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Multi-Valued Mappings of Bounded Generalized Variation

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Abstract—We study the mappings taking real intervals into metric spaces and possessing a bounded generalized variation in the sense of Jordan–Riesz–Orlicz. We establish some embeddings of function spaces, the structure of the mappings, the jumps of the variation, and the Helly selection principle. We show that a compact-valued multi-valued mapping of bounded generalized variation with respect to the Hausdorff metric has a regular selection of bounded generalized variation. We prove the existence of selections preserving the properties of multi-valued mappings that are defined on the direct product of an interval and a topological space, have a bounded generalized variation in the first variable, and are upper semicontinuous in the second variable.

Key words: *multi-valued mapping, bounded generalized variation, selection, Helly selection principle, Hausdorff metric.*

1. INTRODUCTION

A *multi-valued mapping* from a set E into a set X is a mapping F that to each element $t \in E$ assigns some (nonempty) set $F(t) \subset X$ called the *image* of t. The *graph* of F is defined to be the set $\mathrm{Gr}(F) = \{(t, x) \in E \times X \mid x \in F(t)\}$. Any single-valued mapping $f\colon E \to X$ such that $f(t) \in F(t)$ for all $t \in E$ is called a *selection* of the multi-valued mapping F.

Many papers (e.g., [1–5]) deal with continuous and measurable selections under rather general assumptions on E and X. As a rule, continuous selections exist for multi-valued mappings with convex images. Examples [1, 6] show that if the image is not assumed to be convex, then continuous selections need not exist even for continuous mappings from real intervals into compact subsets of $\mathbb{R}^2$ or for Lipschitz mappings from $\mathbb{R}^3$ into compact subsets of the space $\mathbb{R}^3$. Selections inheriting the global (i.e., depending on the entire domain of definition) typical properties of the original multi-valued mapping are of independent interest. Such mappings will be called *regular*.

Let $\mathrm{c}(X)$ denote the family of all nonempty compact subsets of the metric space (X, d). The *Hausdorff metric* on $\mathrm{c}(X)$ is determined by the relation $D(A, B) = \max\{\mathrm{e}(A, B), \mathrm{e}(B, A)\}$, where

$$\mathrm{e}(A, B) = \sup_{x \in A} \mathrm{dist}(x, B) \qquad \text{and} \qquad \mathrm{dist}(x, B) = \inf_{y \in B} d(x, y), \qquad A, B \in \mathrm{c}(X). \tag{1}$$

The existence problem for regular selections of a multi-valued mapping of bounded variation leads to mappings ranging in such a metric space. In what follows, we study the following properties of these mappings: embeddings of functions spaces, the structure of mappings, the jumps of the variation, and the Helly selection principle. In Theorem 10 we show that if the mappings are not convex-valued and no restrictions are imposed on the graphs, then the multi-valued mappings F from intervals of $\mathbb{R}$ into compact subsets of the metric space, having a bounded generalized Φ-variation, possess selections preserving this property. This fact significantly generalizes the results

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about the selections of Lipschitz absolutely continuous mappings of bounded variation obtained in [7–9] (where X is finite-dimensional), as well as in [10, D 1.8] and [11–15] (where X is a Banach space and $\mathrm{Gr}(F)$ is compact) and in [16, 17] (where X is an arbitrary metric space and $\mathrm{Gr}(F)$ is arbitrary). In the last Theorem 15 we prove the existence of (continuous) selections inheriting the properties of a compact-valued multi-valued mapping defined on the direct product of a real interval and some topological space, possessing a bounded generalized Φ-variation in the first argument, and being upper semicontinuous in the second argument.

The results of the present paper were described at the XXIII International Summer Symposium on Real Analysis (June 21–26, 1999, Łódź, Poland) and were announced in [18].

2. φ-FUNCTIONS AND MAPPINGS OF BOUNDED Φ-VARIATION

A function Φ acting from $\mathbb{R}^+ = [0, \infty)$ into $\mathbb{R}^+$ is called a *φ-function* (e.g., [19, Sec. 2]) if it is continuous, nondecreasing, and unbounded and if $\Phi(\rho) = 0$ only for $\rho = 0$. We will say that a φ-function Φ *satisfies the Orlicz condition* or Φ *is an Orlicz φ-function* if $\lim_{\rho\to\infty} \Phi(\rho)/\rho = \infty$. The set of all convex (down) Orlicz φ-functions will be denoted by $\mathcal{N}$.

The *right inverse* of the φ-function Φ is defined by the rule [20, Chap. 1, Sec. 2]

$$\Phi_+^{-1}(r) = \sup\{\rho \ge 0 \mid \Phi(\rho) \le r\}, \qquad r \in \mathbb{R}^+.$$

The function Φ_+^{-1} maps $\mathbb{R}^+$ into $\mathbb{R}^+$, is right-continuous, nondecreasing, and unbounded and vanishes only if the following relations hold: $\Phi(\Phi_+^{-1}(r)) = r$ for $r \ge 0$, $\Phi_+^{-1}(\Phi(\rho)) \ge \rho$ for $\rho \ge 0$, and $\Phi_+^{-1}(\Phi(\rho) - \varepsilon) \le \rho$ for $\rho > 0$ and $0 < \varepsilon < \Phi(\rho)$. The function Φ is the right inverse of Φ_+^{-1}. For the Orlicz φ-function Φ we have

$$\lim_{r\to+0} r\Phi_+^{-1}\left(\frac{\vartheta}{r}\right) = \vartheta \lim_{\rho\to\infty} \frac{\rho}{\Phi(\rho)} = 0, \qquad \vartheta \in \mathbb{R}^+. \tag{2}$$

Any convex φ-function Φ is strictly increasing. Its inverse $\Phi^{-1} = \Phi_+^{-1}$ is hence continuous and concave. Moreover, the functions $\rho \mapsto \Phi(\rho)/\rho$ and $\omega_\Phi(\rho) = \rho\Phi^{-1}(1/\rho)$ do not decrease for $\rho > 0$. Hence the limits

$$\lim_{\rho\to+0} \frac{\Phi(\rho)}{\rho} \in \mathbb{R}^+, \qquad [\Phi] = \lim_{\rho\to\infty} \frac{\Phi(\rho)}{\rho} \in (0, \infty], \qquad \omega_\Phi(0) = \lim_{\rho\to+0} \omega_\Phi(\rho) \in \mathbb{R}^+$$

exist and $\omega_\Phi(0) = 0$ if and only if $\Phi \in \mathcal{N}$.

Suppose that E is an unbounded subset of $\mathbb{R}$, (X, d) is a metric space, and X^E is the set of all mappings $f\colon E \to X$ from E into X.

Let

$$\mathcal{T}(E) = \{T = \{t_i\}_{i=0}^m \subset E \mid m \in \mathbb{N}, t_{i-1} < t_i, i = 1, \dots, m\}$$

be the set of all partitions of E by finite ordered sets of points from E. For a φ-function Φ, $f \in X^E$, and $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we define the variable

$$V_\Phi[f\,;T] = \sum_{i=1}^m U(t_i, t_{i-1}), \qquad \text{where} \qquad U(t,s) = \Phi\left(\frac{d(f(t), f(s))}{t-s}\right)(t-s). \tag{3}$$

A mapping $f \in X^E$ is called a *mapping of bounded Φ-variation* (*in the sense of Jordan–Riesz–Orlicz*) if there exists a constant $C \ge 0$ such that $V_\Phi[f\,;T] \le C$ for all $T \in \mathcal{T}(E)$. The least constant C possessing this property is called the (*total*) *Φ-variation* of the mapping f on E and is denoted by $\mathbf{V}_\Phi(f, E)$. If $E = [a, b]$ is an interval, then $\mathcal{T}(E)$ in the definition of $\mathbf{V}_\Phi(f, E)$ can be replaced by the set $\mathcal{T}_a^b$ of all partitions T of the interval $[a, b]$, i.e., $T = \{t_i\}_{i=0}^m$, where

$m \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$. The set of $f \in X^E$ such that $\mathbf{V}_\Phi(f, E) < \infty$ will be denoted by $BV_\Phi(E\,; X)$.

If $\Phi(\rho) = \rho$, then the above definition of $\mathbf{V}_\Phi(f, E)$ implies the classical notion of variation in the sense of Jordan [21, Chap. 8; 22, Chap. 4, Sec. 9]. In this case $BV_\Phi(E\,; X)$ is denoted by $BV_1(E\,; X)$ and $\mathbf{V}_\Phi(f, E)$ is denoted by $\mathbf{V}_1(f, E)$ and called the (1-)*variation* of the mapping f on E. If $\Phi(\rho) = \rho^q$, where the constant $q > 1$, then $\mathbf{V}_\Phi(f, E)$ gives the notion of q-variation in the sense of F. Riesz [23, Chap. 2, Sec. 3.36]. The case in which Φ is a convex Orlicz φ-function, $E = [a, b]$ is an interval, and $X = \mathbb{R}$ was studied in [24–26].

Recall that if for $f \in X^E$ there exists a constant $C \ge 0$ such that $d(f(t), f(s)) \le C|t - s|$ for all $t, s \in E$, then f is said to be *Lipschitzian* (and is denoted by $f \in \mathrm{Lip}(E\,; X)$), while the least such constant C is denoted by $\mathbf{L}(f)$.

A mapping $f \in X^E$ is said to be *absolutely continuous* if there exists a function $\delta\colon (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0$ and any finite set of points

$$\{a_i, b_i\}_{i=1}^n \subset E, \qquad a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n,$$

the condition $\sum_{i=1}^n (b_i - a_i) \le \delta(\varepsilon)$ implies that $\sum_{i=1}^n d(f(b_i), f(a_i)) \le \varepsilon$. The set of absolutely continuous mappings from E into X will be denoted by $AC(E\,; X)$.

Let us briefly discuss the case in which $(X, \|\cdot\|)$ is a Banach space. In [27] it is shown that a Lipschitzian mapping $f\colon [a, b] \to X$ may happen to be nondifferentiable either strongly (in the norm of X) or weakly (on each functional from the strongly dual space of X) at all points of the interval (a, b). However, if X is reflexive, then any mapping $f \in AC([a, b]\,; X)$ is strongly differentiable almost everywhere on (a, b) and can be represented as the indefinite Bochner integral of its strong derivative f'. For a convex φ-function Φ, by $L^0_\Phi([a, b]\,; X)$ we denote the space of all (equivalence classes of) strongly measurable mappings $f \in X^{[a,b]}$ for which the Lebesgue integral $\int_a^b \Phi(\|f(t)\|)\,dt$ is finite. In [14] the following theorem was proved.

Theorem 1. *Let X be a reflexive Banach space, and let Φ be a convex φ-function. Any mapping $f \in BV_\Phi([a, b]\,; X)$ has the weak derivative $f^\bullet \in L^0_\Phi([a, b]\,; X)$ almost everywhere on (a, b), and*

$$\int_a^b \Phi(\|f^\bullet(t)\|)\,dt \le \mathbf{V}_\Phi(f, [a, b]).$$

Moreover, if $\Phi \in \mathcal{N}$, then $f \in BV_\Phi([a, b]\,; X)$ if and only if $f \in AC([a, b]\,; X)$ and the strong derivative $f' \in L^0_\Phi([a, b]\,; X)$ exists; in this case the following integral formula holds:

$$\mathbf{V}_\Phi(f, [a, b]) = \int_a^b \Phi(\|f'(t)\|)\,dt.$$

For $X = \mathbb{R}$, the criterion given in this theorem is well known due to F. Riesz ([23, Chap. 2, Sec. 3.36], where $\Phi(\rho) = \rho^q$, $q > 1$), Medvedev [24], and Cybertowicz and Matuszewska [25] (for $\Phi \in \mathcal{N}$). In the last paper, a integral formula for the Φ-variation is also obtained. But if X is an arbitrary metric space, $\Phi \in \mathcal{N}$, $f \in X^{[a,b]}$, and $\nu(t) = \mathbf{V}_1(f, [a, t])$, $t \in [a, b]$, then the following criterion (established in a somewhat more general form in [14]) holds: $f \in BV_\Phi([a, b]\,; X)$ if and only if $\nu \in AC([a, b]\,; \mathbb{R})$ and $\nu' \in L^0_\Phi([a, b]\,; \mathbb{R})$. Moreover,

$$\mathbf{V}_\Phi(f, [a, b]) = \int_a^b \Phi(|\nu'(t)|)\,dt.$$

For an arbitrary metric space X, the last criterion implies

$$AC([a, b]\,; X) = \bigcup_{\Phi \in \mathcal{N}} BV_\Phi([a, b]\,; X).$$

In fact, a more general assertion holds. Recall that a function $\Phi \in \mathcal{N}$ *satisfies the* Δ'*-condition* if there exist constants $C > 0$ and $\rho_0 > 0$ such that

$$\Phi(\rho_1 \rho_2) \le C\Phi(\rho_1)\Phi(\rho_2) \qquad \forall \rho_1 \ge \rho_0, \quad \forall \rho_2 \ge \rho_0$$

(see [20, Chap. 1, Sec. 5]). From the last criterion and [20, Chap. 2, Sec. 8.1], we obtain: if X is a metric space, then $f \in AC([a,b]\,;X)$ if and only if one can find a function $\Phi \in \mathcal{N}$ satisfying the Δ'-condition and the condition $f \in BV_{\Phi\circ\Phi}([a,b]\,;X)$. Here $\Phi \circ \Phi$ stands for the repeated composition of the function Φ.

