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Power Index Axiomatics in the Problem of Voting with Quota

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Abstract—An axiomatics of power indices in voting with quota was proposed. It relies on the additivity and dictator axioms. Established was an important property that the player's power index is representable as the sum of contributions of the coalitions in which it is a pivot member. The coalition contributions are independent of the players' weights or the quota. The general theorem of power index representation and the theorem of representation for a power index of anonymous players were formulated and proved.

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1. INTRODUCTION

Measurement of power is an efficient tool for analysis of decision making. The classical methods of power measurement with the use of the Banzhaf and Shapley–Shubik indices are widely used [1, 2]. Axiomatic description of the power indices is of special interest. The existing axiomatics were constructed within the framework of the game-theoretical model of the simple game [3, 4] which is defined by listing the winning coalitions. The power index is considered as a vector function defined on the set of all simple games. The basic problem lies in axiomatic description of the classical power indices.

The present paper proposes a general axiomatics of the power index in the problem of voting with quota. For decision making, this problem is described by defining the player set N, their votes ν_j, $j = 1, \ldots, n$, and the quota q. Under a fixed set N ($|N| = n$), the collection $(\nu_1, \ldots, \nu_n; q)$ is called the **voting situation** [5]. In the model of simple game, different lists of winning coalitions correspond to different situations. The axiomatics is formulated in terms of the voting situations, which makes it sufficiently simple and transparent.

The suggested axiomatics relies on the convention that a player has voting power only if it is a pivot player in a winning coalition, that is, if its exit makes the coalition losing. This convention is reflected in the additivity and dictator axioms which underlie the following fundamental features of the power indices in the problem of voting:

(1) Each winning coalition where the player is a pivotal one makes a certain contribution to its power index independently of the voting situation.

(2) Under an additional convention for anonymity, the contribution of each winning coalition to the power index of any player is independent of the player and the coalition and defined only by the size of this coalition.

These features enable one to formulate and prove the general theorem of representation of the power index and the theorem of representation of the power index of anonymous players. The proposed below axiomatics embraces a wide class of power indices including those of Banzhaf and Shapley–Shubik and the power indices allowing for the players' preferences [5].

2. BASIC DEFINITIONS

The participant of voting is characterized mainly by its weight ν in voting by which usually is meant the number of votes belonging to it such as the number of votes in a parliament fraction or the number of stockowner shares. The present paper considers only voting for one or another decision where each participant may vote only "for" or "against." The decision is regarded as approved if the total weight of the approving voters exceeds a certain quota q ($\sum \nu_i > q$). The two most popular values of q are 50% (simple majority) and 66% or sometimes 75% (qualified majority) [6].

The following notions are used in the definition of the power index:

- Coalition is a set of players.
- Winning coalition is that whose total weight exceeds the quota q.
- Losing coalition is that whose total weight does not exceed the quota q.
- Pivot player in a coalition is that with whom the coalition is winning and becomes losing without it.
- Significant coalition for a player is that for which it is the pivot player.
- Dummy (a term from bridge that was used first in [1]) is the player which is not pivotal in any coalition.

We denote by $\nu(S)$ the total weight of the coalition S

$$\nu(S) = \sum_{i \in S} \nu_i.$$

The Banzhaf and Shapley–Shubik indices are most popular power ones. It is possible to determine for any player a set of coalitions where it is pivotal. The Banzhaf index for the player i obeys the formula

$$\beta_i = \frac{b_i}{\sum\limits_{j=1}^{n} b_j},$$

where b_i is the number of different coalitions where the player i is pivotal and n is the total number of players.

The Shapley–Shubik index for the player i obeys the formula

$$\phi_i = \sum_{S} \frac{(s-1)!(n-s)!}{n!},$$

where summation is carried out over all coalitions S where i is the pivot player and $s = |S|$ is the number of players in the coalition S.

3. GENERAL AXIOMS

The existing approaches to describing the general properties of the power indices are based on the game-theoretical model of the simple game [7] which is defined by the pair (N, u), where N is the set of players and u is the gain function $u : 2^N \to \{0, 1\}$ defining whether any coalition of players S is winning ($u(S) = 1$) or losing ($u(S) = 0$). This function should be monotone

$$\forall S, T \subset N \quad S \subset T \;\Rightarrow\; u(S) \le u(T).$$

The set of all winning coalitions in the game (N, u) is denoted by $W(u)$. The set of all minimal winning coalitions where removal of any player makes them losing is denoted by $M(u)$. The power index is defined as the vector function $\vec{\Phi}(u)$.

The basic axioms of the power indices were formulated in [3], and a new view of the power index axiomatics was presented in [4]. The main axioms of the classical power indices are as follows:

Dummy axiom. In a simple game, the dummy power index is zero.

Anonymity axiom. The equality

$$\Phi_{\pi(i)}(u) = \Phi_i(\pi u), \quad \text{where} \quad \pi u(S) = u(\pi(S))$$

is satisfied for any permutation π of the set N in the simple game (N, u).

Transfer axiom. The equality

$$\Phi(u) + \Phi(\omega) = \Phi(u \vee \omega) + \Phi(u \wedge \omega),$$
$$\textit{where} \quad (u \vee \omega)(S) = \max(u(S),\ \omega(S)), \quad (u \wedge \omega)(S) = \min(u(S),\ \omega(S))$$

is satisfied for any simple games u and ω.

The transfer axiom reflects the transfer of power at merging the lists of the winning coalitions. Various versions of the transfer axiom are discussed in detail in [4].

The problem of voting with quota has its own distinctions within the framework of the game-theoretical model of the simple game. As the following example demonstrates, merging of two lists of winning coalitions corresponding to two different situations of voting may result in a list of winning coalitions which does not correspond to any situation of voting.

Example. Let AB, BC, and ABC be the winning coalitions in game 1 of the players A, B, and C. Such game is voting with quota if one takes, for example, the weights of A, B, and C equal, respectively, to 2, 6, and 2 and quota 7. Let in game 2 the winning coalitions be represented by A, AB, AC, and ABC. This game also is voting with quota if one takes, for example, the weights of A, B, and C equal, respectively, to 6, 2, and 2 and quota 5. The game with the winning coalitions A, AB, AC, BC, and ABC is the union of these games. Yet such voting does not exist because for that the weight of A must exceed the quota. Yet then BC cannot be a winning coalition; otherwise, if A votes "for" and B and C, "against," then the result of voting will be ambiguous.

