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SURVEY PAPERS

# Undistinguishing Statistics of Objectively Distinguishable Objects: Thermodynamics and Superfluidity of Classical Gas[*]

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**Abstract**—In the present paper, we describe an approach to thermodynamics that does not involve Bogolyubov chains or Gibbs ensembles. We present isotherms, isochores, and isobars of various pure gases, as well as binodals, i.e., lines along which gas becomes liquid, and spinodals (endpoints of isotherms). We study supercritical phenomena for values of temperature and pressure above the critical ones. A lot of attention is paid to the region of negative pressures. The superfluid component for supercritical phenomena is described, as well as the thermodynamics of nanostructures and superfluidity in nanotubes.

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*Dedicated to the memory of Boris Zakharovich Milner*

## PREFACE

There are three mathematical approaches to thermodynamics. One of them is based on the Bogolyubov chain and on dynamical systems. This approach is developed by the school of N. N. Bogolyubov and by G. A. Martynov and his disciples. In this approach, the method of Poincaré and Gibbs, brilliantly developed by V. V. Kozlov [1], is used.

Another approach is based on the Gibbs formula for Gibbs ensembles. This formula can be derived rigorously, which was done by the author in previous papers. From the number of elements one computes the asymptotics. They can be constructed for a small number of particles, say two. In this situation one considers the set of identical systems. This approach is close to that of Kolmogorov (in his works on what is now known as Kolmogorov complexity). If the interaction potential is given, the Gibbs formula for a Gibbs ensemble for the potential can yield important results, for example, the correlation radius. This problem was studied by many physicists. Similar results may help solve the very difficult inverse problem, that of finding the interaction potential for a given gas. This problem was also studied by many mathematicians involved in rigorous mathematical study of the properties of the so-called Gibbs fields and related problems of statistical physics and quantum field theory. The best known ones are D. Ruelle, J. Gallovotti, E. Presutti, G. Shpon, I. Frolich, M. Azenman, J. Leibowitz, R. Dobrushin, Ya. Sinai, V. Malyshev, R. Minlos, and others.

Gibbs fields are random fields on a lattice of dimension $\nu$ (or on the continuous space $\mathbb{R}^\nu$) with values in some space (usually called the "spin space") and appearing in the so-called thermodynamical limit $V \to \mathbb{R}^\nu$ from the Gibbs ensemble in a finite volume $V \subset \mathbb{R}^\nu$.

The most interesting mathematical results obtained in this direction during the last decade are:

(1) a criterion for the existence and uniqueness of Gibbs fields for a single interaction potential and fixed temperature;

(2) the existence at low temperatures of several such fields (which it is natural to call phases) for one and the same interaction potential;

(3) the description of the properties of a certain domain occupied by one phase and surrounded by another one (the Wulff droplet);

and several other remarkable results.

We cannot omit mentioning the outstanding seminar that took place in the Mechanics and Mathematics Department of Moscow State University from 1962 to 1994 under the leadership of R. Dobrushin, Ya. Sinai, V. Malyshev, and R. Minlos, whose participants actively worked on the questions indicated above (see [2]).

During the entire second half of the 20th century, mathematical statistical physics was developing in the direction of the study of Gibbs fields, mostly on the lattice $\mathbb{Z}^\nu$. The Gibbs field is determined by the potential, but not always uniquely. In the latter case, one says that different Gibbs fields determine different phases. If the potential depends on one or several parameters, and for some of them the field is the same, and for others is not, then one says that a phase transition of the first kind has occurred. A phase transition of the second kind preserves uniqueness, but separates the domains of parameters with fast or slow decrease of correlation. This direction of study appears in the papers [3], [4], and others.

In the present paper, we describe the third approach to thermodynamics. The author uses neither Bogolyubov chains nor Gibbs ensembles. The asymptotics are written out directly for the number of particles. This asymptotics is of the type proposed in [5]. Such asymptotics are worthless when the number of particles is small. Concerning the corrections required for a smaller number of particles, see the author's papers [6], [7]. In the present paper this question is addressed in Sec. 6.

The present paper shows the isotherms, isochores, and isobars of various pure gases, as well as the binodals (i.e., the lines at which the gas becomes liquid) and the spinodals (the endpoints of the isotherms). We study supercritical phenomena at temperatures and pressures above the critical ones. We also pay a lot of attention to domains of negative pressure. The superfluid component for supercritical phenomena, as well as the thermodynamics of nanostructures and superfluidity in nanotubes is described.

The author mainly bases his study not on the interaction potential, but on four points that are easily found experimentally: the critical point, the triple point, and the two Boyle points. It turns out that the knowledge of these four points allows one determine the thermodynamics of the corresponding gases with sufficient precision.

The main model of the supercritical state is related in spirit to contour theory, which was developed in the papers [2]–[4] as well as in [8], [9], [10], and others. Contour cells bounded above by the number of particles appear on lattice modules at temperatures $T_{cell} < T_c$, i.e., in liquid. It turns out that, in passing through the critical point, they are preserved for temperatures close to $T_c$ and determine a similar model "cells—monomers." Monomers are superfluid through the cellular structure of contours, i.e., they pass through cells without collision, interaction, and without viscosity. Thus, fluids constitute the equilibrium state of two phases: cellular structure and monomers.

Although this is a rigorously mathematical work, we have tried to write it so that it could be easily read by physicists, explaining the derivation of our theorems in a simple language.

## 1. INTRODUCTION

### *1.1. The Force of Habit in Science*

The word "science" (*наука*) in the Russian language, according to the monograph *Constants: a Dictionary of Russian Culture* [1] has the same root as the word "habit" (*привыкание*). Yu. S. Stepanov explains that the ancient meaning of the root "*ук*" corresponds, paradoxically, to the modern understanding of the word "science" (*наука*): "to master a scientific theory" means "to *get used to* (*привыкнуть*) and learn how to use" the theory, which, as Yu. S. Stepanov tells us, sometimes is not an easy thing to do. In particular, in order to master the science of physics, one must get used to it and learn how to use it. And then one does not need to understand it. We have often quoted the words [2] of the well know physicist Ya. I. Frenkel in this connection.

In physical considerations involving asymptotics in several parameters, there is a lot of confusion. For example, in quantum physics, for the attraction of two molecules of noble gas we have asymptotics of the form $A/r^6$ as $r \to \infty$. But the quantity $A$ rapidly tends to zero in the quasi-classical approximation even for a fixed value of $r$. In such cases, theoretical physicists simply guess the order of passage, and in some cases actually get an answer that looks pretty true, especially if a few more errors are involved.

On the one hand, physicists are used to certain methods of analysis, on the other hand they disbelieve rigorous proofs. Mathematicians only notice identical formulas and their consequences. A freshman university student in physics cannot understand that

$$\text{if} \quad \int x\,dx = \frac{x^2}{2}, \qquad \text{then} \quad \int t\,dt = \frac{t^2}{2},$$

because for a physicist these are different quantities: $x$ is distance, $t$ is time. Later they will get used to this.

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[1] Yu. S. Stepanov, *Konstanty: Slovar' Russkoy Kul'tury*, Akademicheskii Proekt, Moscow, 2001, pp. 469–470 [in Russian].

[2] "We easily get used to what is constant and monotonously repetitious, we stop noticing it. Whatever is customary seems natural to us, while the unusual seems unnatural and non-understandable. Basically, we cannot understand, we can only get used to [11, 63]."

Similarly, theoretical physicists sometimes look at identical equations describing different processes and using different notation for the variables, and cannot understand why a mathematician writes, say, Diophantine equations of number theory such as

$$\sum_{i=1} N_i = N, \qquad \sum_{i=1} iN_i = M, \tag{1.1}$$

when considering a Bose condensate. The fact that the equations are the same, they are unable to see, because the letters in them, just as in the above example of the freshman physics student, are different.

The Boltzmann approach, refuted by the Gibbs paradox, which 15 Nobel prize winners, including the great physicist Fermi, as well as the great mathematicians Poincaré and Von Neumann, tried to settle, did not have a sound mathematical foundation. I explain this by the fact that, at the time, experiments were not fine enough to determine the appearance of dimers and clusters, so that the theory of clusters was not adequately described. And so physicists got used to the old erroneous mathematical theory (see [12], [13]) and could not explain a whole series of remarkable experimental effects. They tried to find, using the computer, adequate modifications of the van der Waals model [14], [15].

What is most amazing, is that in the Landau–Lifshits book [16], in § 54 (devoted to the Bose condensate), there is a footnote which describes the well known situation from mathematical combinatorics related to allocating identical balls to boxes and asserts that the number of different allocations (and hence the entropy) is the same as for the quantum Bose gas. In the book, two relations for classical gas appear in § 40, and the same two formulas (in the same notation!) for quantum gas are in § 54. In both cases the problem of maximizing entropy with the same constraints is solved by the Lagrange method. Nevertheless, in tons of books about classical gas, instead of the entropy correctly computed in the footnote in the Landau book, it is the erroneous Boltzmann entropy $-\ln N!$ which is given (see the article "Boltzmann statistics" in the *Mathematical Encyclopedia* [17]).

When this error is indicated to theoretical physicists, they answer that they possess 50 tons of books where this statement is proved, just like Panter from Anatole France's novel *Penguin Island* who said: "In my archives, they (the proofs) occupy seven hundred thirty two square meters so that, counting five hundred kilos per square meter, this gives a total of three hundred sixty six thousand kilograms."

And so the statistics of identical particles, the statistics of Bose–Einstein, Fermi, Dirac, and Gentile [18] are only allowed in such an "unreal" world as quantum physics. The application of the results in everyday life, economics, and sociology only frightens, just as does the sin of learning from the biblical tree of knowledge. The Russian philosopher L. Shestov (1866-1938) wrote: "The great secret of innocence is in that, at the same time, it is also fear," the instinctive fear of the sin of acquiring new knowledge.

The main mistake of theoretical physicists has the same philosophical aspect. Physicists consider as indistinguishable ("absolutely" indistinguishable, of identical essence) only quantum particles. The fact that in a "heap" they should also naturally be regarded as subjectively indistinguishable if the "heap" is measured by a measure invariant with respect to the permutation of particles does not convince them of their error. When I explain this to them and, finally, they understand, they are perplexed, like the driver in the old anecdote, who, having passed the test and obtained his driver's license, said: "I understand everything, except that I can't figure out where do they hide the horse in the machine?"

One of the brilliant experimental physicists, let's call him NN, asked me: "In quantum mechanics, if we fix a particle, then this affects all other particles, even if the particles have no interaction. But what happens in your theory?" I answered that if in Moscow one person is arrested, from the point of view of the city's population density things hardly change: only $0.0\dots01\%$ of the population has been sent to prison. But if it is specifically him (NN) who is arrested, this will spoil the statistics related to the density of people in Moscow, because we have to count and renumber all the city's inhabitants in order to single out precisely NN.

Thus the statistics of indistinguishable Muscovites (Muscovites of identical essence) changes radically when we pass from percentages to personalities.

### *1.2. History of the Question: the Heap Antinomy*

The story behind the Gibbs paradox originates in ancient times.

The philosophical problem related to the “heap antinomy,” which was formulated in the 4th century BC by the Ancient Greeks, is the following. We consider collections of grains of sand, at first a few grains, then we add a few more, and so on. The question arises: At what moment does the collection become a “heap”?

The difference between a small collection of grains of sand from a heap of sand is, first of all, in various methods of measurement. In the first case (few grains), we simply count their number, while in the second (a heap) we measure the heap in spoonfuls, cups, buckets, barrels, or grams, kilograms, tons, etc. Thus these two types of collections differ by their “measure”. However, this does not mean that, by means of present day computer techniques, we cannot count the number of grains in a heap or measure a small collection in milligrams.

Thus the main difference between a small collection of grains of sand and a sand heap is that they involve different “statistics”. Even if we have measured a small collection in milligrams, the total weight of the collection will remain the same if we interchange some of the particles: the total weight does not depend on the position of the grains. But if we are counting the grains, we must number the counted grains so as not to lose count. The sum does not depend on the order in which we count. Therefore, if we are counting, there is no difference in principle between a heap and a small collection. There is another difference, which was apparently noticed by one of the ancient philosophers: if we interchange grains, we must change the numbers assigned to them, and so we obtain a new state. The understanding of this aspect was developed, in accordance to Aristotelian physics and neoplatonic philosophy, by the famous religious thinker Saint Basil the Great (329–379), the Archbishop of Cappadocian Caesaria, in his *Hexaemeron*, which had a considerable influence, in particular, on the Russian Orthodox Church.

First of all, in the very notion of “unique essence of Father and Son” that Saint Basil has left us, the variativity and subjectivity of numeration is clearly established.

One of his followers, a theologian who interpreted the origins of the universe as arising from chaos in six days, proposes, as an example of chaos, the pile of little colored pebbles from which the artist creates the mosaic representing the visage of the Savior. In the heap of mosaic pebbles, the pebbles can be interchanged, the heap will not change. But if, in the finished mosaic, we interchange some elements, we will obtain another picture. Apparently, it is this very difference that the Greek philosopher Eubulidis of Miletus (4th century BC) was stressing when the stated the question of comparing a collection of grains of sand and a heap, and thus presented the “heap antimony”.

Thus we can say that, from the modern point of view, the difference between a heap and an non-heap of sand lies in the choice of statistics, i.e., in the indistinguishability and “unique essence” of the grains in a heap: we can interchange them, and the notion of heap (its identity) will not change.

So we speak of the Boltzmann or Shannon statistics whenever the interchange of two objects changes the notion of heap, and speak of Bose–Einstein, or Fermi–Dirac, or Gentile statistics whenever the interchange of two objects does change it.

As was already mentioned, the contemporary possibilities of the computer allow us to use both types of statistics, and the final computation after averaging the Boltzmann statistics will give the same result. But even with the most powerful computer the heap may be so big, that many years of computer time will be needed.

Let us note once again that for a collection of grains of sand, we can always count them by using the natural numbers, no matter how many grains there are. But we can also choose a different measure, say the weight of the collection, and then, starting from two grains, we can say that this measure admits the transposition of particles: the weight will not change under any permutation. In such considerations both statistics are applicable. However if the transition to a heap has been postulated (“phase transition” has occurred), i.e., the number of grains $K$ after which we have a heap has been fixed, then automatically in our consideration of the heap we introduce the following fundamental notion: the *undistinguishing statistics of distinguishable objects* (UD *statistics*).

Also, there are measuring devices that cannot be replaced by more precise instruments. Such are our senses: eyesight, sense of touch, etc. Thus, when we look at a painting by a pointillist from close up or by using binoculars, we will not see the whole picture, just some points. At the same time, looking at the painting from an appropriate longer distance, our eyes, imagination, memory, associations will allow to perceive the picture in a special way, and to see not only what the artist wanted to depict, but also some additional individual images.

When we speak of of the passage from a small number of grains of sand to the notion of heap, then, obviously, we are not referring to a jump, although the difference is considerable. But it is rather difficult to specify, it depends on the feeling of different people.

The papers [19], [20] address the question: "How can one introduce parameters so as to make this passage equivalent to a phase transition of the first kind?" This is related to a relaxation step by step process, in which a key role is played by the duration of the observation period.

### *1.3. The Relationship of Thermodynamics with Economics and Sociology*

Equilibrium thermodynamics is so general a science that everyone knows about the analogies between thermodynamics and economics (*thermoeconomics*), thermodynamics and sociology (*human thermodynamics*) and so on. These disciplines are related not only by analogies, but also by common mathematical formulas.

The author at first had to face political problems and tried to understand the economic and social problems appearing in Russia, in Vietnam and other countries in the 70-90ies of the previous century (see [21],[22], [23]), in order not to lose his Homeland and his family[3]. But it was only on the basis of equilibrium thermodynamics that the derivation of mathematically established formulas that so well correlated with the latest experiments became possible.

Irving Fisher, a pupil of the great physicist Gibbs, noticed the analogy between the following thermodynamical and economic quantities;

money $M$ — particles $N$,

amount of goods $Q$ — volume $V$,

turnover rate $v$ — temperature $T$.

We extend this analogy to another pair of quantities:

nominal credit rate $R$ — chemical potential $\mu$.

The speed of dissemination and the volume of messages in social networks is a generalization of the notion of temperature. An indirect measurement of such temperatures may be performed by calculating the slowdown of transmission when the volume of messages increases (as they say, when "the internet explodes").

Negative pressure, i.e., negative energy, corresponds in physics to anti-particles, and in economics, to anti-money, i.e., debts (holes).

Nevertheless, time has shown that despite the remarkable coincidence with experimental data, theoretical physicists, even more than people in the humanities, are afraid of falling into "the sin of acquiring new knowledge".

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[3] Perhaps, the way that I will cast my die
Will lead me — oh, child-killer Russia —
Down to your cellars where I'll die
A pool of blood as my last cushion.
M. Voloshin, *At the Bottom of the Underworld*. (Translated by A. Lecsis.)

### *1.4. The Number of Degrees of Freedom*

The author has succeeded in explaining a number of effects by using the undistinguishing statistics of distinguishable objects (UD statistics). Here a significant role was played by one of the most important notions of thermodynamics generalized by the author — the notion of fractional number of degrees of freedom.

For example, we are interested in the average number of degrees of freedom of the population of a given country. So we must take the average over the whole population. But we don't know how to assess that number for one specific person. In Part II, Sec. 13 of his *Metaphysics of Morals* [4], Kant writes:

"Any notion of debt involves an objective enforcement by law (by a moral imperative that limits our freedom) and belongs to the category of practical reason that sets the rules. The awareness of an inner judgement in a human being (during which his thoughts accuse and forgive him) is one's *conscience*. Each human being has a conscience, and is always aware of an inner judge watching him, threatening him, a judge for whom he feels respect (related to fear)."

If in all strata of society there is a moral rule, for instance "Thou shalt not kill," it is always possible to find a way of contradicting the rule, as in M. Gorki's famous phrase "If the enemy refuses to capitulate, he must be destroyed". The word "destroy" is now in the standard lexicon of journalists as a synonym of "kill," nowadays bandits or terrorists are no longer "killed," they are "destroyed".

The biblical Ten Commandments appeared after the heap antinomy, with which we began our consideration of the paradox of the two statistics. Incidentally, the Orthodox ethics of the Russian philosopher S. N. Bulgakov — each must be a monk and an ascetic at heart [5] — also has its roots in Ancient Greece, in particular, in Sparta.

The importance and incomprehensibility of the notion of number of degrees of freedom, just as the "elements" of Pythagoras, which is experimentally computed for any pure gas, can scare one. No wonder Pythagoreans were burned at the stake.

In the charter of his monastery, Basil the Great tried to combine the absence of degrees of freedom of the monk in his cell (monastic isolation) with elements of communal life (monastic togetherness). He based his monastic rules exclusively on togetherness and related the contemplative way of life with socially useful activity.

Robespierre and his followers tried to introduce the principle of interdiction into a state law: the principle forbidding prices above a certain maximum for products of first necessity and forbidding workers' salaries above a certain maximum. Ensuring these limitations of the degree of freedom for a period of time required a policy of terror. Attempts to inculcate this into the "conscience" of the citizen based on the notion of Higher Being did not succeed and therefore an overthrow of power (the "phase transition" of the 8th of Thermidor), which increased the number of degrees of freedom, occurred throughout France very rapidly — after 8 months (see (7.10)).

In the interview *The 20th Century History of Russia in the Mirror of Present Times*, the Russian historian Yu. A. Polyakov said: "Our country underwent epochal metamorphoses: three revolutions, a thermidor that completely transformed it politically, economically, and ideologically, significantly changed its borders, the ethnical character of its population, the logic of its development, the ruling elites." As I have already written, all these metamorphoses I had to foresee and take into consideration so as to survive (see [21], [23]).

Here is another striking example. When Tsar Nicholas II, in order to protect his family from the revolutionaries, sent an elite disciplinary army division under the command of general N. I. Ivanov, who had succeeded in putting down the Kronstadt insurrection, to Gatchino, the soldiers immediately became infected by the absence of limitations leading to the increase of the degree of freedom and shifted their allegiance to the revolutionaries at once, without going through a period of indoctrination. However, had the moral principles of the soldiers been stronger, this would not have occurred. Therefore the critical point, the critical fractional dimension of the number of degrees of freedom of the community of soldiers was close enough.

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[4] I. Kant, *Collected Works in 6 Volumes*, Vol.4, Part 2, Mysl, Moscow, pp. 376–377 [In Russian].

[5] S. N. Bulgakov, *Orthodoxy*: *Essays on the Teachings of the Orthodox Church* (Terra, Moscow, 1991), p. 326 [In Russian].

As mentioned above, the author began the considerations summarized in the present work, devoted mostly to thermodynamics, i.e., to physics and chemistry, by attempting to solve economic and sociological problems and then using the numerous analogies that exist between these sciences related to different statistics.

Let us consider, as an illustration, the old problem of the role of the individual in history, a problem that Marxists claim to have solved. According to our analogy, if the role of the individual in some country during some period of time was significant, then the situation is closer to Boltzmann statistics: one individual cannot be replaced by another.

If one prefers a rougher view of history, then the role of the individual in inessential: one individual may be replaced by another one; this does not seriously affect how history evolves.

Let me give an example when history was significantly changed when one individual replaced another. When Pavel was enthroned in Russia, the war with Prussia was immediately stopped and Russian foreign policy changed. But the new political situation did not last long: Pavel, as we would put it today, was destroyed. When an individual interferes with the evolution of history and is not in resonance with the natural way in which it evolves, then this individual is destroyed.

The Bourbon emigrants from France, headed by the Count of Provence, menaced the Convent and the House of Orleans with revenge, attempted to restore the monarchy and to revoke all the new laws. But they failed. E. Tarle wrote: "*All* (the italics are Tarle's — V. M.) the laws introduced by Napoleon remained after the restoration of the Bourbons."[6]

The solution of the problem of the role of the individual in history in the positive or negative sense cannot be very precise. When one looks deeply into history, just as tropical geometry looks deeply into ordinary geometry, or as fractal dimension gives a deeper view of the British coastline from the altitude of satellites, so could the UD statistics approach give rough predictions of historical evolution for long periods of time.

The relationship of between the number of degrees of freedom and temperature will be discussed below. We shall see that this number $D$ does not change along isotherms and isochores. Do we have the right to carry this rule over to history when the role of the individual is disregarded? Indeed, in the equilibrium situation, the number of degrees of freedom, i.e., the amount of moral principles and prohibitions, is stable, but it changes when the temperature and density in society changes.

In thermodynamics, a phase transition is a change in the number of degrees of freedom. Perhaps, if we disregard the role of the individual in history, this transition will be an abrupt passage to a new "religion" and new moral prohibition rules.

How should one understand the critical point in human society? It is related to endless fluctuations, the gatherings into huge clusters (crowds) that may quickly disintegrate and then regroup into other gatherings. This is the way out of a two-phase equilibrium. Such a point exists, it has a specific temperature, a (small) number of degrees of freedom and so on. But it is only a point, it can be determined only by indirect observations, and it cannot be practically accessed.

The Boyle point is a state of low density and individual ownership. At the triple point, we simultaneously have gas, liquid, and ice. Since a crystal can also be regarded as an ordered state without chaos (a planned economy) such a point is also usually hypothetical, and so it cannot be accessed in practice. But this point indicates the possibility of achieving complete planning.

In thermodynamics, these three points have been determined with sufficient precision for all pure materials. This made it possible for the author to construct equilibrium thermodynamics in the case when no other parameters except these three points are known. This approach is based on UD statistics, which was mentioned above.

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[6] E. B. Tarle, *Collected Works* (Izd. AN SSSR, Moscow, 1959), Vol. VI, p. 381 [in Russian].

### *1.5. The Density Paradox*

By density in thermodynamics, as in other sciences (the density of people in a city, density of fishes in a pond, density of leucocytes in blood, etc.), one understands the ratio of the number of particles to the volume, where the number of particles is very large, and the volume, as compared to the volume of one particle, is also very large. And so, in thermodynamics, we assume that density is the limit of the ratio of the number of particles to the volume when the number of particles tends to infinity. We have become so used to this notion, as well as to the notion of the Avogadro number $10^{23}$, that by joining these two ordinary notions, we come to the so-called paradox of the "two-dimensional Bose condensate". Thousands of books and articles assert that the two-dimensional Bose condensate is only possible at absolute temperature equal to zero. Nevertheless, as the author has written long ago, and repeatedly, it follows from a theorem of Erdös in number theory that the Bose condensate appears at a temperature equal to a certain constant divided by the logarithm of the number of particles

$$T = \frac{C}{\log N}, \tag{1.2}$$

where

$$C = \frac{\hbar^2}{\sqrt{2}m}\left(\frac{N}{\mathfrak{S}}\right),$$

$\hbar$ is the Planck constant, and $\mathfrak{S}$ is the area.

Since the absence of the Bose condensate was not linked to superfluidity, which had been obtained experimentally, physicists introduced a new notion — pseudo-condensate. The author gave another proof of the Erdös theorem, more understandable to physicists. Finally, he succeeded in convincing the leading Russian experts (in particular, Academician Yu. M. Kogan) in the validity of formula (1.2) and in the existence of the Bose condensate in the two-dimensional case. In the present article, we give another, simpler, proof of the Erdös formula and prove the existence of the Bose condensate in the two-dimensional case by a method simpler than the one previously described by us.

The innovative character of my economic approach did not hamper my finding a mutual understanding with the economists V. V. Leontiev, Leo Gurwitz, and B. Z. Milner, all three of whom, unfortunately, are no longer with us. This article is devoted to the memory of the latter.

These remarkable economists told me that they had grasped my theory. Other economists only understood the part of my theory related to tropical mathematics. But the situation in the country at the beginning of *perestroïka* was such that none of the economists whom I had convinced were able to help me bring my ideas to the notice of the ruling elite, on whom the economic policy of the country depended. It was hard to approach the men at the top. I did not even succeed in reaching, before the crisis (default) occurred, a former student of my applied mathematics chair who had moved high up in the national administration. And so I did not succeed in saving the failing national economy, I only managed to rescue my own family.

## 2. THE CORRESPONDENCE PRINCIPLE BETWEEN QUANTUM STATISTICS AND CLASSICAL UD STATISTICS

The following *correspondence principle* holds:

(1) The degeneration temperature $T_0$ (of transition to the Bose condensate) in the quantum case corresponds to the critical temperature $T_c$ in the classical case.

(2) To the Bose condensate corresponds a condensate of the gas (vapor) into a *virtual* liquid without attracting forces, i.e., a dense incompressible ideal gas, which had previously been regarded as a model of the liquid state by some physicists [24], [25].

What objects are to be distinguished is a matter of convention. The simplest example is paper money, i.e., bills. Each bill has a serial number, but from the point of view of the generally accepted convention and even from the point of state laws, bills of the same denomination are interchangeable in the sense that when the salary of an employee is given in rubles, it makes no difference what concrete ruble bills are given to him provided that they exactly add up to his salary. The employee will not sue the bank or the owner of an ATM for giving him bills of the wrong denominations or with serial numbers that he doesn't like.

Let us describe an example from equilibrium thermodynamics regarded as a process.

If we introduce the notion of observable quantity in thermodynamics, then we must keep in mind that the observation itself is performed at discrete moments of time at relatively long intervals of time. From the purely mathematical standpoint, any process leading to equilibrium requires infinite time. Of course, in mathematics, there are notions similar to the notion of "half-life" in physics. For example, one can introduce as such a notion the time required for the difference between the current state and the equilibrium state in a relaxation process to be decreased $e$ times.

In approximation theory and numerical methods, especially after the well-known work of L. I. Mandelshtam and M. A. Leontovich [26], the following relaxation process has often been used: at first the reacting system is taken to equilibrium, then one of the parameters (say the pressure or the temperature) is quickly changed and the evolution towards a new equilibrium is monitored (see, for example, the entry "mean relaxation time" in the Encyclopedia of Oil and Gas at `http://www.ngpedia.ru`).

Since the intervals between steps must be equal to the relaxation time, they are quite large, and such a process may be called a *multistep relaxation process* (MRT). Economic and historical processes, as well as biological processes in living organisms, are the same kind of phenomena, and for this reason thermodynamical models of these processes periodically appear.

Discrete time intervals between observations — that is the first thing to take into account when talking about measurement devices.

If we increase the temperature of a boiler connected to heating devices, then we must wait for a rather long time before an approximately equal temperature will be reached in the room. Similarly, if the pressure in a vessel with a piston is increased, then some time will elapse before the pressure waves are reflected and begin to decay gradually. In the damping process, the main role is played by viscosity, which we neglect, because, finally, an equilibrium is established.

Fluctuations in equilibrium thermodynamics must be sufficiently small. But the transition to equilibrium thermodynamics from nonequilibrium thermodynamics and hydrodynamics is very complex at "catastrophe" points, i.e., at focal and caustic points. This is similar to the transition from quantum mechanics to classical mechanics. In thermodynamics, to turning points (caustic points) correspond the points of the so-called "spinodal," while to a focal point (stronger catastrophe) corresponds a critical point.

Physicists have withdrawn into their niches so far that, as some experimenters (D. Yu. Ivanov, in particular) told me and as the well-known physicist Wagner wrote in a private letter[7], they do not publish in their journals articles that are at odds with the scaling hypothesis and its conclusions.

Closely related inferences follow from the asymptotics at a focal point. They agree with the analogy of the Riemann and Maxwell surface rules. Let us consider some experimental data for the phase transition "gas—liquid" for mercury (see Fig. 1), which was supplied by Professor V. S. Vorob'ev.

We see that the "gas—liquid" is not vertical, but is spread over viscosity, which suggests an analogy with a shock wave. The notion of "dequantization," which has led the author to thermodynamics and tropical mathematics, agrees entirely with the experiment referred to above and is not consistent with the scaling hypothesis and the theory of critical indices derived from it. The critical indices obtained by D. Yu. Ivanov, which are close in value from those obtained from scaling theory, were derived from the tunnel canonical operator applied to Wiener quantization described by Feynman and Gibbs.

Mathematicians and experimental physicists are willing to engage in discussions. Theoretical physicists stand firm, shutting their ears, and do not participate in open discussions. And stand firm all together. The remarks about one of the author's works by an anonymous physicist in a review were answered by the author on the site `http://www.za-nauku.ru`.

The difficulty here is also due to the fact that the theoretical physicist understands only the "science" that he is used to and the logic that follows from the customary set of its rules. My close friend, the late E. G. Maximov, a pupil of V. L. Ginzburg, whose language I had learned to understand, told me the following unforgettable phrase: "You will not convince me by your lemmas!" Nevertheless, he finally intuitively understood me and agreed with my arguments, but wasn't completely sure.

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[7] "Although the "critical point mafia" in the US seems to be overwhelming, it is nevertheless important, I think, to make very clear that there are also other opinions about the thermodynamic behavior of pure fluids in the critical region."

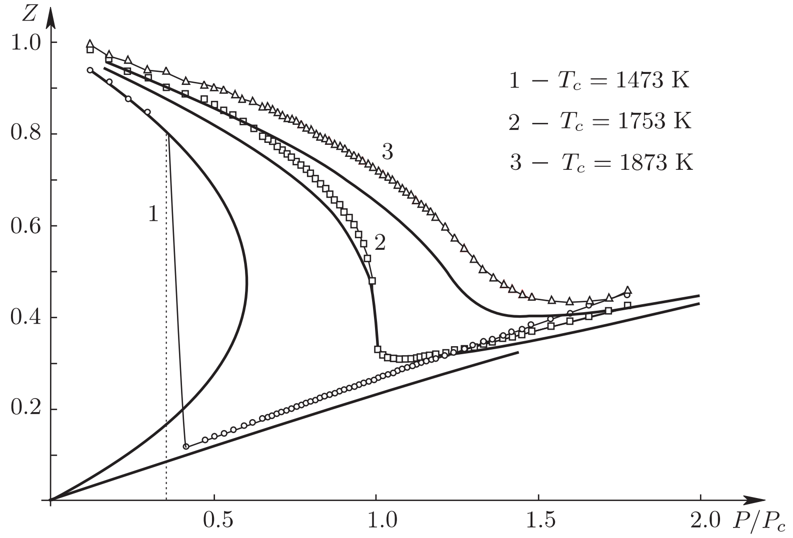


**Fig. 1.** The Hougen–Watson diagram for mercury shows the phase transition "gas–liquid." Experimental data (from the sources W. Gotzlaff, G. Schonherr, F. Hensel, Z. Phys. Chem. Neue Fol. **156** (219) (1988) and W. Gotzlaff, Ph. D. Thesis, University of Marburg, 1988) are equipped with symbols. The thick lines correspond to the van der Waals equation for the related temperatures.

Even the coincidence of my theoretical computations with the results of physical experiments does not convince physicists, on the contrary, it makes them suspicious. This is probably due to the fact that they are used to "fixing up" their "theories" to fit the experiment. Then what language are we speaking?

The very well-known physicist A. A. Vlasov, the author of the famous Vlasov equation in the theory of collective oscillations, asked my pupil V. L. Dubnov, who was defending his PhD thesis, the following question: "Can you draw a straight line through two points?" When Dubnov answered "Yes, I can," Vlasov got up from his seat and loudly declared: "Never, never, no matter how carefully you aim, will you get from one point to the other given point!"

He was right: Dubnov would surely have missed. And Vlasov's shot hit the point, because this is where the main difference between our logical rules and habits lies. L. V. Keldysh, my old and dearly loved friend since our student days, did not doubt the mathematical rigor of my constructions, but would not believe that it has any relationship with physics. I am not a physicist, I am simply constructing, overcoming mathematical errors, a general mathematical equilibrium theory. Unfortunately, the most convenient way of checking this theory in practice is in the case of gas. This is more chemistry than physics, and the coincidence of the theory with experiments has been checked by narrow experts in this field. There is no underlying special physics here, no physical hypotheses. I am correcting the erroneous rules that physicists are used to, and this, as explained in the introduction, is the great sin of learning the truth. But this is fear: "But you will not scare a coward" as Arbenin, a protagonist of Lermontov's drama *The Mascarade* exclaims.

My anonymous opponent (to whom I answer on the above-mentioned site "Moving to the renewal of Russian science"; see the journal *Za Nauku*) asks: "What does number theory have to do with this?". There are rumors that the head of a whole school of theoretical physics asserted that he does not consider that number theory is a science (see [27]). Nevertheless, as we mentioned, Erdös' 1951 theorem refutes the famous "theorem," which physicists have "proved"; it claims that Bose condensate cannot exist in dimension two,

The author is by no means an expert in the theory of gases and chemistry, and cannot be blamed for the fact that his theory leads to a perfect coincidence with experimental data. As a mathematician, the author only answers for the correctness of his theorems, and these theorems show that certain statements put forward by physicists are in error.

The Bolzmann–Maxwell gas also occurs in certain situations, namely when the Bose–Einstein entropy is transformed into the Bolzmann entropy, i.e., in the case of the old Maxwell–Bolzmann ideal gas. The author's theory is also a theory of ideal gas, i.e., of a gas without particle interactions. Nevertheless, it contains the phase transition gas to liquid. But in the theory, liquid is understood as gas under pressure. This point of view on liquids previously appeared in the work of some physicists. We can find the binodal through which this transition takes place. But a real gas under pressure, according to the quantum theory of dipole-dipole interaction, attracts and forms the liquid, to which everyone, especially physicists, are used to. However, the new ideal gas, as a mathematical object, explains a whole series of important phenomena without appealing to particle interactions.

The importance of distinguishing an ideal gas obeying UD statistics from a gas with interactions is as important as distinguishing classical mechanics from quantum mechanics, and of course should be done without denying the quantum theory and the interactions that it involves.

One might ask: How can we take into consideration the interaction of particles in the constructed statistics of identical particles? We must construct the three-dimensional Hamiltonian that contains the interaction of particles, i.e., the self-consistent Hamiltonian or Vlasov Hamiltonian containing the potential of pairwise interactions $V(x_i - x_j)$.

Bogolyubov always maintained that the most important thing derived by Vlasov is the equation of "collective oscillations"; in other words, the linear variational equation for Vlasov's equation. It is this equation that helps discover many secrets of the theory of many interacting particles. The mathematical essence of such secrets is the presence of the minima of the Hamiltonian when there are more than one minimum. If the barrier between these minima is sufficiently high, then this fact leads to different spectra, or, more precisely, to different spectral series. This is very important for the experimenter. It is difficult (practically impossible) to guess the form of the interaction potential, while it is quite feasible to obtain resonance spectral series using the spectrograph.

The important difference between the Boltzmann–Maxwell ideal gas and the quantum ideal gas is that, in the first case, we must consider an important notion, the number of degrees of freedom. For one-atom molecules, this number is 3, for two-atom molecules, it is 5, and for three-atom molecules, it is 6.

For ideal quantum gas, we deal with the three-dimensional and two-dimensional cases. It turns out that these notions are closely related: as $\mu \to -\infty$, the dimension with respect to momenta of the quantum gas coincides with the number of degrees of freedom of the Boltzmann–Maxwell ideal gas.

However, we must always average the number of degrees of freedom over all gas molecules in the given state, and then this number will, in general, be fractional. Denote this number by $D$.

One of the most important notions of modern thermodynamics is the so-called *compressibility factor* $Z$, which is a dimensionless quantity equal to $Z = (PV)/(NT)$. The value of the critical point $Z_c$,

$$Z_c = \frac{P_c V_c}{N_c T_c} = \frac{P_c}{\rho_C T_c},$$

was, until recently, hard to find in tables of critical parameters. For hydrogen, noble gases, and methane, $Z_c = 0.29$. For mercury, $Z_c = 0.39$ (this is the highest value among pure gases). For the van der Waals gas, $Z_c = 0.375$.

For the Bose–Einstein quantum three-dimensional gas,

$$Z = \frac{\mathrm{Li}_{5/2}(a)}{\mathrm{Li}_{3/2}(a)},$$

where $a = e^{\mu/T}$ is the activity and Li is the polylogarithm.

For the general value of $D$,

$$Z = \frac{\mathrm{Li}_{D/2+1}(a)}{\mathrm{Li}_{D/2}(a)}. \tag{2.1}$$

By the correspondence principle, such a value of $Z_c$ is realized for $\mu = 0$, i.e., for $a = 1$. In this case, the polylogarithm is replaced by the Riemann zeta function

$$Z_c = \frac{\zeta(D/2+1)}{\zeta(D/2)}. \tag{2.2}$$

In particular, for $Z_c = 0.29$, we have $D = 2 + 0.444$. Obviously, it is a very small number of degrees of freedom.

The expert in practical chemistry Gekhman, a man of remarkable intuition, was not surprised by the author's communication, but said that he fully agreed with it. Vorob'ev and Apfelbaum compared experimental critical isotherms with the Bose distribution of dimension $D$ given by formula (2.2). Their graphs are shown in Figs. 2–7.

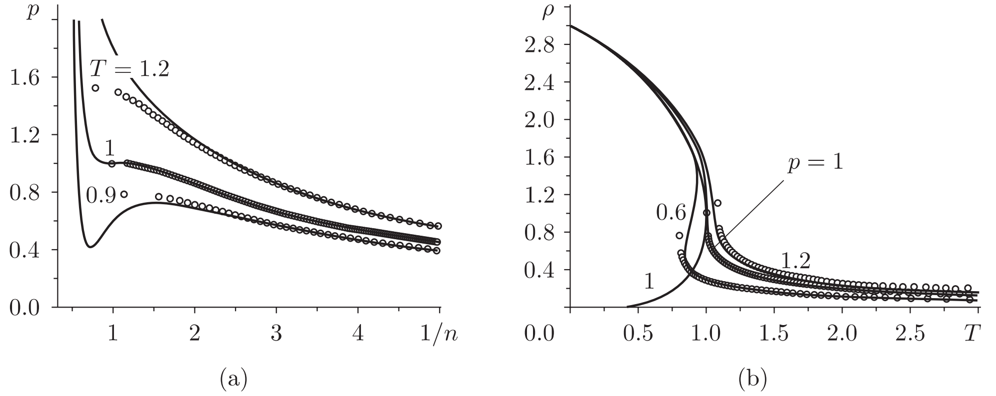


**Fig. 2.** (a) Isotherms of pressure for the van der Waals equation are shown by solid lines. The lines formed by little circles are plotted from calculations for $\gamma = 0.312$ (i.e., the ideal "Bose gas"), $Z_{\mathrm{cr}} = 3/8$. $p = P/P_{\mathrm{c}}$, $n = N/N_{\mathrm{c}}$.
(b) Isobars of density for the van der Waals equation are shown by solid lines. Line 1 is the binodal. The little circles depict the isobars of the "Bose gas" for $\gamma = 0.312$.

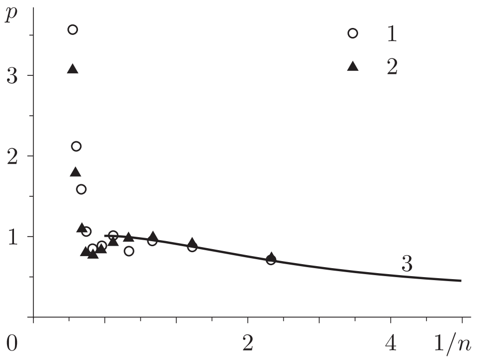


**Fig. 3.** Critical isotherms for the Lennard-Jones system. The symbols 1 and 2 depict the theoretical values. Line 3 corresponds to the ideal Bose gas for $\gamma = 0.24$.

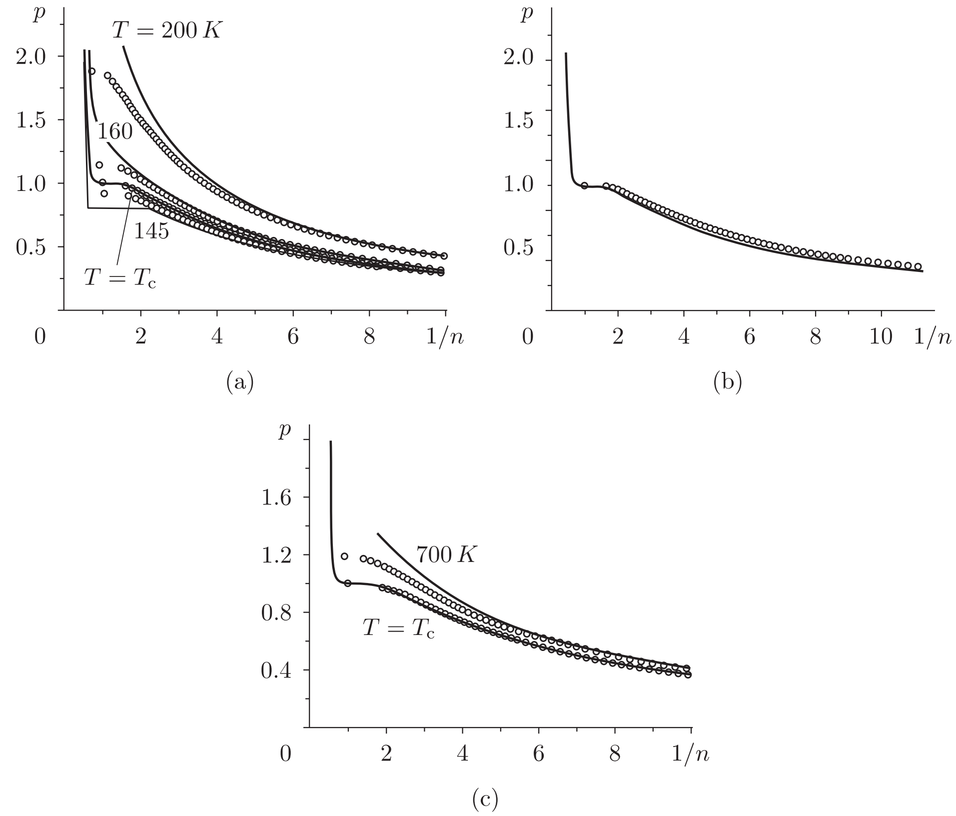
Thus, the critical isotherms of the (nonideal) classical gas are in full agreement with the isotherms of the ideal Bose gas with the corresponding number of degrees of freedom up to the critical point. The continuation of the isotherm will be defined below in Sec. 7 in the case of a positive chemical potential.

