we construct a neighbourhood M_{R_f} of the repeller R_f that is a handlebody of genus $g_{R_f} = 1 + |\Omega_2| - |\Omega_3|$ and for which $M_{R_f} \setminus R_f$ is diffeomorphic to $S_{g_{R_f}} \times (0, 1]$, where $S_{g_{R_f}}$ is an orientable surface of genus g_{R_f}. The set A_f is an attractor, and hence there is a number $n_0 \in \mathbb{N}$ such that $f^{n_0}(M_{A_f}) \subset \operatorname{int} M_{A_f}$. Then the space $K = M_{A_f} \setminus \operatorname{int} f^{n_0}(M_{A_f})$ is diffeomorphic to $S_{g_{A_f}} \times [0, 1]$ (see Theorem 3.3 in [33]). By construction, K is a fundamental domain for the restriction of the diffeomorphism f^{n_0} to V_f. Hence, V_f is diffeomorphic to $S_{g_{A_f}} \times \mathbb{R}$. A similar argument for the diffeomorphism f^{-1} shows that the characteristic manifold V_f is diffeomorphic to $S_{g_{R_f}} \times \mathbb{R}$. Hence, $g_{A_f} = g_{R_f}$. Setting $g = g_{A_f} = g_{R_f}$, we see that

$$\begin{aligned} 2g = g_{A_f} + g_{R_f} &= 1 + |\Omega_1| - |\Omega_0| + 1 + |\Omega_2| - |\Omega_3| \\ &= 2 + |\Omega_1 + \Omega_2| - |\Omega_0 + \Omega_3| = 2 + r_f - l_f = 2g_f, \end{aligned}$$

and thus $g = g_f$.

Consider a natural number $n_1 \in \mathbb{N}$ such that $f^{n_1}(M_{A_f}) \cap M_{R_f} = \varnothing$. Then the manifold $K = M^3 \setminus (f^{n_1}(M_{A_f}) \cup M_{R_f})$ is diffeomorphic to $S_{g_f} \times [0, 1]$. Since the manifolds $f^{n_1}(M_{A_f})$ and M_{R_f} are handlebodies of genus g_f, it follows that $M^3 = f^{n_1}(M_{A_f}) \cup (M_{R_f} \cup K)$ is a splitting of genus g_f of the manifold M^3.

5. Existence of an energy function

The non-wandering set of a diffeomorphism $f \in \mathrm{MS}(M^n)$ is finite, and it is therefore natural to seek a Lyapunov function for it in the class of Morse functions. This leads us to the following definition.

Definition 5.1. A Morse function $\psi\colon M^n \to \mathbb{R}$ is called a *Lyapunov function* for $f \in \mathrm{MS}(M^n)$ if:

1) $\psi(f(x)) < \psi(x)$ for any $x \notin \Omega_f$;

2) $\psi(f(x)) = \psi(x)$ for any $x \in \Omega_f$.

The following assertion ([61], Proposition) shows that the dynamics of a diffeomorphism $f \in \mathrm{MS}(M^n)$ is closely related to properties of the Lyapunov function.

Statement 5.1. *Let $\psi\colon M^n \to \mathbb{R}$ be a Lyapunov function for a diffeomorphism $f \in \mathrm{MS}(M^n)$. Then:*

1) *$-\psi$ is a smooth Lyapunov function for f^{-1};*

2) *if p is a periodic point of f, then $\psi(x) < \psi(p)$ for any $x \in W^u_p \setminus p$ and $\psi(x) > \psi(p)$ for any $x \in W^s_p \setminus p$;*

3) *if p is a periodic point of f, then p is a critical point of ψ;*

4) *the index of a critical point*[11] *p is equal to* $\dim W^u_p$.

By Statement 5.1, any periodic point p is a maximum (respectively, minimum) of the restriction of the Lyapunov function ψ to W^u_p (W^s_p). Also, if an extremum is non-degenerate, then the invariant manifold is transversal to all regular level sets of ψ in some neighbourhood of p. This local property is useful for constructing a (global) Lyapunov function.

Definition 5.2. A Lyapunov function $\psi\colon M^n \to \mathbb{R}$ for a diffeomorphism $f \in \mathrm{MS}(M^n)$ is called a *Morse–Lyapunov function* if any periodic point p is a non-degenerate maximum (respectively, minimum) of the restriction of ψ to the unstable (stable) manifold W^u_p (W^s_p).

Definition 5.3. A Morse–Lyapunov function $\psi\colon M^n \to \mathbb{R}$ for a diffeomorphism $f \in \mathrm{MS}(M^n)$ is called an *energy function* if the set of its critical points coincides with the non-wandering set Ω_f.

The hyperbolicity of any periodic orbit $\mathscr{O}$ of a diffeomorphism $f \in \mathrm{MS}(M^n)$ implies the existence of a Morse–Lyapunov function in some neighbourhood of this orbit (see, for example, Lemma 2.2.1 in [34]). A global Morse–Lyapunov function for f can be constructed using the suspension trick with subsequent use of the results in [44] on the existence of an energy function for a Morse–Smale flow. By constructing 'Pixton's example', Pixton [61] showed that for $n = 3$ the so-constructed function is not an energy function in general, thereby disproving the conjecture of Shub [67] and Takens [72] that there is an energy function for any Morse–Smale diffeomorphism.

We give an argument due to Pixton (see Fig. 28 for clarification). For this we assume that a Pixton diffeomorphism f has an energy function ψ. Then by assertion 2) in Statement 5.1, $\max\{\psi(\omega_1), \psi(\omega_2)\} < \psi(\sigma) < \psi(\alpha)$. By assertion 4) in the same statement together with the Morse lemma, any level curve $\psi^{-1}(c)$ with

[11] By definition, the *index of a critical point* p of a Morse function $\psi\colon M^n \to \mathbb{R}$ is defined as the number of negative eigenvalues of the Hessian matrix $\dfrac{\partial^2\psi}{\partial x_i\,\partial x_j}(p)$. By the *Morse lemma* (see, for example, [46]), in some neighbourhood $V(p)$ of a non-degenerate critical point p of ψ one can choose local coordinates $x_1, \dots, x_n$ ($x_j(p) = 0$ for $j = 1, \dots, n$) called *Morse coordinates* in which ψ has the form $\psi(x) = \psi(p) - x_1^2 - \dots - x_q^2 + x_{q+1}^2 + \dots + x_n^2$, where q is the index of ψ at p.

