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Quantum Geometry and Quantum Mechanics of Integrable Systems. II

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Abstract. For a generic quantum integrable system, we describe the asymptotics of the eigenstate density and of the trace of the evolution operator in all orders of the quantization parameter. This is done by using quantum symplectic geometry, which makes the given quantum system to be equivalent to a deformed classical system with arbitrary accuracy with respect to the quantization parameter. The asymptotics is explicitly given via the deformed symplectic form, deformed Liouville–Arnold tori, and deformed Maslov class.

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1. INTRODUCTION

In 1990, Max Planck theoretically discovered a fundamental fact of jump-like character of the radiation energy and demonstrated that portions of energy are always proportional to the frequency multiplied by a certain universal quantum of action (later called the "Planck constant"). Planck's intuitive conjecture, which was formulated in 1911, declared that the general quantization phenomenon of physical quantities can be described mathematically in terms of discretization of the "action" coordinates in the phase geometry. The importance of such an interpretation was widely recognized after the work of A. Sommerfeld (1915), who successfully applied this idea to the hydrogen atom model. Then K. von Schwarzschild (1916) and A. Einstein (1917) pointed out that the discretization rule in Planck's conjecture can be understood as a topological condition for noncontractible cycles of invariant tori in the phase space. Later, H. Kramers (1926) and J. Keller (1958) introduced half-integer corrections to this rule.

The importance of the phase space geometry in quantum mechanics was clarified from another point of view by H. Weyl (1931) and E. Wigner (1932) who associated the operator trace with the phase space integral and Planck's quantum of action. Then, H. Groenewold (1946) and J. Moyal (1949) rewrote the quantum product of operators in terms of functions on the phase space. Later, P. Argyres (1965) noticed that the fact of spectrum nondegeneracy for one-dimensional systems, together with the Weyl trace formula and the Groenewold–Moyal product, makes it possible to compute the spectrum of such systems analytically with arbitrary power accuracy with respect to the quantization parameter. Argyres' formula represented a deformed discretization rule for one-dimensional systems via an integral over the energy levels of the Hamiltonian, similarly to Planck's original idea.

For systems with many degrees of freedom, in the context of the general theory of global semiclassical approximation, the quantization hypothesis was mathematically cleaned and significantly clarified by V. Maslov [1] in the first two leading terms of the asymptotics in the quantization parameter (and a new homotopic invariant was discovered, the index of paths on Lagrangian submanifolds in the phase space). However, the problem of lower terms in spectral asymptotics for many degrees of freedom remained open. In the note [2] (see [3] for details), it was suggested to use quantum deformation of the classical "action" coordinates to construct higher semiclassical approximations of the discrete spectrum for multidimensional integrable systems. This approach was finally realized in [4], where the following basic statement was established:

The fibration of the phase space by Liouville–Arnold tori and the classical symplectic 2-form can be deformed by means of the quantization parameter (preserving the property of tori to be Lagrangian) in such a way that the usual geometric discretization rule, written out for the new tori and new 2-form, represents the spectrum of the quantum integrable system with arbitrary power accuracy in the quantization parameter.

Let us clarify that, in the discretization rule, instead of the standard integer Maslov class, one has to use here a certain special deformation of this class.

We also note that the deformation of the tori and the symplectic form mentioned in this statement is produced by an explicit and geometrically invariant procedure, and it is unique up to a phase-space diffeomorphism.

Thus, we obtain a universal algorithm for computing spectral asymptotics for multidimensional integrable systems. At the same time, this approach eliminates the phenomenon of destruction (diffusion) of the classical phase space fibration under the quantum evolution. Namely, if one correctly deforms the fibration and preserves the fibers satisfying the discretization rule only, then this new geometric object will not be destructed in the long-time quantum evolution, and only state density diffusion along its fibers will be observed.

The results of [4] demonstrate that, for multidimensional integrable systems, the Planck hypothesis can work not only in the leading two terms but also in all lower terms of the semiclassical asymptotics if one replaces the classical phase space geometry by an appropriate quantum geometry. This deformed geometry is adapted to the original fibration by tori and depends on all higher derivatives of the classical action-angle variables in a very complicated way. Up to arbitrary power accuracy, it allows to distinguish the "à la classical mechanics" component from the given quantum system, in the spirit of Bohr's correspondence and complementarity principles, separating this component from that for which the Heisenberg dispersion and uncertainty are dominating.

The present paper continues [4]. In the framework of multidimensional quantum integrable systems, we obtain two new asymptotic formulas which work in all orders of the semiclassical approximation:

(A) the formula for the eigenstate density (the Wigner function) presented via distributions concentrated on the deformed phase space tori;

(B) the formula for the trace of the evolution operator, presented as a sum over noncontractible cycles on the deformed phase space tori.

Formula (A) demonstrates that the lower terms of the semiclassical asymptotics of the eigenstate density of a quantum system depend on the global phase geometry and rather than on the local properties of Hamiltonians only. Here we meet an essential distinction between the discrete and continuous spectrum cases. The quantum geometry makes it possible to describe the asymptotics of the eigenstate density in all lower orders in the quantization parameter simultaneously.