Therefore, it is only natural to describe the general characteristics of the power indices of the problem of voting with quota in terms of the voting situations. The present paper proposes to use of the dictator and additivity axioms to describe the power indices in the problem of voting with quota. The additivity axiom was formulated in terms of the winning coalitions where the given participant of voting is pivotal (significant coalitions) and is an analog of the general transfer axiom in the model of simple game. To establish the general properties of the power indices from the two basic axioms, we consider the structure of the set of significant coalitions for the player in the problem of voting with quota (Theorems 1 and 2). The result is represented by a general theorem about representation of the power index (Theorem 6). The anonymity axiom added to the two basic axioms simplifies considerably the representation of the power index (Theorem 10).

3.1. Relativity of the Power Indices

The power index is a relative value, that is, not the absolute magnitudes of the power indices but their relations are of importance. Stated differently, the power of players in the two following situations is the same: (1) three players A, B, and C have the power indices Φ_A, Φ_B, and Φ_C; (2) these three players A, B, and C have the power indices $k\Phi_A$, $k\Phi_B$, and $k\Phi_C$.

Definition. Two power indices Φ and Ψ are ***equivalent*** if and only if for any voting situation $(\nu_1, \ldots, \nu_n;\, q)$ both give identical shares of power to the players (at that, the absolute magnitudes of the indices may differ):

$$\forall i = \overline{1, n} \quad \frac{\Phi_i}{\sum\limits_{j=1}^{n} \Phi_j} = \frac{\Psi_i}{\sum\limits_{j=1}^{n} \Psi_j}.$$

3.2. Unambiguity of Voting

Voting unambiguity. If the coalition T is winning and its subcoalition $S \subset T$ is winning as well, then the coalition $T \backslash S$ must be losing:

$$\forall S, T \;\; S \subset T, \quad \nu(S) > q, \quad \nu(T) > q \Rightarrow \boldsymbol{\nu}(T \backslash S) \leq q.$$

This property formulates superadditivity of the simple games in terms the problem of voting. It implies that the votes of the players of any winning coalition define unambiguously the result of voting: if they vote "for," the decision passes, if "against," the decision is rejected.

Example. Let three players A, B, and C have the respective weights 10, 10, and 10. Then, the quota $q = 9$ does not provide unambiguity of voting because in this case $T = ABC$ is a winning coalition and its subcoalition $S = AB$ is winning as well but the coalition $T \backslash S = C$ is again winning. The quota $q = 11$ makes voting unambiguous.

If a quota makes more than 50% of the sum of weights of all n players, that is,

$$q > \frac{1}{2} \sum_{i=1}^{n} \nu_i,$$

then voting is unambiguous. This example demonstrates ($q = 11$) that this condition, although sufficient, is not necessary for voting unambiguity [8].

We assume in what follows that voting is unambiguous.

3.3. Structure of the Set of Significant Coalitions in the Problem of Voting

Theorem 1. *For any coalition which includes the given player and at least one other player, it is always possible to determine a situation where this coalition is a unique significant coalition for the given player:*

$$\forall S, \, i \in S, \, S \neq \{i\} \quad \exists\, (\nu_1, \ldots, \nu_n; \, q): \; \begin{cases} \nu(S) > q \\ \nu(S) - \nu_i \leq q \end{cases}$$

$$\textit{and} \quad \nexists T \neq S, \, i \in T \begin{cases} \nu(T) > q \\ \nu(T) - \nu_i \leq q. \end{cases}$$

Proof. To prove constructively this statement, we denote by S the coalition which must be a single significant coalition for the player i. Let this coalition have $k \geq 2$ players, the total number of players being n.

We take the player weights and the quota as follows:

$$\forall j \notin S \;\; \nu_j = 1,$$
$$\nu_i = 1,$$
$$\forall j \in S, \;\; j \neq i \quad \nu_j = n - k + 1,$$
$$q = (k-1)(n-k+1) + \frac{1}{2}.$$

Such situation is plausible because it features unambiguity of voting (the quota exceeds one half of the total weight of the players):

$$q = (k-1)(n-k+1) + \frac{1}{2},$$

$$\frac{1}{2}\sum_{j=1}^{n}\nu_j = \frac{1}{2}\left(n-k+1+(k-1)(n-k+1)\right) = \frac{k}{2}(n-k+1),$$

$$q = k(n-k+1) - \left(n-k+\frac{1}{2}\right) = \frac{k}{2}(n-k+1) + \frac{k}{2}(n-k+1) - \left(n-k+\frac{1}{2}\right),$$

$$q = \frac{1}{2}\sum_{j=1}^{n}\nu_j + \left(\frac{k}{2}-1\right)(n-k) + \frac{k-1}{2} > \frac{1}{2}\sum_{j=1}^{n}\nu_j, \quad \text{because} \quad k \geq 2.$$

We prove that in this situation the coalition S is indeed a single coalition significant for the player i. The coalition S is significant for i because

$$\begin{cases} \nu(S) = (k-1)(n-k+1) + 1 > q \\ \nu(S) - \nu_i = (k-1)(n-k+1) < q. \end{cases}$$

Let us consider the rest of the winning coalitions that include the player i and prove that none is significant for it. To prove that any winning coalition T including the player i must include all other players from S, we admit the contrary. Let one of these players be not within T. The maximal possible such coalition is that of all players but one from S. Its weight is as follows:

$$\nu(T) = \sum_{j=1}^{n}\nu_j - (n-k+1) = (k-1)(n-k+1) < q.$$

This weight is smaller than the quota, and such coalition cannot be winning. Consequently, any winning coalition T includes all players from S. Apart from the players from S, the coalition T includes at least one player $j \notin S$. The weights of all such players are equal to 1, and the weight of player i also is 1. Consequently,

$$\nu(T) \geq \nu(S) + 1 \;\Rightarrow\; \nu(T) - \nu_i \geq \nu(S) + 1\; - 1 > q.$$

Therefore, the player i is not pivotal in the coalition T, and S is its unique significant coalition.

Example. Let us construct a situation for voting of eight players A, B, C, D, E, F, G, and H where the player A is pivotal only in the coalition of five players $ABCDE$. For that, one has to assume that the weights of A, F, G, and H are equal to 1, the weights of B, C, D, and E are equal to $8 - 5 + 1 = 4$, and the quota is equal to $4 \times 4 + 0.5 = 16.5$.

