ISSN 1064-5624, Doklady Mathematics, 2012, Vol. 86, No. 1, pp. 515–518. © Pleiades Publishing, Ltd., 2012.
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MATHEMATICS

Fixed Points of Modular Contractive Maps[1]

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Presented by Academician I. A. Taimanov February 28, 2012

Received March 6, 2012

DOI: 10.1134/S1064562412040163

Banach's Theorem asserts that any contractive map of a complete metric space into itself admits a unique fixed point. This classical result, having numerous applications [10, 13], has been generalized in the framework of the theory of metric spaces in different directions, among which we mention [2, 11] (and references therein). In the case when norms on linear spaces are given indirectly or implicitly (as, e.g., in the theory of Orlicz spaces or the theory of modular spaces [14, 15]), generalizations of Banach's Theorem have been established in [1, 12].

The purpose of this paper is to present a result on the existence of fixed points of nonlinear maps in the context of the theory of metric modulars [5–7], which extends simultaneously the theory of modular spaces over linear spaces as well as the theory of metric spaces. In the case under consideration a (modular) contraction does not contract the distances between points, but it contracts some generalized average velocities, which correspond to the given modulars. Moreover, a new notion of convergence (modular convergence) can be defined, which is weaker than the metric convergence.

In Section 1 we present basic facts concerning modular (metric) spaces. In Section 2 we define a new notion of the modular convergence and establish a necessary and sufficient condition on the modular, under which the modular convergence is equivalent to the metric convergence (Lemma 2). In Section 3 we introduce the notion of modular contractive maps, study their relationship with Lipschitz continuous maps with respect to the corresponding metrics (Theorem 1) and formulate the main result of the paper on the existence of fixed points of modular contractive maps (Theorem 2). Finally, in the last Section 4 we present an application of Theorem 2 to the existence of solutions to Caratheodory-type differential equations with the right-hand side from the Orlicz space.

[1] The article was translated by the author.

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1. MODULAR SPACES

Let us recall basic definitions, notation and auxiliary facts from [5, 7] needed in the sequel.

A modular on a nonempty set X is a one-parameter family $w = \{w_\lambda\}_{\lambda > 0}$ of maps of the form $w_\lambda: X \times X \to [0, \infty]$ for $\lambda \in (0, \infty)$ satisfying, for all $x, y, z \in X$, the following three conditions: (i) $x = y$ if and only if $w_\lambda(x, y) = 0$ for all $\lambda > 0$; (ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$; and (iii) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$. The modular w on X is said to be: (a) strict if, in addition to (i), condition $w_\lambda(x, y) = 0$ for at least one $\lambda > 0$ implies $x = y$; (b) convex if, instead of the inequality in (iii), for all $\lambda, \mu > 0$ and $x, y, z \in X$, the following inequality holds:

$$\text{(iv) } w_{\lambda+\mu}(x, y) \le \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(y, z).$$

For instance, if (X, d) is a metric space with metric d, then the family $w = \{w_\lambda\}_{\lambda>0}$, given by $w_\lambda(x, y) = \frac{d(x, y)}{\lambda}$ for all $x, y \in X$, is a strict convex modular on X, which can be naturally interpreted as a field of absolute values of average velocities between the points x and y. In the general case a modular is a family of some generalized (nonclassical) average velocities: if $w_\lambda(x, y) = \infty$ for $\lambda \le d(x, y)$, and $w_\lambda(x, y) = 0$ for $\lambda > d(x, y)$, then $w = \{w_\lambda\}_{\lambda>0}$ is a nonstrict modular on X. Numerous examples of (convex) modulars are presented in [4–7], and also in Section 4.

The essential property of any modular w on X is that the function $\lambda \mapsto w_\lambda(x, y)$ is nonincreasing on $(0, \infty)$ for all $x, y \in X$; moreover, if w is convex, then, in addition, the function $\lambda \mapsto \lambda w_\lambda(x, y)$ is also nonincreasing. Thus, in $[0, \infty]$ there exist the limit from the right $w_{\lambda+0}(x, y)$ and the limit from the left $w_{\lambda-0}(x, y)$, for which we have $w_{\lambda+0}(x, y) \le w_\lambda(x, y) \le w_{\lambda-0}(x, y)$ [7].

Let us fix an element $x_0 \in X$ arbitrarily. Modular spaces (around x_0) are the following two sets:

$$X_w = \{x \in X\colon \lim_{\lambda \to \infty} w_\lambda(x, x_0) = 0\}$$

and

$$X_w^* = \{x \in X\colon w_\lambda(x, x_0) < \infty \text{ for some } \lambda = \lambda(x) > 0\}.$$

Clearly, $X_w \subset X_w^*$ (proper inclusion, in general), and in the case of a convex modular w on X these two spaces coincide. It was shown in [5–7] that X_w is a metric space with respect to the (implicitly defined) metric $d_w(x, y) = \inf\{\lambda > 0\colon w_\lambda(x, y) \le \lambda\}$ for all $x, y \in X_w$; if w is a convex modular on X, then a metric on $X_w^* = X_w$ can be defined by the rule $d_w^*(x, y) = \inf\{\lambda > 0\colon w_\lambda(x, y) \le 1\}$ for all $x, y \in X_w^*$.

Generally, the verification of axioms of a modular (i)–(iv) is not difficult, which allows to define efficiently nontrivial metrics in various functional spaces by means of the formulas mentioned above ([5–7]). In the next section we show that, given a modular, one can define a new type of convergence, which is weaker than the convergence in metric.