Formula (B) is somewhat similar to the Gutzwiller trace formula [5], but it makes no use of any dynamical trajectories. Our approach generalizes the well-known work of Berry and Tabor [6] by transforming the classical symplectic geometry to the quantum one. This makes it possible to obtain a very simple geometric representation for the trace of the evolution operator (or for the spectral density) in all orders of asymptotics in the quantization parameter simultaneously.

2. QUANTUM SYMPLECTIC GEOMETRY

In the present paper, we follow the notation introduced in [4]. By $\widehat{A} = A(q, \widehat{p})$ we denote the Weyl symmetrized functions of the generators $q = (q^1, \dots, q^n)$ and $\widehat{p} = -i\hbar\partial/\partial q$, where q^j stand for the Euclidean coordinates on $\mathbb{R}^n$. The symbols $A = A(q,p)$ are functions on $T^*\mathbb{R}^n$, which smoothly depend on the quantization parameter $\hbar \to 0$ (this dependence is not indicated in the notation).

The quantum integrable system under consideration is determined by a set of commuting self-adjoint operators $\widehat{H}_j$ $(j = 1, \dots, n)$, i.e.,

$$[\widehat{H}_j, \widehat{H}_k] = 0. \tag{2.1}$$

In the $\hbar = 0$ limit, the functions

$$H_j^0 \overset{\text{def}}{=} H_j|_{\hbar=0}, \qquad \{H_j^0, H_k^0\} = 0, \tag{2.2}$$

determine a classical integrable system on $T^*\mathbb{R}^n$ with action-angle coordinates

$$s = (s_1, \dots, s_n) \qquad \text{and} \qquad \tau = (\tau^1, \dots, \tau^n), \qquad 0 \leqslant \tau^j \leqslant 2\pi.$$

We assume that the symbols H_j are real and satisfy the usual conditions at infinity [7] which guarantee the self-adjointness of the operators $\widehat{H}_j$ in $L^2(\mathbb{R}^n)$ and the discreteness of their joint spectrum as $\hbar \to 0$ with $\hbar \neq 0$.

Let $\mathcal{D}$ be an open nonempty domain in $T^*\mathbb{R}^n$ whose closure $\overline{\mathcal{D}}$ is determined by the inequalities

$$\alpha_j \leqslant H_j^0(q,p) \leqslant \beta_j \qquad (j = 1, \dots, n).$$

We assume that $\overline{\mathcal{D}}$ is connected and trivially fibered by the classical Liouville–Arnold tori $\{s = \text{const}\}$. The last conditions are just technical and can be generalized; however, in any case, the occurrence of a separatrix is forbidden.

The classical symplectic form on $T^*\mathbb{R}^n$ is

$$\omega \overset{\text{def}}{=} dp \wedge dq = \frac{1}{2} J dX \wedge dX, \qquad X \overset{\text{def}}{=} (\tau, s), \tag{2.3}$$

where X is regarded as a $2n$-dimensional vector-function and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ stands for the standard symplectic $2n \times 2n$ matrix with zero and identity $n \times n$ blocks.

The classical Poisson brackets in (2.2) correspond to the form (2.3). The quantum condition (2.1) can also be represented via some deformed Poisson brackets. In [4], a deformation procedure was described,

$$\begin{aligned} &\omega \to \omega^\hbar, \qquad \{\cdot,\cdot\} \to \{\cdot,\cdot\}^\hbar, \\ &H \to H^\hbar, \qquad (s,\tau) \to (s^\hbar, \tau^\hbar), \end{aligned} \tag{2.4}$$

which produces mod $O(\hbar^\infty)$ a new closed 2-form and new Poisson brackets on the domain $\mathcal{D}$, new action-angle coordinates $s^\hbar$, $\tau^\hbar$, and new *energy functions* $H_j^\hbar$ in involution with respect to the new brackets:[1]

$$\{H_j^\hbar, H_k^\hbar\}^\hbar = 0. \tag{2.5}$$

The classical commutativity (2.5) mod $O(\hbar^\infty)$ is related to the quantum commutativity (2.1) in the following way.

Denote by $*$ the product operation $\widehat{A * B} = \widehat{A}\widehat{B}$, which is explicitly given by the Groenewold–Moyal formula

$$A * B = A \exp\left\{ \frac{i\hbar}{2} \underset{\leftarrow}{D} J \underset{\rightarrow}{D} \right\} B$$

Here D stands for the derivatives with respect to the Euclidean coordinates on $T^*\mathbb{R}^n$. The quantum commutator determines the operation on the symbols

$$[A,B]_* \overset{\text{def}}{=} \frac{i}{\hbar}(A * B - B * A), \tag{2.6}$$

in such a way that condition (2.1) can be represented as

$$[H_j, H_k]_* = 0. \tag{2.7}$$

For a set of functions $S = (S_1, \dots, S_n)$ on $\mathcal{D}$ and a function k of n variables, one can define the Weyl symmetrized $*$-composite function $k(S)_*$, as well as the usual composite function $k(S)$. These functions are related to each other by some transformation V_S, i.e.,

$$k(S)_* = V_S(k(S)). \tag{2.8}$$

[1]In the equations below, we do not indicate the accuracy specially if these equations hold mod $O(\hbar^\infty)$ only; however, we always mention the accuracy in the text.