Theorem 2. *For any situation where the player is pivotal in a nonempty set of coalitions W, there always exists a coalition $\omega \in W$ such that it is possible to find another situation where this player is pivotal in the same coalitions save ω.*

Proof. Let the player i enter the coalitions $\omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_{k+m}, \omega_{k+m+1}, \ldots, \omega_l$ ordered in weight:

$$\begin{aligned} &\nu(\omega_1) \leq \ldots \leq \nu(\omega_k) \leq q < \nu(\omega_{k+1}) \leq \ldots \leq \nu(\omega_{k+m}) \\ &\leq\; q + \nu_i < \nu(\omega_{k+m+1}) \leq \ldots \leq \nu(\omega_l) \;\Leftrightarrow\; \nu(\omega_1\backslash\{i\}) \leq \ldots \leq \nu(\omega_k\backslash\{i\}) \\ &\leq\; q - \nu_i < \nu(\omega_{k+1}\backslash\{i\}) \leq \ldots \leq \nu(\omega_{k+m}\backslash\{i\}) \\ &\leq\; q < \nu(\omega_{k+m+1}\backslash\{i\}) \leq \ldots \leq \nu(\omega_l\backslash\{i\}). \end{aligned}$$

Here, the coalitions $\omega_1, \ldots, \omega_k$ are losing, the coalitions $\omega_{k+1}, \ldots, \omega_{k+m}$ are significant for the player i, and the coalitions $\omega_{k+m+1}, \ldots, \omega_l$ are winning but not significant for i.

We prove that the quota and player weights may be modified so that the player i remains pivotal in all coalitions except for ω_{k+1}. For that, we assume that the quota is equal to $q' = q + \Delta$ and increase by some small value Δ_0 the weights of all players except for those included in the coalition ω_{k+1}:

$$\forall j \in \omega_{k+1} \quad \nu'_j = \nu_j,$$
$$\forall j \notin \omega_{k+1} \quad \nu'_j = \nu_j + \Delta_0.$$

It is required to obtain the following weights of coalitions:

$$\begin{aligned} &\max\left(\nu'(\omega_1\backslash\{i\}), \ldots, \nu'(\omega_k\backslash\{i\})\right) \le \nu(\omega_{k+1}\backslash\{i\}) \\ &= q' - \nu'_i < \min\left(\nu'(\omega_{k+2}\backslash\{i\}), \ldots, \nu'(\omega_{k+m}\backslash\{i\})\right) \\ &\le \max\left(\nu'(\omega_{k+2}\backslash\{i\}), \ldots, \nu'(\omega_{k+m}\backslash\{i\})\right) \le q' \\ &= q + \Delta < \min\left(\nu'(\omega_{k+m+1}\backslash\{i\}), \ldots, \nu'(\omega_l\backslash\{i\})\right). \end{aligned}$$

Since the weights of each player, except for those included in the coalition ω_{k+1}, were increased by Δ_0, the weight of each coalition is increased by $t\Delta_0$, where t is the number of players in this coalition whose weights were increased, $1 \le t < n$, n being the number of all players. Therefore,

$$\begin{aligned} &\forall j,\, \omega_j \subset \omega_{k+1} \Rightarrow \nu'(\omega_j) = \nu(\omega_j), \\ &\forall j,\, \omega_j \not\subset \omega_{k+1} \Rightarrow \nu(\omega_j) + \Delta_0 \le \nu'(\omega_j) < \nu(\omega_j) + n\Delta_0, \\ &\max\left(\nu'(\omega_1\backslash\{i\}), \ldots, \nu'(\omega_k\backslash\{i\})\right) < \max\left(\nu(\omega_1\backslash\{i\}), \ldots, \nu(\omega_k\backslash\{i\})\right) + n\Delta_0 \\ &= \nu(\omega_k\backslash\{i\}) + n\Delta_0, \\ &\min\left(\nu'(\omega_{k+2}\backslash\{i\}), \ldots, \nu'(\omega_{k+m}\backslash\{i\})\right) \ge \min\left(\nu(\omega_{k+2}\backslash\{i\}), \ldots, \nu(\omega_{k+m}\backslash\{i\})\right) + \Delta_0 \\ &= \nu(\omega_{k+2}\backslash\{i\}) + \Delta_0 > \nu(\omega_{k+1}\backslash\{i\}), \\ &\max\left(\nu'(\omega_{k+2}\backslash\{i\}), \ldots, \nu'(\omega_{k+m}\backslash\{i\})\right) < \max\left(\nu(\omega_{k+2}\backslash\{i\}), \ldots, \nu(\omega_{k+m}\backslash\{i\})\right) + n\Delta_0 \\ &= \nu(\omega_{k+m}\backslash\{i\}) + n\Delta_0 \le q + n\Delta_0, \\ &\min\left(\nu'(\omega_{k+m+1}\backslash\{i\}), \ldots, \nu'(\omega_l\backslash\{i\})\right) \ge \min\left(\nu(\omega_{k+m+1}\backslash\{i\}), \ldots, \nu(\omega_l\backslash\{i\})\right) + \Delta_0 \\ &= \nu(\omega_{k+m+1}\backslash\{i\}) + \Delta_0. \end{aligned}$$

Then, to obtain the desired weights of coalitions, it suffices to satisfy the following system of inequalities:

$$\begin{cases} \nu(\omega_k\backslash\{i\}) + n\Delta_0 \le \nu(\omega_{k+1}\backslash\{i\}) \\ q + n\Delta_0 \le q + \Delta \\ \nu(\omega_{k+m+1}\backslash\{i\}) + \Delta_0 > q + \Delta \end{cases} \Leftrightarrow \begin{cases} \Delta_0 \le \dfrac{\nu(\omega_{k+1}\backslash\{i\}) - \nu(\omega_k\backslash\{i\})}{n} \\ \Delta_0 \le \dfrac{\Delta}{n} \\ \Delta < \nu(\omega_{k+m+1}\backslash\{i\}) - q + \Delta_0\,. \end{cases}$$

Since $\nu(\omega_{k+1}\backslash\{i\}) > \nu(\omega_k\backslash\{i\})$ and $\nu(\omega_{k+m+1}\backslash\{i\}) > q$, this system has many solutions. Here is one of them:

$$\begin{cases} \Delta = \nu(\omega_{k+m+1}\backslash\{i\}) - q \\ \Delta_0 = \min\left(\dfrac{\Delta}{n}, \dfrac{\nu(\omega_{k+1}\backslash\{i\}) - \nu(\omega_k\backslash\{i\})}{n}\right). \end{cases}$$