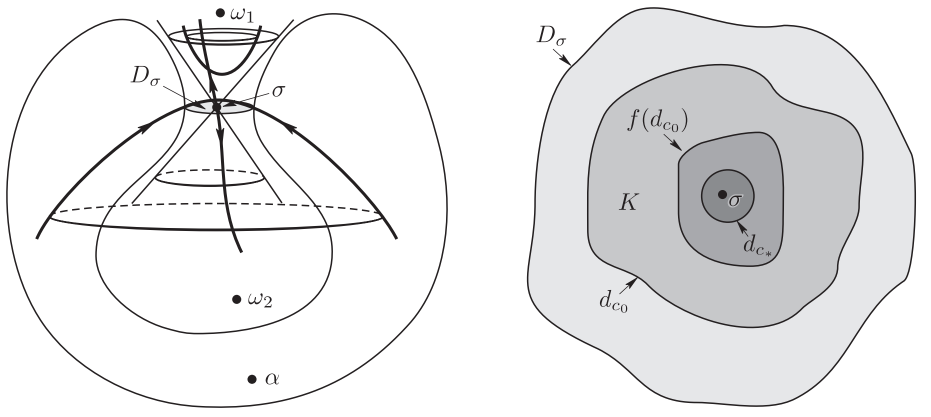


Figure 28. Pixton's arguments.

$c \in (\psi(\sigma), \psi(\alpha))$ is a two-dimensional sphere. Since σ is a non-degenerate minimum of the function $\psi\big|_{W^s_\sigma}$, there exist a two-dimensional disk $D_\sigma \subset W^s_\sigma$ containing σ and a value $c_0 \in (\psi(\sigma), \psi(\alpha))$ such that the set $d_c = \{x \in D_\sigma \colon \psi(x) \leqslant c\}$ is a two-dimensional disk and $\psi^{-1}(c) \cap D_\sigma = \partial d_c$ for any $c \in (\psi(\sigma), c_0)$. By property 1) of the definition of a Lyapunov function, $f(d_{c_0}) \subset \operatorname{int} d_{c_0}$, and hence $K = d_{c_0} \setminus \operatorname{int} f(d_{c_0})$ is a fundamental domain of the restriction of f to $W^s_\sigma \setminus \sigma$. Let us choose $c_* \in (\psi(\sigma), c_0)$ such that $d_{c_*} \subset \operatorname{int} f(d_{c_0})$. Then $\psi(x) > c_*$ for any $x \in K$, and further, by property 1) of the definition of a Lyapunov function, $\psi(x) > c_*$ for any $x \in f^{-k}(K)$, $k \in \mathbb{N}$. Hence, the sphere $\psi^{-1}(c_*)$ intersects W^s_σ in a single circle, contradicting Statement 1.5.

5.1. Quasi-energy function. As we have already observed above, diffeomorphisms of the Pixton class $\mathscr{P}$ fail in general to have an energy function. In this connection, we give the following definition.

Definition 5.4. A Morse–Lyapunov function $\psi \colon M^n \to \mathbb{R}$ is a *quasi-energy function* for a Morse–Smale diffeomorphism $f \colon M^n \to M^n$ if it has the least number of critical points among all Morse–Lyapunov functions for f.

For a Pixton diffeomorphism, the reason for the absence of an energy function is the absence of some f^{-1}-compressible 3-ball B ($f^{-1}(B) \subset \operatorname{int} B$) in W^u_α whose boundary intersects W^s_σ in a single circle. On the other hand, Pixton's example contains a solid torus with the above properties and whose complement is also a solid torus (see Fig. 29). We let $\mathscr{P}_1$ denote the set of diffeomorphisms $f \in \mathscr{P}$ for which there exists an f^{-1}-compressible solid torus $Q^- \subset W^u_\alpha$ whose boundary intersects W^s_σ in a single circle and is a Heegaard surface for $\mathbb{S}^3$. Let $Q^+ = \mathbb{S}^3 \setminus \operatorname{int} Q^-$.

Theorem 5.1. *Each quasi-energy function for a diffeomorphism $f \in \mathscr{P}_1$ has precisely six critical points.*

We describe a scheme for constructing a quasi-energy function for a diffeomorphism $f \in \mathscr{P}_1$.

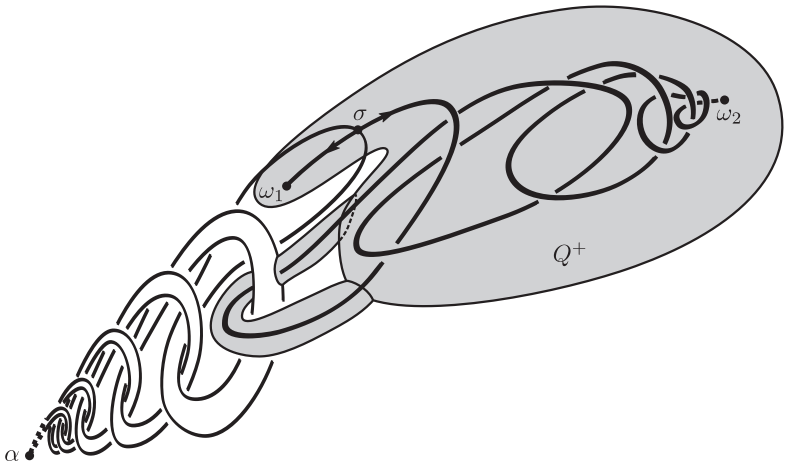


Figure 29. A diffeomorphism in the class $\mathscr{P}_1$.

1. We construct an energy function $\psi_p\colon U_p \to \mathbb{R}$ in a neighbourhood of each fixed point p of f in such a way that $\psi_p(p) = \dim W^u_p$.

2. From the definition of the class $\mathscr{P}_1$ it follows that the set Q^+ is an f-compressible solid torus such that the intersection $W^s_\sigma \cap Q^+$ is a single 2-disk D_σ. We choose a neighbourhood $N(D^+)$ of D_σ such that $Q^+ \setminus N(D_\sigma) = P_{\omega_1} \cup P_{\omega_2}$, where ∂P_{ω_i}, $i = 1, 2$, is a handlebody of genus $i-1$ for which $\omega_i \in f(P_{\omega_i}) \subset \operatorname{int} P_{\omega_i} \subset W^s_{\omega_i}$ and ∂P_{ω_i} is transversal to the regular part of the critical level $C = \psi_\sigma^{-1}(1)$ of ψ_σ. Consequently, there exists an $\varepsilon \in (0, 1/2)$ such that ∂P_{ω_i} intersects each level set of ψ_σ with value in the interval $[1-\varepsilon, 1+\varepsilon]$ transversally in a single circle (see Fig. 30).