Using these $*$-product operations, we can represent explicit formulas for the quantum corrections in (2.4) up to $O(\hbar^4)$. First, we can write

$$\begin{aligned} \omega^\hbar &= \omega + \hbar^2 \varkappa + O(\hbar^4), \\ H^\hbar &= H^0 + \hbar^2 L + O(\hbar^4). \end{aligned} \tag{2.9}$$

The correcting 2-form $\varkappa$ in (2.9) is given by

$$\varkappa = \frac{1}{2}\langle\!\langle s_l, s_j \rangle\!\rangle d\tau^l \wedge d\tau^j + \frac{1}{2}\langle\!\langle \tau^l, \tau^j \rangle\!\rangle ds_l \wedge ds_j + \langle\!\langle s_j, \tau^l \rangle\!\rangle ds_l \wedge d\tau^j, \tag{2.10}$$

where the double angular brackets are taken from the expansion of the quantum commutator (2.6):

$$[A, B]_* = \{A, B\} - \hbar^2 \langle\!\langle A, B \rangle\!\rangle + O(\hbar^4),$$

namely,

$$\langle\!\langle A, B \rangle\!\rangle \overset{\text{def}}{=} -\frac{1}{24} D^3 A \cdot J \otimes J \otimes J \cdot D^3 B. \tag{2.11}$$

The correcting Hamiltonian L in (2.9) is given by

$$L = M + \Delta_s H^0. \tag{2.12}$$

Here M is taken from the expansion

$$H = H^0 + \hbar^2 M + O(\hbar^4), \tag{2.13}$$

and the quantum diffusion operator Δ_s is defined by

$$\Delta_s \overset{\text{def}}{=} \frac{1}{16} D^2 s_l \cdot J \otimes J \cdot D^2 s_k \cdot \frac{\partial^2}{\partial s_l \partial s_k} + \frac{1}{24} D^2 s_l \cdot J \otimes J \cdot (D s_k \otimes D s_m) \frac{\partial^3}{\partial s_l \partial s_k \partial s_m} \tag{2.14}$$

(the summation over repeated Latin indices ranges from 1 to n, here as well as in (2.10)). This is the very operator standing in the general quantum composite function transformation (2.8),

$$V_S = I - \hbar^2 \Delta_S + O(\hbar^4). \tag{2.15}$$

After the symplectic deformed form $\omega^\hbar$ and the deformed tori $\{H^\hbar = \text{const}\}$ are obtained by (2.9), the usual procedure generates the action coordinates

$$s_j^\hbar = \frac{1}{2\pi} \int_{\Sigma_j^\hbar} \omega^\hbar + c_j \qquad (j = 1, \dots, n). \tag{2.16}$$

Here the constants

$$c_j = \frac{1}{2\pi} \oint_{\Gamma_j} p\, dq$$

are defined by basic noncontractible cycles Γ_j in a *fixed torus*. The "membrane" $\Sigma_j^\hbar \subset \mathcal{D}$ is stretched on two cycles, namely, on the jth noncontractible cycle in the torus containing the given phase space point and on the cycle $(-\Gamma_j)$ in the fixed torus (with opposite orientation). By the freedom in the choice of Γ_j, the action coordinate (2.16) is determined uniquely up to an additive constant.

Note that the integral on the right-hand side of (2.16) does not depend on the choice of membrane, since the quantum symplectic form is closed ($d\omega^\hbar = 0$) and the tori $\{H^\hbar = \text{const}\}$ are Lagrangian (i.e., annihilate $\omega^\hbar$) in view of (2.5).

In addition to the new actions, one can also define new angle coordinates $0 \leqslant \tau_j^\hbar \leqslant 2\pi$, according to the representation

$$\omega^\hbar = ds^\hbar \wedge d\tau^\hbar. \tag{2.17}$$

The explicit $\hbar$-expansion of these quantum action-angle coordinates is

$$s^\hbar = s + \hbar^2 a + O(\hbar^4), \qquad \tau^\hbar = \tau + \hbar^2 \phi + O(\hbar^4), \tag{2.18}$$

where

$$a_j = \left(\frac{\partial H^0}{\partial s}\right)^{-1\,l}_{\;j} (L_l - \langle L_l \rangle_j) + \frac{1}{2\pi}\int_{\Sigma_j^0} \varkappa + a_j^0; \tag{2.19}$$