For this modification of the quota and the player weights, we obtain what was required:

$$\begin{aligned}
&\max\left(\nu'(\omega_1\backslash\{i\}),\ldots,\nu'(\omega_k\backslash\{i\})\right) \le \nu(\omega_{k+1}\backslash\{i\}) = q' - \nu'_i\\
&< \min\left(\nu'(\omega_{k+2}\backslash\{i\}),\ldots,\nu'(\omega_{k+m}\backslash\{i\})\right)\\
&\le \max\left(\nu'(\omega_{k+2}\backslash\{i\}),\ldots,\nu'(\omega_{k+m}\backslash\{i\})\right)\\
&\le q' = q + \Delta < \min\left(\nu'(\omega_{k+m+1}\backslash\{i\}),\ldots,\nu'(\omega_l\backslash\{i\})\right)\\
&\Leftrightarrow \max\left(\nu'(\omega_1),\ldots,\nu'(\omega_k)\right) \le \nu(\omega_{k+1})\\
&= q' < \min\left(\nu'(\omega_{k+2}),\ldots,\nu'(\omega_{k+m})\right)\\
&\le \max\left(\nu'(\omega_{k+2}),\ldots,\nu'(\omega_{k+m})\right) \le q' + \nu'_i < \min\left(\nu'(\omega_{k+m+1}),\ldots,\nu'(\omega_l)\right).
\end{aligned}$$

One can easily see that the player i remains pivotal in all coalitions save ω_{k+1}.

3.4. Dictator Axiom

Definition. For the given situation of voting, by the dictator is meant the player whose weight exceeds the quota.

Dictator axiom. If in the situation of voting there is a dictator, then its power is positive, $\Phi > 0$.

3.5. Additivity Axiom

The player actually influences the outcome of voting only if it is pivotal. The more are case where the player is pivotal. the higher must be its power. The following axiom considers the absolute (unnormalized) value of power.

Additivity axiom. If player A is pivotal in situation 1 in some set W^1 of coalitions, in situation 2 A is pivotal in the set W^2 of coalitions, and in situation 3 A is pivotal in the set $W^3 = W^1 \cup W^2$ of coalitions and the sets of coalitions W^1 and W^2 are disjoint ($W^1 \cap W^2 = \varnothing$), then the power of A in situation 3 is equal to the sum of its powers in the two first situations, that is,

$$\Phi^3(A) = \Phi^1(A) + \Phi^2(A).$$

Let us consider an example of the three players A, B, and C with respective weights 34, 33, and 33 and determine the power of player A in three situations differing in the quota q. Let in the first situation $q = 51$, in the second $q = 75$, and in the third $q = 67$. We put down all coalitions where A is the pivot player in the three situation under consideration: (1) AB, AC; (2) ABC; (3) AB, AC, ABC. By the additivity axiom we obtain $\Phi^3(A) = \Phi^1(A) + \Phi^2(A)$. This axiom is satisfied by the unnormalized indices of Banzhaf, $\beta_i = b_i$, and Shapley-Shubik, $\phi_i = \sum\limits_{i\in S}(s-1)!(n-s)!$:

$$\begin{aligned}
&\beta^1(A) = 2, \quad \beta^2(A) = 1, \quad \beta^3(A) = 3 = \beta^1(A) + \beta^2(A),\\
&\phi^1(A) = 2, \quad \phi^2(A) = 2, \quad \phi^3(A) = 4 = \phi^1(A) + \phi^2(A).
\end{aligned}$$

As the above example demonstrates, the power index that depends on the player weight only—$\Phi(A) = \nu_A$, for example,—does not satisfy the additivity axiom. In the example at hand, we obtain for this index that

$$\Phi^1(A) = \Phi^2(A) = \Phi^3(A) = 34.$$

The properties of power index monotonicity, lack of power, equality of powers, and dictatorship, as well as the general representation theorem are derived below from the dictator and additivity axioms.

3.5.1. Property of Lack of Power.

Theorem 3. *If the additivity axiom is satisfied and the player is not pivotal in any coalition, then its power index is* 0.

Proof. We assume the contrary: let the power index of player A be equal to some number $\Phi^1(A) > 0$ in situation 1 where it is not pivotal in any coalition. Let us consider situation 2 where A is pivotal only in one coalition ω and has the power index $\Phi^2(A)$. By Theorem 1 such situation exists.

Then, by the additivity axiom in situation 3 which completely coincides with situation 2 because in situation 1 the set of coalitions where A is pivotal is empty, the power index of A must be equal to the sum $\Phi^3(A) = \Phi^1(A) + \Phi^2(A) > \Phi^2(A)$. Yet since situation 3 coincides completely with situation 2, in these situations the power indices must coincide, $\Phi^3(A) = \Phi^2(A)$, and therefore, the initial assumption is not true and $\Phi^1(A) = 0$.

3.5.2. Property of Dictator.

Theorem 4. *If the dictator and additivity axioms are satisfied, then the dictator has 100% of power in voting, the rest of the players having no power.*

Proof. We denote by V the dictator's weight. Since by definition V is greater than the quota q, by the property of voting unambiguousness the total weight of the rest of the players does exceed q. We consider any such player i and prove that it is not pivotal in any winning coalition, that is,

$$\forall \omega \;\; i \in \omega, \;\; \nu(\omega) > q \quad \Rightarrow \quad \nu(\omega) - \nu_i > q.$$

The dictator must be in any winning coalition because otherwise its weight does not exceed q even if it includes all remaining players. Then,

$$\nu(\omega) \geq \nu_i + V \quad \Rightarrow \quad \nu(\omega) - \nu_i \geq V > q,$$

that is, any player i is not pivotal in any coalition.

By the property of no power (Theorem 3), the power of all players save the dictator is zero. And by the dictator axiom, the dictator has a nonzero power. Therefore, the dictator has all 100% of power in voting.

3.5.3 Property of Equal Power.

Theorem 5. *The dictator and additivity axioms being satisfied, if in two different situations 1 and 2 player A is pivotal in the same set of coalitions, $W^1 = W^2$, then its power index is the same in both situations, $\Phi^1(A) = \Phi^2(A)$.*

Proof. Let us consider situation 0 where A is not pivotal in any coalition. According to the property of no power, its power index is 0: $\Phi^0(A) = 0$. Since the set of coalitions, where A is pivotal, is empty in situation 0, $W^0 = \phi$, in situation 2 the set of coalitions where A is pivotal is in fact equal to $W^2 = W^1 \cup W^0$. By the additivity axiom we obtain $\Phi^2(A) = \Phi^1(A) + \Phi^0(A) = \Phi^1(A)$.