Let us now study the behavior of the Poisson adiabat in its dependence on the number of degrees of freedom $D$ for the Bose distribution as $\mu \to -\infty$. To do this, we expand the formulas for the Bose distribution generalized to dimension $D$

$$P = CT^{D/2+1}\,\mathrm{Li}_{D/2+1}(a), \qquad P = \frac{N}{V}T^{D/2}\,\mathrm{Li}_{D/2}(a) \tag{2.3}$$

and

$$Z = \frac{\mathrm{Li}_{D/2+1}(a)}{\mathrm{Li}_{D/2}(a)} \tag{2.4}$$



(a)

(b)

(c)

**Fig. 4.** (a) Isotherms for argon. The solid lines correspond to experimental data. The lines formed by little circles are plotted from the isotherm of the ideal Bose gas, $Z_{\mathrm{cr}} = \frac{\zeta(\gamma+2)}{\zeta(\gamma+1)} = 0.29$. $p = P/P_{\mathrm{c}}$, and $n = N/N_{\mathrm{c}}$;
(b) The same for water, $Z_{\mathrm{cr}} = 0.23$;
(c) The same for copper, $Z_{\mathrm{cr}} = 0.39$.

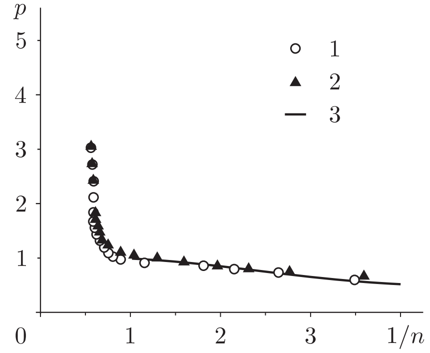


**Fig. 5.** Isotherms for water. The symbols 1 and 2 correspond to experimental data, and line 3 corresponds to the theoretical values for the Bose gas.

as $a \to 0$ (i.e., $\mu \to -\infty$),

$$\mathrm{Li}_{D/2+1}(a) = a + \frac{a^2}{2^{D/2+1}} + O(a^3),$$

$$\mathrm{Li}_{D/2}(a) = a + \frac{a^2}{2^{D/2}} + O(a^3),$$

$$Z(a) = \frac{a + a/2^{D/2+1}}{a + a/2^{D/2}} + O(a^2) = 1 - \frac{a}{2^{D/2+1}} + O(a^2). \tag{2.5}$$

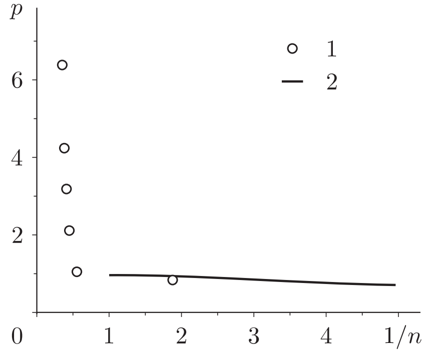
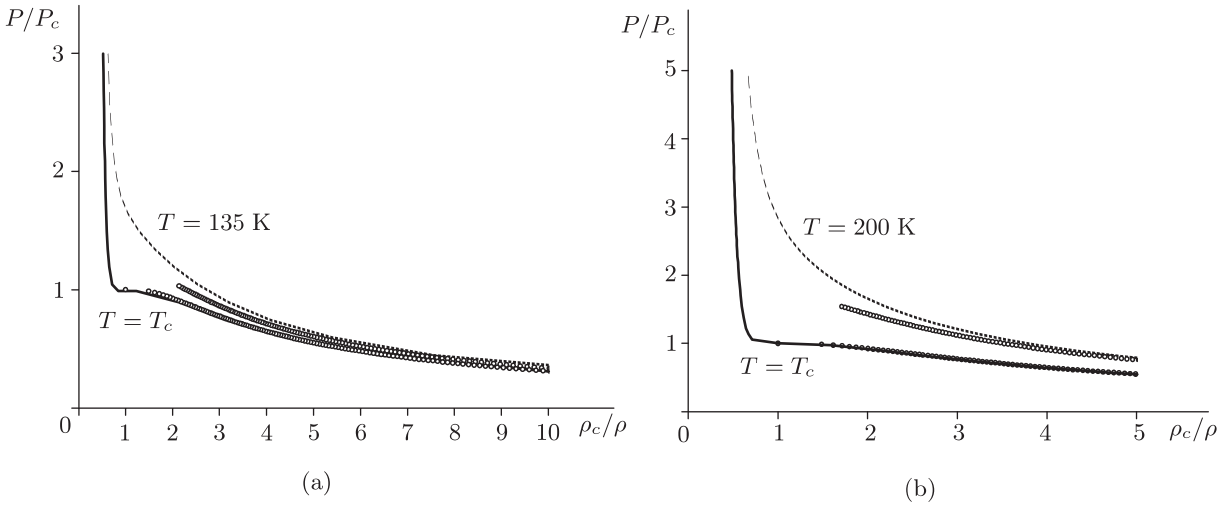


**Fig. 6.** Critical isotherms for mercury. The symbols 1 and 2 correspond to experimental data, and line 3 corresponds to the theoretical values for the Bose gas.



(a) (b)

**Fig. 7.** Isotherms of ideal Bose gas compared to experimental data (lower thick line: (a) isotherms for nitrogen with $\gamma = 0.218$, $Z_c = 0.286$; (b) isotherms for oxygen with $\gamma = 0.219$.

Hence

$$\frac{dP}{dZ} = \mathrm{const}(2T)^{D/2+1} \tag{2.6}$$

as $a \to 0$.

For fractional dimension, the Poisson adiabat generalizes as follows:

$$c_v = \frac{D}{2}, \qquad c_p + c_v = 1, \qquad \tilde{\gamma} = \frac{c_p}{c_v},$$

where $D$ is the number of degrees of freedom. In our case of classical ideal Bose gas, we have

$$\tilde{\gamma} = \frac{D+2}{D}.$$

In view of (2.6),

$$P \sim \frac{PV}{N} T^{\gamma+1}, \qquad \gamma = \frac{D}{2} - 1.$$

In view of formula (43.9) of [16],

$$VT^{1/(\tilde{\gamma}-1)} = VT^{D/2} \sim \mathrm{const}.$$

This Poisson adiabat for ideal Boltzmann—Maxwell gas obtained from the Bose distribution is one of the numerous corroborations of the author's approach treating the Bose distribution as a distribution of classical particles. The resulting coincidence means that the Hougen—Watson isotherm vectors are directed along the isotherms of classical Bose gas.

Theorists in physics are not convinced by the remarkable agreement of the mathematically rigorous approach of the author with the latest experimental data of V. S. Vorob'ev (see [14]). Can this really be an accidental agreement?

## 3. DERIVATION of BACHINSKII'S FORMULA USING THE LENNARD-JONES POTENTIAL

The attraction between particles arises in the quantum mechanical study of the dipole-dipole interaction. In the standard semiclassical limit, if the distance between neutral molecules is fixed (is independent of the parameter $\hbar$, i.e., of a dimensionless parameter proportional to $\hbar$), then the attraction vanishes as $\hbar \to 0$. In this sense, the use of an attractive potential in molecular dynamics based on the classical Newton equation for many particles is not legitimate.

But if the problem under consideration involves other small or large parameters in addition to $\hbar$,then the attractive potential is preserved as $\hbar \to 0$ if there are certain relations between them. For example, this is possible in the short-wave approximation under the Sommerfeld condition at infinity, which leads to a non-self-adjoint problem and essentially uses the fact that viscosity is vanishingly small.

In quantum scattering problems with dissipation, rigorously studied after Kuroda's paper [28] appeared, the semiclassical limit preserves the attractive potential "for nearest neighbors," as physicists used to say. Without presenting these sufficiently complicated estimates, we consider the classical scattering problem involving the attractive potential and, in particular, the Lennard-Jones potential.

As is known, the relation

$$\frac{mv^2}{2} + \frac{M^2}{2mr^2} + \Phi(r) = E. \tag{3.1}$$

holds in the radially symmetric case. We prescribe the individual energy $E$ and the impact parameter $B$ for the initial scattering particles. Their momentum $M$ and the energy $E$ are preserved. It is also known that

$$M^2 = B^2E. \tag{3.2}$$

We express the energy $E$ to obtain

$$E = \frac{(mv^2)/2 + \Phi(r)}{1 - B^2/r^2} \tag{3.3}$$

in the case of attraction in the domain where $r \leq B$.

Figure 8 presents the diagram corresponding to the current state of thermodynamics and was obtained by using molecular dynamics methods (numerical simulation) and by physical experiments.

The comprehensive thermodynamics is based not on the well-known van der Waals empirical relation but on the results obtained by numerical simulation in the new field of science called "molecular dynamics." The results of these experiments are shown in Fig. 8. The solid straight line issuing from the Boyle point is called the Zeno line, while the straight dashed line is called the "binodal" (i.e., the curve separating the two phases) [29]. A jump of the density $\rho$ occurs at a given temperature less than $Tc$; this jump is marked on the abscissa.

As the interaction potential in the scattering problem, we take the Lennard-Jones potential

$$\Phi(r', r'') = 4\varepsilon\biggl(\frac{a^{12}}{||r' - r''||^{12}} - \frac{a^6}{||r' - r''||^6}\biggr), \qquad r = r' - r,'' \tag{3.4}$$

where $\varepsilon$ is the well depth energy, $a$ is the effective radius, and $||r' - r''||$ is the distance between two particles with radius vectors $r'$ and $r''$. If there is no external potential in the two-particle problem, then the problem can be reduced to a one-dimensional radially symmetric problem.

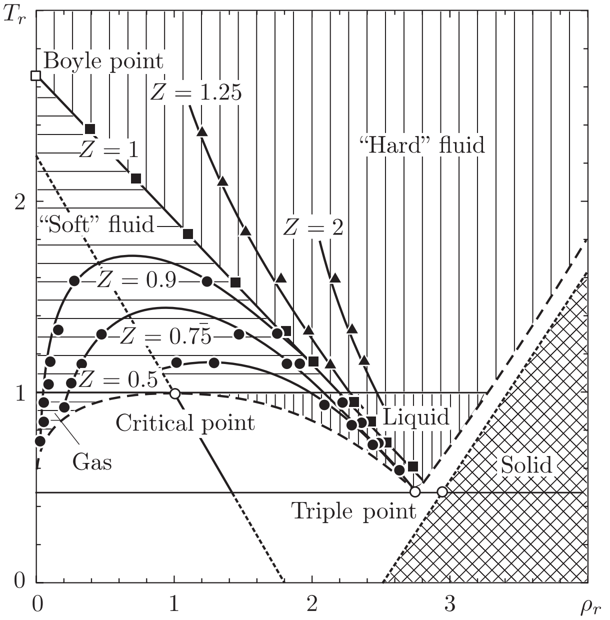


**Fig. 8.** The $(T, \rho)$ diagram for gases corresponding to simple liquids; $T_r = T/T_c$ and $\rho_r = \rho/\rho_c$. The Zeno line (the straight line $Z = PV/kNT = 1.0$) is shown in the phase diagram. The repulsive forces dominate for the states $Z > 1.0$ (hard fluids), and the attractive forces dominate for the states $Z < 1.0$ (soft fluids).

In problem (3.3), other barriers and wells arise for different values of $B$ (see Figs. 9 and 10). The velocity is zero at the stationary points $E_{\min}$ and $E_{\max}$, and hence they can be calculated from the potential term only.

We now consider not a single particle but a pair of particles whose center of mass is trapped. The difference $E_{\max} - E_{\min}$ is therefore the energy required to release this pair (dimer) from the trap. The dimer percentage in a gas can be determined experimentally. It can be seen how dimers are created and split by monomers. Their average number is then calculated. The higher the temperature, the greater the mean energy of monomers, and the fewer the number of dimers.

It is an important characteristic of such an approach that only the two quantities $E_{\max}$ and $E_{\min}$ remain in the skeleton of the scattering problem (cf. the amoeba skeleton in tropical mathematics [30]). The well disappears when$E_{\max} = E_{\min}$. This energy is equal to $0,8\varepsilon$ for the attracting part of the Lennard-Jones potential. It follows from the above isotropy that the mean particle energy is $16\varepsilon/5$. The mean energy corresponds to the temperature $T = 16\varepsilon/5k$. There is no well above this temperature. This is the the so-called Boyle temperature $T_B$ in thermodynamics. According to the data for $\varepsilon$ calculated by this formula, the Boyle temperature is $T_B = 382K$ for argon (Ar) and and $T_B = 547K$ for krypton (Kr); the experimental tables in [31] give $T_B = 392K$ for argon and $T_B = 538K$ for krypton. The discrepancy between the theoretical and experimental values is 2 to 3%.

The critical temperature $E_{\max}$ must correspond to the deepest well, i.e., to the maximum value of the difference $E_{\max} - E_{\min}$ for all values of the impact parameter $B$. This difference shows how the dimer energy drops after its capture by the "trap" and thus what energy the monomer must have for the dimer to be freed from the well (i.e., to cause its disintegration).

The barrier height "protects" the resulting pair (whose reduced mass gets into the trap for the "dimer" and clusters) from "impacts" by monomers. The barrier height decreases with the temperature $T < Tc$, and the clusters must therefore create their own barrier (as an microanalogue of a surface film) in order to survive. Hence, a "domain" that is a three-dimensional cluster (the so-called elementary cluster) with at least one particle protected by other particles must be formed there.

The calculations give $E_{\max} = 0.286\varepsilon/k$ at the point $\max_B(E_{\max} - E_{\min})$. The impact parameter at this point is equal to $B = 2.436$.

**Remark 1.** The determination of the critical temperature and the Boyle temperature from the well depth of the interaction potential $\varepsilon$ is highly inaccurate, and different data are therefore given in different handbooks. In what follows, we therefore consider the dimensionless quantity $T_B/T_c = 2.79$, which agrees better with the experimental data (this value for argon is 2.73).

**Table 1**

| Substance | $\varepsilon$, K | $Tc/4$ | $Ec \cdot \varepsilon/k$ |
|---|---|---|---|
| $Ne$ | 36.3 | 11 | 10.5 |
| $Ar$ | 119.3 | 37 | 35 |
| $Kr$ | 171 | 52 | 50 |
| $N_2$ | 95,9 | 31 | 28 |
| $CH_4$ | 148.2 | 47 | 43 |
| $C_2H_6$ | 243.0 | 76 | 70 |
| $C_4H_{10}$ | 313.0 | 106 | 98 |
| $H_2S$ | 301 | 93 | 87 |
| $PH_3$ | 251.5 | 81 | 73 |

The only dimensionless quantity in our "skeleton" problem is $E_{\min}/E_{\max}$, i.e., we consider the simplified problem with only two stationary points.

Because $E_{\min}/E_{\max}$ is a dimensionless quantity and $E_{\max}$ is the average energy related to the temperature (multiplied by the density $\rho$), it follows that the dimensionless quantity in thermodynamics is the so-called compressibility factor, denoted by $Z$: $Z = PV/RT$.

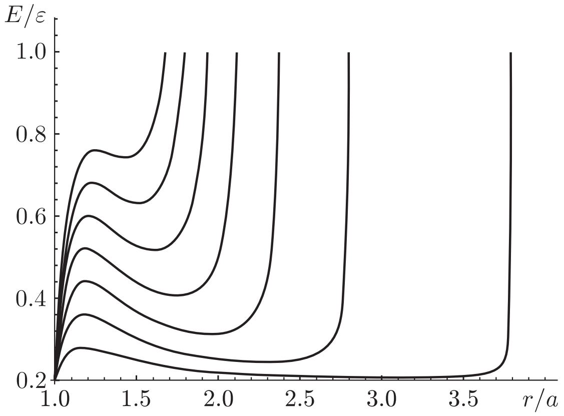


**Fig. 9.** Wells and barriers in the scattering problem for two particles with a Lennard-Jones interaction for different impact parameters $B$.

We now derive analytic formulas for the Zeno line depending on the potential. The initial assumption is that the thermodynamical description is independent of the shape of the vessel. Because only the volume, not the shape, is important for thermodynamical quantities, we can determine the characteristic length from the value of the volume. Such a quantity is the ball radius for a ball-shaped vessel and the tube radius for a long narrow test tube (like the one used by Torricelli). Therefore, according to dimension theory, we generally have some volume measure, and we take the effective radius $a$ for the length measure.

We now use considerations of dimension theory for the scattering problem and the definition of the attracting one-particle (the so-called thermal) potential. The scattering problem is considered for the

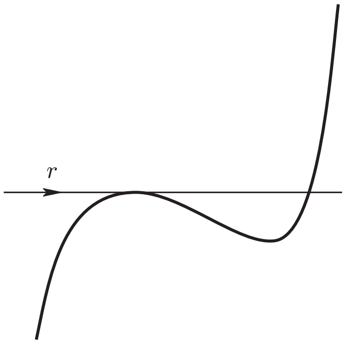


**Fig. 10.** A trap for a fictitious particle (dimer) in the center-of-mass system: $r$ is the dimer radius vector marked on the abscissa, and the particle falls from the left from the point $r = B$.

Lennard-Jones potential, and the problem therefore has an additional length parameter, i.e., the effective radius $a$. The attracting potential arises in the quantum theory of dipole-dipole interaction. If the distance between particles is fixed and the semiclassical parameter $\hbar$ tends to zero, then the attracting potential vanishes. This means that the distance between particles decreases as $\hbar \to 0$. This implies the following assertions:

1. Using the attracting potential in the classical system of Newton equations in molecular dynamics problems is, to say the least, not very well posed.

2. The attracting potential acts only between the nearest neighbors.

It is therefore natural to expand the attracting one-particle potential in a power series in the radius $r$ up to $O(r^3/V)$.

The "thermal" potential $\Psi(r)$ is an attractive potential. Because the volume $V$ is a large parameter, it additionally follows that the expansion of the potential $\Psi(r) = \Psi(ar^2/V)$ in $1/V$ implies that

$$\Psi\left(\frac{ar^2}{V}\right) = C_1 + \frac{C_2 ar^2}{V} + O\left(\frac{1}{V^2}\right). \tag{3.5}$$

We expand

$$r^2 = r_1^2 + r_2^2 = \frac{(r_1 - r_2)^2}{2} + \frac{(r_1 + r_2)^2}{2}, \tag{3.6}$$

and then separate the variables in the two-particle problem, just as in [32], to obtain the scattering problem for a pair of particles and the problem of their cooperative motion (for $r_1 + r_2$). In the obtained scattering problem, the Lennard-Jones interaction potential is then supplemented with the attractive quadratic potential (inverted parabola) $-C_2\rho r^2$, $\rho = a/V$.

The presence of the thermal potential does not follow from the previous and subsequent constructions. It can be obtained by using ideas of the Vlasov–Bogolyubov self-consistent field theory or the fact that the correlation sphere for the $N$-particle Gibbs distribution is finite. As we see below, the dependence on density significantly complements the number theory distributions (see, e.g., Sec. 3 in [32]).

In the scattering problem thus obtained, there are usually two stationary points, the stable point $E_{\min}$ and the unstable point $E_{\max}$. Their ratio is a dimensionless quantity. In thermodynamics, the quantity $Z = PV/NT$ is dimensionless, and because the stable stationary point has the meaning of the temperature, it follows that the relation

$$Z = \frac{PV}{NT} = \frac{E_{\min}}{E_{\max}} \tag{3.7}$$

allows constructing the curves $Z = \text{const}$ on the graph $(T, \rho = N/V)$.

The curve with $Z = 1$ is called the Zeno line (or the *Bachinskii parabola*), and the geometric locus of initial points of the curves for $Z = \text{const}$ (for $C_2 \neq 0$ and $B \to \infty$))is the *binodal*.

The minimum value of the compressibility factor $Z$ for a given $\rho$ and $C_2 \neq 0$ is equal to

$$Z_{\min} = \frac{E_{\min}}{E_{\max}}\bigg|_{B\to\infty}.$$

It corresponds to the Boyle temperature for $Z_{\min} = 0.8\varepsilon$ and $\rho = 0$. As is known,the point on the binodal corresponding to $Z_{\min}$ for $\rho = 0$ corresponds to absolute zero. We let $Z_c$ and $\rho_c$ denote the values of $Z$ and $\rho$ at this point. We also let $\rho_B$ denote the endpoint of the curve $1 - Z_{\min}$ on the axis $\rho$. This point was called the hypothetical Boyle point in [33].

We calculate the value of $Z_c$ and obtain $Z_c = 0.296$, which coincides with the values of $Z_c$ for noble gases up to the third decimal place. The ratio $\rho_c/\rho_B$ also coincides with the values of this quantity for noble gases.

Table 2 presents the data corresponding to the obtained diagram (for $B = 100$ in “molecular” values). We note the deviation between the basic dimensionless relations obtained from data of molecular dynamics and the theoretical relations obtained by physicists using the BBGKY hierarchy of equations and the $N$-particle Gibbs distribution.

**Table 2**

| $Z_c$ | $\rho_c/\rho_B$ | $T_c/T_B$ |
|---|---|---|
| 0.29 | 0.273 | 0.36 |
| 0.308 | 0.285 | 0.38 |
| 0.375 | 0.333 | 0.296 |

The upper row of Table 2 contains the theoretical values of $Z_c$, $\rho_c/\rho_B$ and $T_c/T_B$ obtained using the dimension theory considered above, the second row contains the values of these quantities obtained from the latest data based on molecular dynamics and theoretical physics for the Lennard-Jones potential, and the third row contains the values obtained from the empirical van der Waals equation

The value of $Z_c$ can be obtained experimentally with a very high accuracy and is equal to 0.29 for noble gases, nitrogen, oxygen, and propane. The value of $\rho_c/\rho_B$ (i.e., the ratio of $\rho_c$ to the entire length of the segment of $\rho$, where the Zeno line intercepts the abscissa) coincides with the corresponding values for water, argon, xenon, krypton, ethylene, and a number of other gases.

We now construct the $(Z, \rho)$ diagram in which we show the curve $Z = 1 - (E_{\min}/E_{\max})|_{B=\max}$ and the straight line joining its endpoints ($Z = 1$ and $Z = 0$). If the thermal potential is present, then $B \to \infty$, and the curve has the form shown in Fig. 11.

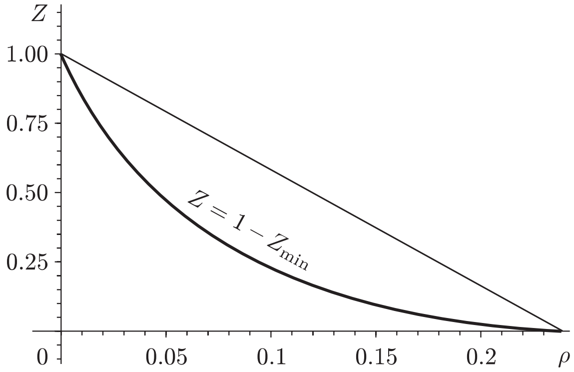


**Fig. 11.** The curve $1 - Z_{\min}$.

We determined the critical temperature for the “pure” Lennard-Jones potential without considering the thermal potential, while $Z_c$ and the Zeno line were calculated with the thermal potential taken into account. We considered the maximal well depth $E_{\max} - E_{\min}$ for the "pure" Lennard-Jones potential to determine the critical temperature.

We now present detailed calculations for determining the Zeno line. We consider the potential

$$E(r) = \frac{-\alpha r^4 + r^2 U(r)}{B^2 - r^2}. \tag{3.8}$$

It first and second derivatives are

$$E'(r) = \frac{r(2B^2U(r) + r(2\alpha r(-2B^2 + r^2) + (B^2 - r^2)U'(r)))}{(B^2 - r^2)^2},$$

$$E''(R) = \frac{1}{(B^2 - r^2)^3}\big(2(B^4 + 3B^2r^2)U(r) + r(-2\alpha r(6B^4 - 3B^2r^2 + r^4) \\ +4(B^4 - B^2r^2)U'(r) + r^2(B^2 - r^2)^2U''(r))\big). \tag{3.9}$$

We obtain the solution of the equation in the form

$$B = \sqrt{-\frac{-r^3U'(r) + r^4U''(r)}{-8U(r) + 2rU'(r) + 2r^2U''(r)}}. \tag{3.10}$$

We substitute $B(r)$ in (3.8) and obtain $E(\alpha)$, which is the Zeno line. We then set $B = 100$ and obtain the first derivative, which we equate to zero to obtain the ratio $(E_{max} - E_{\min})/E_{\max}$.

Thus, Bachinskii's relation has the form

$$\frac{\rho}{\rho_B} + \frac{T}{T_B} = 1.$$

The midpoint

$$\rho = \frac{\rho_B}{2}, \qquad T = \frac{T_B}{2}$$

divides the segment $0 \le P \le P_{\max}$ in half:

$$\frac{P}{2} = \frac{\rho_B T_B}{4},$$

where

$$P_{\max} = \frac{\rho_B T_B}{2}.$$

Bachinskii's relation implies

$$P = \frac{\rho_B}{Z_c}\left(T\left(1 - \frac{T}{T_B}\right)\right). \tag{3.11}$$

Equating (3.11) to $P_{\max}$, we find the extreme temperature.

## 4. THE LAW OF PREFERENCE OF CLUSTER FORMATION OVER PASSAGE TO LIQUID STATE

### *4.1. Phase Transition "Gas–Saturated Vapor" as a Phase Transition of the 3d Kind*

The question that we pose in two-phase thermodynamics is whether clusters or liquid appears first as the temperature reaches the dew point. Debate on this point is still continuing.

We shall show that, in our mathematical theory, the phase transition to liquid is determined by the spinodal in the negative quadrant $-P, -Z$. Negative pressure is equivalent to negative energy, which is related to antiparticles by Dirac's conjecture. The positron is the antiparticle for the electron, or the hole in the negative spectrum filled by electrons. Similar holes in the crystal lattice were discovered by Frenkel.

And what is a “hole” in our case of neutral particles: antimass or the black hole, as in astrophysics? As mentioned above, in economics, it is antimoney, i.e., debts.

In economics, at least in early Middle Ages, associations (clusters) were a better way than taking loans. In Rome, a citizen became a slave because of any unreturned outstanding debt. Associations also provided other advantages, for example, protection against pirates. As was already stated, clusters are “three-dimensional” and also protect molecules at the center of clusters from striking monomers.

In early Middle Ages, the Catholic church strongly objected to providing loans at interest for catholics. This moral prohibition is the usual norm in relations between relatives and good friends.

Judaism forbade to provide loans to its disciples, but did not object against providing loans at high interest to persons of other faiths. Lombards (the tribe from Northern Italy) acted in the same way. Such simple moral prohibition allowed the system to preserve a relative equilibrium for a sufficiently long time. This was a closed society in the sense that transitions from one stratum to another were severely hampered.

Mathematical statistics whose best application is thermodynamics must also explain matters in connection with history if it is considered schematically.

In this section, we demonstrate the law of profitable of association into clusters to transition to liquid state, using pure gases as an example.

As was already mentioned, in economics, by this principle, it is more preferable for people, companies, banks, etc., to form an association rather than fall in debt. In thermodynamics, we shall compare isotherms, binodals, and spinodals in two cases: consideration of the preference principle and pure Bose statistics.

The consideration of the preference principle consists in the maximal increase of the number of degrees of freedom for a given temperature and a given spinodal. Obviously, the increase in the number of dimers leads to an average increase in the number of degrees of freedom. Let us show that this principle is in almost full agreement with the gas part of the van der Waals model, so that the theoretical results approximate the experimental data for nitrogen quite well.

Russian experimental physicists think that, first the number of dimers and clusters maximally increases, and then the liquid is formed at the fixed temperature

$$T_r = \frac{T}{T_c} < 1.$$

This approach changes formulas of the Bose–Einstein statistics for $T_r < 1$. We shall illustrate this fact using graphs.

At the same time, as we shall see later, the occurrence of clusters in the thermodynamics of nanostructures and nanotubes is hampered and the agreement is greater with the ideal Bose distribution (in which the number of degrees of freedom is determined by the value of $Z_c$ at the critical point). And this value is, apparently, related to interaction, i.e., to the fact that the gas under consideration is not ideal. Thus, in our construction, the fact mentioned above occurs only at the critical points, at the triple point, and the Boyle point. Otherwise, the calculation of the number of degrees of freedom allows us to consider an ideal gas, but in UD statistics.

But how does one distinguish clusters from pieces of liquid in the undistinguishing statistics of objectively distinguishable particles, in the case of the “identical essence” of these particles? In equilibrium thermodynamics, only averaged quantities appear: mean energy – temperature, mean density $\rho = N/V$ and their duals: entropy $S$, chemical potential $\mu$; the latter two, due to the fact that the Lagrange submanifold is two-dimensional, depend on the first two quantities. A most important role is played by the number of degrees of freedom $D$, which in the old thermodynamics was considered only in the case of the ideal Boltzmann–Maxwell gas.

It is precisely the trend of increasing $D$, which the author called the “preference principle,” that can be construed as the preference of cluster formation.

We shall call the dual quantity to $D$ the *braking dominant* or *dominant* for short and denote it by the letter $\mathcal{M}$. We would like to stress in this way the role of ancient moral interdictions, written in ancient languages, because it is precisely these moral interdictions that limit the number of degrees of freedom.

Consider the sequence of natural numbers $1, 2, \dots, n, \dots$. Introduce a constant $\Lambda$, which we shall call the *free scaling*, indicating the units in which the ruler with the numbers $1, 2, \dots, n, \dots$ written on it uses to perform measurements.

In accordance with the van der Waals (VdW) model, let us introduce the scales $T_r = T/T_c$, $P_r = P/P_c$ as well as a new quantity — the number of degrees of freedom. Therefore, we must introduce a new scaling $x$. Obviously, if the number of degrees of freedom is $D \neq 1$, then the free scaling must be raised to the power $D$.

**Remark 2.** Under our approach, we use three points, taken from experiments: the critical point, the triple point, and the Boyle point. In general, all the formulas, isotherms, isochores, spinodals, and binodals are obtained only from the statistics of gases without particle interactions (i.e., ideal gas in the sense of Gentile parastatistics).

This approach contradicts the approach of "van der Waals forces" and the consequences of empirical interaction potentials. Actually, the VdW forces, in particular the force of attraction $A/r^6$, are obtained in quantum mechanics as $r \to \infty$. And so, even if $r$ is a finite fixed quantity, the attraction tends to zero as $\hbar \to 0$ and here we come to a contradiction.

Therefore, outside these points, as $\hbar \to 0$, there is no interaction, and we can use UD statistics of identical particles without interaction.

In many cases the answer will coincide with the one based on the VdW forces (the Lennard–Jones potential, see [34]–[36]). But this does not mean that both approaches are applicable here.

In the book [16], the misapprehension mainly concerns the interpretation of the notion of Boltzmann–Maxwell ideal gas. The authors assert that saturated vapor is a particular case of the Boltzmann–Maxwell ideal gas [16, §84]. Yet the book treats saturated vapor as if this notion differs from the notion of "gas". Not only students, but also some teachers of thermodynamics think that vapor is not a gas, and remembering locomotives, regard vapor as a kind of smoke. «Why does the locomotive puff and go, while the sauna puffs and doesn't go?» is a question expressed by a popular character from I. F. Gorbunov's 1861 short story "Scenes of folk life" that has become deeply rooted in our folklore.

In [16, 314-315] the authors write: «Regarding vapor as an ideal gas, let us express its volume according to the formula $v_2 = T/P$».

The assertion "a liquid cannot exist without attraction" is well known. We know that attraction of particles is a quantum phenomenon in which a substantial role is played by exchange interaction, which disappears in the classical limit.

The passage to the Bose condensate in quantum statistics is a phase transition of the 3rd kind. The appearance of density condensation at the temperature of the "binodal" (also called "dew point," although the dew droplets have not yet reached their critical radius) is the classical analog of the quantum Bose condensate, and hence the phase transition "ideal gas → saturated vapor" is a phase transition of the 3rd kind. And under this phase transition, saturated vapor remains an old Boltzmann–Maxwell ideal gas.

According to the correspondence principle indicated below, a density fluctuation "at rest" appears as the analog of the Bose condensate at rest. This fluctuation is at rest only in the same sense as metallic dust, moving so rapidly above a magnet that it does not stick to the magnet, forms a dense cloud "at rest" above the magnet, which can produce a deep shadow. The shadow stays in place, but the dust particles move. However, this sort of density is the *start-up mechanism* of the appearance of quantum forces of attraction which result in the formation of fluctuating droplets, and then as the temperature decreases, the radius of the droplet reaches its critical value and the fluctuating droplet becomes a real droplet.

However, Nature itself does not allow us to get rid of the mathematical model of ideal gas. The thing is that the law of "gas-liquid equilibrium" breaks down, since the pressure inside the droplet is greater than the pressure in saturated vapor. If we wish to remain within the framework of the original mathematical model of the passage from gas to liquid, we must consider the liquid without surface tension, and hence without forces of attraction. Fortunately for the author, certain physicists regard the model of liquid as a model of dense gas [24], [25], and so the author is not completely alone in his efforts to refute the postulate asserting that liquid cannot exist without forces of attraction between particles.

Thus we can assume that between the time when the temperature is that of the binodal and when it drops to the level at which the radius of the droplets attains its critical value, a density fluctuation forms. If our gas is in a spherical vessel in the presence of Earth's gravitational attraction, then a virtual liquid will form near the bottom of the vessel. If, in this situation, we mark a small number of molecules by isotopes (whose positions we can follow), we will see that these marked molecules will move freely from the virtual liquid to the vapor and back and form a dense structure near the bottom of the sphere, so dense that it will produce a shadow if parallel rays of light are directed on it. However, it is impossible to pull out this dense structure from the gas medium. Virtual liquid can be seen, but not felt. It can produce a shadow, but you can't drink it. In that sense I agree with the statement that "a liquid without attraction is not a liquid."

A phase transfer of the 3rd kind described above is actually observed at least for methane ($CH_4$) and tetraftormethane ($CF_4$) in experiments and occurs when part of the molecules associate into dimers, trimers, and other clusters as the temperature decreases (see [37]).

Dimers are formed due to the attraction between molecules. But a dimer lives only $10^{-5}$ seconds, until it is broken up by a monomer. For someone observing the gas discretely at large intervals of time, it is virtual, the observer can only assess the percentage of dimers in the gas. As shown in [38], statistically this phenomenon corresponds to a density condensate dissipated in the vessel. The increase in the density of dimers is accompanied by fluctuations of density as in a virtual liquid.

Just as fluctuations of density help the appearance of quantum attraction, so does the density condensate dissipated in the vessel play the role of start-up mechanism for the quantum forces of attraction that produce dimers. Virtual liquid is a model, it adequately describes the situation of the coexistence of gas-"liquid".

The Clausius condition for the old Boltzmann–Maxwell ideal gas model,

$$Z = \frac{PV}{NT} = 1$$

is not related with the number of degrees of freedom $D$ of the gas molecules. But in fact the passage "gas → saturated vapor" smoothly changes the number of degrees of freedom, because of the appearance of dimers and other clusters. Obviously, one must change the number of particles $N$ in Clausius' old formula, since in the calculation of $N$ in Clausius' time, the measuring devices could not distinguish individual molecules from dimers.

### *4.2. The Van Der Waals Model*

In what follows, using the $P, Z$ diagram, where $Z = PV/(NT)$ is called the *compressibility factor*[8], for the van der Waals equation, we shall show that gas tends to pass into clusters and then into liquid. This means that, for $T_r < 1$, where $T_r = T/T_c$ and $T_c$ is the critical temperature, the number of degrees of freedom $D$ first increases and, therefore, the parameter $\gamma$ also increases to its maximal value at a given temperature $T_r < 1$. (Note that the parameter $\gamma$ is related to the number of degrees of freedom by the relation $D = 2\gamma + 2$). The parameter $\gamma$ increases in such a way that its critical value is not $T_c$, but $T_r$. This effect will possibly occur also for the quantum case in which $T < T_0$, where $T_0$ is the degeneracy temperature of ideal quantum Bose gas.

As soon as experimenters saw dimers and clusters in their atomic microscopes, the old theory of thermodynamics should have undergone a revolution. However, physicists had grown accustomed to the old theory.

It is difficult to get accustomed to new facts rigorously proved and verified by experiments. Some physicists think that old thermodynamics and clusters, percolation, molecular dynamics, scaling conjecture are separate areas. But mathematically rigorous results bear witness to the fact that the Gibbs distribution for the Gibbs ensemble is valid and can be proved rigorously, while the Boltzmann–Maxwell distribution is erroneous from the mathematical point of view. But theoretical physicists do not accept the notion of "mathematical error" in these distributions, because they have their own notion of proof. More precisely, they pay attention to the word "derivation," rather than "proof."

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[8] Do not confuse with the notion of "compressibility."

From the physicists' point of view, the word "erroneous" does not apply here even if they grow accustomed to the Bose–Einstein distribution for classical ideal gas. They will say: "this is valid in a particular region." And this is indeed true: the Boltzmann–Maxwell distribution is valid in a certain region. For physicists, the Gibbs paradox is not a counterexample to the theory, but only another paradox, which can be explained away by means of another hypothesis (see Chap. 1 "Hydrodynamic paradoxes" in [39]).

The Bachinskii condition on the Zeno line on the $Z, P$ diagram uniquely relates $T_c$ to the value of $\gamma$. On the spinodal, for $3 \geq D \geq 2\gamma_0 + 2$, the fact that $\gamma$ is constant on isotherms and isochores implies the univalent dependence $T_c \leftrightarrow \gamma_c$. However, the Bachinskii condition is related to the interaction potential in thermodynamics.

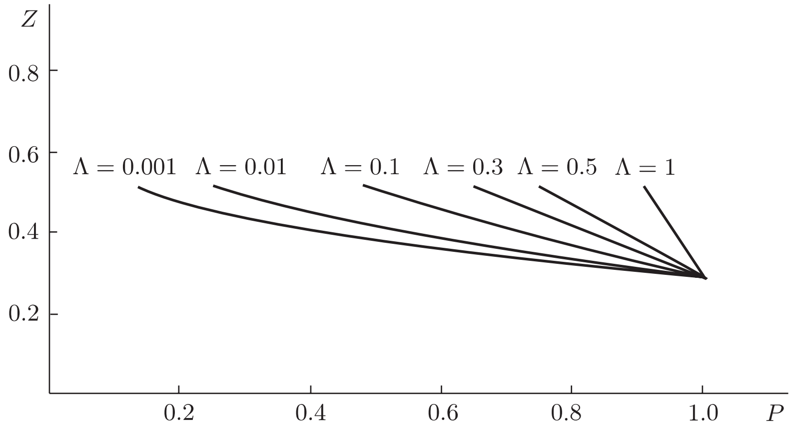

**Fig. 12.** Spinodals for the $\Omega$-potential of ideal Bose gas; $\gamma_c = 0.222$, $0.222 < \gamma < 0.5$.

Can we carry over this condition to economics and socialogy? It is more natural to use the ellipse from the point $Z_c$ to the points corresponding to the triple point.

Figure 12 shows the spinodals for the $\Omega$-potential of ideal Bose gas

$$\Omega = \Lambda^{\gamma-\gamma_c} T^{2+\gamma} \operatorname{Li}_{2+\gamma}(a) \tag{4.1}$$

for different values of the constant $\Lambda$ for

$$P = \Lambda^{\gamma-\gamma_c} \frac{\zeta(\gamma+2)}{\zeta(\gamma_c+2)}, \qquad Z = \frac{\zeta(\gamma+2)}{\zeta(\gamma_c+1)}.$$

The quarter of the ellipse passing from the point $\gamma = 0.222$, $Z = 0.45$ as an empirical spinodal is more natural in problems in social sciences as an analog of Poincaré theory. Incidentally, the ellipse also satisfies, up to 3%, the Bachinskii condition.

Using graphs, let us again show that our concept agrees with the famous empirical van der Waals model. However, we shall use another "derivation" for ideal Bose gas.

The van der Waals model corresponds to our concept of given specific points that define all isotherms, isochores, isobars, binodals, and spinodals of the thermodynamics of pure gases: the critical point the van der Waals model is $Z_c = 0.375$ and, therefore, $\gamma = 0.312$ (i.e., the number of degrees of freedom is $D = 2.624$), $T_c/T_B = 0.296$, and $\rho_c/\rho_B = 0.333$.

Figure 13 shows the graphs of the isotherms $T_r = T/T_c$ at different temperatures. Thin lines depict isotherms constructed according to the Bose distribution (Bose isotherms) for $\gamma = 0.312$. The van der Waals isotherms are denoted by thick dotted lines on which the corresponding temperatures are given. The thin lines pass close to the van der Waals isotherms only for $T_r = 1$. For $T_r = 0.75$ and $T_r = 0.85$, the increase in the number of degrees of freedom for the van der Waals isotherms is marked by the values of $\gamma$ on them.

The van der Waals model strictly obeys the Bachinskii rule. Since the value of $\gamma$ does not change along the isotherm, it follows that the Bachinskii rule immediately leads to a new spinodal shown by the thick line passing from the point $\gamma = 0.312$ to the point $Z = 0.5$. This line passes close to the quarter

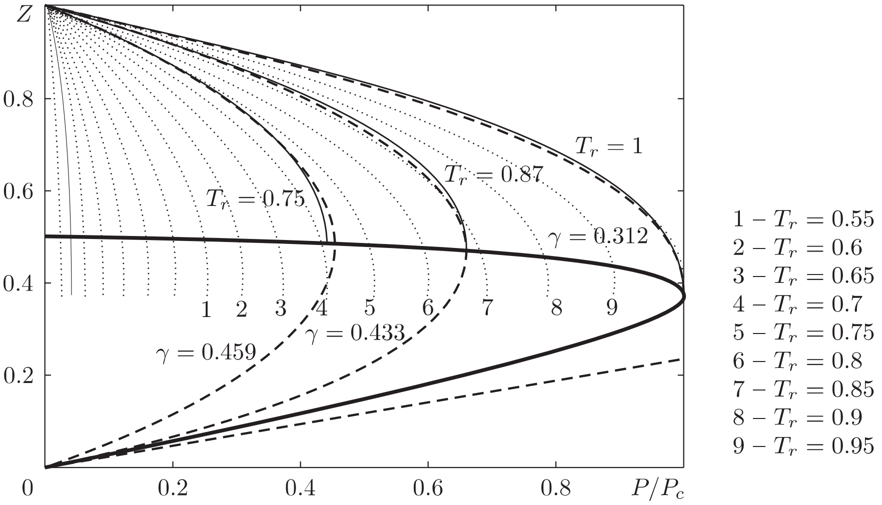

**Fig. 13.** Van der Waals isotherms (thick dotted lines) and Bose–Einstein isotherms (thin solid lines). The thick line is the spinodal.

of the ellipse. Near the point $P = 0$, the tangents of the Bose isotherms and those of the van der Waals isotherms coincide.

Figure 14 shows Bose isotherms with the same tangents and the same ellipse-spinodal as in Fig. 13.

Let us apply the preference principle, i.e., let us increase the value of $\gamma$ for each temperature until this value becomes critical for the given temperature. This leads to an almost full agreement with the van der Waals isotherms; see Fig. 15.

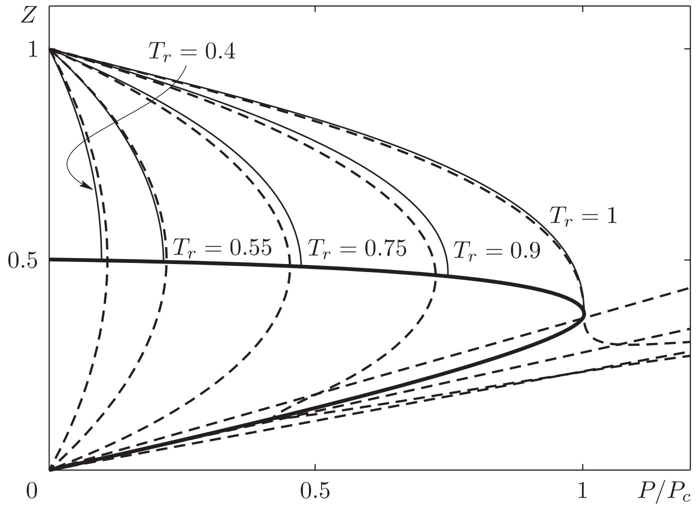

**Fig. 14.** Van der Waals isotherms (dotted lines) and Bose–Einstein isotherms (solid lines). The thick line is the spinodal.