here a_j^0 are some constants,

$$\begin{aligned} \phi^j &= \int_0^\tau \left(\langle\!\langle s_l, \tau^j \rangle\!\rangle - \frac{\partial a_l}{\partial s_j}\right) d\tau^l + \varphi^j(s), \\ \varphi^j(s) &= \int_0^1 \langle\!\langle \tau^j, \tau^l \rangle\!\rangle \Big|_{\tau=0} (s\xi + s^0(1-\xi))(s-s^0)_l \xi \, d\xi + \frac{\partial \psi(s)}{\partial s_j}, \end{aligned} \tag{2.20}$$

s^0 is a chosen point in the s-space, and ψ is an arbitrary function in s-coordinates. The membrane Σ_j^0 in (2.19) spans the jth noncontractible cycle in the classical Liouville–Arnold torus containing the given phase space point and on the cycle Γ_j on the fixed torus (see the remark after (2.16)). The freedom in the choice of the fixed torus generates the freedom in the choice of the constants a_j^0 in (2.19).

One can call $s^\hbar$, $\tau^\hbar$ the *quantum action-angle coordinates* since they obey $\bmod O(\hbar^\infty)$ the canonical commutation relations with respect to the quantum brackets (2.6):

$$[s_j^\hbar, s_k^\hbar]_* = 0, \qquad [\tau^{\hbar j}, \tau^{\hbar k}]_* = 0, \qquad [s_j^\hbar, \tau^{\hbar k}]_* = 0. \tag{2.21}$$

The quantum actions also obey $\bmod O(\hbar^\infty)$ the 2π-periodicity condition for the $*$-exponent,

$$\exp\left(\frac{2\pi i}{\hbar} s_j^\hbar\right)_* = \exp\left\{\frac{i\pi}{2} m_j^\hbar\right\}. \tag{2.22}$$

Here the real numbers $m_j^\hbar$ are some constants on the domain $\mathcal{D}$ which determine the *cohomology class of the deformed tori* $\{s^\hbar = \text{const}\}$. In the classical limit, the class m^0 coincides with the integer Maslov class of the tori $\{s = \text{const}\}$.

The final formula matching the quantum-commuting symbols H_j (2.7) with the classical-commuting energy functions $H_j^\hbar$ (2.5) is

$$H_j = V_{s^\hbar}(H_j^\hbar), \tag{2.23}$$

where $V_{s^\hbar}$ is the transformation (2.8) corresponding to the quantum actions $S = s^\hbar$.

Let $\alpha^\hbar \leqslant H^\hbar \leqslant \beta^\hbar$ be the range of the energy functions corresponding to the range $\alpha \leqslant H^0 \leqslant \beta$ of Hamiltonians on the classical Liouvill–Arnold fibration.

The main result of [4] is derived from (2.22) and (2.23).

Theorem 2.1. *The eigenvalues* $\mathcal{E}^\hbar \in [\alpha^\hbar, \beta^\hbar]$ *of the set of commuting Hamiltonians* $\widehat{H}$ *are determined* $\bmod O(\hbar^\infty)$ *by the dicretization rule*

$$\frac{1}{2\pi}\int_{\Sigma^\hbar} \omega^\hbar + c = \hbar\left(N + \frac{1}{4} m^\hbar\right), \qquad N \in \mathbb{Z}, \tag{2.24}$$

where the membranes $\Sigma^\hbar$ *span arbitrary noncontractible cycles in the deformed tori* $\{H^\hbar = \mathcal{E}^\hbar\}$, *and the set of constants* c *is taken from* (2.16).

The discretization rule (2.24) fixes the “right” values of the quantum actions (2.16) taking into account the global holonomy (2.22) of the quantum flow over the 2π-period.

3. DENSITY OF EIGENSTATES

Denote by $\mathcal{E}^\hbar[N]$ the asymptotic $\bmod O(\hbar^\infty)$ eigenvalues of the set of commuting operators $\widehat{H}$ obtained from the discretization rule (2.24). The corresponding $\bmod O(\hbar^\infty)$ eigenstates are the L^2-normalized eigenstates of the quantum action operators

$$\widehat{s^\hbar}\psi_N = s^\hbar[N]\psi_N, \qquad s^\hbar[N] \overset{\mathrm{def}}{=} \hbar\Big(N + \frac{1}{4}m^\hbar\Big). \tag{3.1}$$

The density (or the Wigner function) ρ_N of the asymptotic eigenstate ψ_N is the phase space distribution defined by the L^2-scalar product:

$$\int_{T^*\mathbb{R}^n} \rho_N A \, d\mathcal{L} \overset{\mathrm{def}}{=} (\widehat{A}\psi_N, \psi_N).$$

Here we denote by $d\mathcal{L} = dq\,dp$ the classical Liouville measure on $T^*\mathbb{R}^n$. The distribution ρ_N is concentrated $\bmod O(\hbar^\infty)$ on the torus[2] $\{s = s^\hbar[N]\}$ (see, e.g., [8, Chap. III]). The leading term of the asymptotics of ρ_N is just the Dirac δ-function on the torus. The following theorem presents all lower terms.