3.5.4. General Theorem of Representation.

Theorem 6. *The dictator and additivity axioms being satisfied, the power index of player A which is pivotal in the coalitions $\omega_1, \ldots, \omega_k$ is situation-independent, depends only on the set of pivotal coalitions, and equals $C_A(\omega_1) + \ldots + C_A(\omega_k)$, where $C_A(\omega) \not\equiv 0$ is the function defining the contribution of the coalition ω to the power index of its pivot player A.*

Proof. According to the property of equal powers in any situation 1 (independently of the player weights and the quota) where A is pivotal only in one coalition ω_i, it has the same power index $\Phi^1(A)$, that is, this index is independent of the player weights and the quota and depends only on the player and the coalition ω_i where it is pivotal. Then, this index is equal to the value of some function $C_A(\omega_i) \geq 0$ which is nonnegative because the power index cannot be negative.

Remark. By Theorem 1, such situation 1 exists for any coalition ω save $\omega = \{A\}$. The case of $\omega = \{A\}$ implies that A is the dictator because its weight exceeds the quota. This case is fully described by the property of dictator (Theorem 4) which shows that the absolute value of the dictator's power is of no importance. It is only important that its power is other than zero. Therefore, if A is a dictator, then one can assume that its power is $C_A(\{A\}) + C_A(\{AB\}) + \ldots + C_A(\{AB\ldots\})$. Since this sum includes the contributions of every possible coalition including A and $C_A(\omega) \not\equiv 0$, this sum is nonzero. Thus, the theorem is satisfied for the case where the player is a dictator.

Let the coalitions $\omega_1, \omega_2, \ldots, \omega_k$ be arranged in the ascending order of weights. By Theorem 2, there exists situation 2 where A is pivotal in the same coalitions save the first coalition: $\omega_2, \ldots, \omega_k$. Then, by the additivity axiom, the power index of A is as follows:

$$\Phi(A) = C_A(\omega_1) + \Phi^2(A),$$

where $\Phi^2(A)$ is the power index of A in the second situation. Now, by Theorem 2 there again exists situation 3 where A is pivotal in the coalitions $\omega_3, \ldots, \omega_k$ and by the additivity axiom

$$\Phi^2(A) = C_A(\omega_2) + \Phi^3(A),$$

and so on until the last coalition for which we obtain

$$\Phi^{k-1}(A) = C_A(\omega_{k-1}) + \Phi^k(A),$$

where $\Phi^k(A)$ is the power index of A in the situation where A is pivotal only in the coalition ω_k. Therefore, $\Phi^k(A) = C_A(\omega_k)$. By adding all terms, we obtain that

$$\Phi(A) = C_A(\omega_1) + C_A(\omega_2) + \ldots + C_A(\omega_k).$$

3.5.5. Property of Monotonicity.

Theorem 7. *The dictator and additivity axioms being satisfied, if the two situations 1 and 2 differ only in that the weight of player A in situation 2 is higher than in situation 1,* $\nu_A^2 > \nu_A^1$, *then its power in the second situation is not less than in the first situation:* $\Phi^2(A) \geq \Phi^1(A)$.

Proof. We denote by W_1 and W_2 the sets of all coalitions where the player A is pivotal, respectively, in situation 1 and situation 2 and prove that $W_1 \subset W_2$. Let us consider some coalition $\omega \in W_1$ where the player A is pivotal in the first situation which means that ω is the winning coalition and $\omega\backslash A$ is the losing coalition.

$$\begin{cases} \sum\limits_{i\in\omega\backslash A} \nu_i < q \\ \sum\limits_{i\in\omega\backslash A} \nu_i + \nu_A^1 \geq q. \end{cases}$$

The player weight in the second situation exceeds

$$\nu_A^2 > \nu_A^1 \quad \Rightarrow \quad \sum_{i\in\omega\backslash A} \nu_i + \nu_A^2 > \sum_{i\in\omega\backslash A} \nu_i + \nu_A^1 \geq q.$$

Consequently,

$$\begin{cases} \sum\limits_{i\in\omega\backslash A} \nu_i < q \\ \sum\limits_{i\in\omega\backslash A} \nu_i + \nu_A^2 \geq q, \end{cases}$$

that is, in situation 2 ω is the winning coalition and $\omega \backslash A$ is the losing coalition. Therefore, A is the pivot player of the coalition ω in the second situation as well. We obtain

$$\forall \omega \in W_1 \Rightarrow \omega \in W_2.$$

Consequently, $W_1 \subset W_2$, and then either $W_2 = W_1$ and $\Phi^2(A) = \Phi^1(A)$ or $W_2 = W_1 \cup \{\omega_1, \ldots, \omega_k\}$. It follows from the general theorem of representation that

$$\Phi^2(A) = \Phi^1(A) + C_A(\omega_1) + \ldots + C_A(\omega_k) \geq \Phi^1(A).$$

3.6. Anonymity Axiom

The power indices of anonymous players should be dependent only on the quota and the player weights. Therefore, the players become depersonalized in a sense, and two players with the same weights do not differ one from the other. Any permutation of the player weights leads to the same permutation of their power indices.

Anonymity axiom. If the situations 1 and 2 differ only in that the weights of the players A and B changed their places, $\nu_A^2 = \nu_B^1$, $\nu_B^2 = \nu_A^1$, then the power indices of these players also change their places: $\Phi^2(A) = \Phi^1(B)$, $\Phi^2(B) = \Phi^1(A)$.

The properties of independence of coalition contribution and dependence of coalition contribution on the size follow from this axiom.

3.6.1. Property of Independence of the Coalition Contribution.

Theorem 8. *The dictator, additivity, and anonymity axioms being satisfied, the contribution of coalition to the power index of its pivot player is independent of the player and its weight. Stated differently, if A and B are the pivot players in a coalition ω, then its contributions to their power indices are equal, that is, $C_A(\omega) = C_B(\omega) = C(\omega)$.*

Proof. Let us consider a case where A and B are pivot players only in one coalition ω. By the general representation theorem, then $\Phi(A) = C_A(\omega)$, $\Phi(B) = C_B(\omega)$. Now we interchange the weights of the players A and B: $\nu'_A = \nu_B$, $\nu'_B = \nu_A$. By the anonymity axiom, their power indices also interchange: $\Phi'(A) = \Phi(B) = C_B(\omega)$ and $\Phi'(B) = \Phi(A) = C_A(\omega)$. We prove that after changing the weights the players A and B remained pivotal only in the coalition ω. Obviously, the weight ν_ω of the coalition ω did not change when A and B interchanged their weights. A was a pivot player in ω; consequently, $\nu_\omega - \nu_A \leq q$, $\nu'_B = \nu_A \Rightarrow \nu_\omega - \nu'_B \leq q$. Therefore, after the change of weights B remains pivotal in ω. Similarly, B was pivotal in ω: $\nu_\omega - \nu_B \leq q$, $\nu'_A = \nu_B \Rightarrow \nu_\omega - \nu'_A \leq q$, that is, after the change of weights A also remains pivotal in ω.