Let us pass to the law of preference for the association of clusters before transition to liquid. Recall that the preference consists in that, at the same temperature $T_r < 1$ less than $T_c$, the gas particles tend to increase the number of degrees of freedom with from the original value $\gamma = \gamma_c$ to the value $\gamma(T_r)$ critical for the given $T_r$.

Recall again that the locus of points at which the transition ‘gas–liquid” occurs is called a *binodal*. The locus of points at which the tangent to the isotherm is perpendicular to the axis $P$ on the $Z, P$ diagram, is called a *spinodal*.

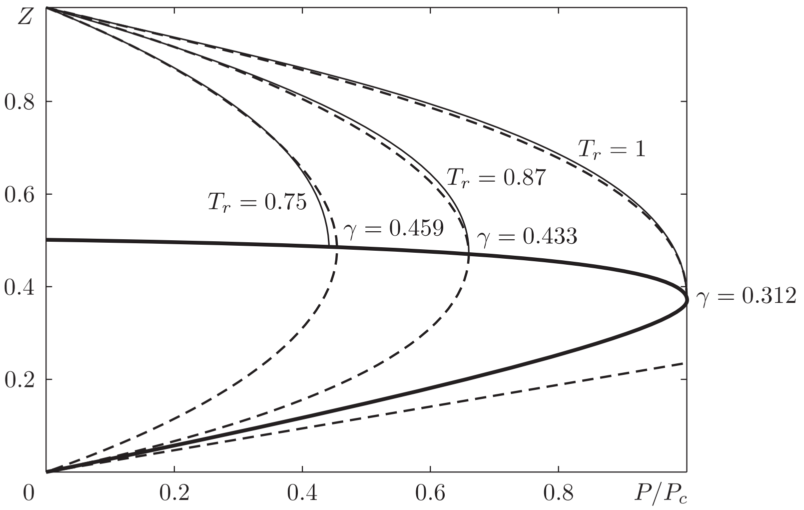

**Fig. 15.** Van der Waals isotherms (dotted lines) and Bose–Einstein isotherms (solid lines). Here the law of preference for the association into clusters is taken into account. The thick line is the spinodal.

Let $\gamma_c$ correspond to the critical isotherm for the given gas, i.e.,

$$Z_c = \frac{P_c V_c}{N_c T_c} = \frac{\zeta(\gamma_c + 2)}{\zeta(\gamma_c + 1)}. \tag{4.2}$$

The point $T_{tr}$ lying on the binodal and at which three phases: liquid, gas, and solid can simultaneously exist is called the *triple point*. Thus, this point is defined on the binodal and, at temperatures below this point, experimental isotherms are not observed. In the van der Waals model, $T_{tr} = 0$.

The isotherm $T_r$ issues from each point $P = 0, Z = 1$. Let us find the value of $\gamma(T_r)$ that is critical for $T_r$. In this case, $T_r = 1$ and $\gamma = \gamma(T_r)$. In the germ near $P = 0, Z = 1$, we have the relation (see Sec. 5):

$$\frac{dP}{dZ} = \frac{4C(\gamma(T_r))2^{\gamma(T_r)}}{\zeta(\gamma(T_r) + 2)} = \frac{4 \cdot 2^{\gamma_c} T_r^{(2+\gamma_c)}}{\zeta(\gamma_c + 2)}, \qquad T_r = \frac{T}{T_c}. \tag{4.3}$$

For a fixed $\gamma = \gamma(T)$, the equation of the isotherm is of the form

$$Z = \frac{\mathrm{Li}_{\gamma+2}(a)}{\mathrm{Li}_{\gamma+1}(a)}, \qquad P = C(\gamma)\frac{\mathrm{Li}_{\gamma+2}(a)}{\zeta(\gamma_c + 2)}, \qquad \gamma = \gamma(T), \tag{4.4}$$

where $a$ is the activity, $\infty > a \geq 0$, $\mathrm{Li}(a)$ is the polylogarithm, and $C(\gamma)$ is determined from the coincidence of the relation

$$\begin{cases} Z = \frac{\zeta(2+\gamma)}{\zeta(1+\gamma)}, \qquad P = C(\gamma)\frac{\zeta(2+\gamma)}{\zeta(2+\gamma_c)}; \\ 2 > \gamma \geq \gamma_c \end{cases} \tag{4.5}$$

with the spinodal constructed above. Here $C(\gamma)$ is uniquely defined.

Indeed, let, for $0 \leq P \leq 1$, the equation of the quarter ellipse be of the form $P = f(Z)$.

Then

$$C(\gamma) = f\left(\frac{\zeta(\gamma + 2)}{\zeta(\gamma + 1)}\right)\frac{\zeta(\gamma_c + 2)}{\zeta(\gamma + 2)}.$$

If $\gamma = \gamma(T_r)$, $T/T_r = 1$, then we finally obtain

$$\frac{C(\gamma(T_r))2^{\gamma(T_r)}}{\zeta(\gamma(T_r) + 2)} = \frac{2^{\gamma_c} T_r^{(2+\gamma_c)}}{\zeta(\gamma_c + 2)}, \qquad T_r = \frac{T}{T_c}. \tag{4.6}$$

Now the relation for $\gamma(T)$ takes the form

$$\frac{2^{\gamma(T)}}{\zeta^2(\gamma(T)+2)}\, f\left(\frac{\zeta(\gamma(T)+2)}{\zeta(\gamma(T)+1)}\right) = \frac{2^{\gamma_c}T^{2+\gamma_c}}{\zeta^2(\gamma_c+2)}. \tag{4.7}$$

The final main equations of the "critical" isotherms of the pure gas corresponding to the value $Z_c$ (4.2) are of the form

$$\begin{cases} Z = \frac{\mathrm{Li}_{\gamma(T)+2}(a)}{\mathrm{Li}_{\gamma(T)+1}(a)}, \\ P = \mathrm{Li}_{\gamma(T)+2}(a)\frac{T^{(2+\gamma_c)}\cdot 2^{(\gamma_c-\gamma(T))}\zeta(\gamma(T)+2)}{\zeta^2(\gamma_c+2)}. \end{cases} \tag{4.8}$$

For the van der Waals equation corresponding to the ellipse passing through the points ($Z = 3/8$, $P = 1$) and ($Z = 1/2, P = 0$) the isotherms pass so close to each other (see Fig. 15) that they are indistinguishable on the graph, in contrast to Fig. 14. As Vorob'ev showed, the divergence with experimental data for nitrogen is greater if the law of preference stated above is not taken into account.

Since, as $T_r \to T_{tr}^0$, where $T_{tr}^0$ is the triple point corresponding to the gas with the given value of $Z_c$ (4.2) the isotherms condense, it follows that, at this point, the derivative of the spinodal with respect to $P$ on the $(P, Z)$ diagram is zero. Therefore, the quarter of the ellipse terminates not on the line $P = 0$, but at the point $\gamma = \gamma_{tr}$ for the value $P_{tr}$ ($\mu \to 0$) corresponding to this point. This uniquely determines the elliptic spinodal of the gas branch.

Thus, the agreement with the latest experiments is very good (see Figs. 16 and 17). The vector of Hougen–Watson isotherms near the point of the old ideal Boltzmann–Maxwell gas is determined by the isotherm of the Bose–Einstein type distribution, while the famous empirical van der Waals model corresponds to the distribution for ideal Bose–Einstein gas. The correspondence with the mathematical statements is thus rigorously proved with mathematical exactitude inherent in number theory.

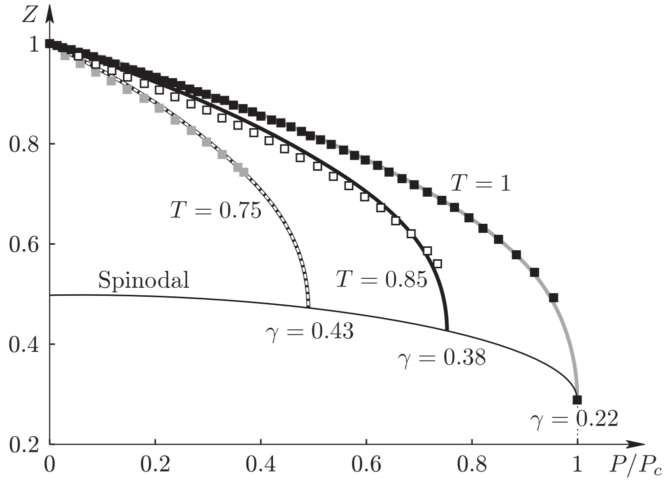

**Fig. 16.** The line marked as "Spinodal (0.5, 0.29)" is an ellipse. The thin line over it is the theoretical spinodal. The thin lines are theoretical isotherms. The lines marked with squares are the latest data for nitrogen drawn by Prof. Vorob'ev.

## 5. NEGATIVE VALUES OF THE PARAMETER $\gamma$ AND NEGATIVE PRESSURE

The gas spinodal, which is defined as the geometric locus of the endpoints of isotherms of the new ideal gas, is formed in the case of maximal entropy at the points where the chemical potential $\mu$ is zero. On the diagram $(Z, P_r)$, the spinodal is therefore the segment $P_r \le 1$, $Z = Z_c$ in the case of the van der Waals normalization $T_r = T/T_c$, $P_r = P/P_c$. For $T_r \le 1$, the Bose condensate arises and hence the quantity $N = T_r^{\gamma_c+1}\zeta(\gamma_c+1)$ remains constant on the liquid phase isotherm for the liquid phase on the

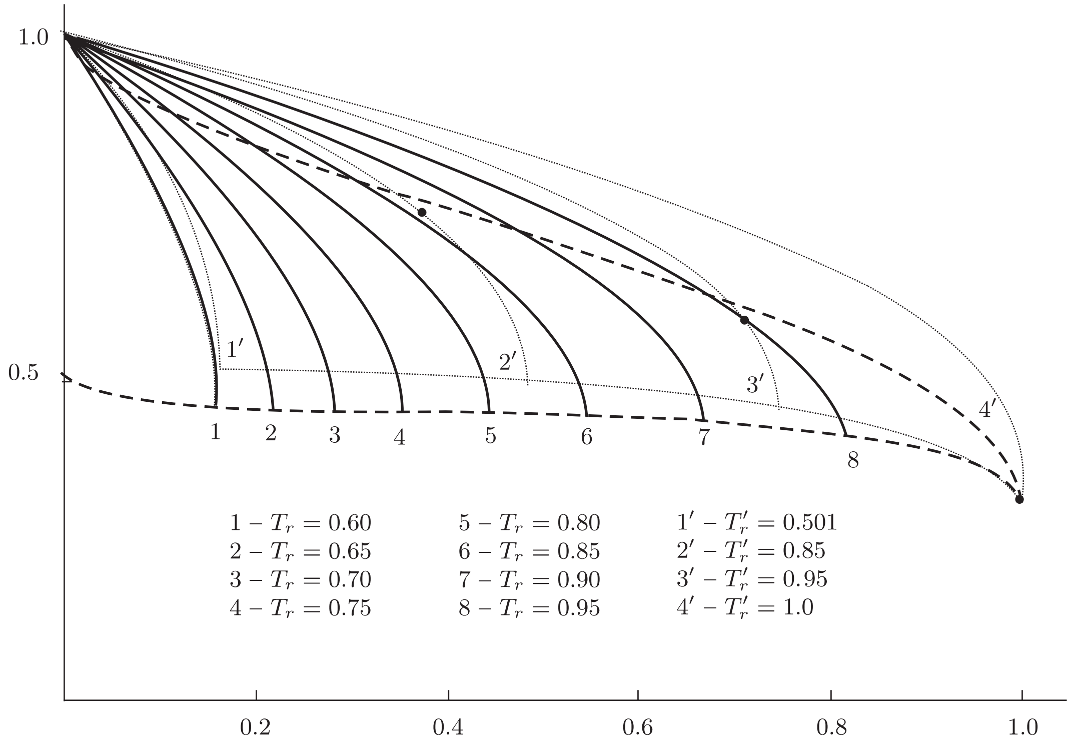

**Fig. 17.** Isotherms shown by thick lines and the corresponding temperatures are taken from [40]. Thin lines depict theoretical isotherms coinciding with experimental isotherms for nitrogen and corresponding to temperatures $T'_r$. The points on the isotherms denote the end of the experimental isotherms and the transition to liquid. See Sec. 5.

spinodal. This means that the isotherm of the liquid phase corresponding to the temperature $T$ has the form

$$Z = \frac{P_r}{T_r N} = \frac{P_r}{T_r^{\gamma_c+2}\zeta(\gamma_c+1)}. \tag{5.1}$$

All the isotherms of the liquid phase (including the critical isotherm at $T_r = 1$) pass through the origin $Z = 0$, $P_r = 0$ and further pass into the negative domain (or into the "second sheet"). The point $Z = 0$ corresponds to the parameter $\gamma = 0$ and hence to the extension of the isotherm for $\gamma < 0$ because the pressure for $\mu = 0$,

$$P_r = T_r^{2+\gamma}\frac{\zeta(2+\gamma)}{\zeta(2+\gamma_c)} \tag{5.2}$$

can be extended for $0 > \gamma > -1$.

Below, we see that as $\mu/T \to o(1/\ln N)$, the quantity $Z$ is also positive, and the spinodal for $0 > \gamma > -1$ therefore gives the second sheet in the diagram $(Z, P)$, which it is more convenient to map into the negative quadrant.

We recall the relation for the $\Omega$-potential:

$$\Omega = -\Lambda^{\gamma-\gamma_c} T_r \sum_i \ln\left(G_i \frac{1-\exp\frac{\mu-\varepsilon_i}{T_r}N}{1-\exp\frac{\mu-\varepsilon_i}{T_r}}\right), \tag{5.3}$$

where $\varepsilon_i = i$ and $G_i = i^{\gamma+1}$.

**Lemma.** *Consider the integral*

$$I = B\int_0^A \left(\frac{1}{e^{bx-b\mu}-1} - \frac{k_0}{e^{k_0(bx-b\mu)}-1}\right)x^\gamma\,dx \tag{5.4}$$

*where* $-1<\gamma<0$*, B is a constant depending on b, and A is a constant much greater than* 1.

*Then*

$$I=-\frac{B}{b^{\gamma+1}}c^{(Ab)}_{b\mu,\gamma}+\frac{Bk_0^{-\gamma}}{b^{\gamma+1}}c^{(k_0Ab)}_{k_0b\mu,\gamma}, \tag{5.5}$$

*where*

$$c^{(A)}_{\mu,\gamma}=\int_0^A\biggl(\frac{1}{\xi-\mu}-\frac{1}{e^{\xi-\mu}-1}\biggr)\xi^\gamma\,d\xi. \tag{5.6}$$

**Proof.** Note that the value of $B$ does not affect calculations in any way (it appears as a multiplier both in the original and final relations); therefore, in the proof, we set $B=1$.

Let us make the change of variable $\xi=bx$ in the integral (5.4). Then

$$I=\frac{1}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{1}{e^{\xi-b\mu}-1}-\frac{k_0}{e^{k_0(\xi-b\mu)}-1}\biggr)\xi^\gamma\,d\xi.$$

Adding and subtracting the summands, we express the integral as the sum of the following three integrals:

$$\begin{aligned}&\frac{1}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{1}{e^{\xi-b\mu}-1}-\frac{1}{\xi-b\mu}\biggr)\xi^\gamma\,d\xi+\frac{1}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{1}{\xi-b\mu}-\frac{1}{(\xi-b\mu)(1+\frac{k_0}{2}(\xi-b\mu))}\biggr)\xi^\gamma\,d\xi\\ &\qquad-\frac{k_0^{-\gamma}}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{k_0^{\gamma+1}}{e^{k_0(\xi-b\mu)}-1}-\frac{k_0^{\gamma+1}}{k_0(\xi-b\mu)(1+\frac{k_0}{2}(\xi-b\mu))}\biggr)\xi^\gamma\,d\xi.\end{aligned} \tag{5.7}$$

Introducing the notation (5.6), we find that the first integral from (5.7) equals $-c^{(Ab)}_{b\mu,\gamma}/b^{\gamma+1}$.

In the second integral (5.7), we subtract the fractions and then make the change $\eta=k_0\xi$, obtaining

$$\begin{aligned}\frac{1}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{1}{\xi-b\mu}-\frac{1}{(\xi-b\mu)(1+\frac{k_0}{2}(\xi-b\mu))}\biggr)\xi^\gamma\,d\xi&=\frac{1}{b^{\gamma+1}}\int_0^{Ab}\frac{1+\frac{k_0}{2}(\xi-b\mu)-1}{(\xi-b\mu)(1+\frac{k_0}{2}(\xi-b\mu))}\xi^\gamma\,d\xi\\ =\frac{1}{b^{\gamma+1}}\frac{k_0}{2}\int_0^{Ab}\frac{1}{(1+\frac{k_0}{2}(\xi-b\mu))}\xi^\gamma\,d\xi&=\frac{1}{b^{\gamma+1}}\frac{(k_0)^{-\gamma}}{2}\int_0^{Abk_0}\frac{1}{1+\frac{\eta-k_0b\mu}{2}}\eta^\gamma\,d\eta=\frac{k_0^{-\gamma}c_1}{b^{\gamma+1}},\end{aligned}$$

where we have introduced the notation

$$c_1=\int_0^\infty\frac{1}{2(1+\frac{\eta-k_0b\mu}{2})}\eta^\gamma d\eta.$$

In the third integral from (5.7), we make the same change $\eta=k_0\xi$, obtaining

$$\begin{aligned}&\frac{k_0^{-\gamma}}{b^{\gamma+1}}\int_0^{Ab}\biggl(\frac{k_0^{\gamma+1}}{e^{k_0(\xi-b\mu)}-1}-\frac{k_0^{\gamma+1}}{k_0(\xi-b\mu)(1+\frac{k_0}{2}(\xi-b\mu))}\biggr)\xi^\gamma\,d\xi\\ &\quad=\frac{k_0^{-\gamma}}{b^{\gamma+1}}\int_0^{Abk_0}\biggl(\frac{1}{e^{\eta-k_0b\mu}-1}-\frac{1}{(\eta-k_0b\mu)(1+\frac{\eta-k_0b\mu}{2})}\biggr)\eta^\gamma\,d\eta\\ &\quad=\frac{k_0^{-\gamma}}{b^{\gamma+1}}\int_0^{Abk_0}\biggl(\frac{1}{e^{\eta-k_0b\mu}-1}-\frac{1}{\eta-k_0b\mu}+\frac{1}{2(1+\frac{\eta-k_0b\mu}{2})}\biggr)\eta^\gamma\,d\eta\\ &\quad=\frac{k_0^{-\gamma}}{b^{\gamma+1}}\biggl(\int_0^{Abk_0}\biggl(\frac{1}{e^{\eta-k_0b\mu}-1}-\frac{1}{\eta-k_0b\mu}\biggr)\eta^\gamma\,d\eta+\int_0^{Abk_0}\frac{1}{2(1+\frac{\eta-k_0b\mu}{2})}\eta^\gamma\,d\eta\biggr)\\ &\quad=\frac{-k_0^{-\gamma}c^{(Abk_0)}_{k_0b\mu,\gamma}}{b^{\gamma+1}}+\frac{k_0^{-\gamma}c_1}{b^{\gamma+1}}.\end{aligned}$$

Collecting all the expressions together, we find that the sum of the three integrals (5.7) is

$$-\frac{1}{b^{\gamma+1}}c^{(Ab)}_{b\mu,\gamma}+\frac{k_0^{-\gamma}}{b^{\gamma+1}}c_1+\frac{k_0^{-\gamma}}{b^{\gamma+1}}c^{(Abk_0)}_{k_0b\mu,\gamma}-\frac{k_0^{-\gamma}}{b^{\gamma+1}}c_1=-\frac{1}{b^{\gamma+1}}c^{(Ab)}_{b\mu,\gamma}+\frac{k_0^{-\gamma}}{b^{\gamma+1}}c^{(Abk_0)}_{k_0b\mu,\gamma},$$

i.e., as a result, the integral $c_1$ cancels out. ☐

We now apply the Euler-Maclaurin formula, take the parameter $\gamma$ into account, and obtain

$$\sum_{j=1}^{n}\left(\frac{j^\gamma}{e^{bj+\varkappa}-1}-\frac{kj^\gamma}{e^{bkj+\varkappa}}\right)=\frac{1}{\alpha}\int_0^\infty\left(\frac{1}{e^{bx+\varkappa}-1}-\frac{k}{e^{bkx+\varkappa}-1}\right)dx^\alpha+R,$$

where $\alpha=\gamma+1$, $k=N$ and $b=1/T$. The remainder $R$ here satisfies the estimate

$$|R|\le\frac{1}{\alpha}\int_0^\infty|f'(x)|\,dx^\alpha,\qquad\text{where}\quad f(x)=\frac{1}{e^{bx+\varkappa}-1}-\frac{k}{e^{k(bx+\varkappa)}-1}.$$

We calculate the derivative and obtain

$$f'(x)=\frac{bk^2e^{k(bx+\varkappa)}}{(e^{k(bx+\varkappa)}-1)^2}-\frac{be^{bx+\varkappa}}{(e^{bx+\varkappa}-1)^2},\quad |R|\le\frac{b^{-\alpha}}{\alpha}\int_0^\infty\left|\frac{k^2e^{k(y+\varkappa)}}{(e^{k(y+\varkappa)}-1)^2}-\frac{e^{y+\varkappa}}{(e^{y+\varkappa}-1)^2}\right|dy^\alpha. \tag{5.8}$$

We also have

$$\frac{e^y}{(e^y-1)^2}=\frac{1}{y^2}+\psi(y),$$

where $\psi(y)$ is smooth and $|\psi(y)|\le C(1+|y|)^{-2}$. We substitute this formula in (5.8), obtaining

$$|R|\le\frac{b^{-\alpha}}{\alpha}\int_0^\infty\bigl|k^2\psi\bigl(k(y+\varkappa)\bigr)-\psi(y+\varkappa)\bigr|\,dy^\alpha\le k^{-\alpha}b^{-\alpha}\int_{k\varkappa}^\infty|\psi(y)|\,dy^\alpha+\int_\varkappa^\infty|\psi(y)|\,dy\le Cb^{-\alpha}$$

where $C$ is a constant. For example, if we have $\varkappa\sim(\ln k)^{-1/4}$, then $|R|$ satisfies the estimate $|R|\sim O((\ln k)^{(2-\alpha)/4})$. We therefore have $k=N_c$ and $T=T_c$, and obtain the formula for the integral at $\mu=0$:

$$M=\frac{\Lambda^{\gamma-\gamma_c}}{\alpha\Gamma(\gamma+2)}\int\frac{\xi\,d\xi^\alpha}{e^{b\xi}-1}=\frac{\Lambda^{\gamma-\gamma_c}}{b^{1+\alpha}}\int_0^\infty\frac{\eta d\eta^\alpha}{e^\eta-1},$$

where $\alpha=\gamma+1$. Consequently,

$$b=\frac{1}{M^{1/(1+\alpha)}}\left(\frac{\Lambda^{\gamma-\gamma_c}}{\alpha\Gamma(\gamma+2)}\int_0^\infty\frac{\xi\,d\xi^\alpha}{e^\xi-1}\right)^{1/(1+\alpha)}.$$

We obtain (see[22])

$$\begin{aligned}N\Lambda^{\gamma-\gamma_c}=&\int_0^\infty\left\{\frac{1}{e^{b\xi}-1}-\frac{k}{e^{kb\xi}-1}\right\}d\xi^\alpha=\frac{1}{b^\alpha}\int_0^\infty\left(\frac{1}{e^\xi-1}-\frac{1}{\xi}\right)d\xi^\alpha+O(b^{-\alpha})\\&+\frac{1}{b^\alpha}\int_0^\infty\left(\frac{1}{\xi}-\frac{1}{\xi(1+(k/2)\xi)}\right)d\xi^\alpha-\frac{k^{1-\alpha}}{b^\alpha}\int_0^\infty\left\{\frac{k^\alpha}{e^{k\xi}-1}-\frac{k^\alpha}{k\xi(1+(k/2)\xi)}\right\}d\xi^\alpha\\&+O(b^{-\alpha})\\=&\frac{c(\gamma)}{b^\alpha}(k^{1-\alpha}-1)+O(b^{-\alpha}).\end{aligned}$$

Setting $k=N|_{\tilde\mu/T_r=o(1)}$, we have

$$N|_{\tilde\mu/T_r=o(1)}=(\Lambda^{\gamma-\gamma_c}c(\gamma))^{1/(1+\gamma)}T_r,\quad\text{where}\quad c(\gamma)=\int_0^\infty\left(\frac{1}{\xi}-\frac{1}{e^\xi-1}\right)\xi^\gamma\,d\xi. \tag{5.9}$$

For $P<0$, the Bachinskii relation for $\rho_c$ is consistent with $A(\gamma)T_r$, where

$$A(\gamma)=(\Lambda^{\gamma-\gamma_c}c(\gamma))^{1/(1+\gamma)}.$$

We can normalize the activity $a$ at the point $T_c$ and then find $a_0$ by matching the liquid and gas branches at $T_c$ for the pressure (i.e., for the quantity $M$) such that there is no phase transition on the critical isotherm at $T_r = 1$.

We further normalize the activities at $T_r < 1$ by the value $a_0$ calculated below. Then the chemical potentials (the thermodynamic Gibbs potentials for the liquid and gas branches in thermodynamics) coincide, and there is therefore no “gas-liquid” phase transition at $T_r = 1$.

For $\gamma < 0$, the density satisfies the formula

$$\rho = \Lambda^{\gamma-\gamma_c}\int_0^\infty \left(\frac{1}{e^{b(x-\mu)}-1} - \frac{N}{e^{Nb(x-\mu)}-1}\right) x^\gamma\,dx. \tag{5.10}$$

Hence, by Lemma 5, we have

$$\rho = T(\Lambda^{\gamma-\gamma_c}C(\gamma))^{1/(1+\gamma)}; \tag{5.11}$$

see Fig. 18.

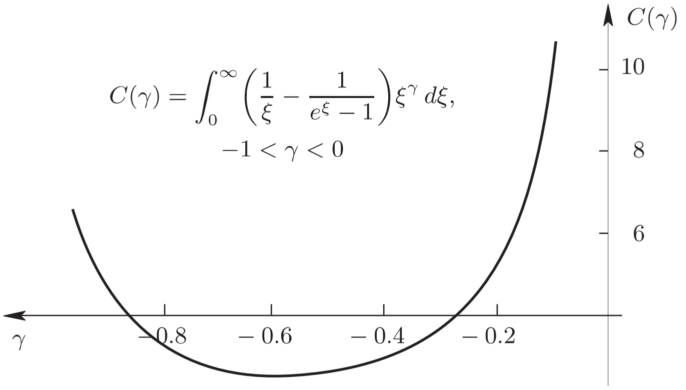

**Fig. 18.** The value of $C(\gamma)$.

The liquid is assumed incompressible (i.e., $\rho$ does not depend on the pressure $P$, but depends only on the temperature $T$). Thus, we assume that all the isochores pass through the origin $(P, Z) = (0, 0)$.

Let us express the density from Bachinskii’s condition (for $Z = 1$):

$$\frac{\rho}{\rho_B} + \frac{T}{T_B} = 1; \tag{5.12}$$

hence we obtain

$$\left(1 - \frac{T}{T_B}\right)\rho_B = T(\Lambda^{\gamma-\gamma_c}C(\gamma))^{1/(1+\gamma)}. \tag{5.13}$$

Therefore, we can express $T$ as a function of $\gamma$:

$$T = \frac{\rho_B}{\rho_B/T_B + (\Lambda^{\gamma-\gamma_c}C(\gamma))^{1/(1+\gamma)}}. \tag{5.14}$$

To a temperature $T < 1$ corresponds two values of $\gamma$. Further, we use the least of them (the largest in absolute value), and denote it by $\tilde\gamma(T)$.

On the spinodal, in the negative region, we have

$$P = \Lambda^{\tilde\gamma(T)-\gamma_c}T^{2+\tilde\gamma(T)}\frac{\zeta(2+\tilde\gamma(T))}{\zeta(2+\gamma_c)}, \qquad \tilde\gamma(T) < 0. \tag{5.15}$$

Let us construct the gas isotherms $T < 1$ by the preference principle. Since there is no phase transition on the critical isotherm, we equate the activities $a$ and $\tilde a$ on it.

Choose $\Lambda = 2.4$. Then $\tilde{\gamma}(1) = 0.742$. Comparing the activities on the gas and liquid isotherms, we obtain

$$\tilde{a} = \frac{1}{1.00271}, \qquad \tilde{\mu} = \mu - 0.0027. \tag{5.16}$$

The phase transitions on the other isotherms is determined by the difference of $a$ and $\tilde{a}$.

For the chosen value of $\Lambda$, we obtain a good agreement at the triple point. Namely, for the isotherm $T = 0.501$ corresponding to the triple point for nitrogen, we have $\tilde{\gamma}(0.50) = 0.742$. The pressure on the gas isotherm at the triple point is 0.0037. The pressure on the liquid isotherm at the triple point is 0.00376.

If, for each of the temperatures $T$ the least (in absolute value) $\gamma$ (and, accordingly, the least (in absolute value) pressure on the negative spinodal) is taken, then, comparing the activities on the critical isotherm, we obtain $\tilde{a} > a$. The pressure $P$ on the gas isotherm for $a = 1$ is the critical pressure $P = 1$, while, on the liquid isotherm for $\tilde{a} = 1$, we obtain $P = 0.90 < 1$.

Therefore, we use only the lower branch in Fig. 19 shown by the solid line.

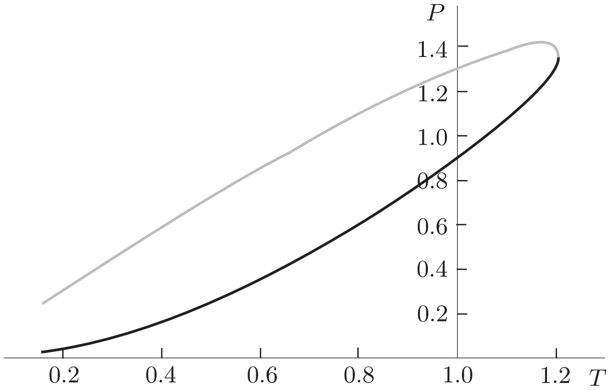

**Fig. 19.** Spinodal on the $(P, T)$ diagram in the negative quadrant. On the vertical axis, the absolute value of pressure is plotted.

Taking into account the correction (5.16) and the fact that $\mu$ and $\tilde{\mu}$ are identical, we obtain the binodal point shown in Fig. 20 and, carrying out the same procedure for the other isotherms, we draw the binodal, shown in Fig. 21, through these points.

**Remark 3.** The critical values $T_c$, $P_c$, and $\rho_c$ were obtained experimentally for almost all gases in thermodynamics, and the critical number of degrees of freedom can, therefore, be given beforehand. The parameter $\Lambda$ $(1.6 < \Lambda < 3,\ T_r > 1/3)$ is determined by the condition that the binodal coincides at the triple point obtained experimentally.

Now we consider the problem of holes in the Bose condensate.

Molecules of an ideal gas can be regarded as tiny balls. We regard holes or excitons in glass as spherical but empty (without any matter) molecules. Obviously, if such balls are chaotically mixed in glass, then chaos also increases in the glass, and this means that the entropy also increases in the presence of holes. Therefore, to attain maximum entropy, more holes must be "mixed" into the glass. In our approach, holes appear for $\gamma < 0$.

We ignore attraction in the model of an ideal liquid, and this means that a liquid can be "expanded" without any resistance to produce holes (just as sand is incompressible when pressure is applied, but can still be easily "expanded"; cf. the appendix in [41]).

If there is no attraction, then there is also no negative pressure under "expansion," and hence there are no holes. If $\gamma < 0$, then the plane $(Z, P)$ is again positive and is therefore covered by another sheet. It is easy to see that the straight lines arriving at the point $Z = 0, P = 0$ (i.e., at the point $\gamma = 0$) are reflected from this second sheet in the reverse direction along the same line. This means that it is geometrically

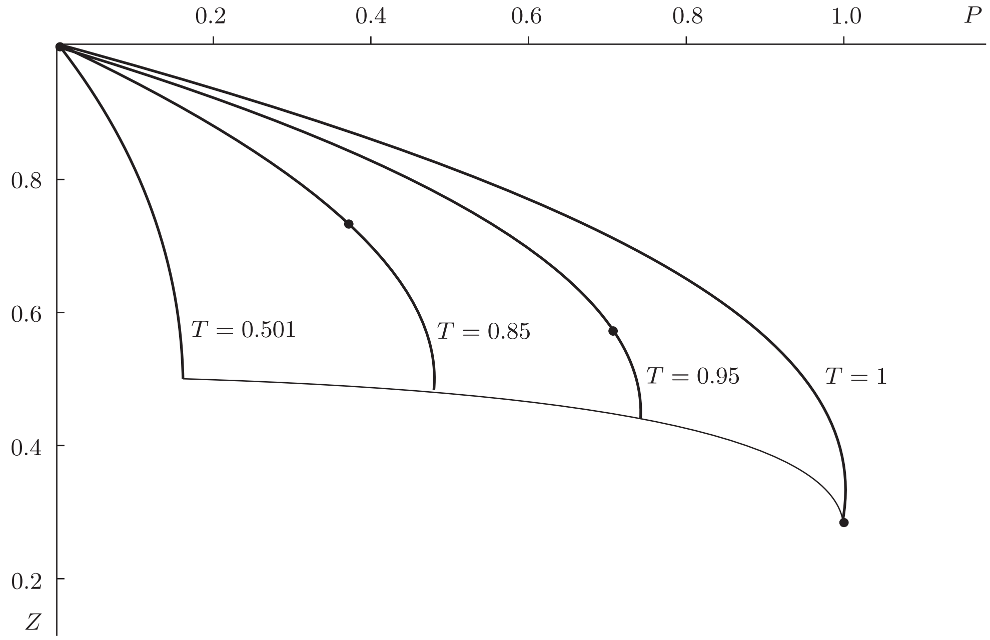

**Fig. 20.** Theoretical points on the binodal for $T = 0.85$ and $T = 0.95$.

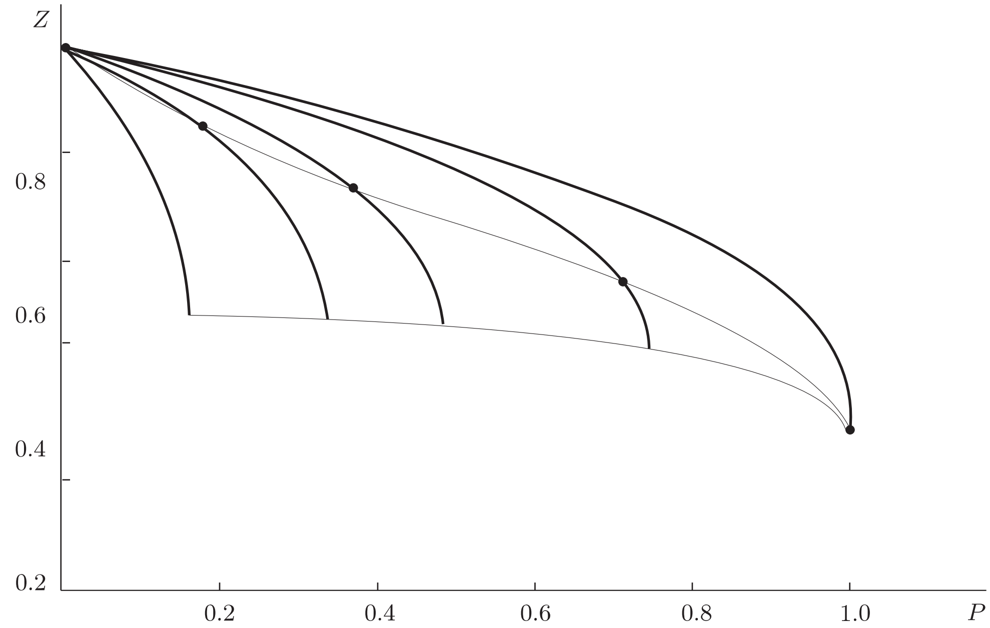

**Fig. 21.** The theoretical binodal is shown by the thin line passing through four points.

convenient to arrange reflection of the vectors from the second sheet by using the matrix $-I$, where $I$ is a two-dimensional unit matrix, i.e., to mirror the sheet $\gamma < 0$ into the negative quadrant.

We note that this procedure corresponds to the concept of Dirac holes, but, conversely, the holes are associated with negative pressure, i.e., with negative energy. The straight lines now pass through the origin into the negative quadrant, although the pressure does not actually change sign. This is only a convenient geometric “uniformization.”

We also note that an ideal liquid is completely plastic because there is no attraction, namely, the liquid does not attempt to return to the original (pre-expansion) state. In this sense, the Bose condensate for $\gamma < 0$, which leads to this “form” of liquid, can also be visualized as glass or as an amorphous substance.

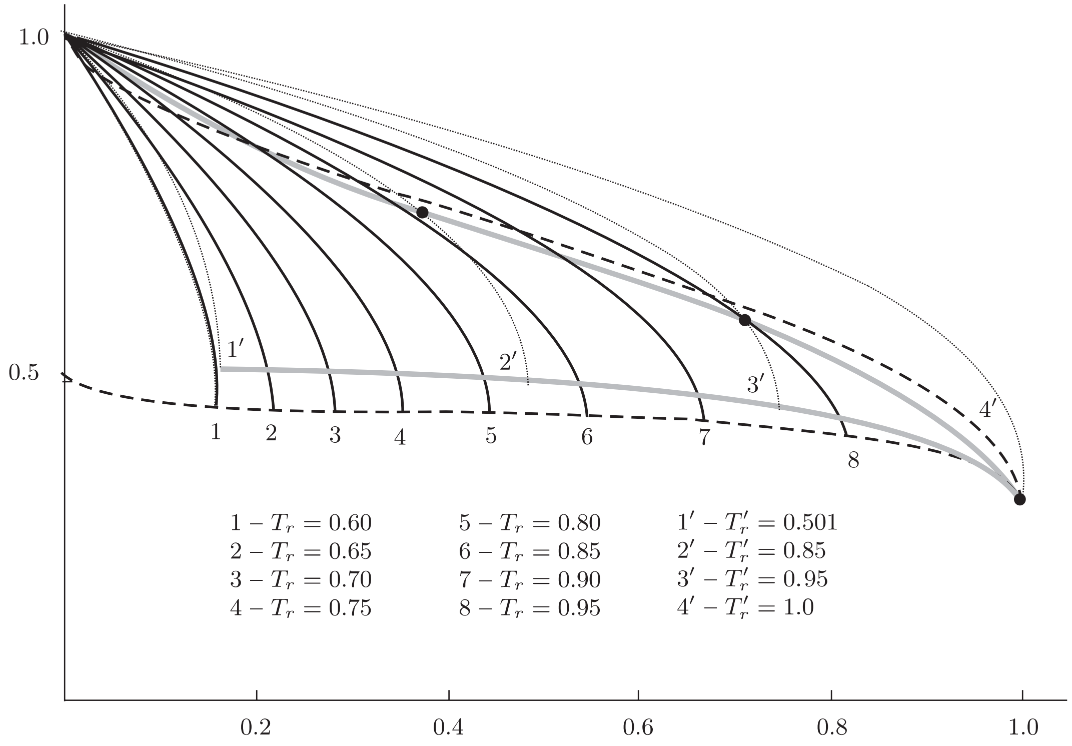

**Fig. 22.** The dotted line shows the binodal given in [40]. The grey solid line corresponds to the theoretical binodal. Thin lines depict the theoretical isotherms, coinciding with the experimental isotherms for nitrogen. They correspond to lines 1', 2', 3' and 4' and, accordingly, to the temperatures denoted by $T_r'$.

**Remark 4.** As mentioned above, we started to revise thermodynamics when studying the process in economics in which money bills can be treated as particles according to the correspondence principle derived by Irving Fisher. Fisher himself did not call his observation the correspondence principle, but he was a disciple of Gibbs, and his relation of the main law in economics

$$PQ = Mv, \tag{5.17}$$

where $Q$ is the volume of goods, $M$ is the amount of money, $v$ is the turnover rate, and $P$ is the price of goods, is therefore obviously related to the correspondence between the economic and thermodynamic quantities: the volume of goods $Q$ corresponds to the volume $V$, the amount of money $M$ corresponds to the number of particles $N$, and the turnover rate $v$ corresponds to the temperature $T$. The price of goods $P$ is not so closely related to pressure, but is still denoted by the same letter.

This correspondence principle naturally associates holes with debts, and the repayment of debts with annihilation.

As previously noted, the geometric locus of points where the chemical potential is zero is the set of points of maximal entropy, which we call the “new spinodal.” In economics, this new spinodal means a certain debt limit.

The above relations can thus be used to obtain a two-sheeted covering of the plane $\{Z, P\}$ for $\gamma \geq 0$ and $-1 \leq \gamma < 0$. The meaning of the second sheet is that the chaotic state of the liquid (as a phenomenon related to the Bose condensate) is enhanced with an increasing number of Frenkel exciton-type holes for $-1 \leq \gamma < 0$, and the holes are located in the liquid, which is fluctuationally concentrated on a rather slowly evolving domain (from the standpoint of the instrument discussed above), where chaotic nanoholes appear, i.e., the structure of the liquid is chaotically expanded [42].

These hole-excitons, just as particles, cannot be labeled by our instrument, and we can only speak about the density of holes. It is more convenient to place this second sheet in the quadrant $[-Z, -P]$, continuing straight lines (5.1) beyond the singular point $Z = 0, P = 0$ into the negative quadrant. In other words, reflection can be performed using the matrix $-I$, where $I$ is the unit matrix.

It thus turns out that it is convenient to introduce “negative pressure,” although the attraction between the particles is neglected, and hence no negative pressure can be there. The pressure, just as the temperature, is usually assumed to be positive. When we expand a liquid, it plastically solidifies in its expanded state and does not try to compress back to its previous state.

It is therefore clearer and geometrically more convenient to speak about “negative pressure,” although this pressure is in fact related to the presence of holes in the Bose condensate. The new spinodal treated as the set of points where the entropy of the liquid attains its maximum at a given temperature can be obtained experimentally by equilibrium expansion of the liquid. The expansion of a plastic structure is not usually called negative pressure, but it is convenient in our case, as already noted, just as in Dirac’s hole theory, to consider negative energy (i.e., pressure) instead of holes, although, as is seen, in contrast to the Dirac equation, the “negative energy” in any substance does not approach $-\infty$, but is bounded in absolute value. In this sense, we can move the energy origin as is usually done in the theory of semidefinite operators. But if the potential at infinity is equal to a constant, then this constant is usually taken as the energy origin (e.g., for the Lennard-Jones potential). In the van der Waals theory, negative pressure can approach infinity, but this is not admissible from the physical standpoint.

Here, neglecting the attraction is just as “legitimate” as in the theory of the “vapor-liquid” equilibrium, where the condition that the pressures are equal is possible only if the surface tension is neglected.

This also explains the smooth transition (without a first-order phase discontinuity) of this structure into ice, which means that the solidified glass is being crystallized.

## 6. CRITICAL INDICES

In passing from quantum mechanics to classical theory, we have seen that quantum mechanics, which is apparently unrelated to classical gas theory, unexpectedly furnishes the missing important term corresponding to superfluidity in classical physics. As to the passage with viscosity tending to zero, we have already used this asymptotics in Sec. 3. It would be strange if the consideration of viscosity and the subsequent passage to the limit with viscosity tending to zero as the number of particles $N \to \infty$ were of no importance at the critical point. As pointed out above, at this point, the main feature of thermodynamics, the relative smallness of fluctuations, is violated.