For each $i = 1, 2$, we consider a handlebody $\widetilde{P}_{\omega_i}$ of genus $i-1$ with the following properties:

a) $f(P_{\omega_i}) \subset \widetilde{P}_{\omega_i} \subset \operatorname{int} P_{\omega_i}$;

b) $\partial\widetilde{P}_{\omega_i}$ intersects any level set of ψ_σ with value in the interval $[1-\varepsilon, 1+\varepsilon]$ transversally in a single circle;

c) $P_{\omega_i} \setminus \operatorname{int} \widetilde{P}_{\omega_i}$ is diffeomorphic to $\partial P_{\omega_i} \times [-\varepsilon, \varepsilon]$ (below we identify these manifolds so that $P_{\omega_i} = \partial P_{\omega_i} \times \{\varepsilon\}$).

3. For $t \in [-\varepsilon, \varepsilon]$ we set (see Fig. 30)

$$P_{i,t} = \widetilde{P}_{\omega_i} \cup \partial P_{\omega_i} \times [-\varepsilon, t], \qquad H_t = \{x \in U_\sigma\colon \psi_\sigma(x) \leqslant 1 + t\},$$
$$Q_t = P_{1,t} \cup P_{2,t} \cup H_t, \qquad E_\varepsilon = (Q_\varepsilon \setminus \operatorname{int} Q_{-\varepsilon}) \cap (H_\varepsilon \setminus \operatorname{int} H_{-\varepsilon}).$$

There is no loss of generality in assuming that ε is so small that, for any $t \in [-\varepsilon, \varepsilon]$, the surface $\partial P_{i,t}$, $i = 1, 2$, intersects the sets ∂H_t transversally in a single two-dimensional disk and that $f^{-1}(E_\varepsilon) \cap H_\varepsilon = \varnothing$ (this is possible because if ψ_σ is a Lyapunov function for $f\big|_{U_\sigma}$, then $\psi_\sigma(f^{-1}(\psi_\sigma^{-1}(1) \setminus \sigma)) > 1$, and hence $(H_0 \setminus \sigma) \subset \operatorname{int} f^{-1}(H_0 \setminus \sigma))$. Then on the set $K = Q_\varepsilon \setminus \operatorname{int} Q_{-\varepsilon}$ the relation $\psi_K(x) = 1 + t_x$, $x \in Q_{t_x}$, defines an energy function for f.

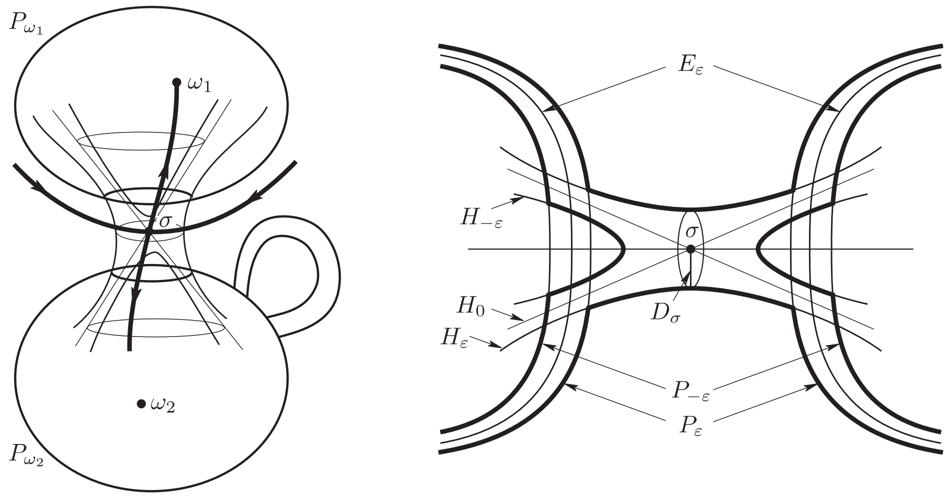


Figure 30. Construction of a quasi-energy function for a diffeomorphism $f \in \mathscr{P}_1$.

4. Since $P_{1,-\varepsilon}$ is a 3-ball such that $\omega_1 \in f(P_{1,-\varepsilon}) \subset \operatorname{int} P_{1,-\varepsilon} \subset W^s_{\omega_1}$, there exists an energy function $\psi_{P_{1,-\varepsilon}}\colon P_{1,-\varepsilon} \to \mathbb{R}$ of f which takes the value $1-\varepsilon$ on $\partial P_{1,-\varepsilon}$ and agrees with ψ_{ω_1} in some neighbourhood of the sink ω_1. That such a function exists follows from the fact that we can replace some already constructed level curve S of the energy function in a manifold diffeomorphic to $S \times \mathbb{R}$ by any incompressible surface S'. Technically this is achieved by successive modifications of the surface S that reduce the number of connected components in the intersection $S' \cap (\bigcup_{k\in\mathbb{Z}} f^k(S))$, $k \in \mathbb{Z}$. The theoretical justification of this process is based on the annulus conjecture when S is a 2-sphere, and on Waldhausen's theorem when S is a surface of positive genus.[12]

Since $P_{2,-\varepsilon}$ is a solid torus and $\omega_2 \in f(P_{2,-\varepsilon}) \subset \operatorname{int} P_{2,-\varepsilon} \subset W^s_{\omega_2}$, there exists a 3-ball B_{ω_2} such that $f(P_{2,-\varepsilon}) \subset B_{\omega_2} \subset \operatorname{int} P_{2,-\varepsilon}$. Such a ball can be constructed as follows. We choose a meridian disk D of the torus $P_{2,-\varepsilon}$ such that $\omega_2 \notin D$. Without loss of generality it can be assumed that D is transversal to $G = \bigcup_{k\in\mathbb{Z}} f^k(\partial P_{2,-\varepsilon})$ and that the intersection $D \cap G$ does not contain curves that are homotopic to zero on the torus $f^k(\partial P_{2,-\varepsilon})$. Then there exists a curve $c \subset (D \cap f^k(\partial P_{2,-\varepsilon}))$ bounding

[12] A smooth surface F embedded in a manifold X is said to be *compressible* in X if either of the following two conditions is satisfied:

1) there exist a non-contractible simple closed curve $c \subset \operatorname{int} F$ and a smoothly embedded 2-disk $D \subset \operatorname{int} X$ such that $D \cap F = \partial D = c$;

2) there exists a 3-ball $B \subset \operatorname{int} X$ such that $F = \partial B$.