It remains to prove that after the change of weights no other coalition where A or B becomes a pivot player can occur. Let us assume that on the contrary there exists a coalition $\omega^* \neq \omega$ where A becomes pivot player:

$$\exists \omega^* \neq \omega \quad \begin{cases} \nu_{\omega^*} > q \\ \nu_{\omega^*} - \nu'_A \leq q \end{cases} \quad \Rightarrow \quad \begin{cases} \nu_{\omega^* \backslash A} + \nu'_A > q \\ \nu_{\omega^* \backslash A} \leq q. \end{cases}$$

If $B \notin \omega^*$, then since $\nu'_A = \nu_B$, we obtain

$$\begin{cases} \nu_{\omega^* \backslash A} + \nu_B > q \\ \nu_{\omega^* \backslash A} \leq q \end{cases} \quad \Rightarrow \quad \begin{cases} \nu_{\omega^* \backslash A \cup B} > q \\ \nu_{\omega^* \backslash A} \leq q, \end{cases}$$

that is, B was pivotal before the change of weights in the coalition $\omega^* \backslash A \cup B$ which cannot coincide with ω because it does not include the player A. Yet B was the pivot player only in ω, and consequently, the assumption of $B \notin \omega^*$ is no true. Let us consider the case of $B \in \omega^*$. We get

$$\begin{cases} \nu_{\omega^*} > q \\ \nu_{\omega^*} - \nu'_A \leq q \end{cases} \Rightarrow \begin{cases} \nu_{\omega^*} > q \\ \nu_{\omega^*} - \nu_B \leq q, \end{cases}$$

which means that before the change of weights B was pivotal in the coalition $\omega^* \neq \omega$. Again we get a contradiction, and consequently, the original assumption that there exists a coalition $\omega^* \neq \omega$ where A becomes pivotal after the change of weights is not true. The corresponding statement for the player B is proved in a similar way.

Therefore, A and B remain pivot players only in one coalition ω. According to the general representation theorem, their power indices become then $\Phi'(A) = C_A(\omega)$, $\Phi'(B) = C_B(\omega)$. As was established before, $\Phi'(A) = C_B(\omega)$, $\Phi'(B) = C_A(\omega)$. Consequently, $C_A(\omega) = C_B(\omega)$. Since $\forall A, B \in \omega$, $C_A(\omega) = C_B(\omega)$, this contribution of the coalition ω to the power indices of its pivot players may be denoted just as $C(\omega)$.

3.6.2. Property of Dependence of the Coalition Contribution on Size.

Theorem 9. *If the dictator, additivity, and anonymity axioms are satisfied, then any coalitions with identical number of their participants make the same contribution to the power indices of their pivot players:* $\forall v, \omega,\ |v| = |\omega| = k \Rightarrow C(v) = C(\omega) = C(k)$.

Proof. Let there exist a losing coalition ω which is made winning by the players A and B, that is, the player A is pivotal in the coalition $\omega \cup A$ and B, in the coalition $\omega \cup B$. Let also there be no other such coalitions, that is, A and B are pivotal only in one coalition. By the general representation theorem and the property of independence of the coalition contribution, the power indices of these players are equal: $\Phi_A = C(\omega \cup A)$, $\Phi_B = C(\omega \cup B)$.

Now we interchange the weights of A and B: $\nu'_A = \nu_B$, $\nu'_B = \nu_A$. By the anonymity axiom, the power indices interchange as well: $\Phi'(A) = \Phi(B) = C(\omega \cup B)$, $\Phi'(B) = \Phi(A) = C(\omega \cup A)$.

A was the pivot player in the coalition $\omega \cup A$ which means that

$$\begin{cases} \nu_\omega \leq q \\ \nu_\omega + \nu_A > q \end{cases} \Rightarrow \begin{cases} \nu_\omega \leq q \\ \nu_\omega + \nu'_B > q. \end{cases}$$

Consequently, B remains pivotal in the coalition $\omega \cup B$. There are no other coalitions where A is pivotal:

$$\forall v \neq \omega \left[\begin{array}{l} \nu_v > q \\ \nu_v + \nu_A \leq q \end{array} \right. \Rightarrow \left[\begin{array}{l} \nu_v > q \\ \nu_v + \nu'_B \leq q. \end{array} \right.$$

Then, B cannot be pivotal in any coalition save $\omega \cup B$. Similarly, A remains pivotal after the change of weights only in the coalition $\omega \cup A$. By the general representation theorem and the property of independence of the coalition contribution, the power indices of A and B become $\Phi'(A) = C(\omega \cup A)$ and $\Phi'(B) = C(\omega \cup B)$, respectively. Yet, it was established above that $\Phi'(A) = C(\omega \cup B)$, $\Phi'(B) = C(\omega \cup A)$. Therefore, $C(\omega \cup A) = C(\omega \cup B)$.

This result means that in any coalition ω' any its player A ($\omega' = \omega \cup A$) can be replaced by another player B without changing the contribution of this coalition to the power indices of its pivot players: $C(\omega') = C(\omega' \backslash A \cup B)$. Then, for any two coalitions v and ω with the same number of participant all players of v can be replaced by the players of ω without changing $C(v)$, that is, $C(v) = C(\omega)$.

Whence it follows that in fact the contribution of a coalition to the power indices of its pivot players depends only on the number of the players in this coalition. Then, the contribution of any coalition ω of k players ($|\omega| = k$) may be denoted just as $C(k)$.

3.7. Theorem of Representation at Satisfaction of the Anonymity Axiom

The power index satisfying the dictator, additivity, and anonymity axioms is representable as $\Phi(A) = \sum_S C(s)$, where $s = |S|$, S being the coalition significant for the player A. The function $C(s) \not\equiv 0$ is the contribution of the coalition of s players to the power indices of its pivot participants. The inverse is true as well: the power index given by $\Phi(A) = \sum_S C(s)$, $C(s) \not\equiv 0$, satisfies the dictator, additivity, and anonymity axioms. To prove the corresponding theorem, two lemmas are proved first.