Throughout the paper (unless otherwise specified), (X,d) is a metric space.

We list the main properties of the Φ-variation, which we shall use below. We set $E_t^- = E \cap (-\infty, t]$, $E_t^+ = E \cap [t, \infty)$, and $E_s^t = E \cap [s,t]$, where $t, s \in E$, $s \le t$.

Lemma 2. *Suppose that Φ is a φ-function and $f\colon E \to X$. We have*

(a) *if $\varnothing \ne A \subset B \subset E$, then $\mathbf{V}_\Phi(f,A) \le \mathbf{V}_\Phi(f,B)$;*
(b) *if $t, s \in E$ and $s < t$, then $d(f(t), f(s)) \le (t-s)\Phi_+^{-1}(\mathbf{V}_\Phi(f,E)/(t-s))$;*
(c) *if $t \in E$, then $\mathbf{V}_\Phi(f,E_t^-) + \mathbf{V}_\Phi(f,E_t^+) \le \mathbf{V}_\Phi(f,E)$; moreover, if Φ is a convex φ-function, then $\mathbf{V}_\Phi(f,E_t^-) + \mathbf{V}_\Phi(f,E_t^+) = \mathbf{V}_\Phi(f,E)$;*
(d) *if the subsequence $\{f_n\}_{n=1}^\infty$ is contained in X^E and $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for all $t \in E$, then $\mathbf{V}_\Phi(f,E) \le \liminf_{n\to\infty} \mathbf{V}_\Phi(f_n,E)$;*
(e) $\mathbf{V}_\Phi(f,E) = \sup\{\mathbf{V}_\Phi(f,E_s^t) \mid s,t \in E,\ s<t\}$;
(f) *if $s = \sup E \in \mathbb{R}\cup\{\infty\}$ and $s \notin E$, then $\mathbf{V}_\Phi(f,E) = \lim_{E\ni t\to s} \mathbf{V}_\Phi(f,E_t^-)$;*
(g) *if $i = \inf E \in \mathbb{R}\cup\{-\infty\}$ and $i \notin E$, then $\mathbf{V}_\Phi(f,E) = \lim_{E\ni t\to i} \mathbf{V}_\Phi(f,E_t^+)$;*
(h) *if s and i are the same as in* (f) *and* (g)*, $s \notin E$ and $i \notin E$, then we also have*

$$\mathbf{V}_\Phi(f,E) = \lim_{\substack{E\ni a\to i\\ E\ni b\to s}} \mathbf{V}_\Phi(f,E_a^b) = \lim_{E\ni b\to s}\lim_{E\ni a\to i} \mathbf{V}_\Phi(f,E_a^b) = \lim_{E\ni a\to i}\lim_{E\ni b\to s} \mathbf{V}_\Phi(f,E_a^b).$$

Proof. Assertions (a) and (b) follow from the definition of Φ-variation.

(c) For the partitions $T_1 \in \mathcal{T}(E_t^-)$ and $T_2 \in \mathcal{T}(E_t^+)$, we set $\widetilde{T}_i = T_i \cup \{t\}$, $i = 1,2$. Then $\widetilde{T}_1 \cup \widetilde{T}_2 \in \mathcal{T}(E)$ and the inequality in (c) is satisfied, since

$$V_\Phi[f\,;T_1] + V_\Phi[f\,;T_2] \le V_\Phi[f\,;\widetilde{T}_1] + V_\Phi[f\,;\widetilde{T}_2] = V_\Phi[f\,;\widetilde{T}_1 \cup \widetilde{T}_2] \le \mathbf{V}_\Phi(f,E).$$

To establish the equality, we use the fact that the function Φ is convex. Let us consider the partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$. For $t \in T$, $t < t_0$ or $t_m < t$, it is clear that we have $V_\Phi[f\,;T] \le \mathbf{V}_\Phi(f,E_t^-) + \mathbf{V}_\Phi(f,E_t^+)$. Now if $t_{k-1} < t < t_k$ for some $k \in \{1,\dots,m\}$, then

$$V_\Phi[f\,;T] = \left(\sum_{i=1}^{k-1} U(t_i,t_{i-1})\right) + U(t_k,t_{k-1}) + \left(\sum_{i=k+1}^{m} U(t_i,t_{i-1})\right) \tag{4}$$

(here the first sum vanishes for $k=1$ and the last sum vanishes for $k=m$). Using the triangle inequality for d, the fact that Φ is increasing, and the Jensen inequality for sums, we obtain

$$U(t_k,t_{k-1}) \le U(t_k,t) + U(t,t_{k-1}). \tag{5}$$

Hence relations (4) and (5) imply

$$V_\Phi[f\,;T] \le V_\Phi[f\,;\{t_i\}_{i=0}^{k-1}\cup\{t\}] + V_\Phi[f\,;\{t\}\cup\{t_i\}_{i=k}^m] \le \mathbf{V}_\Phi(f,E_t^-) + \mathbf{V}_\Phi(f,E_t^+).$$

(d) Taking into account the fact that the metric d and the function Φ are continuous and f_n converges to f pointwise, for any partition $T = \{t_i\}_{i=0}^m$ of the set E, it suffices to pass to the lower limit as $n \to \infty$ in the inequality $V_\Phi[f_n\,;T] \le \mathbf{V}_\Phi(f_n, E)$ and to note that

$$\lim_{n\to\infty} V_\Phi[f_n\,;T] = V_\Phi[f\,;T].$$

(e) Assertion (a) implies that the expression on the left is larger than or equal to that on the right. Conversely, for any number $\alpha < \mathbf{V}_\Phi(f, E)$ there exists a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ such that $V_\Phi[f\,;T] \ge \alpha$, but $T \in \mathcal{T}(E_{t_0}^{t_m})$. Therefore, we have $\mathbf{V}_\Phi(f, E_{t_0}^{t_m}) \ge V_\Phi[f\,;T] \ge \alpha$.

(f) Since $s \notin E$, the point s is a limit point of the set E. Assertion (a) implies that the function $t \mapsto \mathbf{V}_\Phi(f, E_t^-)$ from E into $[0,\infty]$ does not decrease. Hence the limit written in item (f) exists in $[0,\infty]$ and does not exceed $\mathbf{V}_\Phi(f, E)$. On the other hand, it follows from assertion (e) that, for any number $\alpha < \mathbf{V}_\Phi(f, E)$, there exist numbers $a, b \in E$, $a < b < s$, such that $\mathbf{V}_\Phi(f, E_a^b) \ge \alpha$. Whence, for any $t \in E \cap [b, s) \ne \varnothing$, by virtue of (a), we obtain $\mathbf{V}_\Phi(f, E_t^-) \ge \mathbf{V}_\Phi(f, E_a^b) \ge \alpha$. Assertion (f) is thus proved.

Assertion (g) and the first relation in (h) can be proved similarly to assertion (f). For the second relation, we have

$$\begin{aligned}\mathbf{V}_\Phi(f, E) &= \lim_{E\ni b\to s} \mathbf{V}_\Phi(f, E_b^-) = \lim_{E\ni b\to s}\,\lim_{E\ni a\to i} \mathbf{V}_\Phi(f, (E_b^-)_a^+)\\ &= \lim_{E\ni b\to s}\,\lim_{E\ni a\to i} \mathbf{V}_\Phi(f, E_a^b).\end{aligned}$$

The last relation in (h) can be proved in the same way. □

The inequality in Lemma 2(d) can be strict: the sequence of functions

$$f_n(t) = \frac{1}{2\pi n}\cos(2\pi n t)$$

converges as $n \to \infty$ to the function $f \equiv 0$ uniformly in $t \in [0,1]$, and the integral formula for the Φ-variation implies

$$\mathbf{V}_\Phi(f_n, [0,1]) = \frac{2}{\pi}\int_0^{\pi/2} \Phi(\sin t)\,dt > 0,$$

where $n \in \mathbb{N}$ and Φ is a convex φ-function.

Lemma 3. *Suppose that Φ and Ψ are convex φ-functions. If the set E is bounded and the Δ_Ψ^Φ-condition*

$$\limsup_{\rho\to\infty} \frac{\Psi(\rho)}{\Phi(\rho)} < \infty$$

is satisfied, then $BV_\Phi(E\,;X) \subset BV_\Psi(E\,;X)$. Conversely, if $E = [a, b]$ is an interval, $(X, \|\cdot\|)$ is a linear normed space, and $BV_\Phi(E\,;X) \subset BV_\Psi(E\,;X)$, then the Δ_Ψ^Φ-condition is satisfied.

Proof. The Δ_Ψ^Φ-condition is equivalent to that $\Psi(\rho) \le C\Phi(\rho)$ for all $\rho \ge \rho_0$, where $C > 0$ and $\rho_0 > 0$ are some constants. Then, for $f \in BV_\Phi(E\,;X)$ and $T \in \mathcal{T}(E)$, we have

$$V_\Psi[f\,;T] \le \Psi(\rho_0)(\sup E - \inf E) + C\mathbf{V}_\Phi(f, E).$$

Recall that the convex φ-function Φ has the nondecreasing right-continuous right-hand derivative $\Phi'_+(\rho)$, $\rho \ge 0$, and $\Phi'_+(\rho) > 0$ for $\rho > 0$. Hence we have

$$\exists \rho(\Phi) > 0 \qquad c(\Phi) > 0: \qquad \forall \rho \ge \rho(\Phi) \quad \Phi(\rho) \ge c(\Phi)\rho. \tag{6}$$

Now if the Δ^{Φ}_{Ψ}-condition is violated, then there exists an infinitely increasing sequence of positive numbers $\{\rho_n\}_{n=1}^{\infty}$ such that $\Psi(\rho_n) > 2^n\Phi(\rho_n)$, $n \in \mathbb{N}$. We define the sequence $\{a_n\}_{n=0}^{\infty}$ in $[a, b]$ recurrently as follows:

$$a_0 = a, \qquad a_n - a_{n-1} = 2^{-n}(b-a)\frac{\Phi(\rho_1)}{\Phi(\rho_n)}, \quad n \in \mathbb{N}.$$

We set

$$f(t) = \begin{cases} (\rho_n(t - a_{n-1}) + S_{n-1})x & \text{if } a_{n-1} \le t < a_n,\ n \in \mathbb{N}, \\ S_\infty x & \text{if } \lim_{n\to\infty} a_n \le t \le b, \end{cases}$$

where

$$S_0 = 0, \qquad S_k = \sum_{i=1}^{k} \rho_i(a_i - a_{i-1}), \quad k \in \mathbb{N} \cup \{\infty\},$$

$S_\infty < \infty$ by (6), $x \in X$, $\|x\| = 1$. We show that $f \in BV_\Phi([a, b]\,; X)$, but $f \notin BV_\Psi([a, b]\,; X)$. Indeed,

$$\mathbf{V}_\Phi(f, [a, b]) = \sum_{n=1}^{\infty} U(a_n, a_{n-1}) = \sum_{n=1}^{\infty} \Phi(\rho_n)(a_n - a_{n-1}) = (b-a)\Phi(\rho_1) < \infty.$$

On the other hand, for an arbitrary $m \in \mathbb{N}$ and $T_m = \{a_n\}_{n=0}^{m}$, we have

$$\mathbf{V}_\Psi(f, [a, b]) \ge V_\Psi[f\,; T_m] = \sum_{n=1}^{m} \Psi(\rho_n)\frac{(b-a)\Phi(\rho_1)}{2^n\Phi(\rho_n)} \ge m(b-a)\Phi(\rho_1). \quad \square$$

Lemma 3 and assertions (6) for a convex φ-function Φ imply that the set $BV_\Phi(E\,; X)$ is embedded in $BV_1(E\,; X)$. Moreover, if the value of $[\Phi]$ (equal also to the supremum $\sup_{\rho>0}\Phi(\rho)/\rho$) is finite, then $BV_\Phi(E\,; X) = BV_1(E\,; X)$, which follows from the inequality $\mathbf{V}_\Phi(f, E) \le [\Phi]\mathbf{V}_1(f, E)$ for $f \in BV_1(E\,; X)$.