A surface F is said to be *incompressible* in X if it is not compressible in X.

The annulus conjecture for $n = 3$ (see [47]). *Let S^2_1 and S^2_2 be two disjoint 2-spheres that are tamely embedded in $\mathbb{S}^3$. Then the closure of the domain in $\mathbb{S}^3$ bounded by S^2_1 and S^2_2 is a three-dimensional annulus.*

Waldhausen's theorem (see Proposition 3.1 in [75]). *Let G be an orientable surface (possibly with non-empty boundary ∂G) that is not a 2-sphere. Then for any properly embedded ($\partial X \cap F = \partial F$) incompressible surface F in $G \times [0,1]$ satisfying $\partial F \subset G \times \{1\}$ there exists a surface $F_1 \subset G \times \{1\}$ which is homeomorphic to F and such that $\partial F = \partial F_1$ and $F \cup F_1$ bounds a domain Δ in $G \times [0,1]$ whose closure is homeomorphic to $F \times [0,1]$.*

a 2-disk e_c in D whose interior does not contain any curve in $D\cap G$. Two cases are possible: (a) $e_c\subset f^k(P_{2,-\varepsilon})$ and (b) $\operatorname{int} e_c\cap f^k(P_{2,-\varepsilon})=\varnothing$.

In case (a) the disk e_c is a meridian disk in $f^k(P_{2,-\varepsilon})$, and hence $d=f^{-k}(e_c)$ is a meridian disk in $P_{2,-\varepsilon}$ such that $f(P_{2,-\varepsilon})\cap d=\varnothing$. Indeed, $\operatorname{int} e_c\cap G=\varnothing$ by construction, and thus $\operatorname{int} d\cap G=\varnothing$. Consequently, we can find a required 3-ball B_{ω_2} inside the open 3-ball $\operatorname{int} P_{2,-\varepsilon}\setminus d$.

In case (b) there exists a tubular neighbourhood $V(e_c)\subset \operatorname{int} P_{2,-\varepsilon}$ of the disk e_c such that $G\cap \operatorname{int} V(e_c)=\varnothing$ and $B_k=f^k(P_{2,-\varepsilon})\cup V(e_c)$ is a 3-ball. Hence, $f^k(P_{2,-\varepsilon})\subset B_k\subset \operatorname{int} f^{k-1}(P_{2,-\varepsilon})$. Then $B_{\omega_2}=f^{1-k}(B_k)$ is the required 3-ball.

As above, there exists an energy function $\psi_{B_{\omega_2}}\colon B_{\omega_2}\to\mathbb{R}$ for f that assumes the value $1/2$ on ∂B_{ω_2} and agrees with ψ_{ω_2} in some neighbourhood of the sink ω_2.

5. It is known that a solid torus can be obtained from a 3-ball by identifying a pair of disjoint 2-disks on its boundary. There exists a unique (up to isotopy) 3-ball interior to a solid torus, and hence on the manifold $R=P_{2,-\varepsilon}\setminus \operatorname{int} B_{\omega_2}$ there exists a Morse function ψ_R having precisely one critical point of index 1 and such that $\psi_R(\partial B_{\omega_2})=1/2$ and $\psi_R(\partial P_{2,-\varepsilon})=1-\varepsilon$.

6. Consider the smooth function $\psi^+\colon Q_\varepsilon\to\mathbb{R}$ defined by

$$\psi^+(x)=\begin{cases}\psi_K(x), & x\in K,\\ \psi_{P_{1,-\varepsilon}}(x), & x\in P_{1,-\varepsilon},\\ \psi_{B_{\omega_2}}(x), & x\in B_{\omega_2},\\ \psi_R(x), & x\in R.\end{cases}$$

Then ψ^+ is a Morse–Lyapunov function for $f\big|_{Q_\varepsilon}$ with one additional critical point.

7. By construction, Q^-_ε is a solid torus such that $\alpha\in f^{-1}(Q^-_\varepsilon)\subset \operatorname{int} Q^-_\varepsilon\subset W^u_\alpha$. Since α is a sink for f^{-1}, it can be checked that, as in the previous item, there exists a Morse–Lyapunov function $\psi_{Q^-_\varepsilon}$ for f^{-1} with precisely one critical point of index 1 and such that $\psi_{Q^-_\varepsilon}(\partial Q^-_\varepsilon)=2-\varepsilon$ and $\psi_{Q^-_\varepsilon}(\alpha)=3$.

Consider the smooth function $\psi^-\colon Q^-_\varepsilon\to\mathbb{R}$ defined by $\psi^-(x)=3-\psi_{Q^-_\varepsilon}(x)$. Then ψ^- is a Morse–Lyapunov function for $f\big|_{Q^-_\varepsilon}$ with one additional critical point.

Finally, the function $\psi\colon \mathbb{S}^3\to\mathbb{R}$ with $\psi\big|_{Q^+_\varepsilon}=\psi^+$ and $\psi\big|_{Q^-_\varepsilon}=\psi^-$ is the required quasi-energy function for the diffeomorphism f.

5.2. Self-indexing energy function.

Definition 5.5. An energy function ψ for a diffeomorphism $f\in \mathrm{MS}(M^n)$ is said to be *self-indexing* if $\psi(p)=\dim W^u_p$ for any point $p\in \mathrm{Per}_f$.[13]

It is easily verified that a Morse–Smale diffeomorphism $f\colon M^3\to M^3$ with a self-indexing energy function is gradient-like. Indeed, assuming that f is not gradient-like and has a self-indexing energy function $\psi\colon M^3\to\mathbb{R}$, we can find points $x,y\in\mathrm{Per}_f$ $(x\neq y)$ such that $W^u_x\cap W^s_y\neq\varnothing$ and $\dim W^s_x\geqslant \dim W^s_y$. Let $\dim W^u_x=k$, $\dim W^u_y=m$, and $z\in W^u_x\cap W^s_y$. Since $n-k=\dim W^s_x\geqslant \dim W^s_y=n-m$, we have $k\leqslant m$. By Statement 5.1, $\varphi(z)<\varphi(x)=\dim W^u_x=k$ and $\varphi(z)>\varphi(y)=\dim W^u_y=m$, so $k>m$, a contradiction.

[13] A function with similar properties was constructed by Smale [69] for gradient-like vector fields.

The next theorem gives necessary and sufficient conditions for the existence of a self-indexing energy function in terms of a special Heegaard splitting of M^3. We shall also need the following definition.