Theorem 3.1. *The eigenstate density has the representation* $\bmod O(\hbar^\infty)$,

$$\rho_N = \frac{1}{(2\pi)^n} V_{s^\hbar} \delta(s^\hbar - s^\hbar[N]), \tag{3.2}$$

where $s^\hbar$ are the quantum actions (2.16) *and the transformation $V_{s^\hbar}$ is defined as in* (2.8) *for $S = s^\hbar$. The first two terms of asymptotics of the eigenstate density are*

$$\rho_N = \frac{1}{(2\pi)^n}\left[\delta(s - s^\hbar[N]) + \hbar^2\left(a_j \frac{\partial}{\partial s_j}\delta(s - s^\hbar[N]) - \Delta_s \delta(s - s^\hbar[N])\right) + O(\hbar^4)\right], \tag{3.3}$$

where s are the classical action functions, the functions a_j are the first quantum corrections (2.19), *and the third-order differential operator Δ_s is determined by* (2.14).

Proof. Denote by $\widehat{U} = (\widehat{U}_1, \dots, \widehat{U}_n)$ the set of commuting $\bmod O(\hbar^\infty)$ unitary operators generated by the quantum angles: $\widehat{U}_k = \exp\{i\widehat{\tau^{\hbar k}}\}$ or, if the transformation (28) is used,

$$U_k = V_{\tau^\hbar}(\exp\{i\tau^{\hbar k}\}). \tag{3.4}$$

It follows from (2.23) that

$$\widehat{s^\hbar}\widehat{U}^M = \widehat{U}^M(\widehat{s^\hbar} + \hbar M), \qquad M \in \mathbb{Z}^n,$$

and therefore the function $\widehat{U}^M\psi_N$ is the eigenstate of the quantum actions corresponding to the eigenvalues $\hbar(N + M + \frac{1}{4}m^\hbar)$. Thus, this function is orthogonal to ψ_N,

$$(\widehat{U}^M\psi_N, \psi_N) = 0, \qquad M \neq 0. \tag{3.5}$$

Let us now define the action-angle pseudodifferential operators

$$g(\widehat{\tau^\hbar}, \widehat{s^\hbar})\psi_N \overset{\mathrm{def}}{=} \frac{1}{(2\pi)^n} \sum_{M \in \mathbb{Z}^n} \widetilde{g}_{M,N} \widehat{U}^M \psi_N, \tag{3.6}$$

[2]Note that $\bmod O(\hbar^\infty)$ all our objects are localized in the domain $\mathcal{D}$, and thus the continuation of the quantum action functions $s^\hbar$ in (3.1) outside $\mathcal{D}$ (needed for the operators $\widehat{s^\hbar}$ to be correctly defined) plays no role. The same is true for the definition of the unitary operators $\widehat{U} = \exp\{i\widehat{\tau^\hbar}\}$ generated by the quantum angles, see (3.4) below.

where $\widetilde{g}_{M,N}$ is the Fourier transform of the symbol g by the quantum angle coordinates:

$$\widetilde{g}_{M,N} = \int_{\mathbb{T}^n} \exp\{iM\tau^\hbar\} g(\tau^\hbar, s^\hbar[N])\, d\tau^\hbar.$$

Then it follows from (3.5) that

$$\big(g(\widehat{\tau^\hbar}, \widehat{s^\hbar})\psi_N, \psi_N\big) = \frac{1}{(2\pi)^n}\widetilde{g}_{0,N} = \frac{1}{(2\pi)^n}\int g(\tau^\hbar, s^\hbar[N])\, d\tau^\hbar = \frac{1}{(2\pi)^n}\int g\cdot\delta(s^\hbar - s^\hbar[N])\, d\mathcal{L}^\hbar, \tag{3.7}$$

where

$$d\mathcal{L}^\hbar = ds^\hbar\, d\tau^\hbar = \frac{1}{h!}|\underbrace{\omega^\hbar\wedge\cdots\wedge\omega^\hbar}_{n}| \tag{3.8}$$

is the *quantum Liouville measure* generated by the quantum symplectic form $\omega^\hbar$.

By analogy with (2.8), let us introduce the transformation $V_{\tau^\hbar, s^\hbar}$ relating the action-angle pseudodifferential operators (3.6) to the usual $(q,\widehat{p})$-operators,

$$g(\widehat{\tau^\hbar}, \widehat{s^\hbar}) = \widehat{V_{\tau^\hbar, s^\hbar}(g)}. \tag{3.9}$$

Then

$$\big(g(\widehat{\tau^\hbar}, \widehat{s^\hbar})\psi_N, \psi_N\big) = \int \rho_N V_{\tau^\hbar, s^\hbar}(g)\, d\mathcal{L},$$

and (3.7) implies

$$\int \rho_N A\, d\mathcal{L} = \frac{1}{(2\pi)^n}\int V^{-1}_{\tau^\hbar, s^\hbar}(A)\cdot\delta(s^\hbar - s^\hbar[N])\, d\mathcal{L}^\hbar. \tag{3.10}$$