3. SPACE OF MAPPINGS OF GENERALIZED Φ-VARIATION

Let (X, d) be a metric space, and let $E \subset \mathbb{R}$. For $\lambda > 0$ and a convex φ-function Φ, we set $\Phi_\lambda(\rho) = \Phi(\rho/\lambda)$, $\rho \in \mathbb{R}^+$. For brevity, we write BV_Φ instead of $BV_\Phi(E\,; X)$ if this does not lead to ambiguity. It follows from Lemma 3 that $BV_{\Phi_\lambda} \subset BV_\Phi$ for $0 < \lambda \le 1$ and $BV_\Phi \subset BV_{\Phi_\lambda}$ for $\lambda > 1$. In general, the last inclusion is strict. For instance, if $\Phi(\rho) = e^\rho - 1$, $f(t) = t(1 - \log t)$ for $0 < t \le 1$, and $f(0) = 0$, then, using the integral formula for the Φ-variation in Theorem 1, we obtain $\mathbf{V}_{\Phi_\lambda}(f, [0, 1]) = 1/(\lambda - 1)$ for $\lambda > 1$ and $\mathbf{V}_{\Phi_\lambda}(f, [0, 1]) = \infty$ for $0 < \lambda \le 1$. By Lemma 3, the opposite inclusion $BV_{\Phi_\lambda} \subset BV_\Phi$ for $\lambda > 1$ holds if and only if Φ *satisfies the* Δ_2*-condition* (see [20, Chap. 1, Sec. 4; 19, Sec. 3]): $\limsup_{\rho\to\infty}\Phi(2\rho)/\Phi(\rho) < \infty$. This condition is equivalent to the following one: for any number $\lambda > 1$ there exist constants $C(\lambda) > 0$ and $\rho_0(\lambda) > 0$ such that $\Phi(\rho) \le C(\lambda)\Phi(\rho/\lambda)$ for all $\rho \ge \rho_0(\lambda)$. (Note that if $[\Phi] < \infty$, then Φ satisfies the Δ_2-condition.)

In the general case, for any convex φ-function Φ, the *space of mappings of bounded generalized* Φ*-variation* is the set

$$GV_\Phi(E\,; X) = \bigcup_{\lambda>0} BV_{\Phi_\lambda}(E\,; X). \tag{7}$$

It follows from the above that if a convex φ-function Φ satisfies the Δ_2-condition, then $GV_\Phi = BV_\Phi$. Conversely, if $E = [a, b]$, X is a linear normed space, and $GV_\Phi = BV_\Phi$, then Φ satisfies the Δ_2-condition: by virtue of Lemma 3 and the embedding $BV_{\Phi_2} \subset BV_\Phi$, there exist $C > 0$ and $\rho_0 > 0$ such that $\Phi(\rho) \le C\Phi(\rho/2)$, $\rho \ge \rho_0$.

If Φ and Ψ are convex φ-functions, the set E is bounded, and there exist constants $C > 0$ and $\rho_0 > 0$ such that $\Psi(\rho) \le \Phi(C\rho)$ for $\rho \ge \rho_0$, then $GV_\Phi \subset GV_\Psi$, since for $\lambda > 0$, $f \in BV_{\Phi_\lambda}$, and $T \in \mathcal{T}(E)$, we have $V_{\Psi_\mu}[f\,;T] \le \Psi(\rho_0)|\sup E - \inf E| + \mathbf{V}_{\Phi_\lambda}(f, E)$, where $\mu = \lambda C$. The converse statement for $E = [a, b]$ and a linear normed space X is established in [15].

For a reflexive Banach space X and $\Phi \in \mathcal{N}$, it follows from Theorem 1 that $f \in GV_\Phi([a, b]\,; X)$ if and only if $f \in AC([a, b]\,; X)$ and $f' \in L_\Phi([a, b]\,; X)$, where $L_\Phi([a, b]\,; X)$ stands for the Orlicz space [20, 19] of mappings $f \in X^{[a,b]}$ such that $\lambda f \in L^0_\Phi([a, b]\,; X)$ for some $\lambda > 0$.

Since the function Φ is convex, the variable

$$p_\Phi(f, E) = \inf\{\lambda > 0 \mid \mathbf{V}_{\Phi_\lambda}(f, E) \le 1\}, \qquad f \in GV_\Phi(E\,; X), \tag{8}$$

is well defined. The following lemma describes the main properties of $p_\Phi(f, E)$.

Lemma 4. *For a convex φ-function Φ and $f \in GV_\Phi(E\,; X)$, we have*

(a) *$d(f(t), f(s)) \le \omega_\Phi(|t - s|)p_\Phi(f, E)$ for all $t, s \in E$;*
(b) *if $p_\Phi(f, E) = \lambda > 0$, then $\mathbf{V}_{\Phi_\lambda}(f, E) \le 1$;*
(c) *if $\lambda > 0$, then $p_\Phi(f, E) \le \lambda$ if and only if $\mathbf{V}_{\Phi_\lambda}(f, E) \le 1$;*
(d) *if $\lambda > 0$ and $\mathbf{V}_{\Phi_\lambda}(f, E) = 1$, then $p_\Phi(f, E) = \lambda$;*
(e) *if $\{f_n\}_{n=1}^\infty \subset GV_\Phi(E\,; X)$ and $\lim_{n\to\infty} d(f_n(t), f(t)) = 0$ for all $t \in E$, then $p_\Phi(f, E) \le \liminf_{n\to\infty} p_\Phi(f_n, E)$;*
(f) *if $t \in E$, then $p_\Phi(f, E) \le p_\Phi(f, E_t^-) + p_\Phi(f, E_t^+)$.*

Proof. (a) For any $t, s \in E$, $s < t$, the definitions imply

$$\Phi\biggl(\frac{d(f(t), f(s))}{(t - s)\lambda}\biggr)(t - s) \le \mathbf{V}_{\Phi_\lambda}(f, E) \le 1 \qquad \text{for} \quad \lambda > p_\Phi(f),$$

whence, dividing by $t - s$ and calculating the inverse function Φ^{-1}, we obtain (a).

(b) Let $p_\Phi(f) = \lambda > 0$. We choose numbers $\lambda(n) > \lambda$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \lambda(n) = \lambda.$$

Since $\mathbf{V}_{\Phi_{\lambda(n)}}(f, E) \le 1$ for all $n \in \mathbb{N}$, we find

$$\mathbf{V}_{\Phi_\lambda}(f, E) = \lim_{n\to\infty} \mathbf{V}_{\Phi_{\lambda(n)}}(f, E) \le 1.$$

(c) It suffices to show that if $0 < p_\Phi(f) < \lambda$, then $\mathbf{V}_{\Phi_\lambda}(f, E) < 1$, which readily follows from the convexity of Φ and assertion (b): by setting $\mu = p_\Phi(f)$, we obtain

$$\mathbf{V}_{\Phi_\lambda}(f, E) \le (\mu/\lambda)\mathbf{V}_{\Phi_\mu}(f, E) \le \mu/\lambda < 1.$$

(d) It should be only noted that, by (c) and the statement just proved, the cases $p_\Phi(f) > \lambda$ and $p_\Phi(f) < \lambda$ are impossible.

(e) We assume that $\lambda = \liminf_{n\to\infty} p_\Phi(f_n) < \infty$. Then $p_\Phi(f_{n_k}) \to \lambda$ as $k \to \infty$ for some subsequence $\{f_{n_k}\}_{k=1}^\infty$ of the sequence $\{f_n\}_{n=1}^\infty$. Hence, for any $\varepsilon > 0$, there exists a number $k_0(\varepsilon) \in \mathbb{N}$ such that $p_\Phi(f_{n_k}) < \lambda + \varepsilon$ for all $k \ge k_0(\varepsilon)$. It follows from the definition of $p_\Phi(f_{n_k})$ that $\mathbf{V}_{\Phi_{\lambda+\varepsilon}}(f_{n_k}) \le 1$ for $k \ge k_0(\varepsilon)$. Since f_{n_k} converges to f pointwise on E as $k \to \infty$, Lemma 2(d) implies $\mathbf{V}_{\Phi_{\lambda+\varepsilon}}(f) \le \liminf_{k\to\infty} \mathbf{V}_{\Phi_{\lambda+\varepsilon}}(f_{n_k}) \le 1$, whence $p_\Phi(f) \le \lambda + \varepsilon$ for any $\varepsilon > 0$.

(f) We set $\lambda = p_\Phi(f, E_t^-)$ and $\mu = p_\Phi(f, E_t^+)$. If at least one of the numbers λ or μ is zero, then, by virtue of (a), the inequality (in fact, the equality) is obvious. Let $\lambda > 0$, and let $\mu > 0$. It follows from (b) that $\mathbf{V}_{\Phi_\lambda}(f, E_t^-) \le 1$ and $\mathbf{V}_{\Phi_\mu}(f, E_t^+) \le 1$. The inequality $p_\Phi(f, E) \le \lambda + \mu$, by virtue of (c), is equivalent to the inequality $\mathbf{V}_{\Phi_{\lambda+\mu}}(f, E) \le 1$. To prove the last inequality, we consider the partition $T = \{t_i\}_{i=0}^m$ of the set E such that $t_{k-1} \le t \le t_k$ for some $k \in \{1, \dots, m\}$ (the cases $t < t_0$ and $t > t_m$ can be considered similarly). By $U_\lambda(t, s)$ we denote the expression $U(t, s)$ in (3) corresponding to the function Φ_λ. For the variable $V_{\Phi_{\lambda+\mu}}[f; T]$ in (3), we have relation (4) with U replaced by $U_{\lambda+\mu}$. Taking into account the fact that the function Φ is convex, we obtain the inequalities

$$U_{\lambda+\mu}(t_i, t_{i-1}) \le \frac{\lambda}{\lambda+\mu} U_\lambda(t_i, t_{i-1}), \qquad i = 1, \dots, k-1,$$

$$U_{\lambda+\mu}(t_k, t_{k-1}) \le U_{\lambda+\mu}(t, t_{k-1}) + U_{\lambda+\mu}(t_k, t) \le \frac{\lambda}{\lambda+\mu} U_\lambda(t, t_{k-1}) + \frac{\mu}{\lambda+\mu} U_\mu(t_k, t),$$

$$U_{\lambda+\mu}(t_i, t_{i-1}) \le \frac{\mu}{\lambda+\mu} U_\mu(t_i, t_{i-1}), \qquad i = k+1, \dots, m,$$

which, together with (4), imply

$$V_{\Phi_{\lambda+\mu}}[f; T] \le \frac{\lambda}{\lambda+\mu} \mathbf{V}_{\Phi_\lambda}(f, E_t^-) + \frac{\mu}{\lambda+\mu} \mathbf{V}_{\Phi_\mu}(f, E_t^+) \le 1. \quad \square$$

One of the advantages of passing to the space $GV_\Phi(E; X)$ is that the notion of a mapping $f \in GV_\Phi(E; X)$ of bounded generalized Φ-variation depends on the metric topology on X rather than on a concrete metric. Indeed, if d and d_0 are equivalent metrics on X, i.e., if $C_0 d(x, y) \le d_0(x, y) \le C_1 d(x, y)$ for some constants $C_0 > 0$ and $C_1 > 0$ and for all $x, y \in X$, and if $f \in GV_\Phi(E; X)$ in the metric d, then $f \in GV_\Phi(E; X)$ in the metric d_0 and the following inequalities hold:

$$C_0 p_\Phi^d(f, E) \le p_\Phi^{d_0}(f, E) \le C_1 p_\Phi^d(f, E),$$

where $p_\Phi^d(f, E)$ denotes the value of (8) calculated in the metric d.