Before passing to a new conjecture, the scaling conjecture, we need to use “semiclassical” asymptotics and its “catastrophes” (the term used by Arnold). From the mathematical point of view, the consideration of viscosity and the passage to viscosity tending to zero is completely similar to the passage from quantum mechanics to classical mechanics in the regions of shadow, of the tunnel effect, and in Euclidean field theory, where instantons, etc., are used.

The pure imaginary Planck parameter $\hbar$ is considered and, in this case, quantization using the Feynman path integral is carried out in the famous paper [43] of Feynman and Hibbs. In mathematics, this was carried out much earlier by Wiener. Therefore, we shall call this quantization *Wiener quantization*, and the pure imaginary quantity $\hbar$ will be called *viscosity*.

The critical point $T_c$ is such that, for $T < T_c$, the jump from the gas region to the liquid region occurs. Usually, it is assumed to be a jump of $\theta$-function type. However, from Fig. 1, it is seen that the experimental curve of such a transition is an inclined line. In addition, this curve can be determined from Maxwell’s “area rule.” This resembles the Riemann wave and Riemann’a area rule for shock waves.

So long as we do not pass to isotherms of Bose distribution type, the value of the chemical potential is determines up to an additive constant as in the old thermodynamics. But now the point $\mu = 0$ is the point on the spinodal at which the point $P = P_c$ on the critical isotherm is critical.

Here it is useful to apply the notion called “Lagrangian manifold” introduced by the author [44]. In our simplest case, this manifold is a Lagrangian surface in 4-dimensional phase space $T, S, \mu, N$, where the temperature $T$ and the chemical potential $\mu$ play the role of coordinates and entropy $S$, while the number of particles $N$ plays the role of momenta. The spinodal (an ellipse) plays the role of turning points and its extreme point (focus) is that of stronger degeneracy.

**Remark 5.** The passage to the preference principle makes each point of the spinodal a focus of second order of degeneracy. Thus, on each isotherm, we obtain an analog of the transition to a “shock wave,” but, with the exception of the critical point, the other points of the spinodal cannot be reached so far.

The quantization by the tunnel canonical operator is performed in the same way as that of the canonical operator in quantum mechanics. This method yields uniform asymptotics near the focal points.

Quantizing a Lagrangian manifold geometrically (see Sec. 11.4 in [45]) is usually related to introducing the constant $\hbar$. We called the case where the number $\hbar$ is pure imaginary "Wiener" (or "tunnel") "quantization" in [46].

We apply Wiener quantization to thermodynamics. The thermodynamic potential $G = \mu N$ is the action $\mathbb{S} = \int p\,dq$ on a two-dimensional Lagrangian manifold $\Lambda^2$ in the four-dimensional phase space $q_1, q_2, p_1, p_2$, where $q_1$ and $q_2$ are the pressure $P$ and temperature $T$ and $p_1$ and $p_2$ are the volume $V$ and the entropy $S$ with the opposite sign. All the other potentials (the inner energy $E$, free energy $F$,and enthalpy $W$ are the results of projecting the Lagrangian manifold onto the coordinate planes $p_1, p_2$:

$$E = -\int \vec{q}\,d\vec{p}, \quad \vec{q} = \{q_1, q_2\}, \vec{p} = \{p_1, p_2\},$$

$$W = -\int (q_2\,dp_2 + q_1\,dp_1), \qquad F = \int (q_1\,dp_1 - q_2\,dp_2). \tag{6.1}$$

For the Wiener quantization, we have

$$N = \varepsilon \frac{\partial}{\partial \mu}, \qquad V = \varepsilon \frac{\partial}{\partial p}, \qquad S = -\varepsilon \frac{\partial}{\partial T}.$$

The role of the time $t$ in the quantization is therefore played by $\ln(-\mu/T)$:

$$G = \mu N \sim \varepsilon \frac{\mu}{T} \frac{\partial}{\partial(\mu/T)} = \varepsilon \frac{\partial}{\partial \ln(\mu/T)}.$$

We note that tunnel quantization of the van der Waals equation implies the Maxwell rule as $\varepsilon \to 0$ (see below).

As is seen below, the critical point and the spinodal points are focal points, and so they do not become "classics," i.e., the van der Waals model, as $\varepsilon \to 0$. Before the application of the preference principle, the spinodal points, just as the turning points in quantum mechanics, are approximated by the Airy function, and the critical point, i.e., the point where two turning points (two Airy functions) arise, is approximated by the Weber function (see [47]). It is precisely the Weber function in terms of which the point of the shock wave origination as $\varepsilon \to 0$ is expressed in the Burgers equation. If the transition as $\varepsilon \to 0$ occurs outside these points, then the van der Waals–Maxwell model is obtained. But there is no passage to the limit at these points, and the so-called Landau "classical" critical exponents [16] strongly differ from their experimental values. The Weber function implies singularities of the form $\varepsilon^{-1/4}$, and the Airy function implies singularities of the form $\varepsilon^{-1/6}$. After the application of the preference principle, the singularity is of the form $\varepsilon^{-1/4}$.

We now consider the Burgers equation in more detail. We consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\varepsilon}{2} \frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{6.2}$$

where $\varepsilon > 0$ is a small parameter.

It is well known that all linear combinations

$$u = \lambda_1 u_1 + \lambda_2 u_2 \tag{6.3}$$

of the solutions $u_1$ and $u_2$ of Eq. (6.2) are solutions of this equation. We replace $u = \exp(-w(x,t)/\varepsilon)$ and obtain the nonlinear equation

$$\frac{\partial w}{\partial t} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{\varepsilon}{2} \frac{\partial^2 w}{\partial x^2} = 0, \tag{6.4}$$

which is called the integrated Burgers equation[9]. It is obvious that any solution $u_i$ of Eq. (6.2) is associated with the solution $w_i = -\varepsilon \ln u_i$, $i = 1, 2$ of Eq. (6.4). Solution (6.3) of Eq. (6.2) corresponds to the solution (6.4)

$$w = -\varepsilon \ln\biggl(\exp\Bigl(-\frac{w_1+\mu_1}{\varepsilon}\Bigr) + \exp\Bigl(-\frac{w_2+\mu_2}{\varepsilon}\Bigr)\biggr),$$

where $\mu_i = -\varepsilon \ln \lambda_i$, $(i = 1.2)$. Because

$$\lim_{\varepsilon\to 0} w = \min(w_1, w_2),$$

we obtain the $(min, +)$ algebra of tropical mathematics [30].

To determine the solutions for $t > t_c$, Hopf proposed considering the Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \frac{\varepsilon}{2}\frac{\partial^2 v}{\partial x^2} = 0, \qquad v|_{t=0} = p_0(x), \tag{6.5}$$

and calling the function $p_{\text{gen}} = \lim_{\varepsilon\to 0} v$ (the Riemann waves) the (generalized) solution of the equation

$$\frac{\partial p}{\partial t} + p\frac{\partial p}{\partial x} = 0, \qquad p|_{t=0} = p_0(x). \tag{6.6}$$

The solution $v$ of the Burgers equation is expressed via the logarithmic derivative

$$v = -\varepsilon\frac{\partial}{\partial x}\ln u \tag{6.7}$$

of the solution $u$ of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\varepsilon}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u|_{t=0} = \exp\biggl\{-\frac{1}{\varepsilon}\int_{-\infty}^{x} p_0(x)\,dx\biggr\}. \tag{6.8}$$

The original problem thus reduces to studying the logarithmic limit of the solution of the heat equation. It is known that the solution of problem (6.8) has the form

$$u = (2\pi\varepsilon t)^{-1/2}\int_{-\infty}^{\infty}\exp\biggl\{-\biggl((x-\xi)^2 + 2t\int_{-\infty}^{\xi} p_0(\xi)\,d\xi\biggr)\Big/2th\biggr\}\,d\xi. \tag{6.9}$$

The asymptotic behavior of integral (6.9) can be calculated by the Laplace method. We obtain

$$u = \bigl(|J|^{-1/2}(\xi(x,t),t) + O(\varepsilon)\bigr)\exp\biggl\{-\frac{1}{\varepsilon}S(x,t)\biggr\} \tag{6.10}$$

for $t < t_c$.

Here

$$S(x,t) = \int_{-\infty}^{r(t)} p\,dx,$$

and the integral can be calculated along the Lagrangian curve $\Lambda^t$, and $r(x)$ is a point on $\Lambda^t$. For $t > t_{cr}$, there are three points $r_1(x)$, $r_2(x)$, and $r_3(x)$ on $\Lambda^t$, whose projections on the axis $x$ coincide or, in other words, the equation $Q(t,\xi) = x$ has three solutions $\xi_1(x,t)$, $\xi_2(x,t)$, $\xi_3(x,t)$ for $x \in (x_1, x_2)$. We let

$$S(x,t) = \int_{-\infty}^{r(x)} p\,dx \qquad \text{for} \quad x < x_1, \quad x > x_2$$

and

$$S(x,t) = \min(S_1, S_2, S_3), \quad S_j = \int_{-\infty}^{r_j(x)} p\,dx, \qquad \text{where} \quad j = 1,2,3 \quad \text{for } x \in [x_1, x_2].$$

---

[9] The usual Burgers equation is derived from (6.4) by differentiating with respect to $x$ and replacing $v = \partial w/\partial x$.

These arguments allow obtaining a generalized discontinuous solution of problem (6.6) for times $t > t_c$. It is determined by the function $p = p(x,t)$ defining the essential domains of the curve $\Lambda^t$ [46]. We note that this in particular implies the equal area rule often used in hydrodynamics for determining the shock wave front whose evolution is described by Eq. (6.6). We also note that this exactly corresponds to the Maxwell rule for the van der Waals equation.

The solution $v = v(x,\varepsilon)$ of the Burgers equation at the critical point $x = p^3$ is calculated by the formula

$$v(x,\varepsilon) = \varepsilon\frac{\partial \ln u(x)}{\partial x} = \frac{\int_0^\infty \exp\{\frac{-x\xi-\xi^4/4}{\varepsilon}\}\xi\, d\xi}{\int_0^\infty \exp\{\frac{-x\xi-\xi^4/4}{\varepsilon}\}d\xi}. \tag{6.11}$$

As $x \to 0$, we replace $\xi/\sqrt[4]{\varepsilon} = \eta$ and obtain

$$v(\varepsilon,x) \to_{x\to 0} \sqrt[4]{\varepsilon}\cdot \mathrm{const}. \tag{6.12}$$

What does this mean from the classical and classical measurement standpoints when the condition called the “semiclassical condition” in [48] is satisfied (and this means that we are outside the focal point)? For the Laplace transformation, this means that we are in a domain where the Laplace asymptotic method is applicable, i.e., where

$$u(x) = \frac{1}{\sqrt{\varepsilon}}\int_0^\infty e^{-\frac{px-\widetilde{S}(p)}{\varepsilon}}\,dp. \tag{6.13}$$

If the solution of the relation

$$x = \frac{\partial \widetilde{S}}{\partial p} \tag{6.14}$$

is nondegenerate, i.e.,

$$\frac{\partial^2\widetilde{S}}{\partial p^2} \neq 0 \qquad \text{at the point} \quad \frac{\partial \widetilde{S}}{\partial p} = x,$$

then the integral (6.13) is bounded as $\varepsilon \to 0$ in this case. For this integral to have a zero of the order of $\varepsilon^{1/4}$, we must integrate it over $x$ using the fractional derivative $D^{-1/4}$. The value of $D^{-1/4}$ applied to 1 gives approximately $x^{1/4}$.

It follows form the uncertainty principle [52] that the correspondence between the differential operator and a small parameter of the form $D \to 1/\varepsilon$ is preserved for the ratio $-\varepsilon\frac{\partial u/\partial x}{u}$, although the leading term of the asymptotic expansion in the difference $\frac{\partial^2 u/\partial x^2}{u}$ between $\frac{(\partial u/\partial x)^2}{u^2}$ cancels.

As we approach the critical point, the multiplicity of scattering of photons at fluctuations increases. Optical instruments do not possess the necessary resolving power and, therefore, it is impossible to study spectra by using them. However, the difference between the frequencies of the incident and scattered waves is contained the range of radio frequencies. The Fourier transform of the scattered spectrum (beats) is taken (see [53, 126]). In all other experiments, the Fourier transformation is also applied. Therefore, the asymptotics with respect to the small viscosity in the vicinity of the critical point becomes, as the result of the Fourier transforms, the shift in the critical index.

Such a spread in a small parameter mathematically blurs the asymptotic pattern and, despite the fact that, mathematically, this coincides exactly with the occurrence of a shock wave, such a spread, to my knowledge, has never been used. Maybe, in times of Riemann, when the Hopf relationship between the heat equation and the Burgers equation was not known, an attempt to develop a theoretic approach along these lines could have been made. One of the physicists had a strong objection to the use of viscosity in equilibrium thermodynamics. But, as was already stated, the critical point is incompatible with the notion of equilibrium thermodynamics, at least, because of huge fluctuations occurring in it. This does not mean at all that we must exclude its neighborhood from consideration.

In view of the preference principle, a function similar to the Weber function (which was discussed above) appears on the whole spinodal and is the source generating the two-phase state “saturated vapor—liquid.” However, experiments do not reach the spinodal, except for the critical point and, therefore, we actually observe this pattern of passage from the Weber function to the step $\theta$-function only on the critical isotherm. The author considers it more expedient to consider not the jumps of the critical indices, but the exact asymptotics of tunnel canonical operator type depending on small viscosity, in particular, on the Weber function. This is a natural approach, because, using the Wiener quantization of equilibrium thermodynamics, we can obtain important corrections to it, at least, in the direction of increasing viscosity and the passage to nonequilibrium thermodynamics.

**Remark 6.** We can pass to the following correction terms with a view to decrease the number of particles (mesoscopic physics) only for dimensions greater than 4. For example, for photons, we can obtain the following correction to the Stefan–Boltzmann formula (see [6], [7]):

$$F = -\frac{4\delta}{3c}T^4V - \frac{12\hbar\delta}{k}\cdot\frac{\zeta(3)}{\zeta(4)}T^3V^{2/3}, \tag{6.15}$$

where $F$ is the free energy,

$$\delta = \frac{\pi^2k^4}{60\hbar^3c^2}$$

is the Stefan–Boltzmann constant, $V$ is the volume, $c$ is the speed of light, $\hbar$ is the Planck constant, $k$ is the Boltzmann constant, and $T$ is the temperature.

## 7. TRANSITION TO A POSITIVE CHEMICAL POTENTIAL IN THE SUPERCRITICAL REGION

As we have seen, the value of the dimension $D$ at a critical point largely determines the distribution that the corresponding relaxation process obeys.

The paradox of our approach is that it is exactly at the critical point that our main axiom (concerning fluctuations), on which the distribution is based, breaks down. It is precisely in the neighborhood of the critical point that the fluctuations are so large that they no longer obey that axiom: they can be considerably greater than $\sqrt{N}$. This implies that we should avoid the neighborhood of the critical point. There a kind of "phase transition" occurs at the point $\mu = 0$ [49]–[51]. The Bose–Einstein distribution, as well as the parastatistical distribution, take place outside some neighborhood of the point $\mu = 0$, and this allows us to avoid it.

According to the Large Encyclopedic Dictionary of the Russian Language (T. E. Efremova, Editor), fluctuations are random deviations of a physical quantity from its average value. The average observation depends on the interval of time during which the observation is made. If this interval is sufficiently large (because the observer must wait for the system to reach equilibrium), the fluctuation can "live" for a fairly long time, and for the observer it will remain a fluctuation. Further, when the life span of the object that we regarded as a fluctuation will be of the order of the observation time, then this object can no longer be considered to be a fluctuation.

Let us note first of all that there are domains where fluctuations and clusters are essentially the same things. And if we consider clusters consisting of $K_c = 200$ particles or less, then from the parastatistical formula we can find admissible fluctuations.

Thus, at the critical point, besides the restriction on the number of degrees of freedom, we establish an upper bound on the number $K$, the number of particles at any energy level. This leads to a version of UD statistics which is called *parastatistics* or *Gentile statistics* [18] (see Fig. 23).

For each value of $\gamma_c$, $\gamma_c$, $T$, we can choose a number $K(T)$ that allows the critical isochore to reach the Zeno line at the point determined by the Bachinskii rule.

In the case of negative $\mu$, we can use the following parastatistical formulas for pressure and density (where $b = 1/T$, $\gamma = D/2 - 1$).

$$\begin{aligned} P &= T^{\gamma+2}C(\mathrm{Li}_{\gamma+2}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma+1}}\mathrm{Li}_{\gamma+2}(e^{b\mu(K+1)})) \\ &= \frac{T^{\gamma+2}C}{\Gamma(\gamma+2)}\int_0^\infty(\frac{1}{e^{x-b\mu}-1} - \frac{K+1}{e^{(K+1)(x-b\mu)}-1})x^{\gamma+1}dx, \end{aligned} \tag{7.1}$$

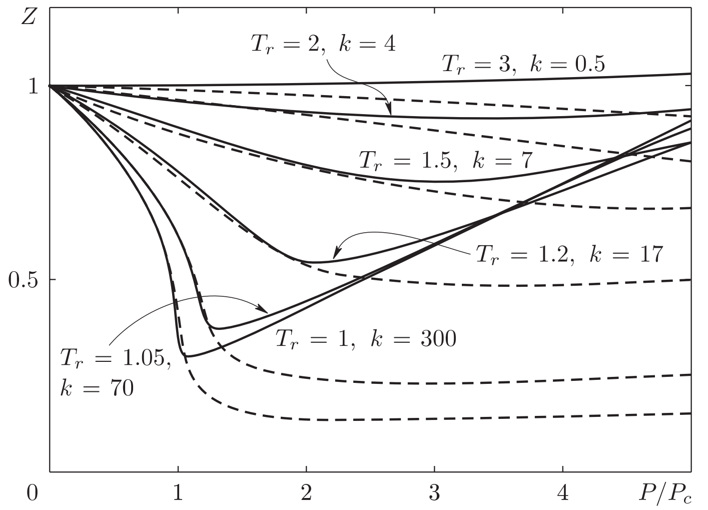

**Fig. 23.** Isotherms for $\gamma = 0.312$ coincide with the van der Waals isotherms for $\mu \le 0$. The Solid lines are van der Waals isotherms and the dotted lines are the parastatistical isotherms corresponding to different values of $K$.

$$\begin{aligned}\rho &= T^{\gamma+1}C(\mathrm{Li}_{\gamma+1}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma}}\mathrm{Li}_{\gamma+1}(e^{b\mu(K+1)})) \\ &= \frac{T^{\gamma+1}C}{\Gamma(\gamma+1)}\int_0^\infty (\frac{1}{e^{x-b\mu}-1} - \frac{K+1}{e^{(K+1)(x-b\mu)}-1})x^{\gamma}dx. \end{aligned} \tag{7.2}$$

For the constant $C$ we can take

$$C = \frac{\Lambda^{\gamma-\gamma_0}}{\zeta(\gamma+2)}, \tag{7.3}$$

where the constant $\Lambda$ is determined by the value of the triple point when constructing the binodal.

For the compressibility factor we have

$$Z = \frac{P}{\rho T} = \frac{\mathrm{Li}_{\gamma+2}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma+1}}\mathrm{Li}_{\gamma+2}(e^{b\mu(K+1)})}{\mathrm{Li}_{\gamma+1}(e^{b\mu}) - \frac{1}{(K+1)^{\gamma}}\mathrm{Li}_{\gamma+1}(e^{b\mu(K+1)})} \tag{7.4}$$

When $\mu = 0$, we obtain

$$P = T^{\gamma+2}C\zeta(\gamma+2)(1 - \frac{1}{(K+1)^{\gamma+1}}) = T^{\gamma+2}(1 - \frac{1}{(K+1)^{\gamma+1}}), \tag{7.5}$$

$$\rho = T^{\gamma+1}C\zeta(\gamma+1)(1 - \frac{1}{(K+1)^{\gamma}}) = T^{\gamma+1}\frac{\zeta(\gamma+1)}{\zeta(\gamma+2)}(1 - \frac{1}{(K+1)^{\gamma}}), \tag{7.6}$$

$$Z = \frac{\zeta(\gamma+2)(1 - \frac{1}{(K+1)^{\gamma+1}})}{\zeta(\gamma+1)(1 - \frac{1}{(K+1)^{\gamma}})}. \tag{7.7}$$

In parastatistics, the critical point is obtained for $K = \infty$. Approaching this point from the supercritical side, we assume $K = \mathfrak{R}$. For example, if $\mathfrak{R} = 100$, we consider an almost critical isotherm, denoting it by $\tilde{T}_c$. For $\gamma = 0.222$, for this isotherm, we obtain $Z|_{\mu=0} = 0.45$ and $P|_{\mu=0} = 0.996$.

Reducing to $\tilde{T}_c$: $T_r = T/\tilde{T}_c$ and imposing the following condition on $K(T_r, \gamma)$, we write

$$\rho_c^{(\gamma)} = \frac{N_c^{(\gamma)}}{V} = C(\gamma)\zeta(1+\gamma) = T_r^{1+\gamma}C(\gamma)\left[1 - \left(\frac{1}{1+K}\right)^{\gamma}\right]\zeta(1+\gamma). \tag{7.8}$$

This condition yields $K(T_r, \gamma)$ for $\mu = 0$ (see (7.9)).

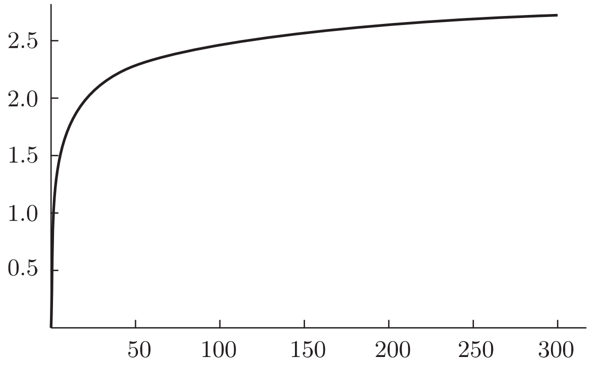

**Fig. 24.** Pressure on the Zeno line. The reduced temperature $T_r$ is plotted along the ordinate.

As a result, we obtain the dependence $K(T_r)$, which, for $\mu = 0$, is given by the formula

$$\frac{T}{T_{cell}} = (1 - (K+1)^{-\gamma})^{-1/(\gamma+1)}, \qquad \gamma = \gamma_c, \tag{7.9}$$

where $T_{cell}$ is the temperature, less than $T_c$, at which contour cells appear (see [3], [10], and others). The temperature $T_{cell}$, is, probably, related to the Frenkel line dividing "soft" and "stiff" liquids. For an experimental criterion for this line, one can use, in particular, the observation of the disappearance of the positive dispersion of acoustic waves in liquid [49], [55]–[58].

These contour cells are preserved in passing through the critical point. They are bounded above by some number determined for each pure gas. This model is, in principle, close to that of the above-mentioned authors, because monomers pass through the cellular structure of the contours without collisions and viscosity. This model serves as a two-phase model "gas–liquid" in the papers [3], [10]. In the present paper, supercritical fluid is described by a similar model.

On other isotherms, both $T_{cell}$ and $\gamma(T)$ are determined from the Bachinskii condition for $Z = 1$.

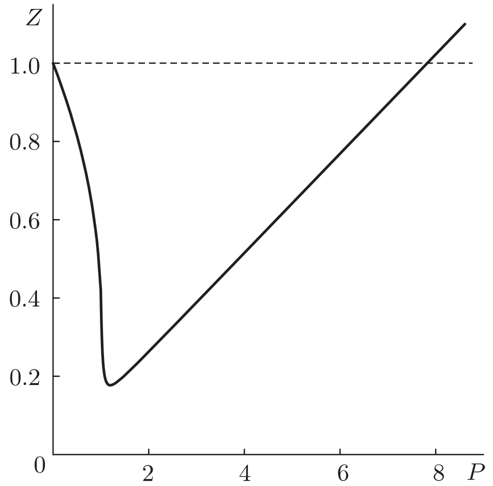

**Fig. 25.** Theoretical critical isotherm for methane.

The value of $K(T)$ may be determined from the critical isochore for $\mu = 0$ (see (7.9)).

From the point of view of economics, this critical isochore is a curve on the $(Z, P)$-plane for the nominal interest equal to zero. Hence we can determine the value $K(T)$ in economics only approximately (see Fig. 27).

Let us derive formulas for the value of $\mu$ from the interval $0 \le \mu \le 1/K$. $0 \le \mu \le 1/K(T_c)$. This interval of thermodynamics is not equilibrium and it should be "sewn" onto equilibrium thermodynamics

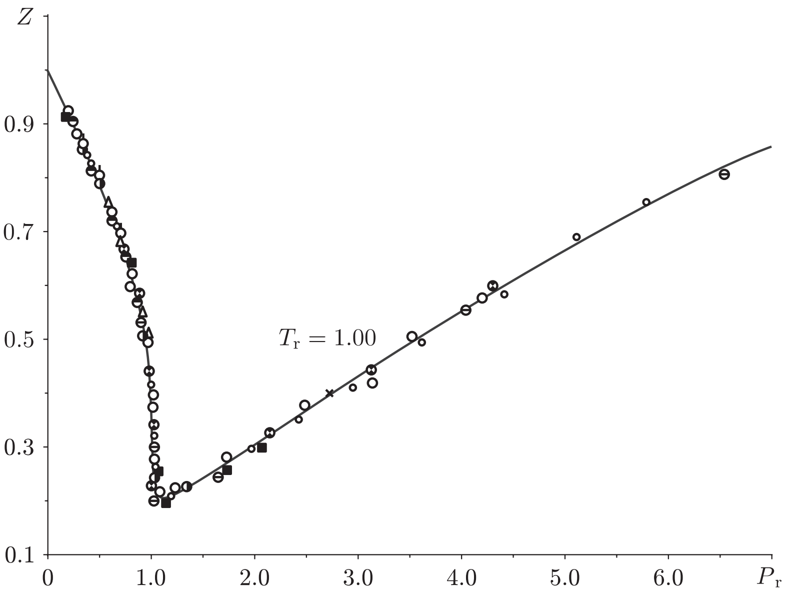

**Fig. 26.** Experimental isotherm for methane, propane, $CO_2$, and other gases; they are depicted by different symbols.

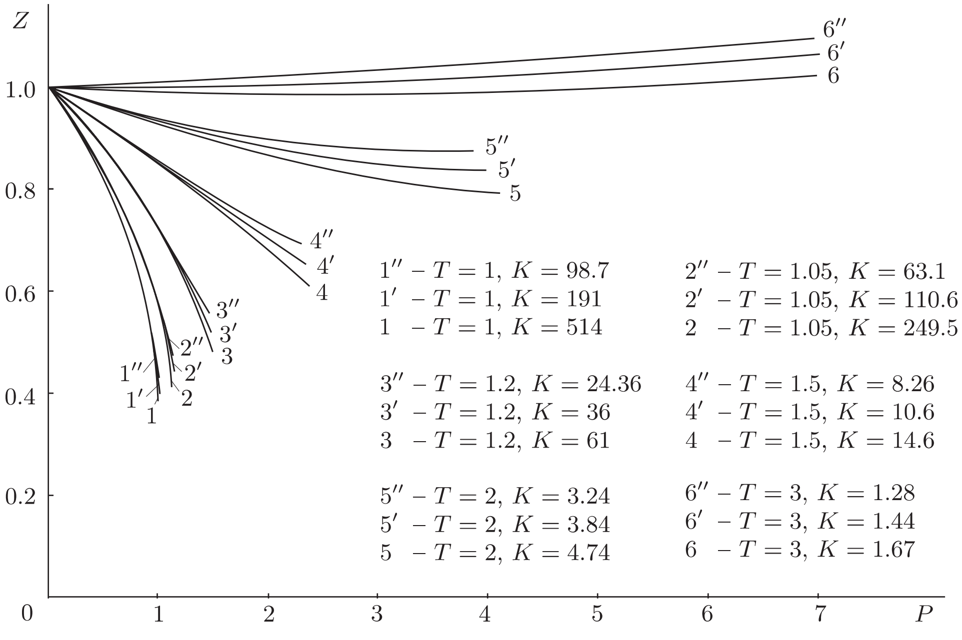

**Fig. 27.** Three families of isotherms for the initial values $K_c = 98.7$, $K_c = 191$, $K_c = 514$ for $\mu \le 0$.

for $\mu \ge 1/K(T_c)$. In this case, the free parameter appearing in the potential $\Omega$ should be taken as

$$\frac{C\Lambda^{\gamma-\gamma_0}}{(K(T_c)+1)\zeta(2+\gamma)}. \tag{7.10}$$

For $0 \le \mu K(T_c) \le 1$, we can sew together these two distributions by taking the free parameter as

$$\frac{C\Lambda^{\gamma-\gamma_0}}{(K(T_c)+1)^{\mu K(T_c)}\zeta(2+\gamma)} \qquad \mu K(T_c) \le 1.$$

For $\mu = 1/K(T_c)$, the free parameter coincides with (7.10).

*In thermodynamics, we need not leave out this interval of* $\mu$ *(*$0 \le \mu \le 1/K(T_c)$*), but can* *determine the constant* $C(\gamma, K)$ *on the Zeno line from Bachinskii's condition on the whole interval* $0 \le P \le \rho_B T_B/2$ *(see* (3.11)*).*

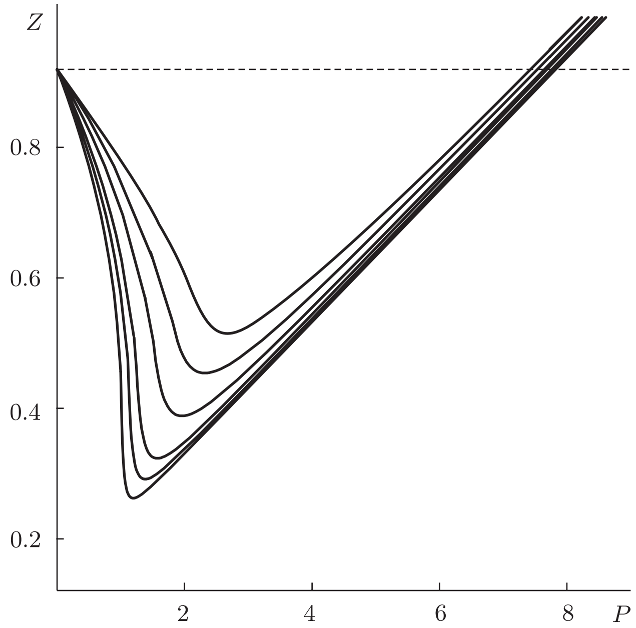

**Fig. 28.** Theoretic isotherms of methane with decreasing parameter $T_r \le 1$.

Let us consider the economic analogy. In the case of economics, it is necessary to leave ot the interval indicated above, because, in economics, there is no condition of Bachinskii type (although the Zeno line corresponds to the GDP; see [59]).

As the authors of [59] have already indicated, to the chemical potential corresponds, in economics, minus the nominal credit rate. The nominal credit rate in Islamic countries equals zero: Islam forbids any gains from lending money. Situations in which the nominal credit rate is negative are very rare. An example is the situation when one has to pay the bank for storing your money, for instance for renting a box in the bank's safe. The amount of money lying in such boxes is not necessarily known to the bank, but the understanding that there is money in those boxes may influence the financial policy of the bank.

As another example of a negative nominal credit rate, we can consider a fairly reasonable way of calculating the purchasing power of very poor population (which is, so to speak, in a Bose condensate) if we accept Irving Fisher's point of view and agree that the turnover rate of money corresponds to temperature and energy in thermodynamics.

The giving of credit (lending money) for buying real estate or other goods in order to begin farming and have a regular monthly income could be a reasonable measure for raising the enthusiasm of agricultural workers and increasing their purchasing power. But if the farmer is unable to return his debt in time (within the duration of the loan), this can be regarded as a crime, and the farmer can be punished, e.g. some soft form of compulsory labor.

If there are repulsive interactions between the particles of gas, then the potential in the case of the most general Hamiltonian [60] leads to a model of cell-like structure for the "normal" supercritical liquid-fluid through which monomers pass in a superfluid way. But despite the very general character of the Hamiltonian considered in [60], this model cannot be used for ideal noninteracting objects. The general Hamiltonian considered below in Sec. 10 possibly allows to construct the interaction between subjects of society, where small businesses will move in a superfluid way among the "cells" of big business. The less will this depend on the interactions and the more it will depend on uniting the two statistics, the more feasible will such a scenario be.

In Sec. 10, it will be shown that, in the case of interactions, a two-liquid fluid consisting of superfluid monomers and cell-like clusters appears.

## 8. DERIVATION OF VLASOV'S EQUATION FROM UD STATISTICS

There are two aspects of the many-body problem. The bodies can be in arbitrary motion in our three-dimensional space, interacting and sometimes colliding with one another. All this occurs in three-dimensional space. But we can treat this phenomenon differently. Namely, we can consider these $N$ bodies (regarded as $N$ material points) as one point in $3N$-dimensional space. And in this $3N$-dimensional space, this single point is in a motion that we can observe. Depending on the case considered, it turns out that one of these two approaches can be used. Thus, from the mathematical point of view, it is simpler to consider one point in space, while, from the physical point of view, it is of interest to know how these material points behave in the three-dimensional space in which we, in fact, live. This will constitute a more realistic "picture". For example, we can follow the motion in three-dimensional space of one particle which is in the field of all other particles. In that theory, we assume that all particles are approximately identical and have identical distributions. We can assume that one particle interacts with another one. And that other particle is actually the same one, i.e., it is distributed exactly like the first one.

From the mathematical point of view, it is important to have not an intuitive equation, but rather the exact equation in three-dimensional space that corresponds to the equation of motion of a particle in $3N$-dimensional space.

It turns out that this can be done by using the method of second quantization. In other words, we consider the behavior in three-dimensional space of operators, not functions, namely, of creation and annihilation operators. This was first done by Dirac in quantum mechanics. However, it is really not necessary to invoke the ideas of quantum mechanics. Thus, Schoenberg (approximately in 1953–1956) considered classical mechanics and applied the method of second quantization to classical objects and to classical statistical physics.

In fact, in the mathematics literature, he first used the notion that we call UD statistics, i.e., statistics for distinguishable objects regarded as indistiguishable, for the classical Hamiltonian. It so happened that physicists did not take any note of these papers. They are really hard to understand from the physical point of view: The particles in the Newton multiparticle system are considered indistiguishable, so that the system does not change under the interchange of two particles. Nevertheless, the present-day derivation of the Vlasov equation can be obtained by using the method of second quantization for classical particles.

Using this method as $N \to \infty$, we obtain the system [61]

$$\begin{aligned}
\dot{u}(p,q,t) &= \left(\frac{\partial U}{\partial q}\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) u(p,q,t) \\
&\qquad + \int dp'\,dq'\,v(p',q',t)\left(\frac{\partial V(q,q')}{\partial q}\frac{\partial}{\partial p} + \frac{\partial V(q,q')}{\partial q'}\frac{\partial}{\partial p'}\right) u(p',q',t)u(p,q,t), \\
\dot{v}(p,q,t) &= \left(\frac{\partial U}{\partial q}\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) v(p,q,t) \\
&\qquad + \int dp'\,dq'\,u(p',q',t)\left(\frac{\partial V(q,q')}{\partial q}\frac{\partial}{\partial p} + \frac{\partial V(q,q')}{\partial q'}\frac{\partial}{\partial p'}\right) v(p',q',t)v(p,q,t), \qquad (8.1)
\end{aligned}$$

where $U(q_i)$ is the external field and $V(q_i, q_j)$ is the pair interaction.

If we replace $u$ and $v$ by the creation and annihilation operators $\hat{u}$ and $\hat{v}$, respectively, in Fock space, then system (8.1) will be equivalent to the N-particle problem for the Newton system. Namely, the projection from the Fock space to the $3N$-dimensional space of N particles will, indeed, give the system of Newton equations. This implies that, "in the framework of the operator approach, the equation for the characteristics for these equations is the Vlasov equation" [62, 10]. This approach allows us to obtain corrections to the Vlasov equation and to the expansion of N-particle Newton equations in powers of $1/N$.

Note that, by substituting

$$u(p,q,t) = \sqrt{\rho(p,q,t)}\,e^{i\pi(p,q,t)}, \qquad v(p,q,t) = \sqrt{\rho(p,q,t)}\,e^{-i\pi(p,q,t)} \qquad (8.2)$$

system (8.1)) can be reduced to the form

$$\dot\rho(p,q,t)=\left(\frac{\partial W^t}{\partial q}\frac{\partial}{\partial p}-p\frac{\partial}{\partial q}\right)\rho(p,q,t), \tag{8.3}$$

$$\dot\pi(p,q,t)=\left(\frac{\partial W^t}{\partial q}\frac{\partial}{\partial p}-p\frac{\partial}{\partial q}\right)\pi(p,q,t)+\int dp'\,dq'\frac{\partial V(q,q')}{\partial q'}\frac{\partial\pi(p',q',t)}{\partial p'}\,\rho(p',q',t); \tag{8.4}$$

here

$$W^t(q)=U(q)+\int dq'V(q,q')\rho(p',q',t)\,dp'\,dq'.$$

The first equation (8.3) of the above system is the Vlasov equation [63], where $\rho$ is the distribution function used in physics for the study of many-particle systems; the second, new, equation (8.4) is linear in the function $\pi(p,q,t)$. The functions $u$ and $v$ are called *semidensities* and $\pi(p,q,t)$,the phase. Semidensity plays an important role in the semiclassical approximation of the wave function in quantum mechanics. The phase $\pi(p,q,t)$ plays an important role when the potential $V(x-y)$ has a singularity and the solution has a vortex line. Then, the conditions for the existence of a unique value of $e^{i\pi}$ yield conditions for the quantization of $\pi(p,q,t)$, i.e., for the quantization of vortices arising in classical problems.

Since the equation for $\pi(p,q,t)$ is linear, we can divide $\pi$ by any constant. It follows from dimension considerations that $\pi(p,q,t)$ must be divided by the Planck constant $\hbar$. Thus, $\hbar$ is injected into a purely classical problem. The quantization of the vortices of dimers of the gas argon can be verified experimentally by short-wave irradiation of the gas and the presence of resonance absorption.

Traditionally, classical physics is considered "historically," i.e., up to a certain point in history. In point of fact, we must revise classical physics from the point of view of the classical limit of quantum mechanics. Under such an approach, new important objects (notions) may arise in classical physics. Thus, the notion of polarization in optics is regarded as a derivation from wave optics. However, this notion is preserved in the short-wave limit. Therefore, it is natural to consider polarization within the framework of geometric optics.

Objects appearing in quantum mechanics in the semiclassical limit and not previously known in classical mechanics are particularly important, because they often change fixed notions in this firmly established science. We have obtained the new equation (8.4) in addition to Vlasov's equation. In what follows, we shall obtain a most important complement to Vlasov's equation of collective oscillations, which is preserved in the semiclassical limit, i.e., for classical particles.

## 9. REPRESENTATION BY THE SEQUENCE OF NATURAL NUMBERS

Consider the following natural conditions:

$$\begin{gathered}\sum N_i=\sum G_i\bar n_i=N,\\ \sum\varepsilon_iN_i=\varepsilon_iG_i\bar n_i=E.\end{gathered} \tag{9.1}$$

These conditions for the number of particles $N$ and the energy $E$ are, a particular case of Diophantine equations (1.1) in number theory.

Physicists can easily interpret the numbers $G_i$ in the way explained in Sec. 40 of the book [16] of Landau and Lifshits. It is simpler for mathematicians to call $G_i$ the multiplicity of the eigenvalue $\varepsilon_i$ (physicists use the term "degeneracy"). This is a particular case of the general pattern to which the physicists are more accustomed. However, this particular case is, at the same time, also a general one, because mathematical formulas do not vary in passing from the general pattern to the standard representation of the multiplicity of eigenvalues and can be uniquely carried over to the pointillistic pattern depicted in [16].

Note that, after passing to integral form [16], it is natural to use the measure $d\varepsilon$, where $\varepsilon$ is energy, considering, as a preliminary, the "phase space element"

$$dp_x\,dp_y\,dp_z\,dV,$$

which is a more habitual term for physicists ($p_x, p_y, p_z$ are the momenta and $V$ is the volume).

In the present section, we use the measure $d\varepsilon$ in the original formulas. If we pass from the integral over $d\varepsilon$ to the Riemann integral sum $\Delta\varepsilon i$, then we approach expression (1.1) of number theory, for which the remarkable theory due to Ramanujan, Harley, and Erdös was developed in a rigorous way; however, it can be said that this theory was not derived from any rational source: Ramanujan declared that the goddess Radha revealed the formula to him while he was asleep.

By the symbol $\varepsilon_i$ physicists mean the energy levels of the self-adjoint operator $A(\widehat{H})$ of the Hamiltonian in the Hilbert space $L_2(R^3)$. According to the *Mathematical Encyclopedic Dictionary* [64] (see item "Spectral Analysis") and the *Mathematical Encyclopedia* [17] (see item "Self-Adjoint Operator") the following equality is valid:

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda),$$

where $E(\lambda)$ is a partition of unity, i.e., a family of projection operators satisfying the spectral conditions. In the discrete case,

$$E(\lambda) = \sum_{\lambda_k < \lambda} E_k, \qquad A = \sum_{k=1}^{\infty} \lambda_k E_k.$$

If the spectral measure is absolutely continuous, then

$$A = \int_{-\infty}^{\infty} \lambda \frac{dE(\lambda)}{d\lambda} \, d\lambda. \tag{9.2}$$

It is this case that we shall consider, passing to the Riemann integral sum $d\lambda = \Delta\lambda i$, $i = 1, 2, \dots, n$.

Let us now give a more rigorous statement. We assume that the restriction of the operator on a subspace has a homomorphic mapping of Euler–Maclaurin type onto an operator with absolute continuous spectral measure. To each such restriction corresponds a spectral series. To each series we assign the sequence of natural numbers. Further, we shall take into account the semiclassical asymptotics for the quantum operator. This means that the invariance of the restriction of the operator will be considered approximately with respect to $\hbar$, tunnel transitions will be prohibited, etc.

Each series will be associated with complex germ theory generalizing the Bogolyubov $u - v$ transformation, in which the variational equation plays an essential role. The remarkable fact is that the term containing the Planck constant $\hbar$ is preserved in the variational equation for a Hartree-type equation; this fact is related to the ratio of two small parameters: the parameter $\hbar$ and the linearization parameter.

For these reasons, such a representation is said to be *natural*. From our point of view, the coefficients of $i$ are the multiplicities of the eigenvalue $i$. The multiplicities can be fractional if averaging is performed over the group of near eigenvalues according to the "pointillistic pattern" of Landau and Lifshits.

Let us now pass to the natural representation of self-consistent operators for the case in which the interaction between two particles is of the form

$$V(x-y), \qquad x \in R^3, \qquad y \in R^3.$$

We shall consider the self-consistent Hamiltonian

$$\widehat{H}[\psi] = -\frac{h^2}{2}\Delta\psi(x) + V_0(x)\psi(x) + \psi(x) \iint \psi(y)\overline{V(x-y)\psi(y)} \, dy, \tag{9.3}$$

where the bar denotes complex conjugation.

The nonlinear symbol is given by the formula

$$H[\rho](x,\, p) = \frac{|p|^2}{2} + V_0(x) + \iint \overline{V(x-x')}\rho(x',\, p') \, dx' \, dp',$$

where $V$, $V_0 \in S^\infty(\mathbf{R}^n)$, $\rho(x', p')$ is of the form (9.8).