Lemma 1. *Let A be the pivot player in the coalitions $\omega_1^A, \ldots, \omega_m^A$ and the weights of the players A and B be interchanged: $\nu'_A = \nu_B$, $\nu'_B = \nu_A$. Then, B is the pivot player only in the coalitions ${\omega'}_1^B, \ldots, {\omega'}_m^B$, where*

$$ {\omega'}_i^B = \begin{cases} \omega_i^A & \text{if } B \in \omega_i^A \\ \left(\omega_i^A \backslash A\right) \cup B & \text{if } B \notin \omega_i^A. \end{cases} $$

For example, if A was a pivot player in the coalitions ABE and AC, then after the change of weights B becomes the pivot player in the coalitions ABE and BC.

Proof. The fact that A was the pivot player in the coalitions $\omega_1^A, \ldots, \omega_m^A$ before the change of weights implies that

$$ \forall i = \overline{1, m} \quad \begin{cases} \nu_{\omega_i^A} > q \\ \nu_{\omega_i^A} - \nu_A \le q. \end{cases} $$

The fact that there are no other coalitions where A is a pivot player means that

$$ \forall \omega \;\; \forall i = \overline{1, m} \quad \omega \cup A \ne \omega_i^A \left[\begin{array}{l} \nu_\omega + \nu_A \le q \\ \nu_\omega > q. \end{array} \right. \tag{1} $$

We consider one of the coalitions ω_i^A where A was the pivot player before the change of weights ω_i^A and prove that after the change of weights B became the pivot player in

$$ {\omega'}_i^B = \begin{cases} \omega_i^A, & \text{if } B \in \omega_i^A \\ \left(\omega_i^A \backslash A\right) \cup B & \text{if } B \notin \omega_i^A. \end{cases} $$

Let us consider two cases: $B \in \omega_i^A$ and $B \notin \omega_i^A$.

(1) For $B \in \omega_i^A$,

$$ \begin{cases} \nu_{\omega_i^A} > q \\ \nu_{\omega_i^A} - \nu_A \le q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega_i^A} > q \\ \nu_{\omega_i^A} - \nu'_B \le q. \end{cases} $$

Consequently, after the change of weights B became the pivot player in ω_i^A and, therefore, ${\omega'}_i^B = \omega_i^A$.

(2) For $B \notin \omega_i^A$,

$$ \begin{cases} \nu_{\omega_i^A} > q \\ \nu_{\omega_i^A} - \nu_A \le q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega_i^A \backslash A} + \nu_A > q \\ \nu_{\omega_i^A \backslash A} \le q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega_i^A \backslash A} + \nu'_B > q \\ \nu_{\omega_i^A \backslash A} + \nu'_B - \nu'_B \le q \end{cases} \Leftrightarrow \begin{cases} \nu_{(\omega_i^A \backslash A) \cup B} > q \\ \nu_{(\omega_i^A \backslash A) \cup B} - \nu'_B \le q, \end{cases} $$

that is, B became the pivot player in $\left(\omega_i^A \backslash A\right) \cup B$, and ${\omega'}_i^B = \left(\omega_i^A \backslash A\right) \cup B$. Therefore, after the change of weights B became the pivot player in the coalitions ${\omega'}_1^B, \ldots, {\omega'}_m^B$, where

$$ {\omega'}_i^B = \begin{cases} \omega_i^A & \text{if } B \in \omega_i^A \\ \left(\omega_i^A \backslash A\right) \cup B, & \text{if } B \notin \omega_i^A. \end{cases} $$

Now we prove that there are no other coalitions distinct from $\omega'^B_1, \ldots, \omega'^B_m$ where B becomes the pivot player. We admit the inverse: let there be a coalition $\exists\omega\ \ B \in \omega$, $\forall i = \overline{1,m}\ \ \omega \neq \omega'^B_i$ where B becomes the pivot player:

$$\begin{cases} \nu_\omega > q \\ \nu_\omega - \nu'_B \leq q. \end{cases}$$

We consider two cases: $A \in \omega$ and $A \notin \omega$.

(1) In the case $A \in \omega$, for those i for which $\omega'^B_i = \left(\omega^A_i \backslash A\right) \cup B$ (for $B \notin \omega^A_i$), the condition $\omega \neq \omega'^B_i$ is satisfied mechanically because $A \in \omega$, but $A \notin \omega'^B_i = \left(\omega^A_i \backslash A\right) \cup B$. For such i, $\omega \neq \omega^A_i$ because $B \in \omega$, but $B \notin \omega^A_i$. For the rest of i, $\omega'^B_i = \omega^A_i$, and since by assumption $\omega \neq \omega'^B_i$, $\omega \neq \omega^A_i$. Then, the following assertion is true:

$$\exists\omega\ \ A \in \omega \qquad \forall i = \overline{1,m}\ \ \omega \neq \omega^A_i \quad \begin{cases} \nu_\omega > q \\ \nu_\omega - \nu'_B \leq q \end{cases}$$

$$\begin{cases} \nu_\omega > q \\ \nu_\omega - \nu'_B \leq q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega\backslash A} + \nu_A > q \\ \nu_{\omega\backslash A} \leq q. \end{cases}$$

By denoting $\omega^* = \omega\backslash A$, we obtain that

$$\exists\omega^*\ \ \forall i = \overline{1,m}\ \ \ \omega^* \cup A \neq \omega^A_i \quad \begin{cases} \nu_{\omega^*} + \nu_A > q \\ \nu_{\omega^*} \leq q \end{cases}$$

which contradicts condition (1) of the lemma according to which there is no coalition distinct from $\omega^A_1, \ldots, \omega^A_m$ where A was the pivot player before the change of weights.

(2) In the case of $A \notin \omega$, for the i for which $\omega'^B_i = \omega^A_i$ (for $B \in \omega^A_i$), the condition $\omega \neq \omega'^B_i$ is satisfied mechanically because $A \in \omega'^B_i = \omega^A_i$, but $A \notin \omega$. For such i, $(\omega \cup A)\backslash B \neq \omega^A_i$ because $B \in \omega^A_i$, but $B \notin (\omega \cup A)\backslash B$. For the rest of the values of i, $\omega'^B_i = \left(\omega^A_i \backslash A\right) \cup B$, and since by assumption $\omega \neq \omega'^B_i = \left(\omega^A_i \backslash A\right) \cup B$, we get that $(\omega \cup A)\backslash B \neq \omega^A_i$. As the result,

$$\exists\omega\ \ A \notin \omega\ \ \forall i = \overline{1,m}\ \ (\omega \cup A)\backslash B \neq \omega^A_i \quad \begin{cases} \nu_\omega > q \\ \nu_\omega - \nu'_B \leq q \end{cases}$$

$$\begin{cases} \nu_\omega > q \\ \nu_\omega - \nu'_B \leq q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega\backslash B} + \nu'_B > q \\ \nu_{\omega\backslash B} \leq q \end{cases} \Leftrightarrow \begin{cases} \nu_{\omega\backslash B} + \nu_A > q \\ \nu_{\omega\backslash B} \leq q. \end{cases}$$