It follows from the above that we have the following embeddings of function spaces for a bounded set E and a convex φ-function Φ:

$$\operatorname{Lip}(E; X) \subset BV_\Phi(E; X) \subset GV_\Phi(E; X) \subset BV_1(E; X). \tag{9}$$

If, in addition, $\Phi \in \mathcal{N}$, then $GV_\Phi(E; X) \subset AC(E; X)$. (Note that if Φ is not necessarily a convex Orlicz φ-function, then, by Lemma 4(a) and relation (2), any mapping $f \in GV_\Phi(E; X)$ is continuous.) Moreover, the following *Jensen inequalities for variation* hold:

$$\Phi\left(\frac{\mathbf{V}_1(f, E)}{|E|}\right) \le \frac{\mathbf{V}_\Phi(f, E)}{|E|}, \qquad f \in BV_\Phi(E; X), \tag{10}$$

$$\mathbf{V}_1(f, E) \le \omega_\Phi(|E|) p_\Phi(f, E), \qquad f \in GV_\Phi(E; X), \tag{11}$$

where $|E| = \sup E - \inf E < \infty$. Indeed, if $f \in BV_\Phi(E; X)$ and T is a partition of E of the form $\{t_i\}_{i=0}^m$, then, using the symbol U from (3), by virtue of the Jensen inequality for sums, we have

$$\Phi\left(\frac{\sum_{i=1}^m d(f(t_i), f(t_{i-1}))}{\sum_{i=1}^m (t_i - t_{i-1})}\right) \le \frac{\sum_{i=1}^m U(t_i, t_{i-1})}{\sum_{i=1}^m (t_i - t_{i-1})} \le \frac{\mathbf{V}_\Phi(f, E)}{\sum_{i=1}^m (t_i - t_{i-1})}.$$

Whence, taking into account the fact that $\sum_{i=1}^m (t_i - t_{i-1}) = t_m - t_0 \le |E|$, we obtain (10). But if $f \in GV_\Phi(E; X)$, then, by setting $\lambda = \mathbf{V}_1(f, E)/\omega_\Phi(|E|)$ and assuming that $\lambda > 0$, by virtue of (10), we have $\mathbf{V}_{\Phi_\lambda}(f, E) \ge |E| \Phi_\lambda(\mathbf{V}_1(f, E)/|E|) = 1$. Therefore, the argument used in the proof of Lemma 4(c) implies $p_\Phi(f, E) \ge \lambda$.

Structurally, the mappings of bounded Φ-variation are closely related to Lipschitzian mappings and to real functions of bounded (generalized) Φ-variation. Let a function $\nu\colon E\to\mathbb{R}$ satisfy the conditions: $J=\nu(E)$ is the image of ν, $g\in \operatorname{Lip}(J\,;X)$, $\mathbf{L}(g)\le 1$, and $f=g\circ\nu$, where $(g\circ\nu)(t):=g(\nu(t))$, $t\in E$. The following assertions hold for a convex φ-function Φ: if $\nu\in BV_\Phi(E\,;\mathbb{R})$, then $f\in BV_\Phi(E\,;X)$ and $\mathbf{V}_\Phi(f,E)\le\mathbf{V}_\Phi(\nu,E)$; if $\nu\in GV_\Phi(E\,;\mathbb{R})$, then $f\in GV_\Phi(E\,;X)$ and $p_\Phi(f,E)\le p_\Phi(\nu,E)$. In Lemma 5 we show that the converse statements also hold.

A mapping $g\colon J\to X$ is said to be *natural* if $\mathbf{V}_1(g,J_s^t)=t-s$ for all $s,t\in J$, $s\le t$. Such a mapping is Lipschitzian and $\mathbf{L}(g)\le 1$, since $d(g(t),g(s))\le\mathbf{V}_1(g,J_s^t)$ by Lemma 2(b) for $\Phi(\rho)=\rho$.

Lemma 5. *Suppose that $f\in BV_1(E\,;X)$, $\nu(t)=\mathbf{V}_1(f,E_t^-)$, $t\in E$, and $J=\nu(E)$. Then there exists a natural mapping $g\colon J\to X$ such that $f=g\circ\nu$ on E and $\mathbf{V}_1(g,J)=\mathbf{V}_1(f,E)$. If E is bounded and Φ is a convex φ-function, we have, in addition, the following assertions*:

(a) *if $f\in BV_\Phi(E\,;X)$, then $\nu\in BV_\Phi(E\,;\mathbb{R})$ and $\mathbf{V}_\Phi(\nu,E)=\mathbf{V}_\Phi(f,E)$;*
(b) *if $f\in GV_\Phi(E\,;X)$, then $\nu\in GV_\Phi(E\,;\mathbb{R})$ and $p_\Phi(\nu,E)=p_\Phi(f,E)$.*

Proof. By (9) and Lemma 2(a), the function ν is well defined and does not decrease on E. If $s\in J$ so that $s=\nu(t)$ for some $t\in E$, then we set $g(s)=f(t)$. Since $d(f(t),f(s))\le|\nu(t)-\nu(s)|$, $t,s\in E$, the mapping $g\colon J\to X$ is well defined and satisfies all the assumptions of our lemma (for detail, see [12, Sec. 3]). Under the assumptions of item (a) on the partition $T=\{t_i\}_{i=0}^m\in\mathcal{T}(E)$, from Lemma 2(c) with $\Phi(\rho)=\rho$ and inequality (10) for $i=1,\dots,m$, we obtain

$$\nu(t_i)-\nu(t_{i-1})=\mathbf{V}_1(f,E_{t_{i-1}}^{t_i})\le(t_i-t_{i-1})\Phi^{-1}\biggl(\frac{\mathbf{V}_\Phi(f,E_{t_{i-1}}^{t_i})}{t_i-t_{i-1}}\biggr),$$

whence, calculating $V_\Phi[\nu\,;T]$ by formula (3) and using Lemma 2(c) once more, we obtain the inequality $V_\Phi[\nu\,;T]\le\mathbf{V}_\Phi(f,E)$. But the equality $\mathbf{V}_\Phi(\nu,E)=\mathbf{V}_\Phi(f,E)$ follows from the expansion $f=g\circ\nu$ and the remarks preceding this lemma.

If $f\in GV_\Phi(E\,;X)$, then we set $\lambda=p_\Phi(f,E)$ (let $\lambda>0$) and, by assertion (a) proved above and Lemma 4(b), obtain $\mathbf{V}_{\Phi_\lambda}(\nu,E)=\mathbf{V}_{\Phi_\lambda}(f,E)\le 1$, whence $p_\Phi(\nu,E)\le\lambda$. The opposite inequality follows from the remarks preceding this lemma. $\square$

4. JUMPS OF FUNCTIONS OF Φ-VARIATION AND CONTINUITY

For $t\in E$, the sets of all limit points of E_t^- and E_t^+ will be denoted by $E_t^{-\prime}$ and $E_t^{+\prime}$, respectively. For the limits of the form $\lim_{E\ni\alpha\to t\pm 0}$ used below, we briefly write $\lim_{\alpha\to t\pm0}$. In this section Φ is a convex φ-function such that $[\Phi]<\infty$, $f\in BV_\Phi(E\,;X)$, and $\phi(t)=\mathbf{V}_\Phi(f,E_t^-)$, $t\in E$, is the *function of Φ-variation* of the mapping f. We shall study the continuity properties of f; we show that the discontinuity points of f coincide with the discontinuity points of ϕ; we establish relations between the jumps of f and the jumps of ϕ and find formulas for the Φ-variation of f on the set E without the limit point, which is removed.

If U is defined in (3), then, as was already noted (see (5) and Lemma 2(b,c)), for all $\alpha,\beta,\gamma\in E$, $\alpha<\beta<\gamma$, we have the inequalities

$$U(\gamma,\alpha)\le U(\gamma,\beta)+U(\beta,\alpha),\qquad U(\beta,\alpha)\le\phi(\beta)-\phi(\alpha). \tag{12}$$

The monotonicity and the boundedness of the function ϕ imply that the finite right-hand and left-hand limits $\phi(t\pm0)=\lim_{\alpha\to t\pm0}\phi(\alpha)$ exist at the points $t\in E_t^{\pm\prime}$, respectively. Since the function ν in Lemma 5 possesses the same property, it follows from (3) that the following (one-sided) limits are finite:

$$U(t, t-0) = \lim_{\alpha \to t-0} U(t, \alpha) = [\Phi] \lim_{\alpha \to t-0} d(f(t), f(\alpha)), \qquad t \in E_t^{-\prime},$$
$$U(t+0, t) = \lim_{\beta \to t+0} U(\beta, t) = [\Phi] \lim_{\beta \to t+0} d(f(\beta), f(t)), \qquad t \in E_t^{+\prime},$$
$$U(t+0, t-0) = \lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} U(\beta, \alpha) = [\Phi] \lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} d(f(\beta), f(\alpha)), \qquad t \in E_t^{-\prime} \cap E_t^{-\prime}.$$

Theorem 6. *The following relations hold*:

(a) $U(t, t-0) = \phi(t) - \phi(t-0) = \mathbf{V}_\Phi(f, E_t^-) - \mathbf{V}_\Phi(f, E_t^- \setminus \{t\}) = \lim_{\alpha \to t-0} \mathbf{V}_\Phi(f, E_\alpha^t)$;
(b) $U(t+0, t) = \phi(t+0) - \phi(t) = \mathbf{V}_\Phi(f, E_t^+) - \mathbf{V}_\Phi(f, E_t^+ \setminus \{t\}) = \lim_{\beta \to t+0} \mathbf{V}_\Phi(f, E_t^\beta)$;
(c) $U(t+0, t-0) = \lim_{\substack{\alpha \to t-0 \\ \beta \to t+0}} \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\})$;
(d) $\mathbf{V}_\Phi(f, E) = \mathbf{V}_\Phi(f, E \setminus \{t\}) + \phi(t+0) - \phi(t-0) - U(t+0, t-0)$,

where a point $t \in E_t^{-\prime}$ *is considered in* (a); *a point* $t \in E_t^{+\prime}$ *in* (b); *and a point* $t \in E_t^{-\prime} \cap E_t^{+\prime}$ *in* (c) *and* (d).

Proof. (a) Let us prove the first relation. Passing to the limit as $\alpha \to t-0$ in the second inequality in (12), where $\beta = t > \alpha \in E$, we obtain $U(t, t-0) \le \phi(t) - \phi(t-0)$. The opposite inequality follows in the limit as $\alpha \to t-0$ from the following assertion: for any $\varepsilon > 0$ there exists a $\tau(\varepsilon) \in E$, $\tau(\varepsilon) < t$, such that $\phi(t) - \phi(\alpha) \le U(t, \alpha) + \varepsilon$ for all $\alpha \in E_{\tau(\varepsilon)}^t \setminus \{t\}$. From the definition of $\phi(t)$, with respect to $\varepsilon > 0$, we find a ε-dependent partition $T = \{t_i\}_{i=0}^m \cup \{t\} \in \mathcal{T}(E_t^-)$, where $t_m < t$. For this partition we have $\phi(t) \le U(t, t_m) + V_\Phi[f; T] + \varepsilon$. Since $T \in \mathcal{T}(E_{t_m}^-)$, using the first inequality in (12) and Lemma 2(b,c), we obtain

$$\phi(t) \le U(t, \alpha) + \mathbf{V}_\Phi(f, E_{t_m}^\alpha) + \mathbf{V}_\Phi(f, E_{t_m}^-) + \varepsilon = U(t, \alpha) + \phi(\alpha) + \varepsilon$$

for all $\alpha \in E$, $t_m < \alpha < t$. It remains to set $\tau(\varepsilon) = t_m$.