Lemma 3.1. *The transformation $V_{\tau^\hbar, s^\hbar}$ (3.9) is $\operatorname{mod} O(\hbar^\infty)$ unitary on smooth functions localized in $\mathcal{D}$:*

$$\int V_{\tau^\hbar, s^\hbar}(g')\,\overline{V_{\tau^\hbar, s^\hbar}(g'')}\, d\mathcal{L} = \int g'\,\overline{g''}\, d\mathcal{L}^\hbar. \tag{3.11}$$

Proof of the lemma. The left-hand side of (3.11) coincides $\operatorname{mod} O(\hbar^\infty)$ with the trace

$$\begin{aligned}(2\pi\hbar)^n \operatorname{tr}\big(\widehat{V_{\tau^\hbar, s^\hbar}(g'')}^*\cdot\widehat{V_{\tau^\hbar, s^\hbar}(g')}\big) &= (2\pi\hbar)^n \operatorname{tr}(g''(\widehat{\tau^\hbar}, \widehat{s^\hbar})^*\cdot g'(\widehat{\tau^\hbar}, \widehat{s^\hbar}))\\ &= (2\pi\hbar)^n\sum_N \big(g'(\widehat{\tau^\hbar}, \widehat{s^\hbar})\psi_N, g''(\widehat{\tau^\hbar}, \widehat{s^\hbar})\psi_N\big) = \Big(\frac{\hbar}{2\pi}\Big)^n\sum_{N,M}\widetilde{g'_{M,N}}\cdot\overline{\widetilde{g''_{M,N}}}\\ &= \hbar^n\sum_N\int g'(\tau^\hbar, s^\hbar[N])\overline{g''(\tau^\hbar, s^\hbar[N])}\, d\tau^\hbar,\end{aligned}$$

where, in the last two equalities, we have used (3.5), (3.6), and the Parseval identity for the Fourier transform by angles. The last sum over N thus obtained can be rewritten $\operatorname{mod} O(\hbar^\infty)$ by using the Poisson summation formula, as the integral by $s^\hbar$, and we obtain the right-hand side of (3.11). The lemma is proved.

Now, by using (3.11), we transform (3.10) $\operatorname{mod} O(\hbar^\infty)$ as follows:

$$\int \rho_N A\, d\mathcal{L} = \frac{1}{(2\pi)^n}\int A\cdot V_{\tau^\hbar, s^\hbar}\delta(s^\hbar - s^\hbar[N])\, d\mathcal{L}.$$

Since the function $\delta(s^\hbar - s^\hbar[N])$ does not depend on the angle coordinates, the transformation $V_{\tau^\hbar, s^\hbar}$ coincides with $V_{s^\hbar}$ on this function, and we obtain (3.2).

The asymptotic expansion (3.3) immediately follows from (3.2) if we take expansions (2.15) and (2.8) into account. The theorem is proved.

Corollary 3.1. *In the nondegenerate case*

$$\det \partial H^\hbar/\partial s^\hbar \neq 0, \tag{3.12}$$

formula (3.2) *is equivalent to*

$$\rho_N = \frac{1}{c[N]} V_{s^\hbar}\delta(H^\hbar - \mathcal{E}^\hbar[N]), \tag{3.13}$$

where

$$c[N] \overset{\text{def}}{=} (2\pi)^n \left| \det \frac{\partial H^\hbar}{\partial s^\hbar}(s^\hbar[N]) \right|^{-1} = \int \delta(H^\hbar - \mathcal{E}^\hbar[N])\, d\mathcal{L}^\hbar. \tag{3.14}$$

In view of (2.23), one can also obtain the following chain of identities:

$$\widehat{V_H g(H)} = g(\widehat{H}) = g(H^\hbar(\widehat{s^\hbar})) = \widehat{V_{s^\hbar} g(H^\hbar)},$$

and thus, by $\operatorname{mod} O(\hbar^\infty)$, we have

$$V_H g(H) = V_{s^\hbar} g(H^\hbar)$$

for any function g localized inside the $[\alpha^\hbar, \beta^\hbar]$ interval. Then (3.13) implies the following assertion.

Corollary 3.1a. *In the nondegenerate case* (3.12), *the eigenstate density is given* $\operatorname{mod} O(\hbar^\infty)$ *by the formula*

$$\rho_N = \frac{1}{c[N]} V_H \delta(H - \mathcal{E}^\hbar[N]). \tag{3.12a}$$

Note that, by (2.9) and (2.10), we obtain the *asymptotics of the quantum Liouville measure*,

$$d\mathcal{L}^\hbar = (1 + \hbar^2 \langle\!\langle s_j, \tau^j \rangle\!\rangle + O(\hbar^4))\, d\mathcal{L}. \tag{3.15}$$

For the Hamiltonians H, one has the representation (2.13) and, for the transformation V_N, one can use (2.15). Thus, using (3.12a) and (3.14) in the nondegenerate case, we can derive the asymptotics

$$\rho_N = \frac{1}{(2\pi)^n}\left[\delta(H^0 - \mathcal{E}^\hbar[N]) + \hbar^2\left(\gamma + M\frac{\partial}{\partial H^0} - \Delta_{H^0}\right)\delta(H^0 - \mathcal{E}^\hbar[N]) + O(\hbar^4)\right]. \tag{3.16}$$

Here

$$\gamma \overset{\text{def}}{=} \left[\frac{\partial}{\partial s_j}\langle L_j\rangle - \langle\, \langle\!\langle s_j, \tau^j \rangle\!\rangle\, \rangle\right]\Big|_{s = s^\hbar[N]},$$

where the angular brackets stand for the averaging over the classical angles, and the operator Δ_{H^0} is given by (2.14) (s must be replaced by H^0).