By denoting $\omega^* = \omega\backslash B$, we obtain

$$\exists\omega^*\ \ \forall i = \overline{1,m}\ \ \ \omega^* \cup A \neq \omega^A_i \quad \begin{cases} \nu_{\omega^*} + \nu_A > q \\ \nu_{\omega^*} \leq q \end{cases}$$

which again contradicts condition (1). As the result, the assumption that there exists a coalition distinct from $\omega'^B_1, \ldots, \omega'^B_m$ where B is the pivot player after the change of weights turned out to be erroneous. Therefore, B became the pivot player in the coalitions $\omega'^B_1, \ldots, \omega'^B_m$, where

$$\omega'^B_i = \begin{cases} \omega^A_i, & \text{if}\ \ B \in \omega^A_i \\ \left(\omega^A_i \backslash A\right) \cup B, & \text{if}\ \ B \notin \omega^A_i, \end{cases}$$

which proves Lemma 1.

Lemma 2. *Let A be a pivot player in m coalitions and the number of participants in each of these coalitions be $s_1, \ldots, s_m$ and player B be a pivot player in k coalitions with $t_1, \ldots, t_k$ participants. Let the weights of the players A and B be interchanged: $\nu'_A = \nu_B$, $\nu'_B = \nu_A$. Then, B is the pivot player in m coalitions of sizes $s_1, \ldots, s_m$ and A is the pivot player k coalitions of sizes $t_1, \ldots, t_k$.*

Proof. The proof follows from Lemma 1. Obviously, $\left|\omega'^B_i\right| = \left|\omega^A_i\right| = s_i$ because $\omega'^B_i = \omega^A_i$ or $\omega'^B_i = \left(\omega^A_i \backslash A\right) \cup B$. The players A and B do not differ one from the other. Therefore, the same holds for the coalitions where A becomes the pivot player: $\left|\omega'^A_i\right| = \left|\omega^B_i\right| = t_i$.

Theorem 10. *The power index of the player A is representable as $\Phi(A) = \sum\limits_S C(s), C(s) \not\equiv 0$ (summation is carried out over all coalitions S that are significant for A, $s = |S|$) if and only if the dictator, additivity, and anonymity axioms are satisfied.*

Proof. Lemma 2 will be used to prove necessity, the general representation theorem and the property of dependence of the coalition contribution on the size, to prove sufficiency.

Necessity. We prove validity of the dictator axiom. The dictator is the pivot player in every possible coalitions which include it. The number of different coalitions of s players including the dictator is equal to $\binom{n-1}{s-1}$, therefore its power index is as follows:

$$\Phi(A) = \sum_{s=1}^{n} \binom{n-1}{s-1} C(s) > 0, \quad C(s) \not\equiv 0,$$

which means that the dictator axiom is satisfied.

We prove validity of the additivity axiom. Let in situation 1 A be the pivot player in the set of coalitions $W^1 = \{S_1, \ldots, S_m\}$, in situation 2 A be the pivot player in the set of coalitions $W^2 = \{S_{m+1}, \ldots, S_k\}$, and in situation 3 A be the pivot player in the set $W^3 = W^1 \cup W^2 = \{S_1, \ldots, S_k\}$. Let the size of the coalition S_i be s_i. We prove that in the third situation the power index is equal to the sum of indices in the first and second situations: $\Phi^3(A) = \Phi^1(A) + \Phi^2(A)$,

$$\Phi^1(A) = C(s_1) + \ldots + C(s_m),$$
$$\Phi^2(A) = C(s_{m+1}) + \ldots + C(s_k),$$
$$\Phi^3(A) = C(s_1) + \ldots + C(s_k) = \Phi^1(A) + \Phi^2(A).$$

Now we prove validity of the anonymity axiom. Let the weights of two players A and B be interchanged: $\nu'_A = \nu_B$, $\nu'_B = \nu_A$. We prove that their power indices interchange as well: $\Phi'(A) = \Phi(B)$, $\Phi'(B) = \Phi(A)$. Let the sizes of the coalitions where A was the pivot player before the change of weights be $s_1, \ldots, s_m$ and where B was the pivot player, be $t_1, \ldots, t_k$. Then, their power indices were as follows:

$$\Phi(A) = C(s_1) + \ldots + C(s_m),$$
$$\Phi(B) = C(t_1) + \ldots + C(t_k).$$

By Lemma 2, after the change of weights A becomes the pivot player in the coalitions of sizes $t_1, \ldots, t_k$, and B, in the coalitions of sizes $s_1, \ldots, s_m$, that is, their power indices were as follows:

$$\Phi'(A) = C(t_1) + \ldots + C(t_k),$$
$$\Phi'(B) = C(s_1) + \ldots + C(s_m).$$

Therefore, the power indices of the players interchanged: $\Phi'(A) = \Phi(B)$, $\Phi'(B) = \Phi(A)$.

Sufficiency. As was shown above, the general representation theorem and the property of dependence of coalition contribution on size follow from the dictator, additivity, and anonymity axioms. According to the general representation theorem, the power index of A which is a pivot player in the coalitions $\omega_1, \ldots, \omega_k$ is as follows:

$$\Phi(A) = C_A(\omega_1) + \ldots + C_A(\omega_k) = \sum_S C_A(S).$$

In virtue of the dependence of the coalition contribution on size

$$C_A(S) = C(s), \quad \text{where} \quad s = |S|\,.$$

Therefore, the power index is as follows:

$$\Phi(A) = \sum_S C(s),$$

which is what we set out to prove.

Remark. For the Banzhaf index, the coalition contribution function is given by $C(s) = 1$, for the Shapley–Shubik index, by $C(s) = (s-1)!(n-s)!$. Other variants of representation of $C(s)$ such as $C(s) = s$ and $C(s) = 1/s$ make sense. In the first case, the contribution of a coalition to the power indices of its pivot players is directly proportional to the coalition size. In the second case, the contribution of a coalition to the power indices of its pivot players is inversely proportional to the size of this coalition.

4. CONCLUSIONS

The paper considered a general approach to determination of the power indices in the problem of voting with quota. The power of each participant was shown to be defined by the sum of contributions of the winning coalitions where the participant plays the pivot role.

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