The other relations follow from Lemma 2(f) that implies

$$\mathbf{V}_\Phi(f, E_t^- \setminus \{t\}) = \lim_{\alpha \to t-0} \mathbf{V}_\Phi(f, (E_t^- \setminus \{t\})_\alpha^-) = \lim_{\alpha \to t-0} \mathbf{V}_\Phi(f, E_\alpha^-) = \phi(t-0).$$

(b) The inequality $U(t+0, t) \le \phi(t+0) - \phi(t)$ is obtained in the limit as $\beta \to t+0$ from the second inequality in (12), where $E \ni \beta > t = \alpha$. It remains to show that, for any $\varepsilon > 0$, there exists a $t_0 = t_0(\varepsilon) \in E$, $t_0 > t$, such that $\phi(\beta) - \phi(t) \le U(\beta, t) + \varepsilon$ for all $\beta \in E_t^{t_0} \setminus \{t\}$. Since $\mathbf{V}_\Phi(f, E_t^+) < \infty$, for $\varepsilon > 0$, there exists a $T = \{t_i\}_{i=0}^m \cup \{t\} \in \mathcal{T}(E_t^+)$ with $t_0 > t$ such that $\mathbf{V}_\Phi(f, E_t^+) \le U(t_0, t) + V_\Phi[f; T] + \varepsilon$. Since $T \in \mathcal{T}(E_{t_0}^+)$, for $\beta \in E$, $t < \beta < t_0$, the first inequality in (12) implies

$$\mathbf{V}_\Phi(f, E_t^+) \le \mathbf{V}_\Phi(f, E_\beta^{t_0}) + U(\beta, t) + \mathbf{V}_\Phi(f, E_{t_0}^+) + \varepsilon = \mathbf{V}_\Phi(f, E_\beta^+) + U(\beta, t) + \varepsilon.$$

Whence, by Lemma 2(c), we obtain

$$\phi(\beta) - \phi(t) = \mathbf{V}_\Phi(f, E_\beta^-) - \mathbf{V}_\Phi(f, E_t^-) = \mathbf{V}_\Phi(f, E_t^+) - \mathbf{V}_\Phi(f, E_\beta^+) \le U(\beta, t) + \varepsilon.$$

The second and the third relations follow from Lemma 2(g,c). Hence we have

$$\begin{aligned}\mathbf{V}_\Phi(f, E_t^+ \setminus \{t\}) &= \lim_{\beta \to t+0} \mathbf{V}_\Phi(f, (E_t^+ \setminus \{t\})_\beta^+) = \lim_{\beta \to t+0} \mathbf{V}_\Phi(f, E_\beta^+) \\ &= \mathbf{V}_\Phi(f, E) - \lim_{\beta \to t+0} \mathbf{V}_\Phi(f, E_\beta^-) = \mathbf{V}_\Phi(f, E_t^+) + \phi(t) - \phi(t+0).\end{aligned}$$

(c) First, we note that

$$\lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) = \inf_{\alpha,\beta} \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}),$$

where the infimum is taken over $\alpha \in E_t^-$ and $\beta \in E_t^+$, $\alpha < t < \beta$. We show that for any $\varepsilon > 0$, there exist $\alpha_0 = \alpha_0(\varepsilon)$ and $\beta_0 = \beta_0(\varepsilon) \in E$, $\alpha_0 < t < \beta_0$, such that

$$\mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) \le U(\beta, \alpha) + \varepsilon \qquad \forall \alpha \in E_{\alpha_0}^t \setminus \{t\}, \quad \forall \beta \in E_t^{\beta_0} \setminus \{t\}. \tag{13}$$

Using the definition of $\mathbf{V}_\Phi(f, E\setminus\{t\})$, for $\varepsilon > 0$, we find a partition $T = \{t_i\}_{i=0}^m$ of the set $E\setminus\{t\}$ such that $t_0 < t_1 < \dots < t_{k-1} < t < t_k < \dots < t_{m-1} < t_m$ for some $1 \le k \le m$ and

$$\mathbf{V}_\Phi(f, E \setminus \{t\}) \le \sum_{i=1}^m U(t_i, t_{i-1}) + \varepsilon.$$

We set $T_1 = \{t_i\}_{i=0}^{k-1}$, $T_2 = \{t_i\}_{i=k}^m$, $\alpha_0 = t_{k-1}$, and $\beta_0 = t_k$. Now if $\alpha, \beta \in E$ satisfy the inequalities $\alpha_0 < \alpha < t < \beta < \beta_0$, then, taking $T_1 \cup \{\alpha\} \in \mathcal{T}(E_\alpha^-)$ and $T_2 \cup \{\beta\} \in \mathcal{T}(E_\beta^+)$ into account, from the first inequality in (12) we obtain

$$\begin{aligned}\mathbf{V}_\Phi(f, E \setminus \{t\}) &\le V_\Phi[f\,; T_1] + U(\alpha, t_{k-1}) + U(\beta, \alpha) + U(t_k, \beta) + V_\Phi[f\,; T_2] + \varepsilon\\ &\le \mathbf{V}_\Phi(f, E_\alpha^-) + U(\beta, \alpha) + \mathbf{V}_\Phi(f, E_\beta^+) + \varepsilon.\end{aligned}$$

This proves (13) if we take into account that Lemma 2(c) implies

$$\mathbf{V}_\Phi(f, E \setminus \{t\}) = \mathbf{V}_\Phi(f, E_\alpha^-) + \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}) + \mathbf{V}_\Phi(f, E_\beta^+). \tag{14}$$

It remains to pass to the limit as $\alpha \to t-0$, $\beta \to t+0$ in (13) and in the inequality $U(\beta, \alpha) \le \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\})$, which holds for all $\alpha \in E_t^-$ and $\beta \in E_t^+$, $\alpha < t < \beta$.

(d) From Lemma 2(c) and relation (14) for $\alpha, \beta \in E$, $\alpha < t < \beta$, we obtain

$$\mathbf{V}_\Phi(f, E) - \mathbf{V}_\Phi(f, E \setminus \{t\}) = \phi(\beta) - \phi(\alpha) - \mathbf{V}_\Phi(f, E_\alpha^\beta \setminus \{t\}).$$

Hence, passing to the limit as $\alpha \to t-0$, $\beta \to t+0$, we obtain relation (d). $\square$

Corollary 7. *A mapping $f \in BV_\Phi(E\,; X)$ is*

(a) *right-continuous at a point $t \in E \setminus \{\sup E\}$ or left-continuous at a point $t \in E \setminus \{\inf E\}$ if and only if its function ϕ of Φ-variation possesses the same property at the point t;*

(b) *continuous on E except for some at most countable set of points.*

Corollary 8. *Let $f \in BV_\Phi(E\,; X)$, $t \in E$, and let $[\Phi] = \lim_{\rho\to\infty} \Phi(\rho)/\rho < \infty$. In this case,*

(a) *if $t \in E_t^{-\prime}$, then*

$$\mathbf{V}_\Phi(f, E_t^-) = \mathbf{V}_\Phi(f, E_t^- \setminus \{t\}) + [\Phi] \lim_{\alpha\to t-0} d(f(t), f(\alpha))\,;$$

(b) *if $t \in E_t^{+\prime}$, then*

$$\mathbf{V}_\Phi(f, E_t^+) = \mathbf{V}_\Phi(f, E_t^+ \setminus \{t\}) + [\Phi] \lim_{\beta\to t+0} d(f(\beta), f(t))\,;$$

(c) *but if $t \in E_t^{-\prime} \cap E_t^{+\prime}$, then, in addition to* (a) *and* (b), *we have*

$$\begin{aligned}\mathbf{V}_\Phi(f, E) = \mathbf{V}_\Phi(f, E \setminus \{t\}) &+ [\Phi]\Big(\lim_{\alpha\to t-0} d(f(t), f(\alpha))\\ &+ \lim_{\beta\to t+0} d(f(\beta), f(t)) - \lim_{\substack{\alpha\to t-0\\ \beta\to t+0}} d(f(\beta), f(\alpha))\Big).\end{aligned}$$

In the case of a complete metric space X, the one-sided limits $f(t \pm 0) = \lim_{\alpha \to t \pm 0} f(\alpha)$ exist at the points $t \in E_t^{\pm\prime}$, respectively. Hence, the limit signs in Corollary 8 can be "pulled" under the sign of the metric d.

For $f \in GV_\Phi(E\,;X)$, formula (a) in Corollary 8 takes the form: if $t \in E_t^{-\prime}$, then

$$p_\Phi(f, E_t^-) \le p_\Phi(f, E_t^- \setminus \{t\}) + [\Phi] \lim_{\alpha \to t-0} d(f(t), f(\alpha)).$$

This can be proved in the same way as Lemma 4(f). The other formulas in Corollary 8 can be modified similarly.

Corollary 8 was established under different assumptions of generality in the papers [11, 12].

5. A GENERALIZED HELLY SELECTION PRINCIPLE

Here we generalize the Helly selection principle [28] to the space $GV_\Phi(E\,;X)$. Special cases of Theorem 9 considered below were studied in [21, 11, 12, 14] and [17].

Theorem 9. *Suppose that Φ is a convex φ-function, X is a complete metric space, the family $\mathcal{F} \subset X^{[a,b]}$ is infinite, and for any $t \in [a,b]$ the set $\{f(t) \mid f \in \mathcal{F}\}$ is precompact in X. If $\sup_{f \in \mathcal{F}} p_\Phi(f, [a,b]) < \infty$, then $\mathcal{F}$ contains a sequence that converges pointwise on $[a,b]$ to some mapping from $GV_\Phi([a,b]\,;X)$. If, in addition, $\Phi \in \mathcal{N}$, then in $\mathcal{F}$ this sequence can be chosen so as it converges uniformly on $[a,b]$.*

Proof. For $f \in \mathcal{F}$ and $t \in [a,b]$ we set $\nu_f(t) = \mathbf{V}_1(f, [a,t])$. By (11), the family of nondecreasing functions $\{\nu_f \mid f \in \mathcal{F}\}$ is infinite and bounded. Hence it follows from the Helly selection principle for monotone functions (see [28; 21, Chap. 8, Sec. 4]) that there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ and a nondecreasing function $\nu\colon [a,b] \to \mathbb{R}^+$ such that $\lim_{n\to\infty} \nu_{f_n}(t) = \nu(t)$ for all $t \in [a,b]$. Since the set of discontinuity points of the function ν is at most countable and, for any $t \in [a,b]$, the set $\{f_n(t)\}_{n=1}^\infty$ is precompact in X, without loss of generality, we can (if necessary, we choose a subsequence $\{f_n\}_{n=1}^\infty$ by using the standard diagonal process) assume that $f_n(s)$ converges in X to the element $f(s)$ at all rational points $s \in [a,b]$, at all discontinuity points s of the function ν, and at the points $s = a$ and $s = b$.

It remains to show that $f_n(t)$ has a limit in X at any irrational point $a < t < b$ of continuity of the function ν. For $\varepsilon > 0$, we find a rational number $a < s < t$ such that $0 \le \nu(t) - \nu(s) \le \varepsilon/9$ and choose a number $N_0 \in \mathbb{N}$ so that $|\nu_{f_n}(t) - \nu(t)| \le \varepsilon/9$ and $|\nu_{f_n}(s) - \nu(s)| \le \varepsilon/9$ for $n \ge N_0$. Then we have

$$d(f_n(t), f_n(s)) \le |\nu_{f_n}(t) - \nu(t)| + (\nu(t) - \nu(s)) + |\nu(s) - \nu_{f_n}(s)| \le \frac{\varepsilon}{3}, \qquad n \ge N_0.$$

If $N_1 \in \mathbb{N}$ is a number such that $d(f_n(s), f_m(s)) \le \varepsilon/3$ for $n, m \ge N_1$, then for all $n, m \ge \max\{N_0, N_1\}$ we have

$$d(f_n(t), f_m(t)) \le d(f_n(t), f_n(s)) + d(f_n(s), f_m(s)) + d(f_m(s), f_m(t)) \le \varepsilon.$$

Thus the sequence $\{f_n(t)\}_{n=1}^\infty$ is fundamental in X. It remains to set $f(t) = \lim_{n\to\infty} f_n(t)$ and to use Lemma 4(e).