4. TRACE FORMULAS

The computations of trace for functions in commuting operators with discrete spectrum is made by summing over the grid of eigenvalues. The basic instrument here is the Poisson summation formula relating the sum to an integral with some oscillating factors. In our case, where the grid of eigenvalues is represented by the grid of phase space tori obeying the discretization rule, the Poisson formula can be represented via integration with respect to the Liouville measure and via phase factors generated by the symplectic form over the tori. If one wants to apply this technique up to $O(\hbar^\infty)$, then one needs to use quantum symplectic geometry (given by the quantum form $\omega^\hbar$ and the quantum Liouville measure $d\mathcal{L}^\hbar$) rather than the classical one. The summation formula adapted to quantum geometry is expressed in the following lemma.

Lemma 4.1. *Let φ be a smooth function localized in $\mathcal{D}$ and constant along the deformed Liouville–Arnold tori. Then*

$$\sum_N \varphi\Big|_{s^\hbar=s^\hbar[N]} = \frac{1}{(2\pi\hbar)^n} \sum_{M\in\pi_1(\mathbb{T}^n)} \int_{T^*\mathbb{R}^n} \exp\left\{\frac{i}{\hbar}\int_{\sigma_M}\omega^\hbar\right\}\varphi\, d\mathcal{L}^\hbar, \tag{4.1}$$

where σ_M is a membrane in $\mathcal{D}$ spans the two closed curves M and $(-M)$ provided that M is a cycle in the homotopy group of the deformed torus containing the given phase space point and $(-M)$ is topologically the same cycle with opposite orientation that belongs to the fixed torus $\{s^\hbar = s^\hbar[N_0]\}$ on which the discretization rule (2.24) *holds.*

Identifying the cycles M in the homotopy group $\pi_1(\mathbb{T}^n)$ of the deformed torus with elements $M \in \mathbb{Z}^n$, one can say that the summation on the right-hand side of (4.1) is taken over membranes whose boundaries wind around the torus generatrices M times.

Note that formula (4.1) is exact, i.e., it is not asymptotical. However, if the function φ in (4.1) does not oscillate as $\hbar \to 0$, then all summands with $M \neq 0$ in (4.1) are of order $O(\hbar^\infty)$. In this case, one obtains much more simple asymptotic formula

$$\sum_N \varphi\Big|_{s^\hbar=s^\hbar[N]} = \frac{1}{(2\pi\hbar)^n} \int_{T^*\mathbb{R}^n} \varphi\, d\mathcal{L}^\hbar + O(\hbar^\infty). \tag{4.2}$$

Let us now apply these formulas to our set of commuting operators $\widehat{H}$. Recall that the joint spectrum of $\widehat{H}$ is given $\operatorname{mod} O(\hbar^\infty)$ by the discretization rule (2.24),

$$\mathcal{E}^\hbar[N] = H^\hbar\big|_{s^\hbar=s^\hbar[N]}. \tag{4.3}$$

Formulas (4.2) and (4.3) imply the following assertion.

Theorem 4.1. *Let a smooth function g be localized in the $[\alpha^\hbar, \beta^\hbar]$ interval. Then*

$$\operatorname{tr} g(\widehat{H}) = \frac{1}{(2\pi\hbar)^n} \int g(H^\hbar)\, d\mathcal{L}^\hbar + O(\hbar^\infty), \tag{4.4}$$

where the symbols $H^\hbar$ stand for the energy functions corresponding to the family of commuting operators $\widehat{H} = (\widehat{H}_1, \dots, \widehat{H}_n)$ and $d\mathcal{L}^\hbar$ for the quantum Liouville measure (3.8).

Note that the trace $\operatorname{tr} g(\widehat{H})$ can also be computed in another way. By applying the quantum composite function transformation (2.8), we obtain

$$g(\widehat{H}) = \widehat{V_H(g(H))},$$

and therefore,

$$\operatorname{tr} g(\widehat{H}) = \frac{1}{(2\pi\hbar)^n} \int V_H(g(H))\, d\mathcal{L}^\hbar. \tag{4.5}$$

This is actually an asymptotical formula rather than an exact one, since the transformation V_H is known only asymptotically. For instance, (2.15) gives $V_H = I - \hbar^2\Delta_H + O(\hbar^4)$.