Definition 5.6. Let S_g be a closed orientable two-dimensional surface of genus g. A subset D of M^3 is called an *(f, S_g)-compressible product* if there exists a diffeomorphism $q\colon S_g \times [0,1] \to D$ such that $q^{-1}(S_g \times \{t\})$ bounds an f-compressible handlebody for any $t \in [0,1]$.

Theorem 5.2. *A gradient-like diffeomorphism $f\colon M^3 \to M^3$ has a self-indexing energy function if and only if M^3 can be represented as a union of three sets with disjoint interiors, $M^3 = P^+ \cup N \cup P^-$, where:*

1) P^+ and P^- are handlebodies of genus g_f such that $A_f \subset f(P^+) \subset \operatorname{int} P^+$ and $R_f \subset P^- \subset \operatorname{int} f^{-1}(P^-)$;

2) for any saddle point $\sigma_1 \in \Omega_1$ (respectively, $\sigma_2 \in \Omega_2$) the intersection $W^s_{\sigma_1} \cap P^+$ ($W^u_{\sigma_2} \cap P^-$) is a single two-dimensional closed disk;

3) N is an (f, S_{g_f})-compressible product.

5.3. Dynamically ordered energy function. In this subsection we introduce for diffeomorphisms in $\mathrm{MS}(M^n)$ the concept of a dynamically ordered energy function, which is closely related to the dynamics of a diffeomorphism, and we investigate conditions for its existence. A key role here is played by the characteristic manifolds V_i introduced in § 3.2.

Definition 5.7. Let $\mathscr{O}_1, \dots, \mathscr{O}_{k_f}$ be a dynamical ordering of the orbits of a diffeomorphism $f \in \mathrm{MS}(M^n)$. A Morse–Lyapunov function ψ for f is said to be *dynamically ordered* if $\psi(\mathscr{O}_i) = i$ for $i \in \{1, \dots, k_f\}$.

To get a better understanding of the three-dimensional case it will be instructive to comment on Pixton's result [61] on the existence of an energy function for any Morse–Smale diffeomorphism on a surface. The construction is based on the fact that for any cascade $f \in \mathrm{MS}(M^2)$ each connected component of the characteristic manifold V_i, $i = 1, \dots, k_f - 1$, is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and, for $i = k_0 + 1, \dots, k_1$, it contains a non-contractible circle that intersects any stable separatrix of a saddle point of the orbit $\mathscr{O}_i$ in at most one point. This makes it possible to reduce the construction of a global energy function to an inductive (with respect to i) process of confluence of local Morse–Lyapunov functions. In view of the above, we give the following definition.

Definition 5.8. Let $f \in \mathrm{MS}(M^3)$. A two-dimensional stable (unstable) manifold of a saddle orbit $\mathscr{O}_i$, $i \in \{k_0 + 1, \dots, k_1\}$ ($i \in \{k_1 + 1, \dots, k_2\}$), is said to be *simply embedded* if each connected component of the manifold V_i (respectively, V_{i-1}) contains an incompressible closed orientable surface that intersects W^s_i (W^u_i) in at most one closed curve.

Using Pixton's arguments, one can show that for the existence of a dynamically ordered energy function ψ for a diffeomorphism $f \in \mathrm{MS}(M^3)$ it is necessary that two-dimensional manifolds of saddle orbits admit a simple embedding. Indeed, in this case there exists an $\varepsilon > 0$ such that the connected components of the level curve $\psi^{-1}(i+\varepsilon)$ (respectively, $\psi^{-1}(i+1-\varepsilon)$) are surfaces satisfying the conditions of Definition 5.8.

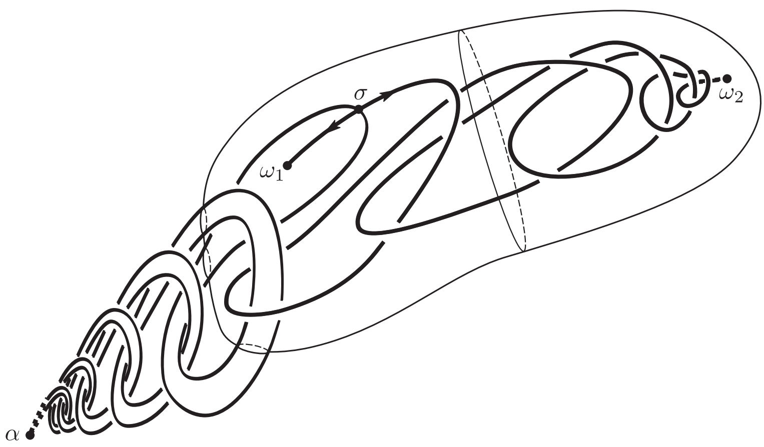


Figure 31. Improperly embedded two-dimensional manifold of a saddle point.

As an example of a non-simple embedding of a two-dimensional manifold of a saddle point we can take Pixton's example with $\mathscr{O}_1 = \omega_1$, $\mathscr{O}_2 = \omega_2$, $\mathscr{O}_3 = \sigma$, and $\mathscr{O}_4 = \alpha$. Here the characteristic space V_3 coincides with the manifold $W^u_\alpha \setminus \alpha$, and hence is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. By Waldhausen's theorem, any incompressible surface in $\mathbb{S}^2 \times \mathbb{R}$ is a sphere. On the other hand, any such sphere in V_3 intersects W^s_σ in more than one connected component (see Fig. 31), and hence W^s_σ is not simply embedded.

Theorem 5.3. *Assume that any characteristic manifold of a diffeomorphism $f \in \mathrm{MS}(M^3)$ is homeomorphic to the direct product of a closed orientable surface and a line, and that the two-dimensional manifold of any saddle orbit is simply embedded. Then f admits a dynamically ordered energy function.*

Under the conditions of Theorem 5.3 the situation is similar to the two-dimensional case, and the construction is in essence reduced by induction on i to the confluence of local energy functions, as in § 5.1 (see Fig. 30).

The next result extends Theorem 4.2 to any diffeomorphism $f \in \mathrm{MS}(M^3)$.