If it is known that $\Phi \in \mathcal{N}$, then the interval $[a,b]$ can be replaced by a compact set E in $\mathbb{R}$. In this case the uniform convergence of some sequence in $\mathcal{F}$ follows from the Arzellà–Ascoli theorem, since, by Lemma 4(a) and relation (2), the family $\mathcal{F}$ is equicontinuous. $\square$

We show that the assumptions of Theorem 9 cannot be weakened. For a convex φ-function φ, by ℓ_φ we denote the space of all sequences $x = \{x_n\}_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}}$ such that there exists a number

$\lambda > 0$ (depending on x) for which $\sum_{n=1}^{\infty} \varphi(\lambda|x_n|) < \infty$ (see [19, Secs. 3, 10]). The space ℓ_φ is an infinite-dimensional Banach algebra with respect to the norm

$$\|x\|_\varphi = \inf\left\{\lambda > 0 \,\middle|\, \sum_{n=1}^{\infty} \varphi\left(\frac{|x_n|}{\lambda}\right) \le 1\right\}$$

and has the standard basis $\{e_n\}_{n=1}^{\infty}$, where $e_n = \{x_i\}_{i=1}^{\infty}$ with $x_i = 0$ for $i \ne n$ and $x_n = 1$. For a convex φ-function Φ, $f \in BV_\Phi([0,1]\,;\mathbb{R})$, $f \ne 0$, and $n \in \mathbb{N}$, we set $f_n(t) = f(t)e_n$, $t \in [0,1]$, so that $f_n\colon [0,1] \to \ell_\varphi$ and $\mathbf{V}_\Phi(f_n, [0,1]) = \mathbf{V}_\Phi(f, [0,1])/\varphi^{-1}(1)$. Thus we see that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly bounded Φ-variation, but the sets $\{f_n(t)\}_{n=1}^{\infty}$ are not precompact in ℓ_φ. Hence the generalized Helly selection principle is violated: none of the subsequences of the sequence $\{f_n\}_{n=1}^{\infty}$ converges in ℓ_φ at any point $t \in [0,1]$.

With the help of the standard diagonal method, the Helly selection principle can be generalized to families of mappings whose domains are bounded or unbounded intervals, or half-intervals.

6. SELECTIONS OF MULTI-VALUED MAPPINGS OF GENERALIZED Φ-VARIATION

A multi-valued mapping $F\colon E \to \mathrm{c}(X)$ is called a *Lipschitz mapping*, or a *mapping of bounded Φ-variation*, or a *mapping of bounded generalized Φ-variation* if it possesses this property (in the sense of Sec. 2) with respect to the Hausdorff metric D generated by the metric d on X. So we write

$$F \in \mathrm{Lip}(E\,; \mathrm{c}(X)), \qquad F \in BV_\Phi(E\,; \mathrm{c}(X)), \qquad F \in GV_\Phi(E\,; \mathrm{c}(X)).$$

Note that on the space $\mathrm{c}(X)$ the Hausdorff topology (i.e., the topology generated by the Hausdorff metric) depends only on the topology on X (but not on the metric on X, e.g., see [4, Corollary II-7]).

Recall that, without the assumption that the images of the multi-valued mapping $F\colon I = [a,b] \to \mathrm{c}(\mathbb{R}^n)$, $n = 1, 2$, be convex, a continuous mapping F (see [1, 6, 29]) and even a mapping F satisfying the Hölder condition $D(F(t), F(s)) \le C|t-s|^\alpha$, $t, s \in I$, for any $0 < \alpha < 1$, whose images are compact in $\mathbb{R}^2$ (see [30]) may have no continuous selections. We also note that, for multi-valued mappings F defined on an interval in $\mathbb{R}$ and ranging in a topological space so that its connected values form a locally connected family of sets, continuous selections exist without any metric restrictions: it only suffices that F be lower semicontinuous (see [5]).

Theorem 10. *Let (X, d) be a metric space, and let Φ be a convex φ-function. Suppose that $F \in GV_\Phi([a,b]\,; \mathrm{c}(X))$ is a compact-valued multi-valued mapping of bounded generalized Φ-variation, $t_0 \in [a,b]$, and $x_0 \in X$. Then F has a selection $f \in GV_\Phi([a,b]\,; X)$ such that*

$$d(x_0, f(t_0)) = \mathrm{dist}(x_0, F(t_0)), \qquad p_\Phi(f, [a,b]) \le p_\Phi(F, [a,b]),$$
$$\mathbf{V}_1(f, [a,b]) \le \mathbf{V}_1(F, [a,b]).$$

If, in addition, F is continuous, then this selection f can also be chosen to be continuous, and if $F \in \mathrm{Lip}([a,b]\,; \mathrm{c}(X))$, then the selection $f \in \mathrm{Lip}([a,b]\,; X)$ can also be chosen so as to have $\mathbf{L}(f) \le \mathbf{L}(F)$. Note that, in particular, if $x_0 \in F(t_0)$, then $f(t_0) = x_0$.

Prior to proving this theorem, we establish the following lemma.

Lemma 11. *If $F \in BV_1(E\,; \mathrm{c}(X))$, where $\varnothing \ne E \subset \mathbb{R}$ and (X, d) is a metric space, then the image $\mathrm{Im}(F) = F(E)$ of the mapping F, determined by the rule $\mathrm{Im}(F) = \bigcup_{t \in E} F(t)$, is a completely bounded separable subset of X; if, in addition, X is complete, then $\mathrm{Im}(F)$ is precompact.*

Proof. For $\varepsilon > 0$, $x \in X$, and a compact set $K \subset X$, we set $B_\varepsilon(x) = \{y \in X \mid d(y,x) < \varepsilon\}$ and $\mathcal{O}_\varepsilon(K) = \{y \in X \mid \operatorname{dist}(y,K) < \varepsilon\}$. To prove that $\operatorname{Im}(F)$ is completely bounded, we must show that, for any $\varepsilon > 0$, the set $\operatorname{Im}(F)$ can be covered by finitely many balls $B_\varepsilon(\cdot)$ centered in $\operatorname{Im}(F)$, i.e., there exist a number N depending on ε and points $t_n \in E$ and $x_n \in F(t_n)$, $n = 1,\dots,N$, such that $\operatorname{Im}(F) \subset \bigcup_{n=1}^N B_\varepsilon(x_n)$. Since the relation $\mathcal{O}_\varepsilon(K) = \bigcup_{x\in K} B_\varepsilon(x)$ holds for the compact set $K \subset X$, the condition that $\operatorname{Im}(F)$ is completely bounded is equivalent to the fact that, for any $\varepsilon > 0$, there exist a number N and points $t_n \in E$, $n = 1,\dots,N$, such that $\operatorname{Im}(F) \subset \bigcup_{n=1}^N \mathcal{O}_\varepsilon(F(t_n))$. Assume the contrary. Then we can assume that $\varepsilon > 0$ is a number for which the last statement is violated. We construct a sequence $\{x_n\}_{n=0}^\infty \subset \operatorname{Im}(F)$ by induction as follows. We choose an arbitrary $t_0 \in E$ and an arbitrary $x_0 \in F(t_0)$. Then we choose $x_1 \in \operatorname{Im}(F)$ so that $x_1 \notin \mathcal{O}_\varepsilon(F(t_0))$, and assume that $x_1 \in F(t_1)$ for some $t_1 \in E$. If the points $x_0, x_1, \dots, x_{n-1} \in \operatorname{Im}(F)$, $n \ge 2$, have been already determined, and $x_j \in F(t_j)$ for $t_j \in E$, $j = 0,1,\dots,n-1$, then we choose a point $x_n \in \operatorname{Im}(F) \setminus \bigcup_{j=0}^{n-1} \mathcal{O}_\varepsilon(F(t_j))$ and assume that $x_n \in F(t_n)$ for some $t_n \in E$. Since $D(F(t_n),F(t_k)) \ge \operatorname{dist}(x_n, F(t_k)) \ge \varepsilon$ for $n > k$, we have $t_n \ne t_k$. Hence, without loss of generality, we can assume that $t_{n-1} < t_n$ for all $n \in \mathbb{N}$. Then for the partition $T_m = \{t_i\}_{i=0}^m$ of the set E, we have

$$\mathbf{V}_1(F,E) \ge \sum_{i=1}^m D(F(t_i), F(t_{i-1})) \ge m\varepsilon.$$

Since m is arbitrary, we conclude that $\mathbf{V}_1(F,E) = \infty$, which contradicts the assumption. It is well known that a completely bounded set is separable in a metric space and is precompact in a complete metric space. $\square$

Proof of Theorem 10. First, we prove this theorem for F from the class $BV_\Phi([a,b]\,;\mathrm{c}(X))$. For $n \in \mathbb{N}$, we consider the partition $T_n = \{t_i^n\}_{i=0}^n \in \mathcal{T}_a^b$ with the following two properties:

1) $t_0 \in T_n$, i.e., $t_0 = t_{k(n)}^n$ for some $k(n) \in \{0,1,\dots,n\}$;
2) if $I_i^n = [t_{i-1}^n, t_i^n]$ and $|I_i^n| = t_i^n - t_{i-1}^n$, $i = 1,\dots,n$, then $\lim_{n\to\infty} \max_{1\le i\le n} |I_i^n| = 0$.

Since the set $F(t_0)$ is compact, we can choose an element $y_0 \in F(t_0)$ so that $d(x_0,y_0) = \operatorname{dist}(x_0, F(t_0))$. We determine elements $x_i^n \in F(t_i^n)$, $i = 0,1,\dots,n$, by induction as follows. First, let $a < t_0 < b$.

a) We set $x_{k(n)}^n = y_0$.

b) If $i \in \{1,\dots,k(n)\}$ and the element $x_i^n \in F(t_i^n)$ has already been constructed, we choose an element $x_{i-1}^n \in F(t_{i-1}^n)$ so that $d(x_i^n, x_{i-1}^n) = \operatorname{dist}(x_i^n, F(t_{i-1}^n))$.

c) If $i \in \{k(n)+1,\dots,n\}$ and the element $x_i^n \in F(t_i^n)$ has already been constructed, we choose an element $x_i^n \in F(t_i^n)$ so that $d(x_{i-1}^n, x_i^n) = \operatorname{dist}(x_{i-1}^n, F(t_i^n))$.

For $t_0 = a$, i.e., for $k(n) = 0$, we determine x_i^n according to a) and c). For $t_0 = b$, i.e, for $k(n) = n$, we follow a) and b). Because of b), c), and (1), we have

$$d(x_i^n, x_{i-1}^n) \le D(F(t_i^n), F(t_{i-1}^n)), \qquad n \in \mathbb{N}, \quad i = 1,\dots,n. \tag{15}$$

It follows from (9) that $\mathbf{V}_1(F,[a,b]) < \infty$. Hence, by Lemma 11, the image $\operatorname{Im}(F)$ of the mapping F is (completely) bounded in X. By $Y = C_b(\operatorname{Im}(F)\,;\mathbb{R})$ we denote the Banach space of all bounded continuous functions from $\operatorname{Im}(F)$ into $\mathbb{R}$. This space is equipped with the uniform norm

$$\|y\| = \sup_{x\in \operatorname{Im}(F)} |y(x)|, \qquad y \in Y.$$

The mapping $\operatorname{Im}(F) \ni x \mapsto R(x) \in Y$, where $R(x)(x') = d(x,x')$ and $x' \in \operatorname{Im}(F)$, determines (according to Kuratowski) an isometric embedding of the set $\operatorname{Im}(F)$ in the space Y so that, in particular, $\|R(x) - R(x')\| = d(x,x')$ for all $x, x' \in \operatorname{Im}(F)$.