The solution $F(x, p, t)$ of Vlasov's equation mentioned above can be regarded as the density of the distribution of classical particles over coordinates and momenta. So, for example, the mean value of the momentum of particles at the instant $t$ is

$$\langle p\rangle = \iint p\cdot F(x, p, t)\,dx\,dp, \tag{9.4}$$

and the mean value of the energy of the particles with the Hamilton function $p^2/2m + V(x)$ is

$$\langle E\rangle = \iint \left(\frac{p^2}{2m} + V(x)\right) F(x, p, t)\,dx\,dp. \tag{9.5}$$

In quantum mechanics, the state of the system is described by some "wave" function $\psi = \psi(x, t)$ continuous with respect to $t$ and belonging to $L^2(\mathbf{R}^n)$ with respect to the argument $x$. In this quantum case, the mean value $\langle\,\widehat{p}\,\rangle_{\text{qu}}$ of the momentum operator $\widehat{p} = -ih\frac{\partial}{\partial x}$ is determined by the integral

$$\langle\,\widehat{p}\,\rangle_{\text{qu}} = \int \psi(x, t)\overline{\widehat{p}\psi(x, t)}\,dx, \tag{9.6}$$

and the mean value of the operator of the energy

$$\widehat{E} = \frac{\widehat{p}^2}{2m} + V(x)$$

by the integral

$$\langle\widehat{E}\rangle_{\text{qu}} = \int \psi(x, t)\overline{\widehat{E}\; psi(x, t)}\,dx. \tag{9.7}$$

It is said that $\langle\,\widehat{p}\,\rangle_{\text{qu}}$ and $\langle\widehat{E}\rangle_{\text{qu}}$ are the mean values of the operators $\widehat{p}$ and $\widehat{E}$ in the state $\psi$.

From the wave function $\psi$, one can construct the quantum distribution function $F_{\text{qu}}(x, p, t)$ so that, for example, the means $\langle\widehat{p}\rangle_{\text{qu}}$ and $\langle\widehat{E}\rangle_{\text{qu}}$ are determined by formulas (9.4) and (9.5) in which $F$ is replaced by $F_{\text{qu}}$. Such a quantum distribution function is of the form

$$(2\pi h)^{-n/2}\psi(x, t)\overline{\widetilde{\psi}(p, t)}e^{-\frac{ixp}{h}} \tag{9.8}$$

and is called the *density function* $\rho_\psi$ corresponding to the wave function $\psi$. Thus, we have

$$F_{\text{qu}} = \rho_\psi.$$

Formula (9.6) and (9.7) (see [65]) can be rewritten as follows:

$$\langle\,\widehat{p}\,\rangle_{\text{qu}} = \iint p\cdot\rho_\psi(x, p, t)\,dx\,dp,$$

$$\langle\widehat{E}\rangle_{\text{qu}} = \iint \left(\frac{p^2}{2m} + V(x)\right)\rho_\psi(x, p, t)\,dx\,dp.$$

In general, the mean of the operator $\widehat{f} = f(\overset{2}{x}, \overset{1}{p})$ in the state $\psi$ is

$$\langle\,\widehat{f}\,\rangle_{\text{qu}} \overset{\text{def}}{=} \int \psi(x, t)\overline{\widehat{f}\psi(x, t)}\,dx = \iint \overline{f(x, p)}\rho_\psi(x, p, t)\,dx\,dp \tag{9.9}$$

$$\text{where}\quad \overset{1}{p}=\overset{1}{\widehat{p}}, \qquad \widehat{p} = ih\,\frac{\partial}{\partial x}. \tag{9.10}$$

In linear theory, the wave function $\psi$ usually obeys some linear partial differential equation. As an example of such equations, we can indicate equations of Hartree type as well as equations obtained in [66] by "quantizing" Vlasov's equations. They all have the following form:

$$H[\rho_\psi](\overset{2}{x}, \overset{1}{p})\psi = 0,$$
$$H[\rho_\psi](x, p) = F(x, p, I_1[\rho_\psi]\dots I_r[\rho_\psi]), \tag{9.11}$$

where $F(x,\, p,\, \lambda)$ is a given function of the variables $x,\, p \in \mathbf{R}^n$, and $\lambda \in \mathbf{R}^r$, while the functionals $I_1 \dots I_r$ have the form of the means (9.10):

$$I_j[\rho_\psi] = \int \overline{k_j(x,\, p,\, x',\, p')}\rho_\psi(x',\, p')\, dx'\, dp' = \int \psi(x')\overline{k_j(x,\, p,\, \overset{2}{x'},\, -i\, \overset{1}{\partial/\partial x'})\psi(x')}\, dx'.$$

We say that equations of type (9.11) are *unitary nonlinear equations*, while the operators $\psi \to H[\rho_\psi](\overset{2}{x},\, \overset{1}{p})\psi$ are unitary nonlinear operators (or, briefly, $UN$-operators). This term is related to the fact that, under unitary transformations in the space of wave functions $\psi$, the symbol of the $UN$-operator is transformed by the unitary canonical transformation.

Consider the Cauchy problem for the $UN$-operator

$$-ih\partial\psi/\partial t + H[\rho_\psi](\overset{2}{x},\, \overset{1}{p})\psi = 0, \quad \psi\big|_{t=0} = \psi_{0,h}(x). \tag{9.12}$$

Under certain assumptions on the symbol $H$, the existence theorem for the solutions of problem (9.12) was proved in [67], [65] using the method of $T$-mappings.

**Definition of unitary nonlinear operators.** We begin by considering the definition of unitary nonlinear operators of Hilbert–Schmidt type.

**Definition 1.** Let $H$ be a smooth mapping of the space $L^2(\mathbf{R}^{2n})$ into itself:

$$\rho(x,\, p) \to H[\rho](x,\, p).$$

By a *unitary nonlinear operator* (or, briefly, *$UN$-operator*) *of Hilbert–Schmidt type* we mean the mapping $\mathcal{H}$ of the space $L^2(\mathbf{R}^n)$ into itself defined by the formula

$$\mathcal{H}[\psi] = H[\rho_\psi](\overset{2}{x},\, \overset{1}{p})\psi, \tag{9.13}$$

where $\rho_\psi$ is the density function corresponding to $\psi$. We say that the mapping $H$ is the *nonlinear symbol* of the operator $\mathcal{H}$, and the function $H[\rho_\psi]$ is the *linear symbol* of the operator $\mathcal{H}$ in the state $\psi$.

**Example 1.** Let the nonlinear symbol $H$ be a constant mapping of $H[\rho] = f_0$, where $f_0$ is a fixed function from $L^2(\mathbf{R}^{2n})$. The unitary nonlinear operator $\widehat{H}$ with such a symbol is the linear pseudodifferential operator

$$\mathcal{H}[\psi] = f_0(\overset{2}{x},\, \overset{1}{p})\psi.$$

Thus, the class of $UN$-operators includes the linear pseudodifferential operators.

**Example 2.** Let the symbol $H$ be a linear bounded mapping from $L^2(\mathbf{R}^{2n})$ into itself. Let us express it as the integral operator with generalized kernel $K(x,\, p,\, x',\, p')$, i.e.,

$$H[\rho](x,\, p) = \int \overline{K(x,\, p;\, x',\, p')}\rho(x',\, p')\, dx'\, dp'.$$

The $UN$-operator $\widehat{H}$ with such a symbol acts by the formula

$$\big(\mathcal{H}[\psi]\big)(x) = \left(\int \psi(y)\overline{K\big(\overset{2}{x},\, -ih\partial/\overset{1}{\partial}\, x,\, \overset{2}{y},\, -ih\, \overset{1}{\partial}\, /y\big)\psi(y)\partial y}\right)\psi(x).$$

If $H = 1$ is the unit operator, then

$$\mathcal{H}[\psi] = (2\pi h)^{-n}\|\psi\|^2 \cdot \psi(x).$$

Consider the variation of the nonlinear symbol of the $UN$-operator under the unitary transformation.

The isometry [65]

$$\mu\colon L^2(\mathbf{R}^{2n}) \to H_2, \quad \mu(f) \equiv f(\overset{2}{x},\, \overset{1}{p})$$

is valid for the class $H_2$ of linear Hilbert–Schmidt operators acting in the space $L^2(\mathbf{R}^2)$. Let $V$ be a linear unitary operator in $L^2(\mathbf{R}^2)$. Define the operator $\mathrm{Ad}_V$ in $H_2$ by the formula

$$\mathrm{Ad}_V(T) = VTV^{-1}. \tag{9.14}$$

Let us also assign to the operator $V$ the following linear unitary operator $\widetilde{V}$ in the space $L^2(\mathbf{R}^{2n})$:

$$\widetilde{V} = \mu^{-1} \circ \mathrm{Ad}_V \circ \mu. \tag{9.15}$$

**Theorem 1.** *Let $\mathcal{H}$ be a UN-operator of Hilbert–Schmidt type, and let $V$ be a linear unitary transformation of the space $L^2(\mathbf{R}^n)$. Then the mapping $V^{-1} \circ \mathcal{H} \circ V$ is a UN-operator and its nonlinear symbol $H_V$ is related to the symbol $H$ by the canonical unitary transformation*

$$H_V = \widetilde{V}^{-1} \circ H \circ \widetilde{V}.$$

**Proof.** It is required to show that $V^{-1}\mathcal{H}[V\psi] = \mathcal{H}_V[\psi]$ for any function $\psi \in L^2(\mathbf{R}^n)$, where $\mathcal{H}_V$ is a $UN$-operator with symbol $H_V$. By definition, we have

$$\mathcal{H}_V[\psi] = f(\overset{2}{x}, \overset{1}{p})\psi,$$

where

$$f = H_V[\rho_\psi] = \widetilde{V}^{-1} H[\widetilde{V}\rho_\psi].$$

In view of formulas (9.14) and (9.15), we have

$$f \equiv \mu^{-1}\big(V \cdot H[\widetilde{V}\rho_\psi] \cdot (\overset{2}{x}, \overset{1}{p}) \cdot V^{-1}\big)$$

and

$$\widetilde{V}\rho_\psi = \mu^{-1}\big(V \cdot \rho_\psi(\overset{2}{x}, \overset{1}{p}) \cdot V^{-1}\big).$$

Therefore, it remains to prove that

$$V \cdot \rho_\psi(\overset{2}{x}, \overset{1}{p}) \cdot V^{-1} = \rho_{V_\psi}(\overset{2}{x}, \overset{1}{p}).$$

This equality follows from the fact that the operator $V$ is unitary and from formula (b) from Lemma 1.3 in [65, Chap. II]. The theorem is proved. □

Let us now give the definition of more general unitary nonlinear operators. Consider some mapping

$$r\colon S_{-\infty}(\mathbf{R}^{2n}) \to S^{\infty}(\mathbf{R}^{2n}).$$

Set $S_0 = S^{\infty}(\mathbf{R}^{2n})$ and denote by $\mathcal{L}_k$ the space of all continuous mappings from $S_{-\infty}(\mathbf{R}^{2n})$ to $\mathcal{L}_{k-1}$ (where $k = 1, 2, \dots$). Let us endow $\mathcal{L}_k$ with continuous convergence [68], [43]. The mapping $r$ is said to be *smooth* if $r$ and all of its differentials $D_r^k\colon \mathcal{L}_0 \to \mathcal{L}_k$, are bounded, i.e., take bounded sets to bounded sets.

**Definition 2.** Let $H\colon S_{-\infty}(\mathbf{R}^{2n}) \to S^{\infty}(\mathbf{R}^{2n})$ be a smooth mapping. The operator

$$\mathcal{H}\colon S(\mathbf{R}^n) \to S(\mathbf{R}^n),$$

given by the formula

$$\mathcal{H}[\psi] = H[\rho_\psi](\overset{2}{x}, \overset{1}{p})\psi, \quad \psi \in S(\mathbf{R}^n),$$

is said to be *unitary nonlinear* (or, briefly, a *UN-operator*).

**Remark 7.** Superfluidity in a nanotube is not a dynamic flow: all the particles in the nanotube belong to the same series. Therefore, $\rho_B$ is very large, which yields a small extreme temperature (see (3.11) in Sec. 3).

## 10. SUPERFLUIDITY IN THE CLASSICAL PROBLEM

Consider the following general Hartree-type equations:

$$\left[H_0\left(x,-ih\frac{\partial}{\partial x}\right)+\int dy\psi^*(y)H_1\left(x,-ih\frac{\partial}{\partial x};y,-ih\frac{\partial}{\partial y}\right)\psi(y)\right]\psi(x)=\Omega\psi(x), \tag{10.1}$$

where $x,y\in\mathbf{R}^n$, $\psi\in L^2(\mathbf{R}^n)$ is a complex-valued function, $h>0$, $\Omega\in\mathbf{R}$, are the operators $-ih\partial/\partial x$ and $-ih\partial/\partial y$ acting first; the operators $x$ and $y$ act next. The function $H_1$ satisfies the condition $H_1(x,p_x;y,p_y)=H_1(y,p_y;x,p_x)$. Equation (10.1) generalizes the ordinary Hartree equation (Eq. (1) in [69], where $N=1$). The study of Eq. (10.1) is important, for example, if one is attempting to find a solution to the Hartree equation in the momentum representation,

$$\psi(x)=\int\widetilde{\psi}(p)e^{(i/\hbar)px}\frac{dp}{(2\pi\hbar)^{n/2}}.$$

Let us also discuss the variational system associated with Eq. (10.1), which can be obtained as follows. Along with Eq. (10.1), let us write out the conjugate equation and consider the variations of both equations *assuming that the variations* $\delta\psi=F$ and $\delta\psi^*=G$ are *independent*.

The variational system has the form

$$\begin{gathered}\left[H_0\left(x,-ih\frac{\partial}{\partial x}\right)-\Omega+\int dy\psi^*(y)H_1\left(x,-ih\frac{\partial}{\partial x};y,-ih\frac{\partial}{\partial y}\right)\psi(y)\right]F(x)\\
+\int dy\biggl(G(y)H_1\left(x,-ih\frac{\partial}{\partial x};y,-ih\frac{\partial}{\partial y}\right)\psi(y)\\
+\psi^*(y)H_1\left(x,-ih\frac{\partial}{\partial x};y,-ih\frac{\partial}{\partial y}\right)F(y)\biggr)\psi(x)=-\beta F(x),\\
\left[H_0\left(x,ih\frac{\partial}{\partial x}\right)-\Omega+\int dy\psi(y)H_1\left(x,ih\frac{\partial}{\partial x};y,ih\frac{\partial}{\partial y}\right)\psi^*(y)\right]G(x)\\
+\int dy\biggl(F(y)H_1\left(x,ih\frac{\partial}{\partial x};y,ih\frac{\partial}{\partial y}\right)\psi^*(y)\\
+\psi(y)H_1\left(x,ih\frac{\partial}{\partial x};y,ih\frac{\partial}{\partial y}\right)G(y)\biggr)\psi^*(x)=\beta G(x).\end{gathered} \tag{10.2}$$

Equations (10.1) and (10.2) play an important role in the problem of constructing asymptotic solutions to the $N$-particle Schrödinger equation as $N\to\infty$ [70]-[72].

For example, the spectrum of system (10.2) (possible values of $\beta$) corresponds to the spectrum of quasi-particles. Namely, the difference between the energy of an excited state and the ground state energy is given by the expression

$$\sum_k\beta_k n_k,$$

where the numbers $n_k\in Z_+$, $k=\overline{1,\infty}$, which are equal to zero starting from some $k$, define the eigenfunction and the eigenvalue of the excited state, and $\beta_k\in\mathbf{R}$ are the eigenvalues of system (10.2).

In this section, we are interested in asymptotic solutions to Eqs. (10.1) and (10.2) as the "inner" $h$ tends to zero.

Asymptotic solutions to Eq. (10.1) are given [66] by the canonical operator on a Lagrangian manifold $\Lambda^n=\{x=X(\alpha),p=P(\alpha)\}$ invariant with respect to the Hamiltonian system

$$\dot{x}=\frac{\partial H(x,p)}{\partial p},\quad \dot{p}=-\frac{\partial H(x,p)}{\partial x}, \tag{10.3}$$

where

$$H(x,p)=H_0(x,p)+\int d\mu_\alpha H_1(x,p;X(\alpha),P(\alpha)),$$

$\alpha \in \Lambda^n$, and $d\mu_\alpha$ is an invariant measure on the manifold $\Lambda^n$. The Lagrangian manifold lies on the surface $H(x,p) = \Omega$. If a chart $A$ is projected diffeomorphically in the $x$-plane, then the canonical operator acts as the multiplication by $\exp\{(i/h)S(x)\}/\sqrt{J}$, where

$$S(x) = \int p\,dx \quad \text{on } \Lambda^n \qquad \text{and} \quad J = Dx/D\mu_\alpha.$$

We are interested in finding asymptotic solutions to Eqs. (10.2). Without loss of generality, we can consider only the case of an $x$-chart. Indeed, to obtain similar expressions in the $p$-chart, one must consider the Fourier transformation of Eqs. (10.1) and (10.2) and apply the same technique, since the form of the equations remains unchanged.

Let us seek the asymptotic solutions to Eqs. (10.2) in the $x$-chart in the form

$$F(x) = \widetilde{f}(x)\psi(x), \quad G(x) = \widetilde{g}(x)\psi^*(x), \tag{10.4}$$

where the functions $f$ and $g$, in contrast to $\psi$ and $\psi^*$, have a limit as $h \to 0$. One can consider a more general case, by allowing $f$ and $g$ to be functions of $x$ and $-ih\partial/\partial x$, but in the leading term as $h \to 0$ we have

$$-ih\frac{\partial}{\partial x}e^{(i/h)S} \approx \frac{\partial S}{\partial x}e^{(i/h)S},$$

and so we arrive at functions $f$ and $g$ that depend only on $x$.

The second equation in system (10.2) can be rewritten in the form

$$\begin{aligned}\biggl[H_0\biggl(x, ih\frac{\partial}{\partial x}\biggr) + \int dy\psi(y)H_1\biggl(x, ih\frac{\partial}{\partial x}; y, ih\frac{\partial}{\partial y}\biggr)\psi^*(y); \widetilde{g}(x)\biggr]\psi^*(x)\\ + \int dy\biggl\{\psi(y)\widetilde{c}(y)H_1\biggl(x, ih\frac{\partial}{\partial x}; y, ih\frac{\partial}{\partial y}\biggr)\psi^*(y)\\ + \psi(y)\biggl[H_1\biggl(x, ih\frac{\partial}{\partial x}; y, ih\frac{\partial}{\partial y}\biggr); \widetilde{g}(y)\biggr]\psi^*(y)\biggr\} = \beta\widetilde{g}(x)\psi^*(x),\end{aligned} \tag{10.5}$$

where $[A;B] = AB - BA$ and

$$c(x) = \widetilde{f}(x) + \widetilde{g}(x). \tag{10.6}$$

Equation (10.1) is used in the derivation of Eq. (10.5). We observe that all terms containing the function $\widetilde{g}$ on the left-hand side in Eq. (10.1) are $O(h)$, since the commutator of two operators depending on $x$ and $-ih\partial/\partial x$ is equal in the classical limit to $(-ih)$ times the Poisson bracket of the corresponding classical quantities.

Thus, the function $c$, as well as the eigenvalue $\beta$, is assumed to be $O(h)$. Let us rescale these quantities as follows:

$$c(x) = h\widetilde{c}(x), \quad \beta = h\widetilde{\beta}. \tag{10.7}$$

Now we can derive the equation for $\widetilde{g}, \widetilde{c}$, and $\widetilde{\beta}$ in the leading term in $h$ and the first correction to it from Eq. (10.5), making use of the following relations:

i)

$$\biggl[A\biggl(x, ih\frac{\partial}{\partial x}\biggr); \xi(x)\biggr] = \sum_{a=1}^{n} ih\frac{\partial A}{\partial p_a}\biggl(x, ih\frac{\partial}{\partial x}\biggr)\frac{\partial \xi}{\partial x_a} - \sum_{a,b=1}^{n}\frac{h^2}{2}\frac{\partial^2 A}{\partial p_a \partial p_b}\biggl(x, ih\frac{\partial}{\partial x}\biggr)\frac{\partial^2 \xi}{\partial x_a \partial x_b}, \tag{10.8}$$

where $p_a = ih\partial/\partial x_a$, $A(x,p)$ is a function, $\mathbf{R}^{2n} \to \mathbf{C}$, $\xi : \mathbf{R}^n \to \mathbf{R}$;

ii)

$$\psi(x) = \chi(x,h)e^{(i/h)S(x)},$$

where $\chi = 1/\sqrt{J}$ in the leading term in $h$;

iii)

$$ih\frac{\partial}{\partial x}e^{-(i/h)S(x)} = e^{-(i/h)S(x)}\biggl(\frac{\partial S}{\partial x_n} + ih\frac{\partial}{\partial x}\biggr);$$

iv)

$$\begin{aligned}B\biggl(ih\frac{\partial}{\partial x} + \frac{\partial S}{\partial x}\biggr) &= B\biggl(\frac{\partial S}{\partial x}\biggr) + ih\sum_{a=1}^{n}\frac{\partial B}{\partial p_a}\frac{\partial}{\partial x_a} + \frac{ih}{2}\sum_{a,b=1}^{n}\frac{\partial^2 B}{\partial p_a\partial p_b}\frac{\partial^2 S}{\partial x_a\partial x_b}\\ &\quad+ \frac{(ih)^2}{2}\sum_{a,b=1}^{n}\frac{\partial^2 B}{\partial p_a\partial p_b}\frac{\partial^2}{\partial x_a\partial x_b} + \frac{(ih)^2}{2}\sum_{a,b,c=1}^{n}\frac{\partial^3 B}{\partial p_a\partial p_b\partial p_c}\frac{\partial^2 S}{\partial x_a\partial x_b}\frac{\partial}{\partial x_c}\\ &\quad+ \frac{(ih)^2}{6}\sum_{a,b,c=1}^{n}\frac{\partial^3 B}{\partial p_a\partial p_b\partial p_c}\frac{\partial^3 S}{\partial x_a\partial x_b\partial x_c}\\ &\quad+ \frac{(ih)^2}{8}\sum_{a,b,c,d=1}^{n}\frac{\partial^4 B}{\partial p_a\partial p_b\partial p_c\partial p_d}\frac{\partial^2 S}{\partial x_a\partial x_b}\frac{\partial^2 S}{\partial x_c\partial x_d} + O(h^3),\end{aligned} \tag{10.9}$$

where all derivatives of $B$ are evaluated at the point $p = \partial S/\partial x$.

These relations can easily be obtained for monomial functions $A$ and $B$. An application of formulas i)–iv) yields the equation

$$\begin{aligned}&i\sum_{a=1}^{n}\frac{\partial H}{\partial p_a^x}\frac{\partial \widetilde{g}}{\partial x_a}(X(\alpha)) - \widetilde{\beta}\widetilde{g}(X(\alpha)) + \int d\mu_\beta \widetilde{c}(X(\beta))H_1\\ &\quad+ i\int d\mu_\beta\sum_{a=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\beta))\frac{\partial H_1}{\partial p_a^y} + \frac{h}{2}\sum_{a,b=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\alpha))\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\frac{\partial\ln J}{\partial x_b}(X(\alpha))\\ &\quad- \frac{h}{2}\sum_{a,b,c=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\alpha))\frac{\partial^3 H}{\partial p_a^x\partial p_b^x\partial p_c^x}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\alpha)) - \frac{h}{2}\sum_{a,b=1}^{n}\frac{\partial^2\widetilde{g}}{\partial x_a\partial x_b}(X(\alpha))\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\\ &\quad+ \frac{ih}{2}\int d\mu_\beta\widetilde{c}(X(\beta))\sum_{a,b=1}^{n}\biggl[\frac{\partial^2 H_1}{\partial p_a^x\partial p_b^x}\frac{\partial^2 S}{\partial x_a\partial x_b}(X(\alpha)) + \frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y}\frac{\partial^2 S}{\partial x_a\partial x_b}(X(\beta))\biggr]\\ &\quad- \frac{ih}{2}\int d\mu_\beta\widetilde{c}(X(\beta))\sum_{a=1}^{n}\biggl[\frac{\partial H_1}{\partial p_a^x}\frac{\partial\ln J}{\partial x_a}(X(\alpha)) + \frac{\partial H_1}{\partial p_a^y}\frac{\partial\ln J}{\partial x_a}(X(\beta))\biggr]\\ &\quad+ \frac{h}{2}\int d\mu_\beta\sum_{a=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\beta))\biggl\{\sum_{b=1}^{n}\biggl(\frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y}\frac{\partial\ln J}{\partial x_b}(X(\beta)) + \frac{\partial^2 H_1}{\partial p_a^y\partial p_b^x}\frac{\partial\ln J}{\partial x_b}(X(\alpha))\biggr)\\ &\quad- \sum_{b,c=1}^{n}\biggl(\frac{\partial^3 H_1}{\partial p_a^y\partial p_b^x\partial p_c^x}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\alpha)) + \frac{\partial^3 H_1}{\partial p_a^y\partial p_b^y\partial p_c^y}\frac{\partial^2 S}{\partial y_b\partial y_c}(X(\beta))\biggr)\biggr\}\\ &\quad- \frac{h}{2}\int d\mu_\beta\sum_{a,b=1}^{n}\frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y}\frac{\partial^2 g}{\partial x_a\partial x_b}(X(\beta)) = 0;\end{aligned} \tag{10.10}$$

in this formula the arguments

$$x = X(\alpha),\quad p^x = P(\alpha),\quad y = X(\beta),\quad p^y = P(\beta) \tag{10.11}$$

of the function $H_1$ and of its derivatives, as well as the arguments $x = X(\alpha)$, $p^x = P(\alpha)$ of the function $H$, are omitted.

Let us now find another equation relating $\widetilde{g}$ to $\widetilde{c}$. To this end, let us multiply the first equation in system (10.2) by $\psi^*(x)$ and the second equation by $\psi(x)$. Let us subtract the first product from the second. We obtain

$$\begin{aligned}
\beta\psi^*(x)\psi(x)c(x) &= \psi(x)\biggl[H\biggl(x, ih\frac{\partial}{\partial x}\biggr); \widetilde{g}(x)\biggr]\psi^*(x) \\
&\quad + \psi^*(x)\biggl[H\biggl(x, -ih\frac{\partial}{\partial x}\biggr); \widetilde{g}(x)\biggr]\psi(x) - \psi^*(x)\biggl[H\biggl(x, -ih\frac{\partial}{\partial x}\biggr); \widetilde{c}(x)\biggr]\psi(x) \\
&\quad + \psi(x)\int dy\psi(y)\biggl[H_1\biggl(x, ih\frac{\partial}{\partial x}; y, ih\frac{\partial}{\partial y}\biggr); \widetilde{g}(y)\biggr]\psi^*(y)\psi^*(x) \\
&\quad - \psi^*(x)\int dy\psi^*(y)\biggl[\widetilde{g}(y); H_1\biggl(x, -ih\frac{\partial}{\partial x}; y, -ih\frac{\partial}{\partial y}\biggr)\biggr]\psi(y)\psi(x) \\
&\quad + \psi(x)\int dy\psi(y)c(y)H_1\biggl(x, ih\frac{\partial}{\partial x}; y, ih\frac{\partial}{\partial y}\biggr)\psi^*(y)\psi^*(x) \\
&\quad - \psi^*(x)\int dy\psi^*(y)H_1\biggl(x, -ih\frac{\partial}{\partial x}; y, -ih\frac{\partial}{\partial y}\biggr)c(y)\psi(y)\psi(x). \qquad (10.12)
\end{aligned}$$

Let us use Eqs. (10.8)–(10.10). We find the following equation for $\widetilde{g}$ and $\widetilde{c}$ modulo $O(h^2)$:

$$\begin{aligned}
& i\sum_{a=1}^{n}\frac{\partial H}{\partial p_a^x}\frac{\partial\widetilde{c}}{\partial x_a}(X(\alpha)) - \widetilde{\beta}\widetilde{c}(X(\alpha)) - \sum_{a,b=1}^{n}\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\frac{\partial^2\widetilde{g}}{\partial x_a\partial x_b}(X(\alpha)) \\
&\quad + \sum_{a,b=1}^{n}\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\frac{\partial\widetilde{g}}{\partial x_a}(X(\alpha))\frac{\partial\ln J}{\partial x_b}(X(\alpha)) \\
&\quad - \sum_{a,b,c=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\alpha))\frac{\partial^3 H}{\partial p_a^x\partial p_b^x\partial p_c^x}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\alpha)) - \int d\mu_\beta\sum_{a,b=1}^{n}\frac{\partial^2\widetilde{g}}{\partial x_a\partial x_b}(X(\beta))\frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y} \\
&\quad - \int d\mu_\beta\sum_{a,b,c=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\beta))\biggl(\frac{\partial^2 S}{\partial x_a\partial x_b}(X(\alpha))\frac{\partial^3 H_1}{\partial p_a^y\partial p_b^x\partial p_c^x} + \frac{\partial^2 S}{\partial y_a\partial y_b}(X(\beta))\frac{\partial^3 H_1}{\partial p_a^y\partial p_b^y\partial p_c^y}\biggr) \\
&\quad + \int d\mu_\beta\sum_{a,b=1}^{n}\frac{\partial\widetilde{g}}{\partial x_a}(X(\beta))\biggl(\frac{\partial\ln J}{\partial x_b}(X(\alpha)) - \frac{\partial^2 H_1}{\partial p_a^y\partial p_b^x} + \frac{\partial\ln J}{\partial x_b}(X(\beta))\frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y}\biggr) \\
&\quad - i\int d\mu_\beta\widetilde{c}(X(\beta))\sum_{a=1}^{n}\biggl(\frac{\partial\ln J}{\partial x_a}(X(\alpha))\frac{\partial H_1}{\partial p_a^x} + \frac{\partial\ln J}{\partial x_a}(X(\beta))\frac{\partial H_1}{\partial p_a^y}\biggr) \\
&\quad + i\int d\mu_\beta\sum_{a=1}^{n}\frac{\partial\widetilde{c}}{\partial x_a}(X(\beta))\frac{\partial H_1}{\partial p_a^y} \\
&\quad + i\int d\mu_\beta\widetilde{c}(X(\beta))\sum_{a,b=1}^{n}\biggl(\frac{\partial^2 H_1}{\partial p_a^x\partial p_b^x}\frac{\partial^2 S}{\partial x_a\partial x_b}(X(\alpha)) + \frac{\partial^2 H_1}{\partial p_a^y\partial p_b^y}\frac{\partial^2 S}{\partial y_a\partial x_b}(X(\beta))\biggr) \\
&\quad + \frac{h}{2}\sum_{a,b,c=1}^{n}\frac{\partial\widetilde{c}}{\partial x_a}(X(\alpha))\frac{\partial^3 H}{\partial p_a^x\partial p_b^x\partial p_c^x}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\alpha)) \\
&\quad - \frac{h}{2}\sum_{a,b=1}^{n}\frac{\partial\widetilde{c}}{\partial x_a}(X(\alpha))\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\frac{\partial\ln J}{\partial x_b}(X(\alpha)) + \frac{h}{2}\sum_{a,b=1}^{n}\frac{\partial^2 H}{\partial p_a^x\partial p_b^x}\frac{\partial^2\widetilde{c}}{\partial x_a\partial x_b}(X(\alpha)) \\
&\quad + \frac{h}{2}\int d\mu_\beta\sum_{a,b,c=1}^{n}\frac{\partial\widetilde{c}}{\partial x_a}(X(\beta))\biggl(\frac{\partial^3 H_1}{\partial p_a^y\partial p_b^x\partial p_c^x}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\alpha)) + \frac{\partial^3 H_1}{\partial p_a^y\partial p_b^y\partial p_c^y}\frac{\partial^2 S}{\partial x_b\partial x_c}(X(\beta))\biggr)
\end{aligned}$$

$$+\frac{h}{2}\int d\mu_\beta \sum_{a,b=1}^{n}\biggl[\frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y}\biggl(\frac{\partial^2 \widetilde{c}}{\partial x_a \partial x_b}(X(\beta)) - \frac{\partial \widetilde{c}}{\partial x_a}(X(\beta))\frac{\partial \ln J}{\partial x_b}(X(\beta))$$
$$-\frac{\partial \widetilde{c}}{\partial x_a}(X(\beta))\frac{\partial \ln J}{\partial x_b}(X(\alpha))\frac{\partial^2 H_1}{\partial p_a^y \partial p_b^x}\biggr] = 0. \tag{10.13}$$

Finally, from the general equations we now pass to the case of interest. If

$$H_0(x, p_x) = p_x^2/2 + U(x),$$
$$H_1(x, p_x; y, p_y) = V(x, y),$$

then Eqs. (10.10) and (10.13) become much simpler and acquire the form

$$\begin{gathered}(i\nabla S\nabla - \widetilde{\beta})\widetilde{g} + \int V(x, X(\alpha'))\widetilde{c}(X(\alpha'))\, d\mu_{\alpha'} + \frac{h}{2}(-\Delta\widetilde{g} + \nabla \ln J \nabla \widetilde{g}) = 0,\\ (i\nabla S\nabla - \widetilde{\beta})\widetilde{c} - \Delta\widetilde{g} + \nabla \ln J\nabla\widetilde{g} - \frac{h}{2}(-\Delta\widetilde{c} + \nabla \ln J \nabla \widetilde{c}) = 0.\end{gathered} \tag{10.14}$$

From Eqs. (10.14) one can approximately find the functions $F$ and $G$, which are important for constructing approximate wave functions in the $N$-particle problem as $N \to \infty$ [70].

Let $\widehat{\rho}$ be the projection on the function $\psi$. Its kernel is $\widetilde{\rho}(x, y) = \psi(x)\psi^*(y)$, and its symbol is $\rho(x, p) = \psi(x)\widetilde{\psi}^*(p)e^{(i/\hbar)px}$. The operator $\widehat{\rho}$ satisfies the Wigner equation, which reduces to the Vlasov equation as $h \to 0$. The operator $\widehat{\sigma}$ with the kernel $F(x)\psi^*(y) + \psi(x)G(y)$ is equal to

$$\widehat{\sigma} = \widetilde{f}\widehat{\rho} + \widehat{\rho}\widetilde{g} \tag{10.15}$$

and satisfies the variational equation to the Wigner equation, which is reduced to the variational equation for the Vlasov equation (8.3). In Eq. (10.15) $\widetilde{f}$ and $\widetilde{g}$ are the operators of multiplication by the functions $\widetilde{f}$ and $\widetilde{g}$. We see that in the semiclassical approximation the symbol of $\sigma$ is $O(h)$, since

$$\widehat{\sigma} = [\widehat{\rho}; \widetilde{g}] + \hbar\, \widetilde{c}\, \widehat{\rho}$$

and

$$\sigma(x, p) \simeq \hbar\biggl(-i\sum_{a=1}^{n}\frac{\partial \rho}{\partial p_a}(x, p)\frac{\partial \widetilde{g}}{\partial x_a} + \widetilde{c}\rho\biggr).$$

Since $\rho$ is the $\delta_\Lambda$-function in the semiclassical approximation [69], the function $\sigma$ is actually the sum of the $\delta_\Lambda$-function and its derivative.

## 11. EXAMPLE OF SUPERFLUIDITY

Consider the case in which the ratio of the number of particles to the phase volume is finite (see [73]). In the model under consideration, the number of particles and the entropy are calculated for a finite number of particles and a finite phase volume. We consider the sum of $N$ Hamiltonians not interacting with one another. The formulas obtained provide a new interpretation to well-known formulas for ideal gas.

First, we consider a simple example in which the potential has the form of a well with two minima symmetric with respect to the ordinate axis. (Such a potential is called the "Lifshits double-well potential.") The eigenfunctions in this potential well are symmetric with respect to the origin. They are either even or odd. The squares of these eigenfunctions are symmetric with respect to the barrier separating the minima. The symmetry is preserved as the barrier height tends to infinity. Consequently, the squared eigenfunctions and hence the probabilities of being in a well are the same for both wells.

If, instead of letting the barrier height tend to infinity, we let the parameter $h$ (the Planck constant) tend to zero, i.e., if we consider the semiclassical limit, then the passages will be prohibited in the limit for the eigenfunctions with eigenvalues below the energy barrier. Nevertheless, the classical particle "sits" simultaneously in both wells, i.e., behaves as if it has split into two parts.

This can be understood as follows. Suppose that one of the wells contains a quantum particle and the barrier height is finite. Then in the course of time the particle can penetrate the barrier owing to resonance by virtue of the tunnel effect and eventually distributes itself uniformly in both wells. After that, if we let the barrier height tend to infinity, two limits can be considered, one with respect to time and the other with respect to the barrier height. These two limits do not commute. However, if first time and then the barrier height tend to infinity, then the probabilities that the particle be in either well are the same. Moreover, if the barrier is high, then the distance between the eigenvalues corresponding to resonance eigenfunctions will be very small. For the infinite height of the barrier, the spectrum is multiple (degenerate).

The same situation occurs in the case of a periodic potential. Suppose that the system itself resides on a circle equipped with a periodic potential with, say, $M$ wells. Then the particle sits in each of the $M$ wells with the same probability. Now if we pass to infinitely high barriers between the wells, the probabilities still remain equal. If we assume the eigenfunction to be normalized, so that the probability of sitting in any well is equal to the integral of the squared eigenfunction over the well, then this integral will be equal to $1/M$. In the limit of infinitely high barriers between the wells, the symmetry property is preserved (it continuously depends on the barrier height). Moreover, each eigenvalue has multiplicity $M$. Naturally, the same situation occurs in the three-dimensional case if one deals with a potential periodic in three directions. In other words, we have a finite crystal satisfying the periodic Born–Karman conditions, i.e., wrapped over itself into a torus. There is a periodic lattice on this torus. Thus, in the three-dimensional case, we, in general, obtain a substantially larger multiplicity, which is not necessarily the same in different wells, since in the three-dimensional case each well may possess its own multiplicity. This multiplicity is multiplied by the number of wells, or, as one says in the case of crystals, by the number of crystal cells.

We consider this very model, i.e., assume that the barrier between the cells is infinite and penetration from one cell to another is not possible. Nevertheless, the particle is assumed to sit in all cells simultaneously. In the framework of this model, we neglect the distance between the spectrum points owing to the tunnel phenomenon. These eigenvalues are indeed very close; namely, the distance between them is an exponentially decaying function of the barrier height, and so we can regard the eigenvalues as coinciding.

Suppose that some energy level $E$ is given. Consider the number of eigenvalues (or energy levels) that do not exceed $E$, counted with their multiplicities. This number is equal to the number of such levels in a single cell times the number $M$ of cells. Now we take the mean energy of this system. Neglecting a constant, it is equal to the sum of eigenvalues not exceeding $E$, counted with their multiplicities, i.e., to the "Spur" of the system with respect to the energy level $E$. The multiplicities of eigenvalues are equal to $M$ times the multiplicities of the respective eigenvalues in a single cell. The ratio of the "Spur" to the number of eigenvalues is the internal energy of the system. Since the number $M$ of cells cancels out in the computation of both mean and internal energy, it follows that internal energy is reduced to the internal energy of a single cell. In the classical limit, internal energy is given by the phase integral over the part of the well below the energy level $E$. The internal energy of a cell coincides with the phase volume, which is the phase integral over energies smaller than the given one. Thus, if the cell Hamiltonian has the form $p^2 + v(x)$, then the phase volume coincides with the integral over the domain $p^2 + v(x) \leq E$.

Accordingly, we consider a self-adjoint one-particle operator $\widehat{H}$ with $n$ distinct eigenvalues $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n$ of multiplicities $G_1, \dots, G_n$, respectively, such that $\sum G_i = M$ and construct the frequency probabilities $g_i = G_i/M$. The eigenfunctions of $\widehat{H}$ corresponding to the eigenvalue $\varepsilon_j$ will be denoted by $\varphi^j_{\alpha_j}(\xi)$, where $\alpha_i = 1, \dots, G_i$. We assume that the variable $\xi$ ranges over a finite discrete set.

Consider an ideal gas of $N$ such particles; it is described by the operator

$$\widehat{H}_N = \sum \widehat{H}(\xi_i).$$

Let $N_i$ be the number of particles corresponding to the eigenvalue $\varepsilon_i$, that is, residing at the energy level $\varepsilon_i$. The Hamiltonian $\widehat{H}_N$ acts on functions of the form $\prod_{i=1}^{N} \varphi_i(\xi_i)$ as follows:

$$\widehat{H}_N(\varphi_1(\xi_1) \cdot \dots \cdot \varphi_N(\xi_N)) = \widehat{H}(\varphi_1(\xi_1)) \cdot \varphi_2(\xi_2) \cdot \dots \cdot \varphi_N(\xi_N)$$

$$+\varphi_1(\xi_1)\cdot\widehat{H}(\varphi_2(\xi_2))\cdot\varphi_3(\xi_3)\cdot\dots\cdot\varphi_N(\xi_N)+\dots$$
$$+\varphi_1(\xi_1)\cdot\dots\cdot\varphi_{N-1}(\xi_{N-1})\cdot\widehat{H}(\varphi_N(\xi_N)). \tag{11.1}$$

Thus, functions of the form

$$\Psi_{\{j,\alpha\}}(\xi_1,\dots\xi_N)=\prod_{s=1}^{N}\varphi^{j_s}_{\alpha_{j_s}}(\xi_s), \tag{11.2}$$

where $\{j,\alpha\}$ stands for the finite sequence $(j_1,\alpha_{j_1};j_2,\alpha_{j_2};\dots;j_N,\alpha_{j_N})$ are eigenfunctions of the Hamiltonian operator $\widehat{H}_N$.

Note that the indices $j_s$, as well as $\alpha_{j_s}$, in (11.2) are not necessarily distinct.

Clearly, $\sum\limits_{i=1}^{n}N_i=N$. The eigenfunctions (11.2) correspond to the eigenvalue

$$\lambda_{\{N\}}=\sum_{i=1}^{n}N_i\varepsilon_i, \tag{11.3}$$

where $\{N\}=\{N_1,\dots,N_n\}$.

Now consider symmetric eigenfunctions of an ideal gas of $N$ particles.

The space of symmetric (bosonic) eigenfunctions of the operator $\widehat{H}_N$ corresponding to an eigenvalue $\lambda\{N\}$ is generated by functions of the form $\mathrm{Symm}_{\xi_1,\dots\xi_N}\,\Psi_{\{j,\alpha\}}(\xi_1,\dots\xi_N)$, where Symm is the operation of symmetrization with respect to $\xi_i$.

The order of arguments of the function to be symmetrized is not essential in the computation of the multiplicity of an eigenvalue. Hence it suffices to consider functions $\Psi_{\{j,\alpha\}}$ of the form

$$\widetilde{\Psi}_{\{j,\alpha\}}(\xi_1,\dots\xi_N)=(\varphi^1_{\alpha^1_1}(\xi_1)\cdot\dots\cdot\varphi^1_{\alpha^{K_1}_1}(\xi_{K_1})\varphi^2_{\alpha^2_1}(\xi_{K_1+1})$$
$$\cdot\dots\cdot\varphi^2_{\alpha^{K_2}_2}(\xi_{K_1+K_2})\cdot\dots\cdot\varphi^n_{\alpha^1_n}(\xi_{K_1+K_2+\dots+K_{n-1}+1})\cdot\dots\cdot\varphi^n_{\alpha^{K_n}_n}(\xi_N) \tag{11.4}$$

Note that some of the indices $\alpha^i_j, j=1,\dots n,\ i=1,\dots N_j$, may occur several times, while some of the $\alpha_j=1,2,\dots G_j$ may be absent.

This product may well contain coinciding eigenfunctions of $\widehat{H}$. Consider the product

$$\varphi^1_1(\xi_1)\cdot\varphi^1_1(\xi_2)\cdot\dots\varphi^1_1(\xi_{l_1})\varphi^1_2(\xi_{l_1+1})\cdot\dots\cdot\varphi^1_2(\xi_{l_1+l_2})\varphi^1_3(\xi_{l_1+l_2+1})\cdot\dots\cdot\varphi^1_{g_1}(\xi_{l_1+\dots+l_{g_1}}) \tag{11.5}$$

where $0\le l_i\le N_1$ and $\sum\limits_{i=1}^{g_1}l_i=N_1$.