Proposition 5.1. *If a diffeomorphism $f \in \mathrm{MS}(M^3)$ has a dynamically ordered energy function ψ, then the ambient manifold M^3 admits a Heegaard splitting of genus g_f.*[14]

We note that the requirement that the characteristic manifold be homeomorphic to a direct product is not necessary for the existence of an energy function. Accordingly, Fig. 32 shows a handlebody P^+ of genus 1 consisting of a 3-ball B^+ and a 1-handle C^+. On P^+ there is a diffeomorphism F^+ onto its image for which the non-wandering set consists of two hyperbolic fixed points: a sink ω and a saddle σ^+ with local stable manifold Δ^+. Here the ball B^+ is mapped onto the ball $\widetilde{B}^+$, and the handle C^+ onto a tubular neighbourhood of the arc L^+. One can show that $W^+ = P^+ \setminus \operatorname{int} F^+(P^+)$ is not homeomorphic to $\mathbb{T}^2 \times [0,1]$. Considering that the knots on L^+ are symmetric with respect to the midsphere of the

[14]The level set $\psi^{-1}(k_1 + 1/2)$ is a Heegaard surface of genus g_f.

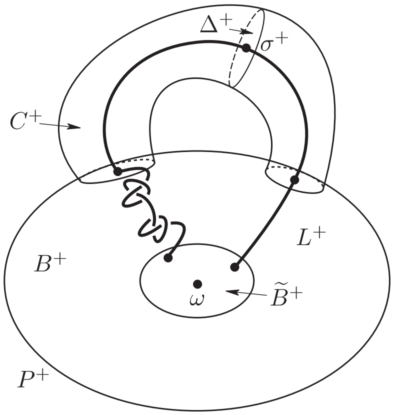


Figure 32. Construction of a diffeomorphism with characteristic manifold that is not a direct product.

annulus $B^+ \setminus \operatorname{int} B^+$, we can glue P^+ together with a copy P^- of it on which the diffeomorphism $F^- = (F^+)^{-1}$ is defined in such a way that the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is obtained with a gradient-like diffeomorphism f on it. In this case one of the characteristic spaces of f is homeomorphic to the space W^+/F^+, and hence the corresponding characteristic manifold is not a direct producct. We note that for the so-constructed diffeomorphism, the two-dimensional manifolds of saddle points are simply embedded in the corresponding characteristic manifolds. Moreover, it is not difficult to construct an energy function for this diffeomorphism.

In the class $\mathrm{MS}_*(\mathbb{S}^3)$ of Morse–Smale diffeomorphisms with no heteroclinic curves on the sphere $\mathbb{S}^3$, it is possible to prove, using Theorem 4.1, that the condition of simple embedding of the two-dimensional manifolds of saddle orbits for a diffeomorphism $f \in \mathrm{MS}_*(\mathbb{S}^3)$ implies that any connected component of any characteristic manifold of f is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. We therefore arrive at the following criterion.

Theorem 5.4. *A Morse–Smale diffeomorphism $f\colon \mathbb{S}^3 \to \mathbb{S}^3$ without heteroclinic curves has a dynamically ordered energy function if and only if, for any $i \in \{k_0 + 1, \dots, k_1\}$ (respectively, $i \in \{k_1+1, \dots, k_2\}$), each connected component of the manifold V_i (V_{i-1}) contains an incompressible two-dimensional sphere that intersects W_i^s (W_i^u) in at most one closed curve.*

In particular, it follows from this theorem that the diffeomorphism $f\colon \mathbb{S}^3 \to \mathbb{S}^3$ whose phase portrait is shown in Fig. 26 has a dynamically ordered energy function. Also, the frame of one-dimensional separatrices of f that contain a sink ω in their closures is not tame, but is rather a mildly wild Debrunner–Fox frame.

6. Embedding in a topological flow

The problem of embeddability of a Morse–Smale cascade in a topological flow goes back to Palis [52], who gave the following necessary conditions for a diffeomorphism $f \in \mathrm{MS}(M^n)$ to be embeddable in a flow X^t:

1) the non-wandering set Ω_f coincides with the set Fix_f of fixed points;
2) the restriction of f to any invariant manifold of any fixed point $p \in \Omega_f$ preserves its orientation;
3) if the intersection $W^s_p \cap W^u_q$ is non-empty for distinct saddle points $p, q \in \Omega_f$, then it does not contain compact connected components.

We explain an easy proof of the necessity of these conditions. For this we assume that f is the time-one map of the flow X^t. If Ω_f contains a periodic point of period $m > 0$, then this point lies in a closed orbit γ of X^t. Hence all the points of this orbit are periodic points of period m for f. However, this is impossible, because Ω_f is finite, and thus condition 1) holds. Condition 2) follows from the equality $\Omega_f = \Omega_{X^t}$ of the non-wandering sets and the equality of the invariant manifolds of the fixed points for the flow and the diffeomorphism. The latter means that the restriction of f to any invariant manifold of any fixed point $p \in \Omega_f$ is isotopic to the identity map, and hence preserves its orientation. Condition 3) holds because each connected component of the intersection $W^s_p \cap W^u_q$ for distinct saddle points $p, q \in \Omega_f$ is a union of trajectories of the flow X^t. In particular, condition 3) implies that any Morse–Smale diffeomorphism embeddable in a flow is gradient-like.

In what follows, conditions 1)–3) will be called the *Palis conditions*. In [52] it is also shown that for $n = 2$ these conditions are sufficient. The proof is based on the fact that the frames of separatrices of a diffeomorphism $f \in \mathrm{MS}_0(M^2)$ have an even stronger embeddability property than tameness (see Definition 4.1). We describe this property in the next definition.

Definition 6.1. Let ω be a sink fixed point for a diffeomorphism $f \in \mathrm{MS}_0(M^n)$. A frame F_ω of one-dimensional unstable separatrices is said to be *trivially embedded* if there exists a homeomorphism $\psi_\omega \colon W^s_\omega \to \mathbb{R}^n$ such that $f\big|_{W^s_\omega} = \psi_\omega^{-1} a^s_{n,+1} \psi_\omega\big|_{W^s_\omega}$ and $\psi_\omega(F_\alpha)$ is a frame of rectilinear rays.

In a similar fashion we define a trivially embedded frame F_α of one-dimensional stable separatrices for a source point α.

The triviality of an embedding of any frame F_ω on a surface is explained by the fact that the orbit space $(W^s_\omega \setminus \omega)/f$ is homeomorphic to the two-dimensional torus, and the projection of F_ω to it is a union of pairwise disjoint non-contractible circles. The topology of the torus makes it possible to carry this union into a family of curves of the form $\{x\} \times \mathbb{S}^1$ (see, for example, [65]). This enables us to embed the diffeomorphism $f\big|_{W^s_\omega}$ in a linear flow in such a way that the separatrices of the frame F_ω become trajectories of the flow. By Statement 1.1, $M^2 = \Omega_1 \cup \bigl(\bigcup_{\omega \in \Omega_0} W^s_\omega\bigr) \cup \bigl(\bigcup_{\alpha \in \Omega_2} F_\alpha\bigr)$. Consequently, the flow on basins of sinks can be modified so that it extends to trivial frames of stable separatrices, thereby proving the Palis result.