For $n \in \mathbb{N}$, we define the mapping $f_n\colon [a,b] \to Y$ by the rule

$$f_n(t) = R(x_{i-1}^n) + \frac{(t - t_{i-1}^n)(R(x_i^n) - R(x_{i-1}^n))}{|I_i^n|}, \qquad t \in I_i^n, \quad i = 1, \dots, n. \tag{16}$$

Note that $f_n(t_0) = R(y_0)$ and $f_n(t_i^n) = R(x_i^n)$, $n \in \mathbb{N}$, $i = 0, 1, \dots, n$. Hence from Lemma 2(c), relations (15) and (16), and the fact that R is isometric for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \mathbf{V}_\Phi(f_n, [a,b]) &= \sum_{i=1}^n \mathbf{V}_\Phi(f_n, I_i^n) = \sum_{i=1}^n \Phi\biggl(\frac{d(x_i^n, x_{i-1}^n)}{|I_i^n|}\biggr)|I_i^n| \\ &\le \sum_{i=1}^n \Phi\biggl(\frac{(D(F(t_i^n), F(t_{i-1}^n))}{|I_i^n|}\biggr)|I_i^n| \le \mathbf{V}_\Phi(F, [a,b]). \end{aligned} \tag{17}$$

Lemma 4(c) implies $p_\Phi(f_n, [a,b]) \le \max\{1, \mathbf{V}_\Phi(F, [a,b])\}$, $n \in \mathbb{N}$, and inequality (17) with $\Phi(\rho) = \rho$ also implies $\mathbf{V}_1(f_n, [a,b]) \le \mathbf{V}_1(F, [a,b])$.

For $t \in [a,b]$, we will show that the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in Y. First, let $\Phi \in \mathcal{N}$ so that F will be (absolutely) continuous on $[a,b]$ with respect to D. For any $n \in \mathbb{N}$, there exists a number $i(n) \in \{1, \dots, n\}$ (also depending on t) such that $t \in I_{i(n)}^n$. Hence condition 2) implies that $t_{i(n)}^n \to t$ and $t_{i(n)-1}^n \to t$ as $n \to \infty$. We choose an element $x_n(t) \in F(t)$ so that $d(x_{i(n)-1}^n, x_n(t)) = \operatorname{dist}(x_{i(n)-1}^n, F(t))$, $n \in \mathbb{N}$. Since $F(t)$ is compact, we can assume (choosing a subsequence) that $x_n(t)$ converges in X to some element $x(t) \in F(t)$. Then, by (16) and (15), as $n \to \infty$, we have

$$\begin{aligned} \|f_n(t) - R(x(t))\| &\le \|f_n(t) - R(x_n(t))\| + \|R(x_n(t)) - R(x(t))\| \\ &\le \|R(x_{i(n)-1}^n) - R(x_n(t))\| + \|R(x_{i(n)}^n) - R(x_{i(n)-1}^n)\| + \|R(x_n(t)) - R(x(t))\| \\ &\le D(F(t_{i(n)-1}^n), F(t)) + D(F(t_{i(n)}^n), F(t_{i(n)-1}^n)) + d(x_n(t), x(t)) \to 0, \end{aligned}$$

which means that

$$f_n(t) \to R(x(t)) \qquad \text{in } Y \quad \text{as } n \to \infty \tag{18}$$

and proves that $\{f_n(t)\}_{n=1}^\infty$ is precompact. For $[\Phi] < \infty$, we renumber the set of discontinuity points of F, since, by Corollary 7(b), this set is at most countable. To each partition T_n considered above, we add the first n discontinuity points of F and again denote the partition thus obtained by T_n. The new partition T_n again satisfies conditions 1) and 2). It follows from the preceding argument that $\{f_n(t)\}_{n=1}^\infty$ is precompact at a point t, where F is continuous. But if $t \in [a,b]$ is a discontinuity point of F, then there exists a number $N(t) \in \mathbb{N}$ such that $t \in T_n$ for all $n \ge N(t)$, i.e., $t = t_{\ell(n)}^n$ for some number $\ell(n) \in \{0, 1, \dots, n\}$ also depending on t. By construction, we have $x_{\ell(n)}^n \in F(t_{\ell(n)}^n) = F(t)$, $n \ge N(t)$. Hence, since the set $F(t)$ is compact, some subsequence of $\{x_{\ell(n)}^n\}_{n=1}^\infty$ (for which we preserve the same notation) converges in X to some point $x(t) \in F(t)$. Since $f_n(t) = R(x_{\ell(n)}^n)$ for $n \ge N(t)$, condition (18) is again satisfied.

Using the generalized Helly selection principle (Theorem 9), in $\mathcal{F} = \{f_n\}_{n=1}^\infty$ we find a subsequence that, in the space Y, converges pointwise on $[a,b]$ to some mapping $g\colon [a,b] \to Y$ of the form (see (18)) $g(t) = R(f(t))$, $t \in [a,b]$, where $f(t) \in F(t)$ for all $t \in [a,b]$, and $g(t_0) = R(y_0)$, while Lemma 2(d) and inequality (17) guarantee that $\mathbf{V}_\Phi(g, [a,b]) \le \mathbf{V}_\Phi(F, [a,b])$ and $\mathbf{V}_1(g, [a,b]) \le \mathbf{V}_1(F, [a,b])$. Since R is an isometry, we conclude that

$$f(t_0) = y_0, \qquad \mathbf{V}_\Phi(f, [a,b]) \le \mathbf{V}_\Phi(F, [a,b]), \qquad \mathbf{V}_1(f, [a,b]) \le \mathbf{V}_1(F, [a,b]).$$

If it is known that $F \in \operatorname{Lip}([a,b]; \mathrm{c}(X))$, then, by (16) and (15), for $n \in \mathbb{N}$ and $i = 1, \dots, n$, it suffices, in addition, to take the inequalities

$$\|f_n(t) - f_n(s)\| \le \frac{|t-s| d(x_i^n, x_{i-1}^n)}{|I_i^n|} \le \mathbf{L}(F)|t-s|, \qquad t, s \in I_i^n,$$

into account. Then the selection $f \in \mathrm{Lip}([a,b]\,;X)$ of the mapping F satisfies the condition $\mathbf{L}(f) \le \mathbf{L}(F)$.

Let $F \in BV_\Phi([a,b]\,;\mathrm{c}(X))$ be continuous. By Lemma 5(a), we have the decomposition $F = G \circ \nu$, where the function $\nu(t) = \mathbf{V}_1(F,[a,t])$, $t \in [a,b]$, belongs to the space $BV_\Phi([a,b]\,;\mathbb{R})$ and is continuous, while the multi-valued mapping G from $\mathrm{Lip}([0,\ell]\,;\mathrm{c}(X))$ possesses the properties:

$$\ell = \mathbf{V}_1(F,[a,b]) = \mathbf{V}_1(\nu,[a,b]), \qquad \mathbf{L}(G) \le 1, \qquad \mathbf{V}_\Phi(\nu,[a,b]) = \mathbf{V}_\Phi(F,[a,b]).$$

We set $\tau_0 = \nu(t_0) \in [0,\ell]$ so that $G(\tau_0) = F(t_0)$. There exists a $g \in \mathrm{Lip}([0,\ell]\,;X)$ such that $d(x_0, g(\tau_0)) = \mathrm{dist}(x_0, G(\tau_0))$, $g(\tau) \in G(\tau)$ for all $\tau \in [0,\ell]$, and $\mathbf{L}(g) \le \mathbf{L}(G) \le 1$. We set $f = g \circ \nu$ and then see that $f \in BV_\Phi([a,b]\,;X)$ is continuous, $f(t_0) = g(\tau_0)$, $f(t) \in F(t)$ for all points $t \in [a,b]$, and the following inequalities hold: $\mathbf{V}_1(f,[a,b]) \le \mathbf{V}_1(F,[a,b])$ and

$$\mathbf{V}_\Phi(f,[a,b]) = \mathbf{V}_\Phi(g \circ \nu,[a,b]) \le \mathbf{V}_\Phi(\nu,[a,b]) = \mathbf{V}_\Phi(F,[a,b]).$$

Finally, let $F \in GV_\Phi([a,b]\,;X)$. We set $\lambda = p_\Phi(F,[a,b])$. Then, by Lemma 4(a), we can assume that $\lambda > 0$ and thus $\mathbf{V}_{\Phi_\lambda}(F,[a,b]) \le 1$ by Lemma 4(b). As was proved above, there exists a selection $f \in BV_{\Phi_\lambda}([a,b]\,;X)$ such that $\mathbf{V}_{\Phi_\lambda}(f,[a,b]) \le \mathbf{V}_{\Phi_\lambda}(F,[a,b])$. Hence, $p_\Phi(f,[a,b]) \le \lambda$. $\square$

Let us consider the Banach space ℓ_1 of summable sequences with the norm $\|x\|_1 = \sum_{n=1}^\infty |x_n|$, $x = \{x_n\}_{n=1}^\infty \in \ell_1$. For $t \in [0,1]$ we introduce the multi-valued mapping

$$F(t) = \{x \in \ell_1 \colon \|x\|_1 = 1 \quad \text{and} \quad x_1 = t\}.$$

Then for the Hausdorff metric we have $D(F(t),F(s)) = 2|t-s|$ for $t,s \in [0,1]$, and any mapping of the form

$$f(t) = (t, (1-t)\{x_n\}_{n=2}^\infty), \quad t \in [0,1], \qquad \text{where} \quad \sum_{n=2}^\infty |x_n| = 1,$$

is a Lipschitzian selection of F. Now if $\nu \in GV_\Phi([a,b]\,;\mathbb{R})$ and $\nu([a,b]) = [0,1]$, then $f \circ \nu$ is a selection of $F \circ \nu$ in the class of mappings $GV_\Phi([a,b]\,;\,\cdot\,)$.

Corollary 12. *Theorem* 10 *holds if in the closed interval* $[a,b]$ *is replaced by* $\mathbb{R}$.

Proof. First, we assume that the multi-valued mapping F belongs to $BV_\Phi(\mathbb{R}\,;\mathrm{c}(X))$. Let $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ be an increasing sequence such that $t_k \to \infty$ and $t_{-k} \to -\infty$ as $k \to \infty$. We set $I_k = [t_k, t_{k+1}]$ for $k \in \mathbb{Z}$, apply Theorem 10 on the interval I_0, and find a selection $f_0 \in BV_\Phi(I_0\,;X)$ of the mapping F (more precisely, of the restriction $F|_{I_0}$ of the mapping F to the interval I_0) such that $d(x_0, f_0(t_0)) = \mathrm{dist}(x_0, F(t_0))$, $\mathbf{V}_\Phi(f_0, I_0) \le \mathbf{V}_\Phi(F, I_0)$ and $\mathbf{V}_1(f_0, I_0) \le \mathbf{V}_1(F, I_0)$. "Moving along the intervals I_k to the right" from the point t_1, we apply Theorem 10 in succession on the interval I_1 with the initial condition $x_0 = f_0(t_1) \in F(t_1)$, ..., and on the interval I_k with the initial condition $x_0 = f_{k-1}(t_k) \in F(t_k)$, $k \in \mathbb{N}$. As a result, for any $k \in \mathbb{N}$, we find a selection $f_k \in BV_\Phi(I_k\,;X)$ of the mapping $F|_{I_k}$ such that

$$f_k(t_k) = f_{k-1}(t_k), \qquad \mathbf{V}_\Phi(f_k, I_k) \le \mathbf{V}_\Phi(F, I_k), \qquad \mathbf{V}_1(f_k, I_k) \le \mathbf{V}_1(F, I_k). \tag{19}$$

In a similar way, we shall "move along the intervals I_k to the left" from the point t_0. Then, for any $k \in \mathbb{Z}$, on the interval I_k there exists a selection f_k of the mapping F such that relations (19) hold. If $t \in \mathbb{R}$ so that $t \in I_k$ for some $k \in \mathbb{Z}$, we set $f(t) = f_k(t)$. Clearly, the mapping $f\colon \mathbb{R} \to X$ is a selection of F on $\mathbb{R}$, $f(t_0) = f_0(t_0)$. By Lemma 2(h,c), we have

$$\begin{aligned}\mathbf{V}_\Phi(f,\mathbb{R}) &= \lim_{k\to\infty} \mathbf{V}_\Phi(f,[t_{-k},t_k]) = \lim_{k\to\infty} \sum_{i=-k}^{k-1} \mathbf{V}_\Phi(f_i, I_i) \\ &\le \lim_{k\to\infty} \sum_{i=-k}^{k-1} \mathbf{V}_\Phi(F, I_i) = \lim_{k\to\infty} \mathbf{V}_\Phi(F,[t_{-k},t_k]) = \mathbf{V}_\Phi(F,\mathbb{R}),\end{aligned}$$

and similarly, $\mathbf{V}_1(f, \mathbb{R}) \le \mathbf{V}_1(F, \mathbb{R})$.