The number of all possible products of this form is equal to the number of ways in which $N_1$ functions can be distributed over $G_1$ classes. This number is equal to

$$C^{N_1}_{G_1+N_1+1}=\frac{(G_1+N_1-1)!}{N_1!(G_1-1)!}, \tag{11.6}$$

**Proposition 1.** *The multiplicity of the eigenvalue $\lambda\{N\}$ of symmetric eigenfunctions of the Hamiltonian operator $\widehat{H}_N$ is equal to*

$$\Gamma_{\{N_i\}\{G_i\}}=\prod_{i=1}^{n}\frac{(G_i+N_i-1)!}{N_i!(G_i-1)!},$$

*which is the frequency (nonnormalized) probability of the eigenvalue $\sum_1^n N_i\varepsilon_i$ of the operator $\widehat{H}_N$ by the above definition.*

As usual, we define the free energy of the ideal gas in question as

$$\min_{N_i}\Big\{\sum N_i\varepsilon_i - \theta\ln\Gamma_{\{N_i\}\{G_i\}}\Big\}, \tag{11.7}$$

where $\theta$ is the temperature and $S = \ln\Gamma_{\{N_i\}\{G_i\}}$ is the entropy. Thus we obtain the distribution of the ideal gas on a crystal base for finite $N$.

Now let us pass to the limit as $N \to \infty$ and $M \to \infty$ assuming that $\lim(N/M) = \rho$, where $\infty > \rho \geq 0$, and apply the Stirling formula.

In this case, if we define the frequency probabilities for $N$ particles as $N_i/N$, then for $p_i = \lim(N_i/N)$ and $g_i = \lim(G_i/M)$ we obtain the following value of specific entropy:

$$S = \sum_i (p_i + q_i\rho^{-1})\ln(p_i + q_i\rho^{-1}) - p_i\ln p_i - q_i\rho^{-1}\ln q_i\rho^{-1}. \tag{11.8}$$

Likewise, the specific entropy of a Fermi gas on a crystal base is equal to

$$S_F = \sum_i q_i\rho^{-1}\ln q_i\rho^{-1} - p_i\ln p_i - (q_i\rho^{-1} - p_i)\ln(q_i\rho^{-1} - p_i), \tag{11.9}$$

Here $p_i$ and $q_i$ are probabilities: $\sum p_i = 1$ and $\sum q_i = 1$. For $\rho \to 0$ they pass, modulo the constant $\ln\rho$, into the Shannon entropy $-p_i\ln(p_i/q_i)$. This limit shows the role of the probabilities $p_i$ and $q_i$ in information theory [74].

Instead of formula (11.7) for free energy, the author suggests considering a free energy *operator* of the form displayed below [27], [75]. Let $\delta(\lambda_n)$ be the multiplicity of an eigenvalue $\lambda_n$ of a self-adjoint operator $\widehat{A} = \int\lambda\,dE_\lambda$ (where $E_\lambda$ is the spectral family) with discrete spectrum in Hilbert space $H$. The free energy operator $\widehat{F}$ in the same space is defined by the formula

$$\widehat{F} = \int\{\lambda - \theta\ln\delta(\lambda)\}\,dE_\lambda.$$

In our case, $\widehat{A}$ is the operator $\widehat{H_N}$, and the minimum eigenvalue of $\widehat{F}$ coincides with (11.7), i.e., with the usual definition of free energy [16].

An analog of the minimum eigenvalue for the Schrödinger operator is its "lower" level, the "ground state," i.e., the state eventually reached by the system at very large times. (If a pendulum swings for a long time, it finally comes to rest.)

The discovery of superfluidity apparently refuted this postulate. At zero temperature, there still is motion, energy is not zero, and this state is preserved. (According to very recent computations, the motion ceases, but only in millions of years.)

Bogolyubov showed, using the many-particle Schrödinger equation as an example, that as $N \to \infty$ there exist metastable eigenvalues that should "live" sufficiently long and thus determine the superfluid state. This was done for zero temperature.

On the basis of the preceding, we give a model of the free energy operator showing that this operator can also have metastable eigenvalues.

As is customary in the physical theory of metals, we consider only two "zones," namely, two eigenvalues $\varepsilon_1$ and $\varepsilon_2$ of the one-particle Hamiltonian. The interaction will also be chosen to correspond to the physical problem [76].

As was already mentioned, in quantum mechanics the squared modulus of an eigenfunction is the coordinate probability density of a particle.

Since we consider distributions over eigenvalues (energy levels), it is important to know level-to-level transition probabilities.

Suppose that there are $N$ particles and

$$V_N^{(s)} = \sum H_i + \frac{1}{N}\sum\sum H_{i_1,i_2} + \frac{1}{N^3}\sum\sum\sum H_{i_1,i_2,i_3} + \cdots + \frac{1}{N^s}\sum\cdots\sum H_{i_1,\dots,i_s}$$

where $s$ is arbitrarily large but much less than $N$ and $H_{i_1,\dots,i_k}$ is an operator affecting only $k$ particles; for example, $N$,

$$H_{i_1 i_2} f(x_1, x_2 \dots, x_N) = H(x_i, x_j) f(x_1, x_2 \dots, x_N),$$

which means that this operator acts as the identity operator with respect to all coordinates except for $x_{i_1}$ and $x_{i_2}$.

*The probability of the transition of a particle from a state* $\lambda_k$ (*corresponding to the eigenvalue* $\lambda_k$ *of the operator* $H_N$) *to a state* $\lambda_{k'}$ (*corresponding to the eigenvalue* $\lambda_{k'}$) *under the action of the operator* $V_N$ *is defined* (*in quantum mechanics*) *as the squared modulus of the inner product* $(\Psi_{\lambda_k}, V_N \Psi_{\lambda_{k'}})$.

This expression is called a *matrix element*. This important physical notion essentially alters our understanding of standard notions concerning the spectrum of an operator in functional analysis: the spectrum is divided into *spectral series* [27], [77], [78].

There is a law in physics referred to as "energetically favorable states" (Bogolyubov): in the course of time, a particle subjected to "friction-like" perturbations loses energy and passes to a lower level. It cannot directly pass from a "state" at level $\lambda_i$ to the level $\lambda_{i-k}$, but it can first pass to the level $\lambda_{i-1}$, then to the level $\lambda_{i-2}$, etc., until it reaches the level $\lambda_{i-k}$ and eventually the least level $\lambda_1$, which is referred to as the "ground state" in physics.

To describe friction-like perturbations mathematically, we should set all matrix elements corresponding to transitions to higher energy levels (larger eigenvalues) equal to zero. Such a perturbation operator $\widetilde{V}_N^{(s)}$ is no longer self-adjoint.

Consider the following simple model: to reach a higher zone, any two particles release an energy quantum $\nu/N$, where

$$2(\varepsilon_2 - \varepsilon_1) > \nu > \varepsilon_2 - \varepsilon_1; \qquad \varepsilon_2 > \varepsilon_1. \tag{11.10}$$

Let $\mathcal{L}$ be a space of functions $\Psi(x_1, \dots, x_N)$ (anti)symmetric with respect to the discrete variables[10] $x_i$, $i = 1, \dots, N$, which range over the values

$$\frac{1}{M}, \frac{2}{M}, \dots, \frac{M-1}{M}, 1.$$

The Hamiltonian corresponding to this mathematical problem acts on elements of $\mathcal{L}$ as follows:

$$\widehat{H}_N \Psi(x_1, \dots, x_N) = \left( \sum_{i=1}^{N} \lambda(x_i) + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=i+1}^{N} V(x_i, x_j) \right) \Psi(x_1, \dots, x_N), \tag{11.11}$$

where $\lambda(x) = \varepsilon_1$ for $1/M \le x \le G_1/M$ and $\lambda(x) = \varepsilon_2$ for $(G_1+1)/M \le x \le 1$, moreover $V(x, x') = -V$,if $(G_1+1)/M \le x \le 1$ and $(G_1+1)/M \le x' \le 1$, while $V(x, x') = 0$ in all other cases.

We introduce functions $\varphi_i(x)$, $i = 1, \dots, G_1$ and $\phi_j(x)$, $j = 1, \dots, G_2$, where $G_2 = M - G_1$, as follows:

$$\varphi_i(x) = \begin{cases} 1 & \text{for } x = i/M \\ 0 & \text{for } x \ne i/M \end{cases}, \qquad \phi_j(x) = \begin{cases} 1 & \text{for } x = (j+G_1)/M \\ 0 & \text{for } x \ne (j+G_1)/M \end{cases}. \tag{11.12}$$

The eigenfunctions of the Hamiltonian (11.11) will be numbered by two sets $\{k\} = k_1, \dots, k_{G_1}$ and $\{l\} = l_1, \dots, l_{G_2}$ of nonnegative integers such that

$$\sum_{i=1}^{G_1} k_i + \sum_{j=1}^{G_2} l_j = N.$$

---

[10] From the viewpoint of physics, the problem is considered in the $p$-representation, and in physical notation we should have written the momentum $p_i$ instead of the coordinate $x_i$.

The eigenfunction $\Psi_{\{k\},\{l\}}$ can be expressed via the functions (11.12) as follows:

$$\begin{aligned}\Psi_{\{k\},\{l\}} = \operatorname{Symm}_{x_1,\dots,x_N}\Big(&\varphi_1(x_1)\dots\varphi_1(x_{k_1})\varphi_2(x_{k_1+1})\dots\varphi_2(x_{k_1+k_2})\dots\varphi_{G_1}(x_{N_1})\\ &\times\phi_1(x_{N_1+1})\dots\phi_1(x_{N_1+l_1})\phi_2(x_{N_1+k_1+1})\dots\phi_2(x_{N_1+k_1+k_2})\dots\phi_{G_2}(x_N)\Big),\end{aligned} \tag{11.13}$$

where $\operatorname{Symm}_{x_1,\dots,x_N}$ is the operator of symmetrization with respect to the variables $x_1,\dots,x_N$, and $N_1$ depends on $\{k\}$ according to the formula

$$N_1 = \sum_{i=1}^{G_1} k_i,$$

For a Fermi gas, we take the antisymmetrization operator Asymm instead of Symm. In what follows, we also use the notation $N_2 = N - N_1$. The eigenvalue of $\widehat{H}_N$ corresponding to the eigenfunction (11.13) has the form

$$\lambda(N_2) = \varepsilon_1 N + (\varepsilon_2 - \varepsilon_1)N_2 - \frac{VN_2(N_2-1)}{2N} \tag{11.14}$$

and its multiplicity in the symmetric case (a Bose gas) is equal to

$$\Gamma_B(N_2) = \frac{(N-N_2+G_1-1)!}{(N-N_2)!(G_1-1)!}\frac{(N_2+G_2-1)!}{N_2!(G_2-1)!}. \tag{11.15}$$

In the antisymmetric case (a Fermi gas), the multiplicity is

$$\Gamma_F(N_2) = \frac{G_1!}{(N-N_2)!(G_1-N+N_2)!}\frac{G_2!}{N_2!(G_2-N_2)!}. \tag{11.16}$$

Let us study the eigenvalues (11.14). It follows from their form that the following inequalities hold:

$$\begin{aligned}\lambda(N_2+1) &> \lambda(N_2) \qquad \text{for} \quad N_2 \le K-1,\\ \lambda(N_2+1) &< \lambda(N_2) \qquad \text{for} \quad N_2 \ge K,\end{aligned} \tag{11.17}$$

where $K$ is given by the formula

$$K = \frac{\varepsilon_2 - \varepsilon_1}{V} N. \tag{11.18}$$

Inequalities (11.17) imply that the eigenvalue $\lambda(0)$ is the (globally) minimal eigenvalue. Moreover, these inequalities imply that the eigenvalue $\lambda(N)$ is a locally minimum eigenvalue, since $\lambda(N_2) < \lambda(N)$ for all $N_2$ such that $K \le N_2 < N$.

Consider perturbations of the Hamiltonian $\widehat{H}_N$ by self-adjoint operators $H_s$ acting in $\mathcal{L}$ as the operators of multiplication by the functions

$$H_s(x_1,\dots,x_2) = \sum_{1\le i_1\ne i_2\ne\dots\ne i_s\le N} U_s(x_{i_1}, x_{i_2},\dots,x_{i_p}), \tag{11.19}$$

where $s \ge 1$, and the $U_s$ are real functions. Such operators $H_s$ are $s$-particle interaction operators. One can readily see that the matrix elements of such operators with respect to the eigenfunctions (11.13) satisfy the relation

$$\big(\Psi_{\{k\},\{l\}}, H_s\Psi_{\{k'\},\{l'\}}\big) = 0 \qquad \text{for} \quad \sum_{i=1}^{G_1}|k_i - k_i'| + \sum_{j=1}^{G_2}|l_j - l_j'| > 2s. \tag{11.20}$$

Since $s = O(1)$, we see that (11.20) and (11.17) imply the following assertion.

*Under $s$-particle perturbations, the eigenfunctions corresponding to global and local minima of eigenvalues can pass only to states with larger eigenvalues.*

By Landau's approach, this means that a local minimum state is a superfluid state.

Proceeding to another "specific" scale, we have

$$\lambda_p = \frac{\lambda_(N_2)}{N} = \varepsilon_1(1-p) + \varepsilon_2 p - \nu\frac{p^2}{2} - \frac{p}{2N}, \qquad p = \frac{N_2}{N}, \quad 0 \le p \le 1.$$

The last term tends to zero as $N \to \infty$; hence we neglect it. The spectrum is continuous in this limit.

The ground state (global minimum) is $\bar{\lambda}_0 = \varepsilon_1$. The maximum $\lambda_p$ corresponds to $p = \frac{\varepsilon_2 - \varepsilon_1}{\nu}$. There is also another minimum $\lambda_1$ for $p = 1$, which is equal to $\varepsilon_2 - \frac{\nu}{2}$.

Let us apply a friction-like perturbation $\widetilde{V}_s$ to this problem. Knowing the eigenfunctions (11.5), one can readily show that under this perturbation some of the eigenvalues (particles at high energy levels) gradually pass via the nearest eigenvalues to the absolute minimum $\lambda_0$, while the others pass to the local minimum $\lambda_1$. The transition from $\lambda_1$ to $\lambda_0$ is forbidden asymptotically (for $N \gg 1$). Thus, a particle can spend a rather long time at the level $\lambda_1 > \lambda_0$. In the language of physics, $\lambda_1$ is a metastable state. In contrast with $\lambda_0$, the level $\lambda_1$ corresponds to nonzero energy, and particles with nonzero (kinetic) energy move at some nonzero velocity.

Now let us assume that the perturbation of the Hamiltonian $\widehat{H}_N$ is an $s$-particle nonself-adjoint operator (friction operator) $\widehat{V}_s$ admitting only transitions to levels with lower eigenvalues. Then it follows from (11.20) and inequality (11.17) that there are two classes of eigenfunctions (11.13).

*The eigenfunctions* $\Psi_{\{k\},\{l\}}$ *satisfying the inequality*

$$N_2 = \sum_{j=1}^{G_2} l_j \le N\lambda_{p_1} - S - 1$$

*are eventually taken by this perturbation to the global minimum state, and the eigenfunctions satisfying the inequality*

$$N_2 = \sum_{j=1}^{G_2} l_j \ge N\lambda_{p_1} + S,$$

(*i.e.,* $N_2/N > \lambda_{p_1} + O(1/N)$) *are taken to the local minimum.*

Let $\lambda(N_2, G_1, G_2)$ be eigenvalues of the free energy operator $\widehat{F}$, let $\Lambda(p,q) = \lambda/N$, and let

$$\Lambda_F = \varepsilon_1(1-p) + \varepsilon_2 p - \nu\frac{p^2}{2} - \theta \ln \Gamma_B, \qquad 0 \le p \le 1.$$

For a Fermi gas, $\ln \Gamma_B$ is replaced by $\ln \Gamma_F$. There are three critical points of $\Lambda_F$, determined by the equation $\partial\Lambda_F/\partial p = 0$, for sufficiently small $\theta$; the first is a point of minimum, the second, a point of local minimum, and the third, a point of maximum:

$$p_1 \approx \frac{q_2}{\rho + 1 - q_2} \exp\Big(-\frac{\varepsilon}{\theta}\Big), \qquad p_2 \approx 1 - \frac{1-q_2}{\rho + q_2} \exp\Big(-\frac{\nu - \varepsilon}{\theta}\Big),$$
$$p_3 \approx \frac{\varepsilon}{\nu} + \frac{\theta}{\nu} \ln \frac{\varepsilon}{\nu - \varepsilon} \frac{\nu(1-q_2) + \rho(\nu - \varepsilon)}{\nu q_2}, \qquad \varepsilon = \varepsilon_2 - \varepsilon_1 \tag{11.21}$$

(for a Bose gas).

The eigenfunctions of the free energy operator coincide with those of the Hamiltonian $H_N$. Consider the eigenvalues of the free energy operator $\widehat{F}$ and the same friction operator $\widetilde{V}_s$, i.e., an operator whose matrix elements with respect to the eigenfunctions (11.5) are zero for the transitions from lower to higher eigenvalues of the free energy operator (see above). Then, as a result of multiple applications of the friction operator, the eigenvalues of the free energy operator for $p > p_3$ tend to a local minimum, which is a metastable superfluid free energy state.

The state $p_2$ is also locally stable (metastable); this means that $\lambda_F$ can only increase under perturbations by the operator $\alpha H_s$, where $\alpha$ is sufficiently small.

The critical temperature values at which phase transitions occur are determined by the equations $\partial f/\partial p = 0$ and $\partial^2 f/\partial p^2 = 0$.

A critical point occurs where the entropy curves $S_1(\theta)$ and $S_2(\theta)$ corresponding to the point of maximum and a point of local minimum meet. On the left of this point $\theta_c$, the electric capacity $\partial S/\partial\theta$ tends to $\infty$ as $(\theta_1 - \theta_0)^{-1/2}$. In the language of physics, one deals with a phase transition with critical exponent 1/2. Note that the curve $\theta(S)$ is smooth and makes a turn at the critical point, where $S$ plays the role of momentum and $\theta$ plays the role of coordinate (see [79]). Since this is a simple turning point, we can follow [79] and apply the tunnel canonical operator, which, in this case, precisely gives the Airy function.

Beyond the critical point, metastability (superfluidity) disappears, deceleration begins, and eventually the particles, influenced by the friction operator, reach the global minimum, i.e., come to a halt.

Thus, it follows from our model that if one heats a system in a metastable (superfluid) state, then a transition to a "normal" state (a state with friction) occurs at temperature $\theta_0$.

Let us summarize.

The semiclassical asymptotics as $h \to 0$ of such a quantum problem already does not coincide with the initial classical self-consistent Vlasov-type equation but contains a semiclassical correction when the complex germ method is applied; this correction was first obtained in Bogolyubov's work about weakly nonideal Bose gas and cannot be neglected as $h \to 0$.

It was shown above that, in the case of a repulsive potential (i.e., in the region of supercritical temperature $T > T_c$), there are particles of two types, namely, "superfluid" and normal particles.

Thus, the two-liquid situation arises in the region $T > T_c$, and moreover, particles of one liquid pass without collisions through particles of the other liquid. Probably, these are clusters and monomers, and the supercritical "liquid-gas" is not transparent because of the clusters, but the monomers pass through them practically without retardation.

One can show that the complex germ method is a generalization of the Bogolyubov $(u - v)$-transformation to the case of, for example, focal points and other singularities. In particular, the author succeeded in generalizing the well-known Bogolyubov relation [80] to the case where an external field $v(x)$ is present (see below). It turned out that the transition as $h \to 0$, i.e., the result of transition to classical mechanics, contains an additional term generalizing the Bogolyubov formula $(6')$ given in [80], and this term remains in the classical limit, because, although the value $h$ is infinitely small, it is multiplied by a constant that is generally unbounded. Thus, in this case, the parameter $h$ is included in the limiting classical picture (also see [62], [61]).

The Lagrangean manifold $\Lambda^n = \{x = X(\alpha),\, p = P(\alpha)\}$ is invariant under the Hamiltonian system

$$\dot{x} = \frac{\partial H(x,p)}{\partial p}, \qquad \dot{p} = \frac{\partial H(x,p)}{\partial x}, \tag{11.22}$$

where

$$H(x,p) = H_0(x,p) + \int d\mu_\alpha\, H_1(x,p;X(\alpha),P(\alpha)).$$

Here $H_0(x,p_x) = p_x^2/2 + V(x)$ and $H_1(x,p_x;y,p_y) = U(x,y)$, $\alpha \in \Lambda^n$, and $d\mu_\alpha$ is an invariant measure on $\Lambda^n$. The Lagrangian manifold lies on the surface $H(x,p) = \Omega$. If $\Lambda^n$ can diffeomorphically be projected on the $x$-plane, then the canonical operator acts as multiplication by $\exp\{(i/h)S(x)\}/\sqrt{J}$, where

$$S(x) = \int p\,dx \qquad \text{on} \quad \Lambda^n$$

and $J = Dx/D\mu_\alpha$. The leading term of the formula in the presence of an external field gives the following system for the functions $\widetilde{g}$ and $\widetilde{c}$ (see Section 10):

$$\begin{gathered}(i\nabla S\nabla - \widetilde{\beta})\widetilde{g} + \int U(x,X(\alpha'))\widetilde{c}(X(\alpha'))\,d\mu_{\alpha'} + \frac{h}{2}(-\Delta\widetilde{g} + \nabla \ln J\nabla\widetilde{g}) = 0,\\ (i\nabla S\nabla - \widetilde{\beta})\widetilde{c} - \Delta\widetilde{g} + \nabla\ln J\nabla\widetilde{g} - \frac{h}{2}(-\Delta\widetilde{c} + \nabla\ln J\nabla\widetilde{c}) = 0.\end{gathered} \tag{11.23}$$

**Superfluidity in the classical problem.** Let us give examples for the notion "series of eigenvalues" defined by the author in [81], [82].

First, consider the following illustrative examples.

**Example 3.** Consider the Laplace operator with Dirichlet conditions on the boundary and assume that the domain consists of two nonsymmetric nonintersecting domains. Let

$$\Delta U(x) = \lambda U(x), \qquad U(x)|_{\Gamma} = 0$$

and

$$\Gamma = \Gamma_1 \oplus \Gamma_2,$$

where $\Gamma$ is the boundary.

Obviously, this operator splits into two different operators. Nevertheless, if they are considered as a single operator, there arise two different series of eigenvalues and corresponding eigenfunctions $U_m$ and $\widetilde{U}_n$. Then, for any function $f(x)$ belonging to the domain that includes both boundaries, we have

$$(U_m, f(x)\widetilde{U}_n) = 0, \qquad f(x) = L_2(R^2). \tag{11.24}$$

Then $\{\widetilde{U}_n\}$ and $\{U_m\}$ are series.

**Example 4.** Consider the one-dimensional Schrödinger equation

$$-\frac{h^2}{2m}\frac{d^2}{dx^2}\Psi + U(x)\Psi = E\Psi \tag{11.25}$$

with potential $U(x)$, which can be regarded as having two nonsymmetric gaps (wells) such that the barrier between them goes to infinity. In this case, the problem can also be divided into two different Schrödinger operators with two different potential gaps,

$$E_n = \{E'_n \cup E''_n\},$$

where $E'_n$ and $E''_n$ are series.

The exact series $E'_n$ and $E''_n$ go to infinity. The asymptotic series $E'_n$ and $E''_n$ do not go to infinity.

**Example 5.** Assume that the barrier between the gaps in the preceding example is not infinite but sufficiently high. Then we can speak of asymptotic series where the asymptotics is taken for the high barrier. For the eigenfunctions corresponding to the eigenvalues lying inside each of the gaps, the semiclassical asymptotics also does not allow them to penetrate through the barrier if there is no resonance between the eigenfunctions.

Thus, we have considered several examples of different series corresponding to the same Hamiltonian.

In the physics encyclopedia [83], the following definition of spectral series appears. "Spectral series are groups of spectral lines arising in allowed transitions from different levels to the same final level. The lines in spectral series tend to a limit or to the boundary of the spectral series. The observed set of spectral lines is most clearly divided into spectral series in the spectra of elements of the first subgroups of the Mendeleev periodic system, i.e., hydrogen, helium, alkaline, and alkaline-earth metals."

One can see that this definition is very close to the Landau concept of the existence of friction. The rigorous mathematical definition of asymptotic spectral series was given above.

## 12. SECOND QUANTIZATION OF DIMERS

Since the two-particle distribution function is described more rigorously by the true symbol (not the pseudosymbol) using the method of ultrasecond quantization [27], we first consider the operator $\widehat{H}$ equivalent to the system of multiparticle Newton equations with pair interaction.

Denote by $\widehat{H}$ such an operator:

$$\begin{aligned}\widehat{H} = i\int dpdq\;\Psi^+(p,q)\left(\frac{\partial U(q)}{\partial q}\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right)\Psi^-(p,q)\\ + i\int\!\!\int dp\,dq\,dp'dq'\;\Psi^+(p,q)\Psi^+(p',q')\left(\frac{\partial V(q,q')}{\partial q}\cdot\frac{\partial}{\partial p} + \frac{\partial V(q,q')}{\partial q'}\cdot\frac{\partial}{\partial p'}\right)\Psi^-(p',q')\Psi^-(p,q).\end{aligned} \tag{12.1}$$

Suppose that $\widehat{\rho}_B$ is the operator introduced in [27] and in [84, formula (19)].

Denote by

$$\mathcal{H}_B(b^*(\cdot),b(\cdot),B^*(\cdot),B(\cdot)) = \frac{Sp\widehat{\rho}_B\widehat{H}}{Sp\widehat{\rho}_B}$$

the pseudosymbol of this two-particle operator.

Here $\widehat{\rho}_B$ depends on the functions $b(x,s)$, $B(y,y')$:

$$\begin{aligned}\widehat{\rho}_B = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{k!M!(k+2M)!}\left(\sum_{s=0}^{\infty}\int\int dxdx'b(x,s)b^*(x',s)\widehat{\psi}^+(x)\widehat{\psi}^-(x')\right)^k\\ \times\left(\iint dy_1dy_2B(y_1,y_2)\widehat{\psi}^+(y_1)\widehat{\psi}^+(y_2)\right)^M\\ \times\left(\iint dy_1'dy_2'B(y_1',y_2')\widehat{\psi}^-(y_1')\widehat{\psi}^-(y_2')\right)^M\exp\left(-\int dz\widehat{\psi}^+(z)\widehat{\psi}^-(z)\right),\end{aligned} \tag{12.2}$$

where $\widehat{\psi}^+(x)$ and $\widehat{\psi}^-(x)$ are Bose creation and annihilation operators ordered in the sense of Wick [85].

Suppose that $\mathcal{F}$ is Fock bosonic space [85], $\widehat{b}^+(x,s)$ is the creation operator of particles with number $s$, $\widehat{b}^-(x,s)$ is the annihilation operator of particles with number $s$ in the space $\mathcal{F}$, $\widehat{B}^+(x,x')$ and $\widehat{B}^-(x,x')$ are the creation and annihilation operators, respectively, of a pair of particles. These operators satisfy the following commutation relations for Bose particles $b^-$ and $b^+$

$$\begin{aligned}&[\widehat{b}^-(x,s),\widehat{b}^+(x',s')] = \delta_{ss'}\delta(x-x'), \qquad [\widehat{b}^\pm(x,s),\widehat{b}^\pm(x',s')] = 0,\\ &[\widehat{B}^-(x_1,x_2)\widehat{B}^+(x_1',x_2')] = \delta(x_1-x_1')\delta(x_2-x_2'),\\ &[\widehat{B}^\pm(x_1,x_2),\widehat{B}^\pm(x_1',x_2')] = 0,\\ &[\widehat{b}^\pm(x,s),\widehat{B}^\pm(x_1',x_2')] = [\widehat{b}^\pm(x,s),\widehat{B}^\mp(x_1',x_2')] = 0.\end{aligned} \tag{12.3}$$

These operators must be substituted into the symbol $\mathcal{H}_B(b^*(\cdot),b(\cdot),B^*(\cdot),B(\cdot))$ to obtain the operator $\widehat{\mathcal{H}}_B$, and then to calculate its true symbol, thus obtaining self-consistent equations for the two-particle distribution functions (semidensities).

Consider the case of ultrasecond quantization only for pairs. In this case, the true symbol is the following functional:

$$\begin{aligned}\sqrt{\mathcal{H}}\left[\Phi^+(\cdot),\Phi(\cdot)\right] = i\int\!\!\int dpdqdp'dq'\;\Phi^+(p,q,p',q')\\ \times\left(\frac{\partial U(q)}{\partial q}\frac{\partial}{\partial p} + \frac{\partial U(q')}{\partial q'}\frac{\partial}{\partial p'} - p\frac{\partial}{\partial q} - p'\frac{\partial}{\partial q'}\right)\Phi(p,q,p',q')\\ + \frac{i\alpha}{2}\int\dots\int dpdqdp'dq'dp_1dq_1dp_1'dq_1'\Phi^+(p,q,p',q')\Phi^+(p_1,q_1,p_1',q_1')\end{aligned}$$

$$
\begin{aligned}
&\quad\times\biggl(\frac{\partial V(q,q_1)}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q_1)}{\partial q_1}\frac{\partial}{\partial p_1}+\frac{\partial V(q,q_1')}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q_1')}{\partial q_1'}\frac{\partial}{\partial p_1'}\\
&\qquad+\frac{\partial V(q',q_1)}{\partial q'}\frac{\partial}{\partial p'}+\frac{\partial V(q',q_1)}{\partial q_1}\frac{\partial}{\partial p_1}+\frac{\partial V(q',q_1')}{\partial q'}\frac{\partial}{\partial p'}+\frac{\partial V(q',q_1')}{\partial q_1'}\frac{\partial}{\partial p_1'}\biggr)\\
&\quad\times\Phi(p_1,q_1,p_1',q_1')\Phi(p,q,p',q')\\
&+\frac{i(1-\alpha)}{2}\int\dots\int dpdqdp'dq'dp_1dq_1dp_1'dq_1'\Phi^+(p,q,p_1',q_1')\Phi^+(p_1,q_1,p',q')\\
&\quad\times\biggl(\biggl(\frac{\partial V(q,q_1)}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q_1)}{\partial q_1}\frac{\partial}{\partial p_1}\biggr)\Phi(p_1,q_1,p_1',q_1')\Phi(p,q,p',q')\\
&+\biggl(\frac{\partial V(q_1',q_1)}{\partial q_1'}\frac{\partial}{\partial p_1'}+\frac{\partial V(q_1',q_1)}{\partial q_1}\frac{\partial}{\partial p_1}\biggr)\Phi(p_1,q_1,p,q)\Phi(p',q',p_1',q_1')\\
&+\biggl(\frac{\partial V(q,q')}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q')}{\partial q'}\frac{\partial}{\partial p'}\biggr)\Phi(p',q',p_1',q_1')\Phi(p,q,p_1,q_1)\\
&+\biggl(\frac{\partial V(q',q_1')}{\partial q'}\frac{\partial}{\partial p'}+\frac{\partial V(q',q_1')}{\partial q_1'}\frac{\partial}{\partial p_1'}\biggr)\Phi(p_1,q_1,p_1',q_1')\Phi(p,q,p',q')\biggr),
\end{aligned}
\tag{12.4}
$$

where $\alpha$ is a temperature-dependent parameter.

The corresponding system of self-consistent equations for the two-particle distribution functions (semidensities) is of the form

$$
\begin{aligned}
\frac{\partial\Phi}{\partial t}(p,q,p',q',t)&=\biggl(\frac{\partial U(q)}{\partial q}\frac{\partial}{\partial p}-p\frac{\partial}{\partial q}+\frac{\partial U(q')}{\partial q'}\frac{\partial}{\partial p'}-p'\frac{\partial}{\partial q'}\biggr)\Phi(p,q,p',q',t)\\
&+2\alpha\iint dp_1dq_1dp_1'dq_1'\Phi^+(p_1,q_1,p_1',q_1',t)\\
&\quad\times\biggl(\frac{\partial V(q,q_1)}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q_1)}{\partial q_1}\frac{\partial}{\partial p_1}+\frac{\partial V(q_1,q')}{\partial q_1}\frac{\partial}{\partial p_1}+\frac{\partial V(q_1,q')}{\partial q'}\frac{\partial}{\partial p'}\biggr)\\
&\quad\times\Phi(p',q',p,q,t)\Phi(p_1',q_1',p_1,q_1,t)\\
&+2(1-\alpha)\iint dp_1dq_1dp_1'dq_1'\Phi^+(p_1,q_1,p_1',q_1',t)\\
&\quad\times\biggl(\biggl(\frac{\partial V(q,q_1)}{\partial q}\frac{\partial}{\partial p}+\frac{\partial V(q,q_1)}{\partial q_1}\frac{\partial}{\partial p_1}\biggr)\Phi(p_1,q_1,p',q',t)\Phi(p,q,p_1',q_1',t)\\
&+\biggl(\frac{\partial V(q',q_1)}{\partial q'}\frac{\partial}{\partial p'}+\frac{\partial V(q',q_1)}{\partial q_1}\frac{\partial}{\partial p_1}\biggr)\Phi(p_1,q_1,p,q,t)\Phi(p',q',p_1',q_1',t)\biggr).
\end{aligned}
\tag{12.5}
$$

The second equation is obtained from (12.5) by replacing $\Phi(p,q,p',q',t)$ and $\Phi^+(p,q,p',q',t)$ by $\Phi^+(p,q,p',q',t)$ and $\Phi(p,q,p',q',t)$,respectively. The functions $\Phi(p,q,p',q',t)$ and $\Phi^+(p,q,p',q',t)$ ) are symmetric with respect to the permutation of the pairs $p,q$ and $p',q'$.

**Proposition 2.** *The integral*

$$\int\Phi(p,q,p',q',t)\Phi^+(p,q,p',q',t)\,dp\,dq\,dp'\,dq'$$

*is independent of time* $t$.

The proof is obtained by a direct verification.

Of course, $\int\rho(p,q,t)\,dp\,dq$ is also preserved. However, after "condensation," the number of particles in the gas is not equal to the integral over all energies, and only over energies ranging from some $E_l$ to infinity; hence the one-particle "gas" formula for the distribution and the number of particles in the gas may depend on time. On the other hand, the one-particle distribution in the liquid does not coincide with the distribution in the gas because of the presence, in the liquid, of a great number of quantized vortices (the rotational components of the molecules increase essentially), and, in view of Proposition 2, we can only guarantee the conservation of the total number of particles.

The most important property of the distribution corresponding to UD statistics is that it makes possible the construction of creation and annihilation operators for individual particles as well as creation and annihilation operators for dimers.

Essentially, Vlasov's self-consistent field equation implicitly uses UD statistics: the distribution for every given particle is simultaneously the distribution for any particle. The Vlasov equations were derived 10 years after the corresponding Hartree equations of quantum mechanics. This shows that universal statistics appeared earlier than UD statistics. Vlasov's theory was intuitively based on his statement about the straight line passing through 2 points, which was mentioned above. It was strongly objected to by the leading physicists of the time and could have had dire consequences for him.

In his famous work [80] on the weakly nonideal Bose gas, Bogolyubov essentially obtained a term that does not disappear in the quasiclassical limit, but is added to the Vlasov equation of collective oscillations.

**Remark 8.** Bogolyubov's dispersion is of the form (the mass is put equal to 1)

$$\sqrt{k^r\widetilde{V}+\frac{k^4\hbar^2}{4}}, \tag{12.6}$$

where $k$ is the wave vector, $V(x-y)$ is the interaction potential, and $\widetilde{V}$ is the Fourier transform.

Apparently, as $\hbar \to 0$, the second term under the root sign must disappear. However, the value of $k$ can be arbitrarily large and this obstructs the passage to the limit. On the other hand, if the interaction potential $V(x-y)$ is zero, then the term $k^2\hbar/2$ arises and it is further multiplied by $\hbar$. The kinetic energy $k^2\hbar^2/2$ naturally does not disappear in the semiclassical limit. Therefore, the term containing $\hbar$ remains in the classical limit. It is this term that is responsible for Bogolyubov superfluidity. The fact that this term does not disappear in the classical limit is not obvious in Bogolyubov's papers, because there is no external field (external potential $u(x)$) in his work. In this case, as a rule, the semiclassical solution coincides with the exact solution. Therefore, we shall introduce the external potential in its most general form and calculate this correction.

**Remark 9.** In economics, both self-consistent equations and their linearization arise. The method in which we pass to the limit as the parameter $\hbar$ tends to zero can yield the most important correction in the corresponding linearized problems and this shows the way for solving these equations in general form.

## 13. NANODISTRIBUTION

The following question arises: If helium at low temperature consists of normal and superfluid liquid, then its superfluidity also occurs in a capillary (in Bogolyubov's famous paper of 1947) [80], superfluidity was established for a torus without taking the boundary conditions into account.)

In our case, in order to believe in such an effect in the supercritical part of any pure gas, we must establish the existence of the same superfluidity in a capillary. We shall show this to be true and also prove the fact that superfluidity occurs in a very narrow nanocapillary in what follows.

When the author first produced his proof, his friends, specialists in physics, were quite bewildered. It turns out that such superfluidity is observed experimentally. However, this effect was unexpected for physicists, who often do not attach too much significance to experimental data. The fact that, in this case, the theory and experiment are in agreement was not very convincing for physicists (the author and the experimenters did not know each other's works at that time). Since it is hard for experimenters to understand the author's mathematical works, we shall try to explain in a more convincing manner the effect in question.

The important tool used by Bogolyubov in his paper on the superfluidity of weakly nonideal Bose gas was the second quantization method. In this method, Fock space and creation and annihilation operators are used and a chain of equations for $N$ particles, where $N$ varies up to infinity, is considered. This chain is rewritten in the form of creation and annihilation operators as a single operator equation.

The projection operator $\widehat{N}$ of the number of particles is introduced; this operator projects the solution of the operator equation on the solution of the original $N$-particle equation.

In Landau's paper [86], the following remarkable fact is mentioned: in the limit as $N \to \infty$, when the commutators of creation and annihilation operators vanish, the operator equation becomes a unitarily nonlinear equation in three variables. Using the UD statistics, we can carry out this procedure for objectively distinguishable objects. The asymptotics for the obtained unitarily nonlinear equation is derived by using the complex germ method developed by the author in the general case.

It is sufficient to use these methods in order to obtain the superfluid component in supercritical thermodynamics, as described in Sec. 10. But these methods are inadequate for establishing superfluidity in a narrow capillary. It turns out that we must also introduce creation and annihilation operators for vapor dimers. This can be seen in the transition of cooled helium across the $\lambda$-point (see [87], [88]).

This procedure makes the asymptotic transition as $\mu \to \infty$ significantly more difficult, but, on the other hand, it allows us to take into account the boundary conditions for the unitarily nonlinear equation on the boundary of the capillary. A rigorous mathematical solution requires the use of number theory. One of the lemmas was proved by A. Karatsuba at my request; his proof occupied 18 pages. In this paper, we consider a less general problem, which made cumbersome calculations unnecessary.

How to construct the thermodynamics of nanostructures not from the physicist's, but from the mathematician's point of view? Under our approach, we must remove everything that contradicts the state of gas in a nanotube. Obviously, we mean cluster formation. In other words, the preference principle considered in Sec. 4 is no longer suitable here, and we must consider an analog of the Bose distribution without the preference principle. The parastatistics based on the creation of clusters also does not apply here.

Let us compare the Bose distribution with the picture constructed by physicists [40]. Consider Fig. 29.

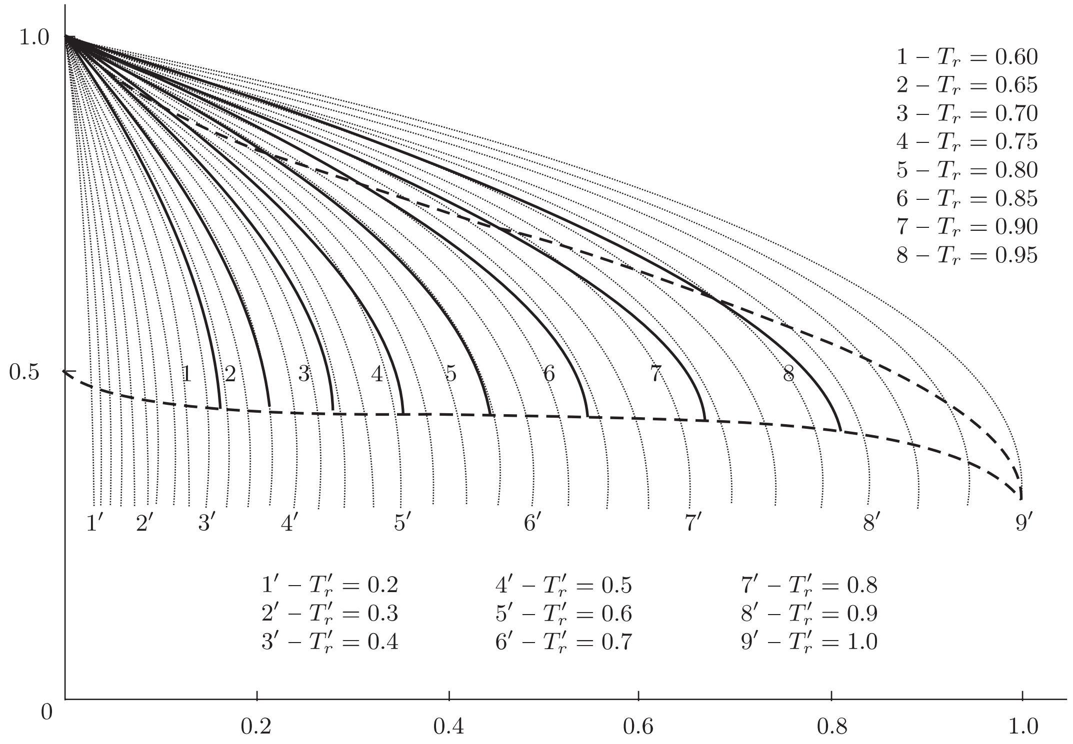

**Fig. 29.** Comparison of isotherms plotted from the Bose distribution (shown by the thin lines corresponding to the temperatures $T_r'$) with isotherms obtained in [40] (shown by the thick lines corresponding to the temperatures $T_r$).

It is seen that there is a very good agreement. However, the triple point in the nanotube is lower, and hence the constant $\lambda$ is very small. The recount of temperatures depends on the unknown spinodal, but the recounting method was described above.

For supercritical states, we shall use Bachinskii's formula for the Zeno line, which was obtained by the author for the Lennard-Jones potential [34]–[36].

Taking into account the Zeno line influences the form of the $\Omega$-potential as follows:

$$\Omega(\mu,T)=-\Lambda^{\gamma+1}V\varphi(V/V_0)\frac{T^{\gamma+2}}{\Gamma(\gamma+2)}\int_0^\infty\frac{t^{\gamma+1}\,dt}{(e^t/y)-1}=-\Lambda^{\gamma+1}T^{\gamma+2}V\varphi(V/V_0)\,\mathrm{Li}_{\gamma+2}(y),\qquad(13.1)$$

where $y=\exp(\mu/T)$ is the activity and $\mu$ stands for the chemical potential.

Let us write out the differential equations for $\varphi(x)$ with regard to the relations on the Zeno line,

$$T_z=T_B\left(1-\frac{\rho_z}{\rho_B}\right),\qquad P_z=\rho_zT_B\left(1-\frac{\rho_z}{\rho_B}\right),\qquad(13.2)$$

where the subscript $z$ means that the corresponding values are taken on the Zeno line, i.e., for $Z=1$.

Let us construct the relation $Z=1$ on the Zeno line. This relation is of the form

$$Z=\frac{\partial\Omega/\partial V}{T_z\partial\Omega/\partial\mu}=\frac{\varphi(V_z/V_0)+(V_z/V_0)\varphi'(V_z/V_0)}{\varphi(V_z/V_0)}\cdot\frac{\mathrm{Li}_{\gamma+2}(y_z)}{\mathrm{Li}_{\gamma+1}(y_z)}=1.\qquad(13.3)$$

It follows from (13.2) on the Zeno line that

$$T_z^{2+\gamma}[\varphi(V_z/V_0)+(V_z/V_0)\varphi'(V_z/V_0)]\,\mathrm{Li}_{\gamma+2}(y_z)=\left(\frac{N}{V_z}\right)T_B\left(1-\frac{N}{V_z\rho_B}\right).\qquad(13.4)$$

Assume that the conditions $N/V=\mathrm{const}$ and $N=\mathrm{const}$ hold on the isochore and on the Zeno line defined by relation (13.2). It follows from (13.3)and (13.4) that

$$T_B^{\gamma+1}\left(1-\frac{N}{V_z\rho_B}\right)^{\gamma+1}\varphi(V_z/V_0)\,\mathrm{Li}_{\gamma+1}(y_z)=\frac{N}{V_z\rho_B}.\qquad(13.5)$$

Here $N/V_z=\mathrm{const}$ and $N=\mathrm{const}$, and hence this equation contains the unknowns $y_z$, $V_z$, and $\varphi(V_z/V_0)$.