Now let us consider more complicated functions of the set of commuting operators $\widehat{H}$. For instance, the multi-time Schrödinger dynamics is given by the exponential function $\exp(-\frac{i}{\hbar}t\widehat{H})$, $t \in \mathbb{R}^n$. Because of the presence of $\hbar$ in the denominator of the exponent, one cannot apply the simple version (4.2) of the summation formula to this function. For such cases of oscillating functions, the complete version (4.1) has to be applied.

Theorem 4.2. *The trace of the Schrödinger evolution generated by the quantum integrable system is given by the following formula*

$$\operatorname{tr}\left(e^{-\frac{i}{\hbar}t\widehat{H}}g(\widehat{H})\right) = \frac{1}{(2\pi\hbar)^n}\sum_{M\in\pi_1(\mathbb{T}^n)}\int \exp\left\{\frac{i}{\hbar}\int_{\sigma_M}\omega^\hbar - \frac{i}{\hbar}tH^\hbar\right\} g(H^\hbar)\,d\mathcal{L}^\hbar + O(\hbar^\infty). \tag{4.6}$$

In this theorem, we use the same notation as in Lemma 4.1 and Theorem 4.1.

In the leading term as $\hbar\to 0$, formula (4.6) is reduced to the Berry and Tabor result [6]. The use of quantum geometry makes it possible to write out the trace formula in all orders of the quantization parameter $\hbar$.

Of course, in (4.6), one can apply the stationary phase method in order to eliminate the integrals. The stationary points are given by the equations

$$t\partial H^\hbar/\partial s^\hbar = 2\pi M, \tag{4.7}$$

where M is regarded as an element of $\mathbb{Z}^n$. Equation (4.7) means that the trajectory of the Hamiltonian system generated by $H^\hbar$ and by the quantum symplectic form $\omega^\hbar$ is periodic. When M is running over $\mathbb{Z}^n$, we obtain all periodic trajectories of this Hamiltonian system by (4.7). Let us stress that this is not the ordinary system generated by H^0 with respect to the classical form ω; this is a deformed Hamiltonian system. We refer to it as the *quantum Hamiltonian system.*

In each integral (4.6), the contribution of the phase $\int_{\sigma_M}\omega^\hbar$ at the stationary point (4.7) is given by the quantum symplectic area of the membrane $\sigma^\hbar$ spanning the periodic trajectories of the quantum integrable system.

Thus, the stationary phase method transforms formula (4.6) as follows:

$$\operatorname{tr}\left(e^{-\frac{i}{\hbar}t\widehat{H}}g(\widehat{H})\right) = \left(\frac{2\pi}{\hbar}\right)^{n/2}\sum_{\substack{\text{periodic}\\ \text{trajectories } \partial\sigma^\hbar\\ \text{of the quantum}\\ \text{Hamiltonian}\\ \text{system}}} \exp\left\{\frac{i}{\hbar}\int_{\sigma^\hbar}\omega^\hbar - \frac{i}{\hbar}tH^\hbar\Big|_{\partial\sigma^\hbar}\right\}\frac{g(H^\hbar)e^{-i\frac{\pi}{2}\nu^\hbar}}{\sqrt{|\det(tD^2H^\hbar)|}} \tag{4.8}$$

$$+\ \textit{next stationary phase method terms.}$$

The amplitudes in the leading terms of the asymptotics (4.8) are taken on the same periodic trajectories, i.e., on the boundaries of the membranes $\sigma^\hbar$. The index $\nu^\hbar = m^\hbar + \operatorname{index}(t\cdot D^2H^\hbar)$ in (4.8) is given by the topological index $m^\hbar$ from the discretization rule (2.24) plus the inertia index of the matrix of second derivatives of the energy functions $H^\hbar$.

In conclusion, note that, using the trace formula (4.8), one readily derives the formula for the spectral density by applying the Fourier transform with respect to t.

Corollary 4.1. *The asymptotics of the spectral density of the set of commuting operators* $\widehat{H} = (\widehat{H}_1,\dots,\widehat{H}_n)$ *is given by*

$$\operatorname{tr}(\mathcal{E}-\widehat{H}) = \frac{1}{(2\pi\hbar)^n}\sum_M\int e^{\frac{i}{\hbar}\int_{\sigma_M}\omega^\hbar}\delta(\mathcal{E}-H^\hbar)\,d\mathcal{L}^\hbar + O(\hbar^\infty), \tag{4.9}$$

where $\omega^\hbar$ *stands for the quantum symplectic form,* $d\mathcal{L}^\hbar$ *for the quantum Liouville measure, and* $H^\hbar$ *for the energy functions corresponding to the Hamiltonians* H.

In the leading term as $\hbar\to 0$, formula (4.9) is again reduced to the Berry and Tabor result [6]. The introduction of the quantum geometrical and dynamical objects $\omega^\hbar$, $d\mathcal{L}^\hbar$, and $H^\hbar$ keeps the structure of this result up to $O(\hbar^\infty)$.

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