Now if $F \in GV_\Phi(\mathbb{R}; X)$, then for $\lambda = p_\Phi(f, \mathbb{R}) > 0$, we have $\mathbf{V}_{\Phi_\lambda}(F, \mathbb{R}) \le 1$, and it remains to apply the result we have just proved. $\square$

A statement similar to Corollary 12 also holds for bounded and unbounded intervals and half-intervals of the real line.

Note that Lipschitzian selections of Lipschitzian multi-valued mappings were established in different contexts in [7, 8, 10–12, 16, 17, 31].

A mapping $f\colon [a, b] \to X$ is said to be *regular* if it has at most countably many discontinuity points at which the left-hand and right-hand one-sided limits exist. A sequence $f_n\colon [a, b] \to X$, $n \in \mathbb{N}$, is called the *Castaing representation* for a multi-valued mapping $F\colon [a, b] \to \mathrm{c}(X)$ if, for almost all $t \in [a, b]$, the sequence $\{f_n(t)\}_{n=1}^\infty$ is dense in $F(t)$. Recall [4, Theorem III.8] that the measurability of a mapping F with closed images is equivalent to the fact that its Castaing representation is contained in the sequence of measurable selections. The following corollary is based on Theorem 10 and can be proved similarly to Theorem D 1.9 in [10].

Corollary 13. *Suppose that X is a complete metric space, Φ is a convex φ-function, and $F \in GV_\Phi([a, b]; \mathrm{c}(X))$. Then*

(a) *for any measurable selection f of the mapping F, there exists a sequence of regular selections of F that converges almost everywhere on $[a, b]$ to f;*

(b) *there exists a sequence of regular selections of F which is the Castaing representation for the mapping F.*

7. SELECTIONS OF MULTI-VALUED MAPPINGS ON THE PRODUCT OF TWO SPACES

Let X and Y be two (Hausdorff) topological spaces. Recall that a multi-valued mapping Γ from X into Y is said to be

lower semicontinuous (briefly, *l.s.c.*) on X if for any open set $\mathcal{U} \subset Y$, its preimage $\Gamma^-(\mathcal{U}) = \{x \in X \mid \Gamma(x) \cap \mathcal{U} \ne \varnothing\}$ is open in X;

upper semicontinuous (briefly, *u.s.c.*) on X if $\Gamma^-(\mathcal{U})$ is closed in X for any closed $\mathcal{U} \subset Y$;

continuous on X if Γ is l.s.c. and u.s.c. on X simultaneously.

A multi-valued mapping Γ from X into a metric space Y is said to be *weakly upper semicontinuous* if for any closed ball $\mathcal{U}$ in Y, the preimage $\Gamma^-(\mathcal{U})$ is closed in X. If the metric space Y has the property that

$$\text{the closure in } Y \text{ of any open ball is a closed ball,} \tag{20}$$

then the weak upper semicontinuity of the multi-valued mapping Γ implies the upper semicontinuity of this mapping. For example, any linear metric space has property (20).

We also recall that a topological space is said to be *extremely nonclosed* (see [32, Sec. 46. VI]) if the closure of any set open in this space is open.

We shall need the following result from [33] (another simpler proof of this result is contained in [34]).

Lemma 14. *Suppose that X is an extremely nonclosed Hausdorff topological space, Y is a regular Hausdorff $\mathcal{T}_3$-space, and Γ is an upper semicontinuous multi-valued mapping from X into Y with compact values. Then Γ has a continuous selection $\gamma\colon X \to Y$.*

Let I be an interval (a connected subset of $\mathbb{R}$), and let $H = H(t, x)$ be a multi-valued mapping from $I \times X$ into Y. For a chosen $x \in X$, we introduce a multi-valued mapping $H(\,\cdot\,, x)$ from I into Y by the rule $H(\,\cdot\,, x)(t) = H(t, x)$, $t \in I$. If $t \in I$ is fixed, then we similarly set $H(t, \,\cdot\,)(x) = H(t, x)$, $x \in X$, so that $H(t, \,\cdot\,)$ maps X into subsets of Y.

The main result of this section is the following assertion.

Theorem 15. *Suppose that $I \subset \mathbb{R}$ is an interval, X is an extremely nonclosed Hausdorff topological space, (Y, d) is a complete metric space satisfying condition* (20), *D is the Hausdorff metric on $\mathrm{c}(Y)$ generated by d, $t_0 \in I$, $\eta\colon X \to Y$ is a continuous mapping, $\Phi \in \mathcal{N}$, and $H\colon I \times X \to \mathrm{c}(Y)$ is a multi-valued mapping such that*

(i) $H(\cdot, x) \in GV_\Phi(I; \mathrm{c}(Y))$ *for all* $x \in X$;
(ii) $H(t, \cdot)\colon X \to \mathrm{c}(Y)$ *is upper semicontinuous for all* $t \in I$.

Then there exists a selection $h\colon I \times X \to Y$ of the mapping H such that

(a) $h(\cdot, x) \in GV_\Phi(I; Y)$ *for all* $x \in X$;
(b) $p_\Phi\big(h(\,\cdot\,, x), I\big) \le p_\Phi(H(\cdot, x), I)$ *for all* $x \in X$;
(c) $d(\eta(x), h(t_0, x)) = \operatorname{dist}(\eta(x), H(t_0, x))$ *for all* $x \in X$;
(d) $h(t, \,\cdot\,)\colon X \to Y$ *continuously for all* $t \in I$.

If, in addition, $\sup_{x\in X} p_\Phi(H(\,\cdot\,, x), I) < \infty$, then the selection h is continuous on $I \times X$. Moreover, property (c) *implies that, in particular, if η is a selection of $H(t_0, \,\cdot\,)$, then $h(t_0, \,\cdot\,) = \eta$.*

Proof. Let $C(I; Y)$ denote the space of all continuous mappings from I into Y with a compactly open topology whose subbasis is formed by the sets $\{f \in C(I; Y) \mid f(J) \subset \mathcal{U}\}$, where $J \subset I$ is compact and $\mathcal{U} \subset Y$ is open. We define a multi-valued mapping Γ from X into $C(I; Y)$ by the rule

$$\Gamma(x) = \{f \in C(I; Y) \mid f(t) \in H(t, x) \quad \forall t \in I, \quad p_\Phi(f, I) \le p_\Phi(H(\,\cdot\,, x), I), \\ d(\eta(x), f(t_0)) = \operatorname{dist}(\eta(x), H(t_0, x))\}, \qquad x \in X.$$

For $x \in X$, we have $\Gamma(x) \ne \varnothing$ by virtue of Theorem 10 and assumption (i). By Lemma 4(e), $\Gamma(x)$ is a closed subset of $GV_\Phi(I; Y)$. We show that Γ is an upper semicontinuous mapping from X into $\mathrm{c}(C(I; Y))$. Indeed, for any $x \in X$, the family $\Gamma(x)$ is equicontinuous by Lemma 4(a) and the condition $\Phi \in \mathcal{N}$, and, for any $t \in I$, the set $\{f(t) \mid f \in \Gamma(x)\}$ is compact in Y, since the set $H(t, x)$ is compact. Hence, by the Arzellá–Ascoli theorem, $\Gamma(x)$ is precompact in $C(I; Y)$ and, since the set $\Gamma(x)$ is closed, it is compact in $C(I; Y)$. To prove that Γ is upper semicontinuous, by the remarks preceding Lemma 14, it suffices to show that Γ is weakly upper semicontinuous. For $\varepsilon > 0$, $g \in C(I; Y)$, and $t \in I$, we set

$$\overline{\mathcal{U}}_\varepsilon(g) = \{f \in C(I; Y) \mid \sup_{t\in I} d(f(t), g(t)) \le \varepsilon\} \qquad \text{and} \qquad \overline{B}_\varepsilon(g(t)) = \{y \in Y \mid d(g(t), y) \le \varepsilon\}.$$

For $x \in X$, $\Gamma_1(x)$ denotes the (compact) set of mappings $f \in C(I; Y)$ satisfying the first two conditions in the definition of $\Gamma(x)$, and $\Gamma_2(x)$ denotes the set f satisfying the third condition in the definition of $\Gamma(x)$. Note that

$$\Gamma_1^-(\overline{\mathcal{U}}_\varepsilon(g)) = \{x \in X \mid \Gamma_1(x) \cap \overline{\mathcal{U}}_\varepsilon(g) \ne \varnothing\} = \bigcap_{t\in I} H(t, \cdot)^-(\overline{B}_\varepsilon(g(t))),$$

and, by condition (ii), any preimage under the intersection sign is closed in X. Hence $\Gamma_1^-(\overline{\mathcal{U}}_\varepsilon(g))$ is closed in X so that the multi-valued mapping $\Gamma_1\colon X \to C(I; Y)$ is upper semicontinuous. Since all the values of $\Gamma_1(x)$ are compact in $C(I; Y)$ and the graph of the multi-valued mapping Γ_2 is closed in $X \times C(I; Y)$, it follows from Theorem 3.1.8 in [35] that the mapping $x \mapsto \Gamma(x) = \Gamma_1(x) \cap \Gamma_2(x)$ is upper semicontinuous.

It follows from Lemma 14 that Γ has a continuous selection $\gamma\colon X \to C(I; Y)$. Let us set $h(t, x) = \gamma(x)(t)$, $t \in I$, $x \in X$. Relations (a)–(c) follow from the definition of $\Gamma(x)$ and the inclusion $\gamma(x) \in \Gamma(x)$, $x \in X$. Let us prove (d). If $t \in I$ and $\mathcal{U} \subset Y$ is open, then, by setting $\mathcal{U}(t) = \{f \in C(I; Y) \mid f(t) \in \mathcal{U}\}$, we obtain

$$\{x \in X \mid h(t, x) \in \mathcal{U}\} = \{x \in X \mid \gamma(x)(t) \in \mathcal{U}\} = \{x \in X \mid \gamma(x) \in \mathcal{U}(t)\}.$$

Since $\mathcal{U}(t)$ is open in $C(I\,;Y)$ and γ is continuous, the latter set is open in X. Hence the mapping $h(t,\cdot)$ is continuous for any $t \in I$.

If, in addition, it is known that $\sup_{x\in X} p_\Phi(H(\,\cdot\,,x),I) < \infty$, then the selection $h\colon I \times X \to Y$ constructed above is continuous. This follows from item (b), Lemma 4(a), the condition $\Phi \in \mathcal{N}$, and the following inequality for (t_0,x_0), $(t,x) \in I \times X$:

$$\begin{aligned} d(h(t,x),h(t_0,x_0)) &\le d(h(t,x),h(t_0,x)) + d(h(t_0,x),h(t_0,x_0)) \\ &\le \omega_\Phi(|t-t_0|)p_\Phi(H(\,\cdot\,,x),I) + d(h(t_0,x),h(t_0,x_0)). \quad \square \end{aligned}$$

Note that the existence of selections h with the property $h(\,\cdot\,,x) \in \mathrm{Lip}(I\,;Y)$ was established in [16, Theorem 5] for the case in which condition (i) of Theorem 15 involves the set $H(\,\cdot\,,x) \in \mathrm{Lip}(I\,;\mathrm{c}(Y))$.

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