In dimension 3 there are non-trivial embeddings of frames of one-dimensional separatrices. As we have already seen, such an effect is observed for a Pixton diffeomorphism (see Fig. 4) and a Debrunner–Fox diffeomorphism (see Fig. 26). However, as will follow from Lemma 6.1 below, the triviality of all frames of one-dimensional separatrices is a necessary condition for embeddability of a diffeomorphism $f \in \mathrm{MS}(M^3)$ in a flow. We note that from results of Kuperberg [38] it follows that a wild arc can be a trajectory of some topological flow on a 3-manifold.

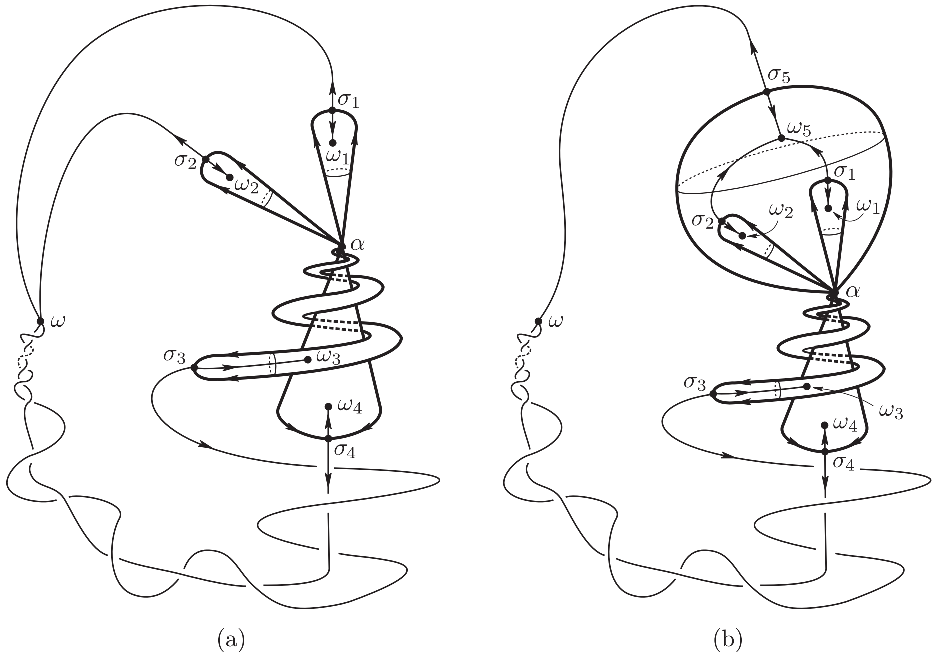


Figure 33. Phase portraits of $\mathrm{MS}\,(S^3)$-diffeomorphisms that do not embed in a topological flow: a) a diffeomorphism for which all frames of one-dimensional separatrices are tame, but the frame F_ω is non-trivial; b) a diffeomorphism for which all frames of one-dimensional separatrices are trivial.

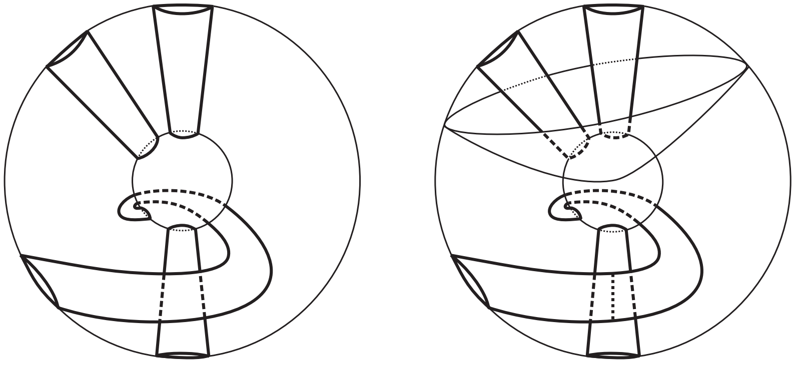
Lemma 6.1. *Assume that a diffeomorphism $f \in \mathrm{MS}(M^3)$ embeds in a topological flow. Then all the frames of its one-dimensional separatrices are trivial.*

A surprising fact here is that augmenting the Palis list by the condition that all frames of one-dimensional separatrices of saddle points of an $f \in \mathrm{MS}(M^3)$ be trivial does not lead to sufficient conditions for embeddability of it in a topological flow. An example to illustrate this fact is given in Fig. 33, which shows the phase portraits of diffeomorphisms that do not embed in any topological flow.

In effect, the key to the Palis problem of embeddability of a Morse–Smale diffeomorphism $f\colon M^3 \to M^3$ in a flow is the scheme of the diffeomorphism f (see Definition 3.1) and the number $g_f = (|\Omega_1 \cup \Omega_2| - |\Omega_0 \cup \Omega_3| + 2)/2$ introduced in § 4.

Let $\mathbb{S}_{g_f}$ denote a closed orientable surface of genus g_f, and let $\widehat{\mathbb{V}}_{g_f} = \mathbb{S}_{g_f} \times \mathbb{S}^1$. The set $\widehat{\lambda} = c_{\widehat{\lambda}} \times \mathbb{S}^1$, where $c_{\widehat{\lambda}}$ is a simple smooth closed curve on $\mathbb{S}_{g_f}$, will be called a *trivial torus* on the manifold $\widehat{\mathbb{V}}_{g_f}$.

Definition 6.2. The scheme S_f of a diffeomorphism $f \in \mathrm{MS}(M^3)$ is said to be *trivial* if there exists a homeomorphism $\widehat{\psi}_f\colon \widehat{V}_f \to \widehat{\mathbb{V}}_{g_f}$ such that every connected component of the sets $\widehat{\psi}_f(\widehat{L}^s_f)$ and $\widehat{\psi}_f(\widehat{L}^u_f)$ is a trivial torus on the manifold $\widehat{\mathbb{V}}_{g_f}$.



(a) (b)

Figure 34. Schemes of diffeomorphisms whose phase portraits are given in Fig. 33.

Figure 34 depicts the schemes of diffeomorphisms whose phase portraits are given in Fig. 33. It is readily checked that both these schemes are non-trivial.

The main result in this subsection is the following theorem.