After finding the value $y_z$ as a function of $V_z$ and $\varphi(V_z/V_0)$, we substitute it into formula (13.3), thus transforming equation (13.3) into a differential equation for the function $\varphi(x)$ depending on the constants $\rho_B$ and $T_B$. The equation for $\varphi(V)$ enables one to find the point $y_z(\rho)$ on the Zeno line, and, after this, the isochore is changed only at the expense of the modification of activity $a=e^{-\mu/T}$ from $y_z(\rho)$ to $a=1$ for the polylogarithm $\mathrm{Li}_{\gamma+1}(a)$. The function $\varphi(V)$ is not reflected in the structure of the isochore, and it is reflected in scaling only (see Fig. 30).

The most important problem in the theory of differential equations, the general existence problem for a solution, remains open. Physicists pay less attention to this problem than mathematicians.

What are conditions for the existence of a solution of equations (13.3)-(13.4)?

Introduce the notation $T_s=T_{\text{standard}}=T/T_m$ and $P_s=P_{\text{standard}}=P/P_m$, where $T_m$ and $P_m$ are defined by the formulas

$$P_m=T_m^{\gamma+2},\qquad V_0=V_m=V_{\max},$$

and $Z_m=Z_{\max}$ and $\rho_m=\rho_{\min}$ are defined below.

Since $N_c$ does not depend on $T$, it follows that $V$ and $N$ are constant along the isochore $V=V_m$.

Let us write out the above relations at the point $T_s=1$, $P_s=1$:

$$[\varphi(V_z/V_0)+(V_z/V_0)\varphi'(V_z/V_0)]\zeta(\gamma+2)=1,\qquad V_m\varphi(1)\zeta(\gamma+1)=N_c,\qquad V_m=V_{\max},\qquad(13.6)$$

Hence

$$Z_m=\frac{V_m}{N_c}\frac{\zeta(\gamma+1)}{\zeta(\gamma+2)}\cdot\frac{\mathrm{Li}_{\gamma+2}(1)}{\mathrm{Li}_{\gamma+1}(1)}=\frac{V_m}{N_c}=\frac{1}{\rho_m}.\qquad(13.7)$$

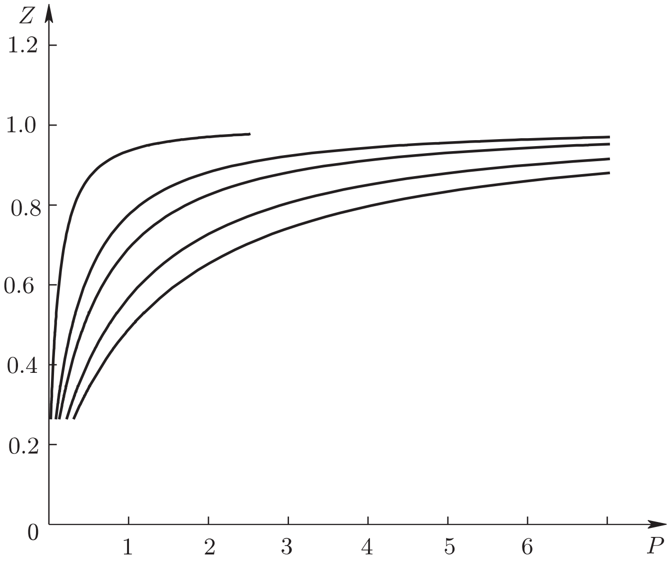

**Fig. 30.** The graph for the isochore of the polylogarithm $\mathrm{Li}_{\gamma+1}(a)$ for $\gamma = 2$.

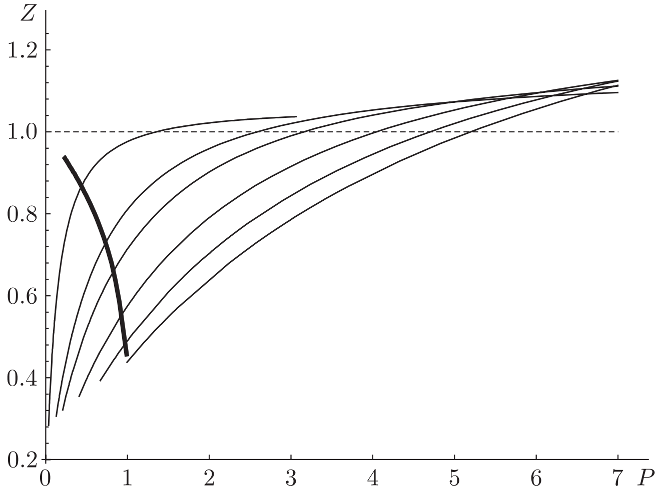

**Fig. 31.** The dotted line shows the Zeno line $Z = 1$. The bold line is the critical isotherm of a real gas (mercury) which is calculated theoretically, and the thin lines are isochores of mercury for $T < T_c$.

Since we construct isochores $V = \text{const}$ on the plane $\{Z, P\}$, it follows that $V_m = V_z$. Eliminating $(V_z\varphi(V_z/V_0)'$ from (13.6) by using (13.5), we obtain

$$T_B^{\gamma+1}\left(1-\frac{1}{Z_m\rho_B}\right)^{\gamma+1}\frac{1}{\zeta(\gamma+2)}\,\mathrm{Li}_{\gamma+2}(y_z)=\frac{1}{Z_m}. \tag{13.8}$$

Since $V_m = V_z$, we see that

$$\varphi(1)=N_c(\zeta(\gamma+1)V_m)^{-1}, \qquad \varphi(1)+\varphi'(1)=(\zeta(\gamma+2))^{-1}, \tag{13.9}$$

and it follows from (13.3) on the Zeno line that the following equation holds:

$$\frac{\mathrm{Li}_{\gamma+2}(y_z)}{\mathrm{Li}_{\gamma+1}(y_z)}\cdot\frac{V_m}{N_c}\cdot\frac{\zeta(\gamma+1)}{\zeta(\gamma+2)}=1. \tag{13.10}$$

Eliminating $y_z$, we find a relation for $Z_m$ and $\gamma$. The maximum value of $Z_m$ depends on the values of $\rho_B$ and $T_B$ only.

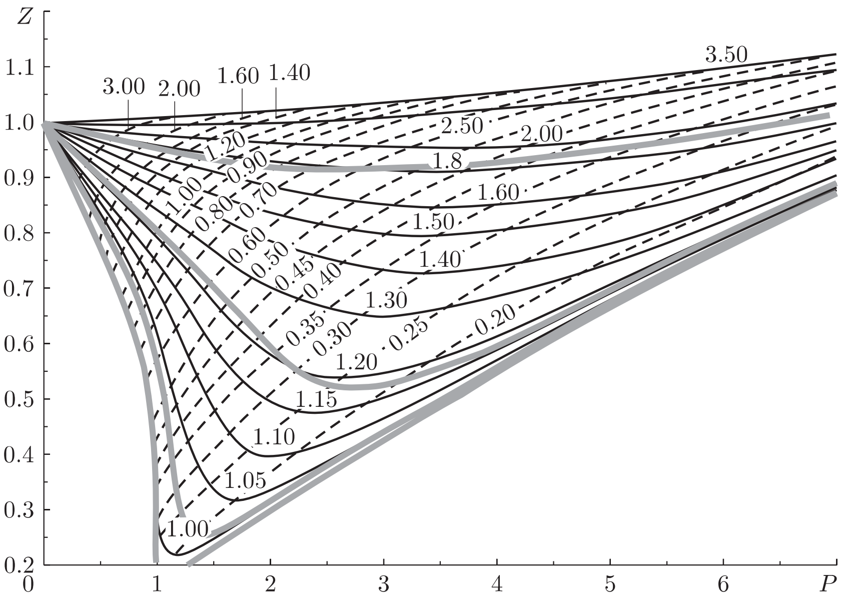

**Fig. 32.** Thin solid lines depict experimental isotherms for methane. Thick solid lines approximating experimental curves are plotted from theoretical data. The dotted lines show experimental isochores.

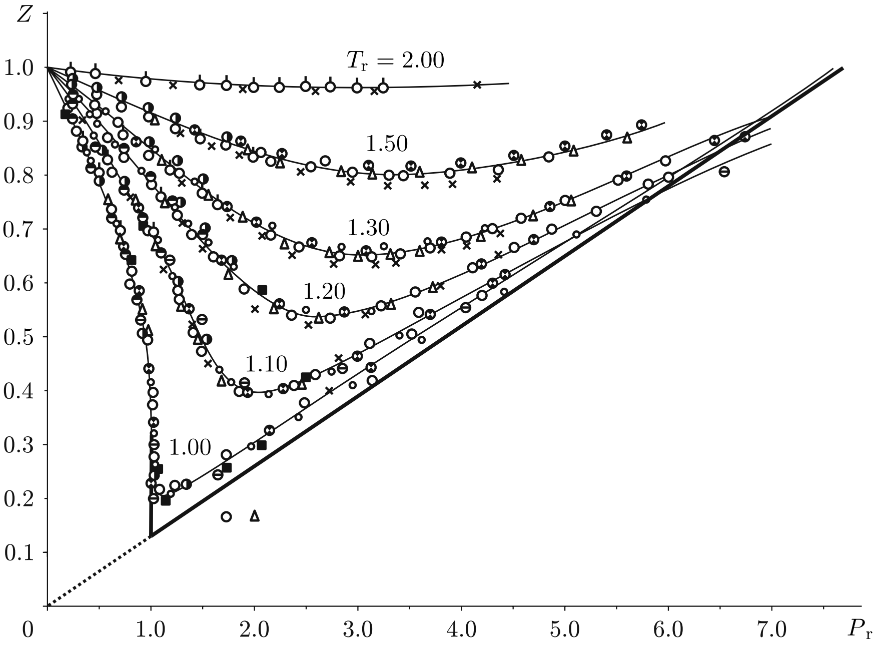

**Fig. 33.** The law of corresponding states for molecules of different gases. Thin lines depict isotherms of methane. Different symbols on the isotherms correspond to argon, carbon dioxide, water, etc. The fact the isotherms of different gases are close to one another illustrates the empirical law of corresponding states. The bold line depicts the theoretical critical isotherm.

The family of isochores, according to system (13.4)-(13.5) with the above initial condition (13.9), is shown in Fig. 31.

The first relation for the limit isochore $V_m/N_c = Z_m$, for $\gamma = \gamma_m$, is of the form

$$Z = Z_m \cdot \frac{\zeta(\gamma+1)}{\zeta(\gamma+2)} \cdot \frac{\mathrm{Li}_{\gamma+2}(y)}{\mathrm{Li}_{\gamma+1}(y)}, \qquad 1 \le y \le y_z(\gamma). \tag{13.11}$$

Since $P_s = T_s^{\gamma+2}\operatorname{Li}_{\gamma+2}(y)/\zeta(\gamma+2)$ and $N = T_s^{\gamma+1}\operatorname{Li}_{\gamma+1}(y)V_m$, it follows that the other relation is

$$P_s = \left(\frac{\zeta(\gamma+1)}{\operatorname{Li}_{\gamma+1}(y)}\right)^{(\gamma+2)/(\gamma+1)} \frac{\operatorname{Li}_{\gamma+2}(y)}{\zeta(\gamma+2)}, \qquad 1 \le y \le y_z(\gamma). \tag{13.12}$$

Equations (13.11) and (13.12) give an almost straight segment of the isochore.

Starting from $Z < 0.4$ (for example, for a van der Waals gas), the phase transition to an ideal liquid occurs.

**Remark 10.** Since the rightmost isochore (which is not shown in Fig. 31) is a segment of a straight line, it follows that all isochores of high density must also be line segments. They pass through the point $\rho > \rho_m$ on the Zeno line and the point $Z_0 = 1/\rho$ on the line $P_s = 1$. We thus obtain (when including the isochores shown in Fig. 31) a complete family of isochores for $Z \le 1$, $P \ge 1$. To any point of an isochore in the plane $\{Z, P\}$ there corresponds some temperature, and we construct isotherms that are well within the spread of gas isotherms satisfying the experimental law of corresponding states (see Fig. 32 and Fig. 33).

## 14. SUPERFLUIDITY OF CLASSICAL LIQUID IN A NANOTUBE FOR EVEN AND ODD NUMBERS OF NEUTRONS IN A MOLECULE

### *14.1. Preliminaries*

**1. The case of an even number of neutrons** We first note that solutions of the variational equation for the Vlasov equation do not coincide with the classical limit for the variational equations corresponding to the mean-field equations in quantum theory. We consider mean-field equations of the form

$$ih\frac{\partial}{\partial t}\varphi^t(x) = \left(-\frac{h^2}{2m}\Delta + W_t(x)\right)\varphi^t(x), \qquad W_t(x) = U(x) + \int dy\, V(x,y)|\varphi^t(y)|^2 \tag{14.1}$$

with the initial condition $\varphi|_{t=0} = \varphi_0$, where $\varphi_0 \in W_2^\infty(\mathbb{R}^\nu)$, $\int dx\,|\varphi_0(x)|^2 = 1$.

To find an asymptotic representation of the complex-germ type [62], we consider a system consisting of the Hartree equation and its conjugate equation. We then take the system of variational equations for it and replace the variations $\delta\varphi$ and $\delta\varphi^*$ with the independent functions $F$ and $G$. For $F$ and $G$, we obtain the system of equations

$$\begin{aligned} i\frac{\partial F^t(x)}{\partial t} &= \int dy\left(\frac{\delta^2 H}{\delta\varphi^*(x)\delta\varphi(y)}F^t(y) + \frac{\delta^2 H}{\delta\varphi^*(x)\delta^*\varphi(y)}G^t(y)\right), \\ -i\frac{\partial G^t(x)}{\partial t} &= \int dy\left(\frac{\delta^2 H}{\delta\varphi(x)\delta\varphi(y)}F^t(y) + \frac{\delta^2 H}{\delta\varphi(x)\delta^*\varphi(y)}G^t(y)\right). \end{aligned} \tag{14.2}$$

Roughly speaking, the classical equations can be obtained from the quantum ones using a substitution of the form $\varphi = \chi e^{iS/h}$ (the WKB method), $\varphi^* = \chi^* e^{-iS^*/h}$, $S = S^*$, $\chi = \chi(x,t) \in C^\infty$, $S = S(x,t) \in C^\infty$.

For variational equations, it is natural to vary not only the limit equation for $\chi$ and $\chi^*$ but also the functions $S$ and $S^*$. This gives an important new term in the solution of the equation for collective oscillations. We consider this fact for the simplest example investigated in the famous work by Bogolyubov on a weakly nonideal Bose gas [80].

Let $U = 0$ for Eq. (14.1) in a three-dimensional box with edge length $L$, and let the $L$-periodicity condition be imposed on the wave functions in this case (i.e., a problem on the torus with generators $L$, $L$, and $L$ is considered). Then the function

$$\varphi(x) = L^{-3/2}e^{i(px-\Omega t)/h} \tag{14.3}$$

with $p = 2\pi n/L$, where $n$ is an integer-valued vector, satisfies Eq. (14.1) with

$$\Omega = \frac{p^2}{2m} + L^{-3}\int dx\, V(x). \tag{14.4}$$

We consider functions $F^{(\lambda)}(x)$ and $G^{(\lambda)}(x)$, where $\lambda = 2\pi n/L$, $n \neq 0$, of the form

$$\begin{aligned} F^{(\lambda)t}(x) &= L^{-3/2}\rho_\lambda e^{i|(p+\lambda)x+(\beta-\Omega)t|/h}, \\ G^{(\lambda)t}(x) &= L^{-3/2}\sigma_\lambda e^{i|(-p+\lambda)x+(\beta+\Omega)t|/h}, \end{aligned} \tag{14.5}$$

where

$$\begin{gathered} -\beta_\lambda\rho_\lambda = \biggl(\frac{(p+\lambda)^2}{2m} - \frac{p^2}{2m} + \widetilde{V}_\lambda\biggr)\rho_\lambda + \widetilde{V}_\lambda\sigma_\lambda, \\ \beta_\lambda\rho_\lambda = \biggl(\frac{(p-\lambda)^2}{2m} - \frac{p^2}{2m} + \widetilde{V}_\lambda\biggr)\sigma_\lambda + \widetilde{V}_\lambda\rho_\lambda, \\ |\sigma_\lambda|^2 - |\rho_\lambda|^2 = 1, \qquad \widetilde{V}_\lambda = L^{-3}\int dx\, V(x)e^{i\lambda x/h}. \end{gathered} \tag{14.6}$$

From system (14.6), we obtain

$$\beta_\lambda = -p\lambda + \sqrt{\biggl(\frac{\lambda^2}{2m} + \widetilde{V}_\lambda\biggr)^2 - \widetilde{V}_\lambda^2}. \tag{14.7}$$

In this example, we have

$$u = e^{is(x,t)/h} \qquad \text{and} \qquad u^* = e^{-is(x,t)/h},$$

where $s(x,t,) = px + \beta t$, and the variation of the action for the vector $(\delta u, \delta u^*)$ is equal to $\lambda x \pm \Omega t$. A more thorough passage to the limit gives

$$\widetilde{V}_\lambda \to V_0 = L^{-3}\int dx\, V(x).$$

In the classical limit, we thus obtain the famous Bogolyubov relation (14.7). In this case, we have $u(x) = 0$, and the exact solution coincides with the classical one as in the linear Schrödinger equation. The situation with $u(x) \neq 0$ was investigated in [77], and it turns out that a relation similar to (14.7) is the classical limit as $h \to 0$ for the variational equation in this general case. The curve for the dependence of $\beta_\lambda$ on $\lambda$ is called the *Landau curve*, and it specifies the superfluid state. The value $\lambda_c$ at which the superfluidity disappears is called the *Landau criterion*. Bogolyubov explains the superfluidity phenomenon as follows: "the 'degenerate condensate' can move without friction relative to elementary excitations with an arbitrary sufficiently small velocity" (p. 210 in [80]).

But this mathematical consideration is not related to the Bose–Einstein condensate; it merely means that the quasiparticle spectrum determined for $\lambda < \lambda_c$ is positive. This means that it is metastable (see [61]). The Bose–Einstein condensate is mentioned here only to disprove the idea that it follows from what was said above that this consideration applies to a classical liquid.

Indeed, for example, the molecules of a classical undischarged liquid are Bose particles if the number of neutrons in the molecule is even. Because every particle (molecule) is neutral and is formed of an even number $l$ of neutrons, an $N$-particle equation can be written for this liquid. Thus, every $i$th particle is $3(2k+l)$-dimensional, where $k$ is the number of electrons; there is a dependence on the potential $u(x_i)$, $x_i \in \mathbb{R}^{6k+3l}$; and an equation for $N$ particles $x_i$, $i = 1, 2, \dots, N$, with a pair interaction potential $V(x_i - x_j)$ can be considered.

But Bogolyubov found only one series for the spectrum of the many-particle problem. As Landau wrote, "N. N. Bogolyubov recently managed to find the general form of the energy spectrum for the Bose–Einstein gas with a weak interaction between the particles using a clever application of second quantization" (p. 43 in [86]). But this series is not unique, i.e., it does not exhaust the whole energy spectrum.

In 2001, we suggested the ultrasecond quantization method [27] (also see [89]–[93]). The ultrasecond-quantized Schrödinger equations, like the second-quantized ones, represent the $N$-particle Schrödinger equation, and this means that the ultrasecond-quantized equation is essentially identical to the original $N$-particle equation: it coincides with the latter on the $3N$-dimensional space. But in contrast to the second-quantized case, replacing the creation and annihilation operators with $c$-numbers does not yet give the correct asymptotic representation; it turns out that its results coincide with those obtained by applying the Schröder variational principle or the Bogolyubov variational method.

For the Bardeen exotic potential, the correct asymptotic solution coincides with the one resulting from applying the abovementioned ultrasecond quantization method. For potentials of general form, in the case of pair interaction for example, the answer turns out to be different. In particular, the ultrasecond quantization method gives certain other asymptotic series of eigenvalues corresponding to the $N$-particle Schrödinger equation, which, in contrast to Bogolyubov series (14.7), are not metastable. They correspond to vortex filaments [94].

It turns out that the decisive factor here is not the Bose–Einstein condensate, but the thickness of the capillary (nanotube) in which the liquid flows. If we consider a liquid in a capillary or nanotube, then the velocity corresponding to metastable states is not small for a sufficiently small radius. Consequently, the liquid flows without friction for a smaller velocity.

The no-flow condition on the boundary of the nanotube (absence of flow) is the Dirichlet boundary condition or the Born–von Karman boundary condition. It generates a standing wave that can be interpreted as a particle–antiparticle pair: a particle with the momentum $p$ orthogonal to the tube wall and an antiparticle with the momentum $-p$.

In the boson case, we consider a short-range interaction potential $V(x_i - x_j)$. This means that only interaction with finitely many particles is possible as $N \to \infty$ ($N$ is the number of particles). Consequently, the potential depends on $N$ as $V_N = V((x_i - x_j)N^{1/3})$. If $V(y)$ is finitely supported in $\Omega_V$, then the number of particles captured by the support is independent of $N$ as $N \to \infty$. As result, superfluidity occurs for velocities less than $\min(\lambda_c, h/(2mR))$, where $R$ is the nanotube radius. The upper bound is determined by the condition that the radius of action of the molecule must be less than the radius of the nanotube.

We now present our own considerations that do not relate to the mathematical presentation. Viscosity is connected with collisions of particles: the higher the temperature, the greater the number of collisions. In a nanotube, there are few collisions because only those with the tube walls occur, which is taken into account by the series obtained below. Precisely this fact, rather than the presence of the Bose condensate, leads to the weakening of viscosity and consequently to superfluidity. In other words, even for liquid $\mathrm{He}^4$, the main factor in the superfluidity phenomenon is not the condensate but the presence of a thin capillary [95], [96].

**2. The case of an odd number of neutrons** In the case of an odd number of neutrons, the attractive, not repulsive, part of the potential is important, and a more precise knowledge of the behavior of the potential at infinity is therefore needed. To consider a system in which the number of neutrons in a molecule is odd and its spin is therefore equal to $1/2$, which leads to antisymmetry of eigenfunctions, we use an interaction potential ensuring repulsion at short distances and attraction at long distances. In particular, the interaction potential in the Fermi liquid $\mathrm{He}^3$ has this property.

The potential $V(\xi)$ is assumed to be spherically symmetric, whence it follows that its Fourier transform $\widetilde{V}(\xi)$ is also spherically symmetric. Next, we set a purely mathematical condition that we call the mirror property. Both the potential $V(\xi)$ and its Fourier transform $\widetilde{V}(\xi)$ must have a branching of the form $\sqrt{|x|} = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is even possible that this condition is unnecessary for the results presented below, but it appears in a wide variety of other problems and is profoundly significant for analytic continuation (see, e.g., [97]). This means that the potential $V(r)$ essentially depends on the first degree of the radius $r$ and its Fourier transform has the same property in relation to $|p|$.

Passing to spherical coordinates leads to the relation[11]

$$V(r) = \frac{4\pi}{r} \int_0^\infty \widetilde{V}(\xi) \sin(\xi r)\xi \, d\xi. \tag{14.8}$$

---

[11] M. V. Karasev comuunicated this formula to us.

Under the assumption that $\widetilde{V}(|\xi|)$ tends to infinity as $\mathrm{const}/|\xi|^2$, we obtain

$$V(r) = \frac{\mathrm{const}}{r} + O(1)$$

as $r \to 0$ and $V(r)$ tends to infinity as $\mathrm{const}/r^4$.

The mirror condition implies that $\widetilde{V}'_{|\xi|}(|\xi|)\big|_{|\xi|=0} \neq 0$. It can be easily verified that

$$\widetilde{V}_{|\xi|}\big|_{|\xi|=0} = -\frac{1}{8\pi}\lim_{r\to\infty} r^4 V(r).$$

As is known, the interaction potential decreases as $r^{-6}$ in a free space, but if an arbitrarily small radially symmetric potential is added, then the decrease is almost certainly of the order of $r^{-4}$ (see [98, 310], formula (6.170)).

### *14.2. Ultrasecond Quantization*

Ultrasecond quantization was introduced in [27], [99]–[101] for problems in quantum mechanics and statistical physics. We recall the notation and the main facts in the case of quantization by pairs of particle and particle index and by pairs of two particles. In what follows, quantization over pairs allows taking the pair correlations of particles into account when constructing asymptotic expansions. The ultrasecond-quantization space is the bosonic Fock space $\mathcal{F}$, $\hat{b}^+(x,s)$ and $\hat{b}^-(x,s)$ are the creation and annihilation operators for particles with the index $s$ in the space $\mathcal{F}$ [85], and $\widehat{B}^+(x,x')$ and $\widehat{B}^-(x,x')$ are the creation and annihilation operators for pairs of particles in $\mathcal{F}$. These operators satisfy the commutation relations

$$\begin{aligned}
&[\hat{b}^-(x,s),\hat{b}^+(x',s')] = \delta_{ss'}\delta(x-x'), \qquad [\hat{b}^\pm(x,s),\hat{b}^\pm(x',s')] = 0,\\
&[\widehat{B}^-(x_1,x_2),\widehat{B}^+(x_1',x_2')] = \delta(x_1-x_1')\delta(x_2-x_2'),\\
&[\widehat{B}^\pm(x_1,x_2),\widehat{B}^\pm(x_1',x_2')] = 0,\\
&[\hat{b}^\pm(x,s),\widehat{B}^\pm(x_1',x_2')] = [\hat{b}^\pm(x,s),\widehat{B}^\mp(x_1',x_2')] = 0.
\end{aligned} \tag{14.9}$$

Let $\Phi_0$ be the vacuum operator in $\mathcal{F}$ with the properties

$$\hat{b}^-(x,s)\Phi_0 = 0, \qquad \widehat{B}^-(x_1,x_2)\Phi_0 = 0. \tag{14.10}$$

The variable $x$ lies on a three-dimensional torus $L\times L\times L$, denoted by $\mathbf{T}$. The discrete variable $s$, $s = 0,1,\dots$, is called the index (number) or statistical spin. Any vector $\Phi$ in $\mathcal{F}$ is uniquely representable in the form

$$\begin{aligned}
\Phi = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}\\
\times \Phi_{k,M}(x_1,s_1;\dots;x_k,s_k;y_1,y_2;\dots;y_{2M-1},y_{2M})\\
\times \hat{b}^+(x_1,s_1)\cdots\hat{b}^+(x_k,s_k)\widehat{B}^+(y_1,y_2)\cdots\widehat{B}^+(y_{2M-1},y_{2M})\Phi_0,
\end{aligned} \tag{14.11}$$

where the function $\Phi_{k,M}(x_1,s_1;\dots;x_k,s_k;y_1,y_2;\dots;y_{2M-1},y_{2M})$ is symmetric under permutations of the pairs of variables $(x_j,s_j)$ and $(x_i,s_i)$ and under permutations of the pairs of variables $(y_{2j-1},y_{2j})$ and $(y_{2i-1},y_{2i})$. In the bosonic case, we introduce the subspace $\mathcal{F}_{k,M}^{\mathrm{Symm}}$ consisting of vectors $\Phi$ such that $\Phi_{k',M'} = 0$ for $(k',M') \neq (k,M)$ and $\Phi_{k,M}$ is a symmetric function of $x_1,x_2,\dots,x_k,y_1,y_2,\dots,y_{2M}$. In the fermionic case, we introduce the subspace $\mathcal{F}_{k,M}^{\mathrm{Asymm}}$ consisting of vectors $\Phi$ such that $\Phi_{k',M'} = 0$ for $(k',M') \neq (k,M)$ and $\Phi_{k,M}$ is an antisymmetric function of $x_1,x_2,\dots,x_k,y_1,y_2,\dots,y_{2M}$. The orthogonal projection operator from the space $\mathcal{F}$ onto the subspace $\mathcal{F}_{k,M}^{\mathrm{Symm}}$ has the form[27], [101]

$$\widehat{\Pi}_{k,M}^{\mathrm{Symm}} = \frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}$$

$$\times\, \hat{b}^{+}(x_1,s_1)\cdots\hat{b}^{+}(x_k,s_k)\widehat{B}^{+}(y_1,y_2)\cdots\widehat{B}^{+}(y_{2M-1},y_{2M})$$

$$\times \operatorname*{Symm}_{x_1\dots x_k y_1\dots y_{2M}}\big(\hat{b}^{-}(x_1,s_1)\cdots\hat{b}^{-}(x_k,s_k)\widehat{B}^{-}(y_1,y_2)\cdots\widehat{B}^{-}(y_{2M-1},y_{2M})\big)$$

$$\times \exp\biggl(-\sum_{s=0}^{\infty}\int dx\,\hat{b}^{+}(x,s)\hat{b}^{-}(x,s)-\iint dy\,dy'\,\widehat{B}^{+}(y,y')\widehat{B}^{-}(y,y')\biggr), \tag{14.12}$$

where $\operatorname{Symm}_{x_1\dots x_k y_1\dots y_{2M}}$ is the symmetrization operator with respect to $x_1,\dots,x_k,y_1,\dots,y_{2M}$ and the operators $\hat{b}^{+}(x,s)$, $\hat{b}^{-}(x,s)$, $\widehat{B}^{+}(y,y')$, and $\widehat{B}^{-}(y,y')$ are Wick ordered [85]. The orthogonal projection operator from the space $\mathcal{F}$ onto the subspace $\mathcal{F}_{k,M}^{\mathrm{Asymm}}$ has the form [27]

$$\widehat{\Pi}_{k,M}^{\mathrm{Asymm}}=\frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}$$

$$\times\, \hat{b}^{+}(x_1,s_1)\cdots\hat{b}^{+}(x_k,s_k)\widehat{B}^{+}(y_1,y_2)\cdots\widehat{B}^{+}(y_{2M-1},y_{2M})$$

$$\times \operatorname*{Asymm}_{x_1\dots x_k y_1\dots y_{2M}}\big(\hat{b}^{-}(x_1,s_1)\cdots\hat{b}^{-}(x_k,s_k)\widehat{B}^{-}(y_1,y_2)\cdots\widehat{B}^{-}(y_{2M-1},y_{2M})\big)$$

$$\times \exp\biggl(-\sum_{s=0}^{\infty}\int dx\,\hat{b}^{+}(x,s)\hat{b}^{-}(x,s)-\iint dy\,dy'\,\widehat{B}^{+}(y,y')\widehat{B}^{-}(y,y')\biggr),$$

where $\operatorname{Asymm}_{x_1\dots x_k y_1\dots y_{2M}}$ is the antisymmetrization operator with respect to $x_1,\dots,x_k,y_1,\dots,y_{2M}$. Here and further on, unless otherwise stipulated, the operators $\hat{b}^{+}(x,s)$, $\hat{b}^{-}(x,s)$, $\widehat{B}^{+}(y,y')$, and $\widehat{B}^{-}(y,y')$ are assumed to be Wick ordered.

We consider a system of $N$ identical particles on the torus $\mathbf{T}$. We assume that the Hamiltonian for $N$ bosons or fermions has the form

$$\widehat{H}_N=-\frac{\hbar^2}{2m}\sum_{j=1}^{N}\Delta_j+\sum_{j=1}^{N}\sum_{l=j+1}^{N}V(x_j-x_l). \tag{14.13}$$

According to [27], the ultrasecond-quantized Hamiltonian of the form

$$\overline{\widehat{H}}_{\mathrm{B}}=\sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}$$

$$\times\, \hat{b}^{+}(x_1,s_1)\cdots\hat{b}^{+}(x_k,s_k)\widehat{B}^{+}(y_1,y_2)\cdots\widehat{B}^{+}(y_{2M-1},y_{2M})\widehat{H}_{k+2M}$$

$$\times \operatorname*{Symm}_{x_1\dots x_k y_1\dots y_{2M}}\big(\hat{b}^{-}(x_1,s_1)\cdots\hat{b}^{-}(x_k,s_k)\widehat{B}^{-}(y_1,y_2)\cdots\widehat{B}^{-}(y_{2M-1},y_{2M})\big)$$

$$\times \exp\biggl(-\sum_{s=0}^{\infty}\int dx\,\hat{b}^{+}(x,s)\hat{b}^{-}(x,s)-\iint dy\,dy'\,\widehat{B}^{+}(y,y')\widehat{B}^{-}(y,y')\biggr) \tag{14.14}$$

corresponds to the operator (14.13) in the bosonic case. And in the fermionic case, the corresponding operator $\overline{\widehat{H}}_{\mathrm{F}}$ is expressed by a similar formula in which the operator Symm is replaced with Asymm. By analogy with (14.13) and (14.14), the ultrasecond-quantized operator $\overline{\widehat{A}}$ is associated [27] with each $N$-particle operator,

$$\widehat{A}_N\biggl(\overset{2}{x}_1,\dots,\overset{2}{x}_N;-i\,\overset{1}{\frac{\partial}{\partial x_1}},\dots,-i\,\overset{1}{\frac{\partial}{\partial x_N}}\biggr).$$

For example, in the bosonic case, the ultrasecond-quantized identity operator of the form

$$\begin{aligned}\overline{\overline{E}}_{\mathrm{B}} = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}\\ \times\hat{b}^{+}(x_1,s_1)\cdots\hat{b}^{+}(x_k,s_k)\widehat{B}^{+}(y_1,y_2)\cdots\widehat{B}^{+}(y_{2M-1},y_{2M})\\ \times\mathop{\mathrm{Symm}}_{x_1\ldots x_k y_1\ldots y_{2M}}(\hat{b}^{-}(x_1,s_1)\cdots\hat{b}^{-}(x_k,s_k)\widehat{B}^{-}(y_1,y_2)\cdots\widehat{B}^{-}(y_{2M-1},y_{2M}))\\ \times\exp\biggl(-\sum_{s=0}^{\infty}\int dx\,\hat{b}^{+}(x,s)\hat{b}^{-}(x,s)-\iint dy\,dy'\,\widehat{B}^{+}(y,y')\widehat{B}^{-}(y,y')\biggr),\end{aligned}\tag{14.15}$$

which is the sum of projections (14.12), is associated with the identity operator. Similarly, the ultrasecond-quantized identity operator in the fermionic case has the form

$$\overline{\overline{E}}_{\mathrm{F}}=\sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\widehat{\Pi}_{k,M}^{\mathrm{Asymm}},$$

and the operator Symm in formula (14.15) is replaced with Asymm.

We consider the eigenvalue problem

$$\overline{\overline{\widehat{H}}}_{\mathrm{B,F}}\Phi=\lambda\overline{\overline{E}}_{\mathrm{B}}\Phi,\qquad\overline{\overline{E}}\Phi\neq0,\tag{14.16}$$

in the bosonic and fermionic cases. The validity of the following assertion was proved in [27]: *on the subspaces* $\mathcal{F}_{k,M}^{\mathrm{Symm}}$ *and* $\mathcal{F}_{k,M}^{\mathrm{Asymm}}$ *of the space* $\mathcal{F}$, *the corresponding operators* $\overline{\overline{\widehat{H}}}_{\mathrm{B}}$ *and* $\overline{\overline{\widehat{H}}}_{\mathrm{F}}$ *coincide with the operator* $\widehat{H}_{k+2M}$ *after they are acted upon by the projection onto the* $N$*-dimensional subspace* (*see formula* (14.22) *below*). Therefore, the eigenvalues $\lambda$ for problem (14.16) in the bosonic and fermionic cases coincide with the corresponding eigenvalues of the operators $\widehat{H}_N$ in (14.13). When the commutators between the operators $\hat{b}^{-}(x,s)$ and $\hat{b}^{+}(x,s)$ and also between $\widehat{B}^{-}(x,y)$ and $\widehat{B}^{+}(x,y)$ are of the order of smallness $1/N$, the asymptotic expressions for the solutions of problem (14.16) are determined by the extremum points of the pseudosymbol corresponding to problem (14.16).

For an arbitrary second-quantized operator $\widehat{A}$, the pseudosymbol of the ultrasecond-quantized operator $\overline{\overline{\widehat{A}}}$ corresponding to the former operator is given by the formula [27]

$$A_{\mathrm{B,F}}[b^*(\,\cdot\,),b(\,\cdot\,),B^*(\,\cdot\,),B(\,\cdot\,)]=\frac{\mathrm{Sp}(\widehat{\rho}_{\mathrm{B,F}}\widehat{A})}{\mathrm{Sp}(\widehat{\rho}_{\mathrm{B,F}})}.$$

In the bosonic case, the pseudosymbol has the form

$$\begin{aligned}\mathcal{H}_{\mathrm{B}}[b^*(\,\cdot\,),b(\,\cdot\,),B^*(\,\cdot\,),B(\,\cdot\,)]=\biggl\{\sum_{k,M=0}^{\infty}\frac{1}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_k\,dy_1\cdots dy_{2M}\\ \times b^*(x_1,s_1)\cdots b^*(x_k,s_k)B^*(y_1,y_2)\cdots B^*(y_{2M-1},y_{2M})H_{k+2M}\\ \times\mathop{\mathrm{Symm}}_{x_1\ldots x_k y_1\ldots y_{2M}}(b(x_1,s_1)\cdots b(x_k,s_k)B(y_1,y_2)\cdots B(y_{2M-1},y_{2M}))\biggr\}\\ \times\biggl\{\sum_{k',M'=0}^{\infty}\frac{1}{k'!M'!}\sum_{s'_1=0}^{\infty}\cdots\sum_{s'_{k'}=0}^{\infty}\int\cdots\int dx'_1\cdots dx'_{k'}\,dy'_1\cdots dy'_{2M'}\\ \times b^*(x'_1,s'_1)\cdots b^*(x'_{k'},s'_{k'})B^*(y'_1,y'_2)\cdots B^*(y'_{2M'-1},y'_{2M'})\end{aligned}$$

$$\times \operatorname*{Symm}_{x'_1\ldots x'_{k'}y'_1\ldots y'_{2M'}} (b(x'_1, s'_1)\cdots b(x'_{k'}, s'_{k'})B(y'_1, y'_2)\cdots B(y'_{2M'-1}, y'_{2M'}))\biggr\}^{-1}. \tag{14.17}$$

In the fermionic case, the pseudosymbol is expressed similarly with the only distinction that Symm in (14.17) is replaced with Asymm. The identity for the pseudosymbol (14.17)

$$\mathcal{H}_{\mathrm{B}}[b^*(\,\cdot\,), b(\,\cdot\,), B^*(\,\cdot\,), B(\,\cdot\,)] = \frac{\mathrm{Sp}(\widehat{\rho}_{\mathrm{B}}\widehat{H})}{\mathrm{Sp}(\widehat{\rho}_{\mathrm{B}})}, \tag{14.18}$$

where $\widehat{H}$ and $\widehat{\rho}_{\mathrm{B}}$ are the second-quantized operators,

$$\widehat{H} = \int dx\, \widehat{\psi}^+(x)\biggl(-\frac{\hbar^2}{2m}\Delta\biggr)\widehat{\psi}^-(x) + \;+\frac{1}{2}\iint dx\, dy\, V(x,y)\widehat{\psi}^+(y)\widehat{\psi}^+(x)\widehat{\psi}^-(y)\widehat{\psi}^-(x), \tag{14.19}$$

holds in the bosonic case. Here $\widehat{\rho}_{\mathrm{B}}$ depends on the functions $b(x, s)$ and $B(y, y')$,

$$\begin{aligned}\widehat{\rho}_{\mathrm{B}} = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty} &\frac{1}{k!M!(k+2M)!}\biggl(\sum_{s=0}^{\infty}\iint dx\, dx'\, b(x,s)b^*(x',s)\widehat{\psi}^+(x)\widehat{\psi}^-(x')\biggr)^k \\ &\times\biggl(\iint dy_1\, dy_2\, B(y_1,y_2)\widehat{\psi}^+(y_1)\widehat{\psi}^+(y_2)\biggr)^M \\ &\times\biggl(\iint dy'_1\, dy'_2\, B(y'_1,y'_2)\widehat{\psi}^-(y'_1)\widehat{\psi}^-(y'_2)\biggr)^M \exp\biggl(-\int dz\, \widehat{\psi}^+(z)\widehat{\psi}^-(z)\biggr),\end{aligned} \tag{14.20}$$

where $\widehat{\psi}^+(x)$ and $\widehat{\psi}^-(x)$ are the respective Wick-ordered bosonic creation and annihilation operators [85]. In the fermionic case, a similar identity holds,

$$\mathcal{H}_{\mathrm{F}}[b^*(\,\cdot\,), b(\,\cdot\,), B^*(\,\cdot\,), B(\,\cdot\,)] = \frac{\mathrm{Sp}(\widehat{\rho}_{\mathrm{F}}\widehat{H})}{\mathrm{Sp}(\widehat{\rho}_{\mathrm{F}})},$$

where $\widehat{H}$ and $\widehat{\rho}_{\mathrm{F}}$ are the second-quantized operators

$$\begin{aligned}\widehat{H} &= \int dx\, \widehat{\psi}^+(x)\biggl(-\frac{\hbar^2}{2m}\Delta\biggr)\widehat{\psi}^-(x) + \frac{1}{2}\iint dx\, dy\, V(x,y)\widehat{\psi}^+(x)\widehat{\psi}^+(y)\widehat{\psi}^-(y)\widehat{\psi}^-(x), \\ \widehat{\rho}_{\mathrm{F}} &= \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{k!M!(k+2M)!}\biggl(\iint dy_1\, dy_2\, B(y_1,y_2)\widehat{\psi}^+(y_1)\widehat{\psi}^+(y_2)\biggr)^M \\ &\quad\times\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\, dx'_1\cdots dx_k\, dx'_k\, (x_1,s_1)b^*(x'_1,s_1)\cdots b(x_k,s_k)b^*(x'_k,s_k) \\ &\quad\times\widehat{\psi}^+(x_1)\cdots\widehat{\psi}^+(x_k)\widehat{P}_0\widehat{\psi}^-(x'_k)\cdots\widehat{\psi}^-(x'_1)\biggl(\iint dy'_1\, dy'_2\, B(y'_1,y'_2)\widehat{\psi}^-(y'_2)\widehat{\psi}^-(y'_1)\biggr)^M,\end{aligned} \tag{14.21}$$

where $\widehat{\psi}^+(x)$ and $\widehat{\psi}^-(x)$ are the fermionic creation and annihilation operators in this case and $\widehat{P}_0$ is the projection operator onto the vacuum vector of the fermionic Fock space.

We introduce the ultrasecond-quantized number-of-particles operators in the space $\mathcal{F}$ [27],

$$\overline{\widehat{N}}_{\mathrm{B}} = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}(k+2M)\widehat{\Pi}_{k,M}^{\mathrm{Symm}}, \qquad \overline{\widehat{N}}_{\mathrm{F}} = \sum_{k=0}^{\infty}\sum_{M=0}^{\infty}(k+2M)\widehat{\Pi}_{k,M}^{\mathrm{Asymm}}. \tag{14.22}$$

In the bosonic case, the pseudosymbol of the operator $\overline{\widehat{N}}_{\mathrm{B}}$ has the form

$$N_{\mathrm{B}} = \biggl\{\sum_{k=0}^{\infty}\sum_{M=0}^{\infty}\frac{k+2M}{k!M!}\sum_{s_1=0}^{\infty}\cdots\sum_{s_k=0}^{\infty}\int\cdots\int dx_1\cdots dx_{k+2M}$$

$$\times\, b^*(x_1,s_1)\cdots b^*(x_k,s_k)B^*(x_{k+1},x_{k+2})\cdots B^*(x_{k+2M-1},x_{k+2M})$$

$$\times \operatorname*{Symm}_{x_1\dots x_{k+2M}}\big(b(x_1,s_1)\cdots b(x_k,s_k)B(x_{k+1},x_{k+2})\cdots B(x_{k+2M-1},x_{k+2M})\big)\Big\}$$

$$\times\Big\{\sum_{k'=0}^{\infty}\sum_{M'=0}^{\infty}\frac{1}{k'!M'!}\sum_{s'_1=0}^{\infty}\cdots\sum_{s'_{k'}=0}^{\infty}\int\cdots\int dz_1\cdots dz_{k'+2M'}\,b^*(z_1,s'_1)\cdots b^*(z_{k'},s'_{k'})$$

$$\times\, B^*(z_{k'+1},z_{k'+2})\cdots B^*(x_{k'+2M'-1},x_{k'+2M'})$$

$$\times \operatorname*{Symm}_{z_1\dots z_{k'+2M'}}\big(b(z_1,s'_1)\cdots B(z_{k'+2M'-1},z_{k'+2M'})\big)\Big\}^{-1}. \tag{14.23}$$

In the corresponding formula for the fermionic case, Symm is replaced with Asymm.

### *14.3. The Symbol of an Ultrasecond-Quantized Operator*

We first note that the above definition of the pseudosymbol does not reflect the thermodynamic asymptotic behavior quite fully, although it in fact agrees with the Bogolyubov–Dirac rule according to which the creation and annihilation operators in the leading asymptotic term should be set equal to $c$-numbers. Therefore, we define the symbol correctly (i.e., introduce the true symbol). Let an operator $\widehat{H}$ have the form

$$\widehat{H}=\sum_{l=1}^{L}\int\cdots\int dx_1\cdots dx_l\,\widehat{\psi}^+(x_1)\cdots\widehat{\psi}^+(x_l)$$

$$\times\, H_l\bigg(\overset{2}{x}_1,\dots,\overset{2}{x}_l;-i\overset{1}{\frac{\partial}{\partial x_1}},\dots,-i\overset{1}{\frac{\partial}{\partial x_l}}\bigg)\widehat{\psi}^-(x_l)\dots\widehat{\psi}^-(x_1). \tag{14.24}$$

Then in the case of ultrasecond quantization without creation and annihilation operators $\widehat{B}^{\pm}(x,y)$ for pairs of particles, the identity

$$\overline{\widehat{H}}=\overline{\widehat{E}\widehat{A}} \tag{14.25}$$

where $\overline{\widehat{A}}$ is an operator in $\mathcal{F}$ defined by the relation

$$\overline{\widehat{A}}=\sum_{l=1}^{L}\sum_{s_1=0}^{\infty}\cdots\sum_{s_l=0}^{\infty}\int\cdots\int dx_1\cdots dx_l\,\hat{b}^+(x_1,s_1)\cdots\widehat{\psi}^+(x_l,s_l)$$

$$\times\, H_l\bigg(\overset{2}{x}_1,\dots,\overset{2}{x}_l;-i\overset{1}{\frac{\partial}{\partial x_1}},\dots,-i\overset{1}{\frac{\partial}{\partial x_l}}\bigg)\hat{b}^-(x_l,s_l)\cdots\hat{b}^-(x_1), \tag{14.26}$$

holds for the operators $\overline{\widehat{H}}$ and $\overline{\widehat{E}}$ defined above.

If ultrasecond quantization also takes the creation and annihilation operators into account for the pairs of particles, then identity (14.25) also holds, but the operator $\overline{\widehat{A}}$ has a more complicated form than (14.26). For example, if the operator $\widehat{H}$ is a particular case of operator (14.24) for $L=2$, then $\overline{\widehat{A}}$ has the form

$$\overline{\widehat{A}}=\sum_{s=0}^{\infty}\int dx\,\hat{b}^+(x,s)\Big(-\frac{h^2}{2m}\Delta\Big)\hat{b}^-(x,s)$$

$$+\iint dx\,dy\,\widehat{B}^+(x,y)\Big(-\frac{h^2}{2m}(\Delta_x+\Delta_y)\Big)\widehat{B}^-(x,y)$$

$$+\frac{1}{2}\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\iint dx\,dy\,V(x,y)\,\hat{b}^+(x,s_1)\hat{b}^+(y,s_2)\hat{b}^-(y,s_2)\hat{b}^-(x,s_1)$$

$$+\sum_{s=0}^{\infty}\iiint dx\,dy\,dz\,(V(x,y)+V(x,z))\hat{b}^+(x,s)\widehat{B}^+(y,z)\widehat{B}^-(y,z)\hat{b}^-(x,s)$$

$$+\iint dx\,dy\,V(x,y)\widehat{B}^+(x,y)\widehat{B}^-(x,y)$$

$$+\frac{1}{2}\iiiint dx\,dy\,dz\,dw\,V(x,y)\widehat{B}^+(x,y)\widehat{B}^+(z,w)$$

$$\times(\widehat{B}^-(y,w)\widehat{B}^-(x,z)+\widehat{B}^-(w,y)\widehat{B}^-(z,x)$$

$$+\widehat{B}^-(y,z)\widehat{B}^-(w,x)+\widehat{B}^-(z,y)\widehat{B}^-(x,w)). \qquad (14.27)$$

If the operators $\widehat{B}^{\pm}(x,y)$ and $\hat{b}^{\pm}(x,y)$ in (14.26) and (14.27) are set equal to $c$-numbers, then we obtain the (true) symbol corresponding to the asymptotic representation in the thermodynamic limit.

### *14.4. The Case of an Even Number of Neutrons*

We consider a system of $N$ identical bosons of mass $m$ located on a torus $\mathbf{T}$ whose diameters are $L_1$, $L_2$, and $L_2$. We assume that the bosons interact between themselves and that the interaction potential has the form

$$V(N^{1/3}(x-y)), \qquad (14.28)$$

where $V(\xi)$ is a finitely supported even function and $x$ and $y$ are the boson coordinates on $\mathbf{T}$. The boundary condition on the side $L_1$ is assumed to be periodic, and the zero-derivative condition is imposed along the side $L_2$. We note that interaction potential (14.28) depends on $N$ whose radius decreases as $N$ increases with the average number of particles with which one particle interacts remaining constant.

For ultrasecond quantization over the pairs, the explicit expression for the ultrasecond-quantized operator $\overline{\overline{\widehat{H}}}$ corresponding to the bosonic system in question is given in (14.24). As discussed above, this ultrasecond-quantized operator satisfies the identity

$$\overline{\overline{\widehat{H}}}=\overline{\overline{\widehat{E}}}\widehat{A}, \qquad (14.29)$$

where $\overline{\overline{\widehat{E}}}$ is the ultrasecond-quantized identity operator and $\widehat{A}$ is an operator in the second-quantization space. It can be easily verified that identity (14.29) is satisfied by an operator of the form

$$\widehat{A}=\iint dx\,dy\,\widehat{B}^+(x,y)\biggl(-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)+V(N^{1/3}(x-y))\biggr)\widehat{B}^-(x,y)+$$

$$+2\iiiint dx\,dy\,dx'\,dy'\,V(N^{1/3}(x-y))\widehat{B}^+(x,y)\widehat{B}^+(x',y')\widehat{B}^-(x,x')\widehat{B}^-(y,y'), \qquad (14.30)$$

where $\widehat{B}^+(x,y)$ and $\widehat{B}^-(x,y)$ are the respective bosonic creation and annihilation operators for a pair of particles in the Fock ultrasecond-quantization space. By identity (14.29), to find the asymptotic expression as $N\to\infty$ for the spectrum of the bosonic system under consideration, we must find the corresponding asymptotic expression for operator (14.30).

Because the product of function (14.28) and $N$ converges weakly to the Dirac delta function as $N\to\infty$, the expression for the operator $\widehat{A}$ in the right-hand side of (14.30) involves the small parameter $1/N$ before the second term in this limit case. This means that to find asymptotic representations for eigenvalues and eigenfunctions of $\widehat{A}$, we can use the classical methods developed in [61]. The asymptotic formulas for eigenvalues and eigenfunctions are determined by the symbol of the operator (14.30), and this symbol is the true symbol for the ultrasecond-quantized problem. The true symbol corresponding to operator (14.30) is the functional defined for the pair of functions $\Phi^+(x,y)$ and $\Phi(x,y)$

$$\begin{aligned}\mathcal{H}[\Phi^+(\,\cdot\,),\Phi(\,\cdot\,)] = &\iint dx\,dy\,\Phi^+(x,y)\biggl(-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\biggr)\Phi(x,y)+\\ &+2\iiiint dx\,dy\,dx'\,dy'\,(NV(N^{1/3}(x-y)))\Phi^+(x,y)\Phi^+(x',y')\Phi(x,x')\Phi(y,y').\end{aligned} \tag{14.31}$$

The conservation law for the number of particles in the system implies the condition

$$\iint dx\,dy\,\Phi^+(x,y)\Phi(x,y)=\frac{1}{2} \tag{14.32}$$

for $\Phi^+(x,y)$ and $\Phi(x,y)$.

According to the asymptotic methods in [61], to each solution of the system of equations

$$\Omega\Phi(x,y)=\frac{\delta\mathcal{H}}{\delta\Phi^+(x,y)},\qquad \Omega\Phi^+(x,y)=\frac{\delta\mathcal{H}}{\delta\Phi(x,y)} \tag{14.33}$$

that also satisfies condition (14.32), there corresponds an asymptotic series of eigenfunctions and eigenvalues for operator (14.30) as $N\to\infty$. It follows from the explicit form of true symbol (14.31) that system of equations (14.33) can be written in the form

$$\begin{aligned}\Omega\Phi(x,y)=&-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\Phi(x,y)\\ &+\iint dx'\,dy'\,(NV(N^{1/3}(x-y))+NV(N^{1/3}(x'-y')))\Phi^+(x',y')\Phi(x,x')\Phi(y,y'),\\ \Omega\Phi^+(x,y)=&-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\Phi^+(x,y)\\ &+2\iint dx'\,dy'\,(NV(N^{1/3}(x-x'))+NV(N^{1/3}(y-y')))\Phi(x',y')\Phi^+(x,x')\Phi^+(y,y').\end{aligned}$$

Let $v_q$ be the coefficients in the Fourier series expansion for the potential $NV(N^{1/3}x)$ on the torus $(L_1,L_2,L_2)$,

$$v_q=\frac{1}{L_1L_2^2}\int_{\mathbf{T}}e^{-iqx}NV(\sqrt[3]{N}x)\,dx,\qquad v_{-q}=v_q. \tag{14.34}$$

The exact solutions of system (14.34) are the functions

$$\Phi^+_{k_1,k_2}=\frac{1}{L_1L_2^2}e^{-ik_1(x+y)}\cos(k_2(x-y)), \tag{14.35}$$

$$\Phi_{k_1,k_2}=\frac{1}{L_1L_2^2}\sum_{l=0}^{\infty}\varphi_{k_2,l}e^{ik_1(x+y)}e^{il(x-y)} \tag{14.36}$$

with the eigenvalue

$$\Omega=\frac{h^2}{m}(k_1^2+k_2^2)+v_0+v_{2k_2}, \tag{14.37}$$

where $\varphi_{k_2,l}$ has the form

$$\varphi_{k_2,l}=\begin{cases}-\dfrac{b_l}{2}+\dfrac{1}{2}\sqrt{b_l^2-1}, & l^2>k_2^2,\\ -\dfrac{b_l}{2}-\dfrac{1}{2}\sqrt{b_l^2-1}, & l^2<k_2^2,\\ \dfrac{1}{2}, & l=k_2.\end{cases} \tag{14.38}$$

Here, we have

$$b_l = \frac{(h^2/m)(l^2-k_2^2)-(v_0+v_{2k_2})}{v_{l-k_2}+v_{l+k_2}},$$

where $b_l = b_{-l}$ and $\varphi_{k_2,l} = \varphi_{k_2,-l}$ (if $V(\xi)\to 0$, then $\varphi_{k_2,l}\to 0$). The vectors $k_1$ and $k_2$ play the role of parameters labeling the Bogolyubov solutions of this system. The vector $\hbar k_1/m$ is equal to the flow velocity along the capillary for the bosonic system, and $k_2$ is the wave vector of the transverse mode.

We note that $b_l\to\infty$ as $|l|\to\infty$ because

$$|v_l| = \frac{1}{L_1L_2^2}\int_{N\mathbf{T}} e^{-il\xi/N}V(\xi)\,d\xi \le \frac{1}{L_1L_2^2}\int_{N\mathbf{T}}|V(\xi)|\,d\xi < \frac{1}{L_1L_2^2}\int_{\mathbb{R}^3}|V(\xi)|\,d\xi. \tag{14.39}$$

(Here, $N\mathbf{T}$ is the torus with diameters $NL_1$, $NL_2$, and $NL_2$.) Therefore, we have

$$\varphi_{k,l}\cong\frac{1}{b_l^2}, \tag{14.40}$$

and series (14.36) hence converges absolutely.

We split series (14.36) into two parts: a sum over $l\le N^{1/6}$ and a sum over $l>N^{1/6}$. To an accuracy of $N^{-1/6}$, the terms in the first sum converge to

$$\begin{aligned} b_l &\to \frac{h^2(l^2-k_2^2)}{2mV_0}-1 \overset{\text{def}}{=} b_l^0,\\ \varphi_{k_2,l} &\to -\frac{b_l}{2}\pm\frac{1}{2}\sqrt{b_l^2-1}\overset{\text{def}}{=}\varphi^0_{k_2,l} \end{aligned} \tag{14.41}$$

as $N\to\infty$, which can be easily obtained using the change of variable $\sqrt[3]{N}x=\xi$ in (14.34).

By (14.40), the other sum tends to zero as $O(N^{-1/6})$. Therefore, the system of equations (14.34) supplemented with condition (14.32) has the family of solutions

$$\begin{aligned} \Phi_k^+(x,y) &= \frac{1}{L_1L_2^2}\cos(k(x-y)),\\ \Phi_k(x,y) &= \frac{1}{L_1L_2^2}\sum_l \varphi_{k,l}e^{il(x-y)} \end{aligned} \tag{14.42}$$

for $k_1=0$ as $N\to\infty$. Here, $k$ and $l$ are three-dimensional vectors of the form $2\pi(0,n_2/L_2,n_3/L_2)$, $n_2$ and $n_3$ are integers, and $\varphi_{k,l}$ in (14.38) become

$$\varphi^0_{k_2,l} = \frac{1}{2V_0}\left(\frac{\hbar^2}{2m}(k_2^2-l^2)+V_0\pm\sqrt{\left(\frac{\hbar^2}{2m}(k_2^2-l^2)+V_0\right)^2-V_0^2}\right) \tag{14.43}$$

in this case, where we take the plus sign by virtue of the selection rules. Finally,

$$V_0=\frac{1}{L_1L_2^2}\int_{\mathbb{R}^3}dx\,V(x). \tag{14.44}$$

The vector $k$ in formulas (14.42) plays the role of a parameter labeling the various solutions of system of equations (14.34), (14.32). Solutions (14.42) are standing waves, and there is no flow in the series that correspond to them.

The leading term of the asymptotic expression for the eigenvalues in the series corresponding to solution (14.35), (14.36) is equal to the value of symbol (14.31) on these functions multiplied by $N$,

$$E_{k_1,k_2}=N\left(\frac{\hbar^2(k_1^2+k_2^2)}{2m}+\frac{V_0}{2}\right). \tag{14.45}$$

Apart from the system of equations (14.34), the asymptotic expressions for the eigenvalues and eigenfunctions, in particular, the subsequent terms after $E_{k_1,k_2}$, are also determined by solutions of the system of variational equations for the system of Hamilton equations. The system of variational equations for (14.34) has the form

$$\begin{aligned}(\Omega-\lambda)F(x,y)&=-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)F(x,y)\\&\quad+2N\iint dx'\,dy'\,\big(V(\sqrt[3]{N}(x-y))+V(\sqrt[3]{N}(x'-y'))\big)\\&\quad\times\big(G(x',y')\Phi(x,x')\Phi(y,y')+\Phi^+(x',y')F(x,x')\Phi(y,y')\\&\quad+\Phi^+(x',y')\Phi(x,x')F(y,y')\big),\\(\Omega+\lambda)G(x,y)&=-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)G(x,y)\\&\quad+2N\iint dx'\,dy'\,\big(V(\sqrt[3]{N}(x-x'))+V(\sqrt[3]{N}(y-y'))\big)\\&\quad\times\big(F(x',y')\Phi^+(x,x')\Phi^+(y,y')+\Phi(x',y')G(x,x')\Phi^+(y,y')+\\&\quad+\Phi(x',y')\Phi^+(x,x')G(y,y')\big).\end{aligned}\tag{14.46}$$

To find the quasiparticle spectrum, among all the solutions of the system of variational equations, we must select the ones satisfying the selection rule stated in [102] for the complex germ in the case of non-self-adjoint operators with a real spectrum.

If $k_2=0$, then the asymptotic series associated with this solution is the Bogolyubov series corresponding to the flow velocity $\hbar k_1/m$. The quasiparticle spectrum for this series is given by the well-known formula

$$\lambda_l=\sqrt{\left(\frac{\hbar^2l^2}{2m}+V_0\right)^2-V_0^2}-\frac{\hbar^2lk_1}{m}.\tag{14.47}$$

We consider the case $k_2\neq0$. Substituting solutions (14.35), (14.36) in (14.46) and taking the symmetry into account, we find that the solutions of the system of variational equations have the form

$$\begin{aligned}G_l(x,y)&=u_{1,l}(e^{i(-k_1+k_2)x+i(-k_1+l)y}+e^{i(-k_1+k_2)y+i(-k_1+l)x})\\&\quad+u_{2,l}\big(e^{i(-k_1-k_2)x+i(-k_1+2k_2+l)y}+e^{i(-k_1-k_2)y+i(-k_1+2k_2+l)x}\big),\\F_l(x,y)&=-v_{1,l}\big(e^{i(k_1+k_2)x+i(k_1+l)y}+e^{i(k_1+k_2)y+i(k_1+l)x}\big)\\&\quad-v_{2,l}\big(e^{i(k_1-k_2)x+i(k_1+2k_2+l)y}+e^{i(k_1-k_2)y+i(k_1+2k_2+l)x}\big)\\&\quad+\sum_{l'\neq l,\,l+2k_2}w_{l,l'}\big(e^{i(k_1+k_2+l-l')x+i(k_1+l')y}+e^{i(k_1+k_2+l-l')y+i(k_1+l')x}\big),\end{aligned}\tag{14.48}$$

where $l\neq-k_2$, and the numerical coefficients $u_{1,l}$, $u_{2,l}$, $v_{1,l}$, $v_{2,l}$, and $w_{l,l'}$ are found from an infinite system of equations. This system contains a closed subsystem of four equations for the coefficients $u_{1,l}$, $u_{2,l}$, $v_{1,l}$, and $v_{2,l}$ that can be written in the standard form

$$\tilde\lambda X=MX,\tag{14.49}$$

where $\tilde{\lambda} = \lambda + \frac{\hbar^2}{m}(k_1, l)$, and $X$ is the column vector of the form

$$X = \begin{pmatrix} u_{1,l} \\ u_{2,l} \\ v_{1,l} \\ v_{2,l} \end{pmatrix}.$$

Using Eqs. (14.48) and (14.49), we obtain the matrix $M$ whose elements are of the form

$$M = \begin{pmatrix} B_l + \frac{v_{l-k_2}}{2} & \frac{v_{2k_2}+v_{l+k_2}}{2} & -\frac{v_{l+k_2}+v_{l-k_2}}{2} & 0 \\ \frac{v_{2k_2}+v_{l+k_2}}{2} & B_{l+2k_2} + \frac{v_{l+3k_2}}{2} & 0 & -\frac{v_{l+k_2}+v_{l+3k_2}}{2} \\ 2(v_0 + v_{l-k_2})\varphi_{k_2,l} & (v_{2k_2} + v_{l+k_2})\,(\varphi_{k_2,l} + \varphi_{k_2,l+2k_2}) & -B_l - \frac{v_{l-k_2}}{2} & -\frac{v_{2k_2}+v_{l+k_2}}{2} \\ (v_{2k_2} + v_{l+k_2})\,(\varphi_{k_2,l} + \varphi_{k_2,l+2k_2}) & 2(v_0 + v_{l+3k_2})\varphi_{k_2,l+2k_2} & -\frac{v_{2k_2}+v_{l+k_2}}{2} & -B_{l+2k_2} - \frac{v_{l+3k_2}}{2} \end{pmatrix},$$

where

$$B_l = \frac{\hbar^2}{2m}(l^2 - k_2^2) + (v_{l-k_2} + v_{l+k_2})\varphi_{k_2,l} - \frac{v_{2k_2}}{2}.$$

The exact solution of relation (14.48) was obtained in [103]. It has the form

$$\lambda_{k_1 k_2,l} = -\frac{\hbar^2}{m}(k_1, l) + \sqrt{\frac{\xi_{k_2,l} + \sqrt{\xi_{k_2,l}^2 - 4\eta_{k_2,l}}}{2}}, \tag{14.50}$$

where $\xi_{k,l}$, $\eta_{k,l}$ is determined below in (14.51)–(14.55).

Below $k_2$ is denoted by $k$. The expansion of $\xi_{k,l}$ with respect to $a = \frac{\hbar^2}{2m}$ is of the form

$$\begin{aligned} \xi_{k,l} = {} & -\frac{1}{2}(v_{l+k} + v_{2k})(v_{l+3k} + v_{l-k} - 2v_{2k}) \\ & + a\left(l^2(v_{l+3k} + v_{l-k} - 2v_{2k}) + k^2(3v_{l+3k} - 2v_{2k} - v_{l-k}) + 4(l,k)(v_{l+3k} - v_{2k})\right) \\ & + 2a^2\left(l^4 + 5k^4 + 8((k,l))^2 + 2l^2k^2 + 4l^2(k,l) + 12k^2(k,l)\right), \end{aligned} \tag{14.51}$$

where $(k, l)$ denotes the inner product of the vectors $k$ and $l$, while the second and third powers of $k$ and $l$ denote the second and fourth powers of the moduli of these vectors. The expansion of $\eta_{k,l}$ with respect to $a$ is of the form

$$\begin{aligned} \eta_{k,l} = {} & -\frac{a}{2}(l^2 + k^2 + 2(k,l))(v_{l+k} + v_{2k}) \\ & \quad \times (2v_{l+3k}v_{l-k} - v_{l+3k}v_{2k} + v_{l+3k}v_{l+k} - v_{2k}v_{l-k} - 2v_{2k}v_{l+k} + v_{l+k}v_{l-k}) \\ & + a^2\eta_{k,l}^{(2)} + a^3\eta_{k,l}^{(3)} + a^4\eta_{k,l}^{(4)}, \end{aligned} \tag{14.52}$$

where

$$\begin{aligned} \eta_{k,l}^{(2)} = {} & -8((l,k))^2 v_{2k}v_{l-k} - 8((l,k))^2 v_{l+k}v_{l-k} + 4k^2(l,k)(v_{2k})^2 \\ & - 4k^2(l,k)(v_{l+k})^2 - \frac{3}{2}k^4 v_{2k}v_{l-k} + 4l^2(l,k)(v_{2k})^2 - \frac{1}{2}l^4 v_{l+k}v_{l+3k} \\ & - \frac{1}{2}l^4 v_{l+k}v_{l-k} - \frac{3}{2}l^4 v_{2k}v_{l+3k} + \frac{5}{2}k^4 v_{2k}v_{l+3k} - \frac{1}{2}k^4 v_{l+k}v_{l+3k} \\ & - l^4 v_{2k}v_{l+k} - 2l^2k^2(v_{l+k})^2 - \frac{9}{2}k^4 v_{l+k}v_{l-k} - 4l^2(l,k)(v_{l+k})^2 - \frac{3}{2}l^4 v_{2k}v_{l-k} \end{aligned}$$

$$
\begin{aligned}
&+ l^4 v_{l-k} v_{l+3k} - 3k^4 v_{l-k} v_{l+3k} + 3k^4 v_{2k} v_{l+k} + 2l^2k^2 (v_{2k})^2 + 2l^2k^2 v_{l-k} v_{l+3k} \\
&- 4l^2(l,k) v_{l+k} v_{l-k} - 4l^2(l,k) v_{2k} v_{l+k} - 4l^2(l,k) v_{2k} v_{l+3k} - 8l^2(l,k) v_{2k} v_{l-k} \\
&+ 4l^2(l,k) v_{l-k} v_{l+3k} - 12k^2(l,k) v_{l+k} v_{l-k} + 4k^2(l,k) v_{2k} v_{l+k} + 4k^2(l,k) v_{2k} v_{l+3k} \\
&- 8k^2(l,k) v_{2k} v_{l-k} - 4k^2(l,k) v_{l-k} v_{l+3k} - 5l^2k^2 v_{2k} v_{l-k} + l^4 (v_{2k})^2 - k^4 (v_{l+k})^2 \\
&+ k^4 (v_{2k})^2 - 4((l,k))^2 (v_{l+k})^2 + 4((l,k))^2 (v_{2k})^2 - l^2k^2 v_{2k} v_{l+3k} - 2l^2k^2 v_{2k} v_{l+k} \\
&- 3l^2k^2 v_{l+k} v_{l-k} + l^2k^2 v_{l+k} v_{l+3k} - l^4 (v_{l+k})^2, \qquad (14.53)
\end{aligned}
$$

$$
\begin{aligned}
\eta^{(3)}_{k,l} &= 16k^2((l,k))^2 v_{2k} - 16k^2((l,k))^2 v_{l-k} + 20k^4(l,k) v_{2k} - 24k^4(l,k) v_{l-k} + 4l^4(l,k) v_{l+3k} \\
&- 16l^2((l,k))^2 v_{2k} + 16l^2((l,k))^2 v_{l-k} - 5l^2k^4 v_{l+3k} + 2l^2k^4 v_{2k} + 3l^2k^4 v_{l-k} + l^4k^2 v_{l+3k} \\
&- 12l^4(l,k) v_{2k} + 8l^4(l,k) v_{l-k} - 6l^4k^2 v_{2k} + 4k^4(l,k) v_{l+3k} + 5l^4k^2 v_{l-k} - 8l^2k^2(l,k) v_{2k} \\
&- 8l^2k^2(l,k) v_{l+3k} + l^6 v_{l+3k} - 9k^6 v_{l-k} + 6k^6 v_{2k} + 3k^6 v_{l+3k} + 16l^2k^2(l,k) v_{l-k} \\
&- 2l^6 v_{2k} + l^6 v_{l-k}, \qquad (14.54)
\end{aligned}
$$

$$
\eta^{(4)}_{k,l} = \bigl(l^4 + 2l^2k^2 + 4l^2(l,k) - 3k^4 - 4k^2(l,k)\bigr)^2 . \qquad (14.55)
$$

Formulas (14.51)–(14.55) with $l \neq -k_2$ determine the quasiparticle spectrum for the series corresponding to solutions (14.35), (14.36). After the application of the selection rules for the complex germ in the non-self-adjoint situation [102], it follows from explicit formulas (14.51)–(14.55) that the quasiparticle spectrum contains only positive eigenvalues. Consequently, the series corresponding to smooth solution (14.35), (14.36) for $k_2 \neq 0$ is metastable. In what follows, we assume that $L_1 \gg L_2$.

We consider the Bogolyubov series corresponding to the flow with velocity $\hbar k_0/m$ along the capillary, where $k_0 = 2\pi(n_1/L_1, 0, 0)$. For the bosonic system under consideration, the leading asymptotic term for the eigenvalues in this series is equal to

$$
N\biggl(\frac{\hbar^2 k_0^2}{2m} + \frac{V_0}{2}\biggr). \qquad (14.56)
$$

We now assume that the relation between $L_1$ and $L_2$ admits a pair $k_1$, $k = 2\pi(0, n_2/L_2, n_3/L_2)$ such that the corresponding value of symbol (14.31) in (14.45) exactly coincides with leading asymptotic term (14.56). This means that there can be resonance between superfluid states in the Bogolyubov series and nonsuperfluid states in the nonmetastable series corresponding to branching solutions [104]. If $L_1$ is very large, then resonance is also possible in the case of the eigenvalue series corresponding to the branching solutions close to expression (14.56) and not necessarily coinciding with it. The existence of this resonance indicates the possibility of transition from the superfluid state to the metastable one, from which the system then passes to a lower energy level, and this indicates the loss of superfluidity.

The minimum energy corresponding to the nonmetastable series relates to the case

$$
k = 2\pi(0, 1/L_2, 0),
$$

and according to formula (14.45), it is given by the expression

$$
E_{\min} = N\biggl(\frac{\hbar^2 (2\pi)^2}{2mL_2^2} + \frac{V_0}{2}\biggr). \qquad (14.57)
$$

Comparing (14.57) and (14.56), we conclude that no resonance is possible if the modulus of the flow velocity $v$ is less than a certain bound,

$$
|v| < v_{\mathrm{c}}(L_2) \equiv \frac{2\pi\hbar}{mL_2} = \frac{h}{mL_2}, \qquad (14.58)
$$

which corresponds to the appearance of a vortex filament. The expression $v_{\rm c}(L_2)$ in the right-hand side of (14.58) increases as the capillary thickness $L_2$ decreases. If $L_2$ is less than $2\pi\hbar/(mv_{\rm cL})$, where $v_{\rm cL}$ is the Landau critical velocity, then it is substantially larger than the Landau critical velocity determined by (14.47), and superfluidity disappears in this case as the Landau velocity is attained. But as $L_2$ increases, the resonance between superfluid and nonmetastable states sets in at velocities less than the Landau velocity, and this accounts for the dependence of the critical velocity on the thickness of the capillary.

As we showed in [77], the Landau curve and the Bogolyubov quasiparticles are preserved in the classical limit, and all the conditions therefore remain applicable for the classical liquid. Consequently, under the condition $v < h/(mL_2)$, where $L_2$ is the nanotube diameter, the following mathematical fact occurs: the classical liquid in a nanotube must be superfluid.

For finite values of $l$, we have obtained series that provide decay. But large values of $l$ play a certain role in the Bogolyubov formula, and we therefore consider the situation with these large values. In this case, $V_l$ does not converge to $V_0$ as $N \to \infty$. Consequently, Eqs. (14.48) and (14.49) lead to a more exact matrix $M$ whose elements have the forms

$$M_{11} = B_l + \frac{v_{l-k_2}}{2}, \qquad M_{12} = M_{21} = \frac{v_{2k_2} + v_{l+k_2}}{2},$$

$$M_{13} = -\frac{v_{l+k_2} + v_{l-k_2}}{2}, \qquad M_{14} = M_{23} = 0, \qquad M_{22} = B_{l+2k_2} + \frac{v_{l+3k_2}}{2},$$

$$M_{24} = -\frac{v_{l+k_2} + v_{l+3k_2}}{2}, \qquad M_{31} = 2(v_0 + v_{l-k_2})\varphi_{k_2,l},$$

$$M_{32} = M_{41} = (v_{2k_2} + v_{l+k_2})(\varphi_{k_2,l} + \varphi_{k_2,l+2k_2}), \qquad M_{33} = -B_l - \frac{v_{l-k_2}}{2},$$

$$M_{34} = M_{43} = -\frac{v_{2k_2} + v_{l+k_2}}{2} \qquad M_{42} = 2(v_0 + v_{l+3k_2})\varphi_{k_2,l+2k_2},$$

$$M_{44} = -B_{l+2k_2} - \frac{v_{l+3k_2}}{2},$$

where

$$B_l = \frac{\hbar^2}{2m}(l^2 - k_2^2) + (v_{l-k_2} + v_{l+k_2})\varphi_{k_2,l} - \frac{v_{2k_2}}{2}.$$

It is clear that

$$v_{l+k_2} + v_{l+3k_2} = v_{l-k_2} + v_{l+k_2} + O\left(\frac{1}{N}\right) \tag{14.59}$$

holds uniformly with respect to $l$ as $l \to \infty$. (To show this, it suffices to make a type-(14.39) change of variable.) Therefore, the matrix $M$ can be represented approximately as a block matrix,

$$M = \begin{pmatrix} C & -V_l E \\ D & -C \end{pmatrix},$$

where $E$ is the 2×2 identity matrix and $V_l = (v_{l-k_2} + v_{l+k_2})/2$. We also introduce the notation $V_l^+ = (v_{l+k_2} + v_{2k_2})/2$ and $V_l^- = (v_{l-k_2} + v_0)/2$. The eigenvalues corresponding to Eq. (14.49) after application of the selection rules become

$$\lambda_{k_1,k_2,l} = -2ak_1(k_2 + l)$$

$$+ \Biggl(\frac{1}{2}(a(l^2 - k_2^2) + V_l - V_l^+)^2 + \frac{1}{2}(a(l_1^2 - k_2^2) + V_l - V_l^+)^2 + {V_l^+}^2 - V_l^2$$

$$+\frac{1}{2}(a(l_1^2+l^2-2k_2^2)+2V_l-2V_l^+)\sqrt{a^2(l_1^2-l^2)^2+4V_l^{+^2}}\Bigg)^{1/2}, \tag{14.60}$$

where $a=\hbar^2/(2m)$ and $l_1=l+2k_2$. In this case, we have

$$\lambda_{k_1,k_2,l}=-2ak_1l+((al^2+V_l-V_l^+)^2+V_l^{+^2}-V_l^2+2(al^2+V_l-V_l^+)|V_l^+|)^{1/2},$$

$$V_l=v_l, \qquad V_l^+=V_l^-=\frac{v_l+v_0}{2}$$

for $k_2=0$. Formally setting $k_2=0$, we obtain the well-known Bogolyubov formula

$$\lambda_{1,l}=-\frac{\hbar^2}{m}k_1l+\sqrt{\left(\frac{\hbar^2l^2}{2m}+v_l\right)^2-v_l^2}, \tag{14.61}$$

where the Fourier transform of the potential $v_l$ tends to zero for $l\gg\sqrt[3]{N}$ and the kinetic term then starts playing the dominant role.

We assume that $L_2$ is much greater than some standard length, for example, the electron radius $r_0$, and although $L_1\gg L_2$, we can choose a sufficiently large integer $n$. In other words, we have $L_2/r_0\to\infty$ and $L_1/r_0\to\infty$, but the vector $k_1=(n/L_1,0,0)$ is finite because $n\to\infty$. In the language of nonstandard analysis, this means that $L_2$ is an infinitely large (nonstandard) number, $L_1$ and $n_1$ are nonstandard numbers of higher order, and $k_1=(n/L_1,0,0)$ is a standard finite number. In this case, $k_2$ is equal to an infinitely small (nonstandard) zero, i.e., $k_2\cong 0$, and $k_1$ is a standard number.

### *14.5. The Case of an Odd Number of Neutrons*

We consider the system of Hamilton equations for fermions

$$\begin{aligned}
\Omega\Phi(x,y)&=-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\Phi(x,y)+\\
&\quad+2N\iint dx'\,dy'\,(V(x-y)+V(x'-y'))\Phi^+(x',y')\Phi(x,x')\Phi(y',y),\\
\Omega\Phi^+(x,y)&=-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\Phi^+(x,y)+\\
&\quad+2N\iint dx'\,dy'\,(V(x-x')+V(y-y'))\Phi(x',y')\Phi^+(x,x')\Phi^+(y',y).
\end{aligned} \tag{14.62}$$

The functions $\Phi^+(x,y)$ and $\Phi(x,y)$ are antisymmetric and satisfy normalization condition (14.32). We represent the interaction potential in the form of a Fourier series,

$$NV(x)=\sum_p v_pe^{ipx}, \qquad v_p=\frac{1}{L_1L_2^2}\int dx\,NV(x)e^{-ipx}, \qquad v_p=v_{-p}. \tag{14.63}$$

We seek the solution of system of equations (14.62), (14.32) in the form

$$\begin{aligned}
\Phi^+_{k_1,k_2}(x,y)&=\frac{1}{L_1L_2^2}e^{-ik_1(x+y)}\sin(k_2(x-y)),\\
\Phi_{k_1,k_2}(x,y)&=\frac{1}{L_1L_2^2}\sum_l\varphi_{k_2,l}e^{il(x-y)+ik_1(x+y)},
\end{aligned} \tag{14.64}$$

where $k_1$, $k_2$, and $l$ are three-dimensional vectors of the form $2\pi(n_1/L_1,n_2/L_2,n_3/L_2)$ and $n_1$, $n_2$, and $n_3$ are integers. The numbers $\varphi_{k_2,l}$ must satisfy the condition $\varphi_{k_2,l}=-\varphi_{k_2,-l}$. After the substitution, the eigenvalue is equal to

$$\Omega = \frac{\hbar^2}{m}(k_1^2 + k_2^2) + v_{2k_2} - v_0,$$

and the numbers $\varphi_{k_2,l}$ are

$$\varphi_{k_2,l} = -\frac{ib_{k_2,l}}{2} \pm \frac{1}{2}\sqrt{1 - b_{k_2,l}^2}, \qquad b_{k_2,l} \equiv \frac{\frac{\hbar^2}{m}(l^2 - k_2^2) + v_0 - v_{2k_2}}{v_{l-k_2} - v_{l+k_2}}.$$

We note that $b_{k_2,l} = -b_{k_2,-l}$. We set

$$\varphi_{k_2,l} = -\frac{ib_{k_2,l}}{2} + \frac{1}{2}\frac{v_{l-k_2} - v_{l+k_2}}{|v_{l-k_2} - v_{l+k_2}|}\sqrt{1 - b_{k_2,l}^2}.$$

Then $\varphi_{k_2,l}$ is equal to $-\varphi_{k_2,-l}$. For the latest work on carbon nanotube transport, see [105]–[107].

We consider the system of variational equations for fermions

$$\begin{aligned}
(\Omega - \lambda)F(x,y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)F(x,y) \\
&\quad + 2N \iint dx'\,dy'\,\bigl(V(\sqrt[3]{N}(x-y)) + V(\sqrt[3]{N}(x'-y'))\bigr) \\
&\quad \times r\bigl(G(x',y')\Phi(x,x')\Phi(y',y) + \Phi^+(x',y')F(x,x')\Phi(y',y) \\
&\quad + \Phi^+(x',y')\Phi(x,x')F(y',y)\bigr), \\
(\Omega + \lambda)G(x,y) &= -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)G(x,y) \\
&\quad + 2N \iint dx'\,dy'\,\bigl(V(\sqrt[3]{N}(x-x')) + V(\sqrt[3]{N}(y-y'))\bigr) \\
&\quad \times \bigl(F(x',y')\Phi^+(x,x')\Phi^+(y',y) + \Phi(x',y')G(x,x')\Phi^+(y',y) \\
&\quad + \Phi(x',y')\Phi^+(x,x')G(y',y)\bigr).
\end{aligned} \tag{14.65}$$

The solutions of system (14.65) are

$$\begin{aligned}
G_l(x,y) &= u_{1,l}\bigl(e^{i(-k_1+k_2)x+i(-k_1+l)y} - e^{i(-k_1+k_2)y+i(-k_1+l)x}\bigr) \\
&\quad + u_{2,l}\bigl(e^{i(-k_1-k_2)x+i(-k_1+2k_2+l)y} - e^{i(-k_1-k_2)y+i(-k_1+2k_2+l)x}\bigr), \\
F_l(x,y) &= v_{1,l}\bigl(e^{i(k_1+k_2)x+i(k_1+l)y} - e^{i(k_1+k_2)y+i(k_1+l)x}\bigr) \\
&\quad + v_{2,l}\bigl(e^{i(k_1-k_2)x+i(k_1+2k_2+l)y} - e^{i(k_1-k_2)y+i(k_1+2k_2+l)x}\bigr) \\
&\quad + \sum_{l' \neq l,\, l+2k_2} w_{l,l'}\bigl(e^{i(k_1+k_2+l-l')x+i(k_1+l')y} - e^{i(k_1+k_2+l-l')y+i(k_1+l')x}\bigr),
\end{aligned} \tag{14.66}$$

where $l \neq -k_2$ and the numerical coefficients $u_{1,l}$, $u_{2,l}$, $v_{1,l}$, $v_{2,l}$, and $w_{l,l'}$ are found from an infinite system of equations. This system of equations contains a closed subsystem of four equations for the coefficients $u_{1,l}$, $u_{2,l}$, $v_{1,l}$, and $v_{2,l}$ that can be written in standard form (14.49).

The matrix $M$ has the elements

$$M_{11} = B_l + \frac{v_{l-k_2}}{2}, \qquad M_{12} = M_{21} = \frac{v_{l+k_2} - v_{2k_2}}{2},$$

$$M_{13} = \frac{v_{l-k_2} - v_{l+k_2}}{2}, \qquad M_{14} = M_{23} = 0,$$

$$M_{22} = B_{l+2k_2} + \frac{v_{l+3k_2}}{2}, \qquad M_{24} = \frac{v_{l+3k_2} - v_{l+k_2}}{2},$$

$$M_{31} = 2i(v_{l-k_2} - v_0)\varphi_{k_2,l}, \qquad M_{32} = M_{41} = i(v_{2k_2} - v_{l+k_2})(\varphi_{k_2,l+2k_2} - \varphi_{k_2,l}),$$

$$M_{33} = -B_l - \frac{v_{l-k_2}}{2}, \qquad M_{34} = M_{43} = -\frac{v_{l+k_2} - v_{2k_2}}{2},$$

$$M_{42} = 2i(v_0 - v_{l+3k_2})\varphi_{k_2,l+2k_2}, \qquad M_{44} = -B_{l+2k_2} - \frac{v_{l+3k_2}}{2},$$

where

$$B_l = \frac{\hbar^2}{2m}(l^2 - k_2^2) + i(v_{l+k_2} - v_{l-k_2})\varphi_{k_2,l} - \frac{v_{2k_2}}{2}.$$

After the selection rules are applied, the eigenvalues corresponding to Eq. (14.49) also contain positive quasiparticles for $k_2 \neq 0$. For $k_2 = 0$, the eigenvalues are given by

$$\lambda_{1,l} = -\frac{\hbar^2}{m} l k_1 \pm \frac{\hbar^2 l^2}{2m}, \qquad \lambda_{2,l} = -\frac{\hbar^2}{m} l k_1 \pm \left|\frac{\hbar^2 l^2}{2m} + v_l - v_0\right|.$$

This means that we take the vectors $k_2 = (0, 1/L_2, 0)$ and $k_1 = (n/L_1, 0, 0)$. If $n \gg 1$ and consequently $k_1 \gg k_2$, then the velocity vector of the liquid is directed predominantly along the tube. As before, let $L_1 = \infty$, $L_2 = \infty$, and $n = \infty$ be nonstandard numbers, let $L_1 \gg L_2$, and let $k_1$ be a standard number. Then $k_2 \cong 0$, and $\varphi_{k_2,l}$ assumes nonstandard values for $k_2 \cong 0$. But if $0 \leq l \leq \sqrt{k_2} M$, where $M \leq \infty$, i.e., $0 \leq l \leq \infty$, then $\varphi_{k_2,l}$ can be regarded as a standard number. For $k_2 \cong 0$, taking the selection rule into account, by analogy with formula (14.61) in the boson case, we obtain

$$\lambda_{2,l} = -\frac{\hbar^2}{m} l k_1 + \left|\frac{\hbar^2 l^2}{2m} + v_l - v_0\right|. \tag{14.67}$$

By Sec. 1.2, we have

$$v_l - v_0 \sim -4\pi |l| \lim_{r\to\infty} r^4 V(r)$$

as $|l| \to 0$, and because $V(r)$ is an attractive potential as $r \to \infty$, we have

$$-4\pi \lim_{r\to\infty} r^4 V(r) = c > 0.$$

The modulus tends to infinity as $\hbar^2 l^2/(2m)$ as $l \to \infty$ because $v_l$ tends to infinity as $1/|l|^2$. Hence, the behavior of the modulus in (14.67) is similar to the variation of the Landau curve for bosons. By analogy with the Landau criterion, the criterion for $k_1$ has the form

$$|k_1| \leq \frac{m}{\hbar^2} \min_l \left|\frac{v_l - v_0}{|l|} + \frac{\hbar^2 |l|}{2m}\right|.$$

Of course, constraint (14.58) also applies here.

Thus, we see that the major role in fermionic superfluidity is played by the attraction at long distances, while the repulsion at short distances is important for the appearance of bosonic superfluidity. In both cases, the transition to the condensate state (for pairs in the Fermi liquid) is a phase transition that has no relation to superfluidity.

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