Theorem 6.1. *A diffeomorphism* $f \in \mathrm{MS}(M^3)$ *embeds in a topological flow if and only if its scheme is trivial.*

Lemma 6.1 and the necessity of the conditions in Theorem 6.1 are proved in the same way. For definiteness, we show how the embeddability of a diffeomorphism $f \in \mathrm{MS}(M^3)$ in a topological flow X^t implies that the scheme S_f is trivial.

We let Y^t denote the restriction of X^t to the set V_f. By the construction of V_f it follows that $\lim_{t\to+\infty} Y^t(x) \in A_f$ and $\lim_{t\to-\infty} Y^t(x) \in R_f$ for any point $x \in V_f$. Thus, for any points $p, q \in V_f$ there exist neighbourhoods $U_p, U_q \subset V_f$ and a constant $T > 0$ such that $Y^t(U_p) \cap U_q = \varnothing$ for any t with $|t| > T$. According to the definition on p. 545 of [22], this means that the flow Y^t is *dispersive*. Then it follows from Theorem 3 in [22] that Y^t is a *parallelizable* flow, that is, there exist a set $\Sigma_f \subset V_f$ and a homeomorphism $\xi_f \colon V_f \to \Sigma_f \times \mathbb{R}$ such that $\bigcup_{t\in\mathbb{R}} Y^t(\Sigma_f) = V_f$ and $\xi_f(Y^t(z)) = (z, t)$ for any $z \in \Sigma_f$ and $t \in \mathbb{R}$.

From [36] it follows that the topological dimension of Σ_f is equal to two. Indeed, $\dim V_f \leqslant \dim \Sigma_f + \dim \mathbb{R}$ by virtue of Theorem III.4 in [36], and hence $\dim \Sigma_f \geqslant 2$. We have $\Sigma_f \subset V_f$, so $\dim \Sigma_f \leqslant 3$. Assuming that $\dim \Sigma_f = 3$, it would follow from Theorem IV.3 in [36] that Σ_f contains an open 3-ball U. But then $\xi_f^{-1}(U \times \mathbb{R})$ would be a four-dimensional subset of the three-dimensional manifold V_f, a contradiction. Thus, $\dim \Sigma_f = 2$. Hence, according to [76], Σ_f is a manifold without boundary (in Theorem 2 in [76] it is shown that if the Cartesian product $A \times B$ of topological spaces A and B is an n-manifold and if $\dim A = 1$ or 2, then A is a manifold, and it does not have a boundary if $A \times B$ does not have a boundary). Consequently, Σ_f is a closed orientable surface. Let ρ_f denote the genus of this surface. We assert that $\rho_f = g_f$.

By construction, the surface Σ_f splits the manifold into two parts. We denote their closures by P_{A_f} and P_{R_f} and assume that $A_f \subset \operatorname{int} P_{A_f}$ and $R_f \subset \operatorname{int} P_{R_f}$.

Since P_{A_f} is a 3-manifold with boundary Σ_f, we have $\chi(\Sigma_f) = 2\chi(P_{A_f})$ (see, for example, Corollary 8.7 in [21]). Further, $\chi(\Sigma_f) = 2 - 2\rho_f$, and hence $\chi(P_{A_f}) = 1 - \rho_f$.

On the other hand, f is isotopic to the identity map, and therefore, by the Lefschetz formula, the Euler characteristic $\chi(P_{A_f})$ equals the sum of the indices of the fixed points $p \in \mathrm{Fix}_f$, where the index of p is $(-1)^{\dim W^u_p}$. Therefore, $\chi(A_f) = |\Omega_0| - |\Omega_1|$, and thus $|\Omega_0| - |\Omega_1| = 1 - \rho_f$. Applying similar arguments to the attractor, we then get that $|\Omega_3| - |\Omega_2| = 1 - \rho_f$. Adding the last two equalities, we have $|\Omega_0| - |\Omega_1| + |\Omega_3| - |\Omega_2| = 2 - 2\rho_f$, giving $\rho_f = (|\Omega_1 \cup \Omega_2| - |\Omega_0 \cup \Omega_3| + 2)/2$, and so $\rho_f = g_f$.

Since every two-dimensional separatrix λ of f is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and is a union of trajectories of the flow Y^t, there exists a simple closed curve $\gamma_\lambda \subset \Sigma_f$ such that $\xi_f(\lambda) = \gamma_\lambda \times \mathbb{R}$. Also, there exists a homeomorphism $h_f \colon \Sigma_f \to \mathbb{S}_{g_f}$ such that $c_\lambda = h_f(\gamma_\lambda)$ is a simple smooth closed curve for any two-dimensional separatrix λ. We define the flow $A^t_{g_f}$ on the manifold $\mathbb{V}_{g_f} = \mathbb{S}_{g_f} \times \mathbb{R}$ by the formula $A^t_{g_f}(s,r) = (s, r+t)$, and we define the homeomorphism $\psi_f \colon V_f \to \mathbb{V}_{g_f}$ by $\psi_f(Y^t(z)) = A^t_{g_f}(h_f(z))$, $z \in \Sigma_f$, $t \in \mathbb{R}$. By construction, ψ_f conjugates the flows Y^t and $A^t_{g_f}$, so it also conjugates their time-one maps. Furthermore, $\psi_f(\lambda) = c_\lambda \times \mathbb{R}$. By construction, $\widehat{\mathbb{V}}_{g_f} = \mathbb{V}_{g_f}/A^1_{g_f}$. We let $p_{g_f} \colon \mathbb{V}_{g_f} \to \widehat{\mathbb{V}}_{g_f}$ denote the natural projection. Then by Statement 1.7 the homeomorphism $\widehat{\psi}_f = p_{g_f}\psi_f p_f^{-1} \colon \widehat{V}_f \to \widehat{\mathbb{V}}_{g_f}$ satisfies the condition in Definition 6.2. Thus, the scheme S_f is trivial.

The sufficiency of the conditions in the theorem can be verified by constructing a gradient-like flow $\widetilde{X}^t$ using the trivial scheme S_f of the diffeomorphism f. Conceptually, this construction generalizes the realization of Pixton diffeomorphisms. The diffeomorphism $\widetilde{f}$ which is the time-one map of the flow $\widetilde{X}^t$ constructed has a scheme $S_{\widetilde{f}}$ which is equivalent to S_f. Hence, by Theorem 3.1 the diffeomorphisms f and $\widetilde{f}$ are topologically conjugate by some homeomorphism $h \colon M^3 \to M^3$ such that $hf = \widetilde{f}h$. It follows that f embeds in the topological flow $X^t = h^{-1}\widetilde{X}^t h$.

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