2. MODULAR CONVERGENCE

It is known ([7]) that if w is a convex modular on a set X, $\{x_n\}$ is a sequence in X_w^* and $x \in X_w^*$, then condition $\lim_{n \to \infty} d_w^*(x_n, x) = 0$ is equivalent to the condition $\lim_{n \to \infty} w_\lambda(x_n, x) = 0$ for all $\lambda > 0$ (a similar assertion is valid for any modular w if in the above we replace X_w^* by X_w and d_w^* —by d_w). The notion of metric convergence can be weakened if we assume the condition on the right in the assertion above to hold only for some $\lambda > 0$ (instead of all $\lambda > 0$).

Let w be a modular on X. A sequence $\{x_n\}$ from X_w^* is said to be modular convergent to an element $x \in X$, provided there exists a number $\lambda = \lambda(\{x_n\}, x) > 0$ such that $\lim_{n \to \infty} w_\lambda(x_n, x) = 0$ (in short, $x_n \xrightarrow{w} x$). Any such element x is called a modular limit of $\{x_n\}$.

Lemma 1. *Given a modular w on X, we have:* (a) *modular spaces X_w and X_w^* are closed with respect to the modular convergence* (*i.e., if $\{x_n\}$ is from X_w or X_w^*, $x \in X$ and $x_n \xrightarrow{w} x$, then $x \in X_w$ or $x \in X_w^*$, respectively*); (b) *if w is a strict modular on X, then the modular limit is determined uniquely* (*if it exists*).

In the next lemma we exhibit conditions, under which the metric convergence (with respect to d_w or d_w^*) is equivalent to the modular convergence.

Lemma 2. *The metric convergence in X_w^** (*with respect to d_w if w is a modular, and with respect to d_w^* if w is a convex modular*) *coincides with the modular convergence if and only if the modular w satisfies the following Δ_2-condition: if $\{x_n\} \subset X_w^*$, $x \in X_w^*$ and $\lambda > 0$ is such that $\lim_{n \to \infty} w_\lambda(x_n, x) = 0$, then $\lim_{n \to \infty} w_{\lambda/2}(x_n, x) = 0$.*

A counterpart of the completeness of a metric space in the context of the modular convergence is the following notion.

The modular space X_w^* is said to be modular complete if the conditions $\{x_n\} \subset X_w^*$ and $\lim_{n, m \to \infty} w_\lambda(x_n, x_m) = 0$ for some $\lambda > 0$ imply the existence of an element $x \in X_w^*$ such that $\lim_{n \to \infty} w_\lambda(x_n, x) = 0$.

3. MODULAR CONTRACTIVE MAPS

First, let us characterize Lipschitz continuous maps $T\colon X_w^* \to X_w^*$ with respect to the metrics d_w and d_w^* in terms of their underlying modulars w on X. Let $k > 0$ be a constant and $x, y \in X_w^*$.

Theorem 1. (a) *Condition $d_w(Tx, Ty) \le k d_w(x, y)$ is equivalent to $w_{k\lambda+0}(Tx, Ty) \le k\lambda$ for all $\lambda > 0$ such that $w_\lambda(x, y) \le \lambda$.*

(b) *Given a convex modular w on X, we have: $d_w^*(Tx, Ty) \le k d_w^*(x, y)$ if and only if $w_{k\lambda+0}(Tx, Ty) \le 1$ for all $\lambda > 0$ such that $w_\lambda(x, y) \le 1$.*

In particular, this implies that if $w_{k\lambda}(Tx, Ty) \le k w_\lambda(x, y)$ for all $\lambda > 0$, then $d_w(Tx, Ty) \le k d_w(x, y)$; and if $w_{k\lambda}(Tx, Ty) \le w_\lambda(x, y)$ for all $\lambda > 0$, where w is a convex modular, then $d_w^*(Tx, Ty) \le k d_w^*(x, y)$.

The following definition extends the notion of a contractive map to the case of maps on modular spaces.

Let w be a modular on X. A map $T\colon X_w^* \to X_w^*$ is said to be modular contractive (strongly modular contractive) if there exist constants $0 < k < 1$ and $\lambda_0 > 0$ such that $w_{k\lambda}(Tx, Ty) \le w_\lambda(x, y)$ ($w_{k\lambda}(Tx, Ty) \le k w_\lambda(x, y)$, respectively) for all $0 < \lambda \le \lambda_0$ and $x, y \in X_w^*$.

The main result of the paper is the following theorem on the existence of fixed points of modular contractive maps.

Theorem 2. *Let w be a strict convex modular on a set X such that X_w^* is modular complete, and $T\colon X_w^* \to X_w^*$ be a modular contractive map such that for each $\lambda > 0$ there exists $x_\lambda \in X_w^*$ such that $w_\lambda(x_\lambda, Tx_\lambda) < \infty$. Then T admits a fixed point, i.e., $Tx_* = x_*$ for some $x_* \in X_w^*$. In addition, if the modular w assumes only finite values on*

$(0, \infty) \times X_w^* \times X_w^*$, then the last hypothesis on T is redundant, the fixed point x_* of T is unique, and for each $\bar{x} \in X_w^*$ the sequence of iterations $\{T^n\bar{x}\}_{n=1}^{\infty}$ is modular convergent to x_*.

This theorem remains valid if we replace the terms "strict convex modular" by "strict modular" and "modular contractive map"—by "strongly modular contractive map." An application of Theorem 2 is presented in the next section.

4. APPLICATION

Let $\varphi: [0, \infty) \to [0, \infty)$ be a convex function, $\varphi(u) = 0$ only at $u = 0$, $\lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty$, $[a, b]$ be a closed interval in $\mathbb{R}$ $(a < b)$ and $(M, \|\cdot\|)$ be a reflexive Banach space (over $\mathbb{R}$ or $\mathbb{C}$) with the norm $\|\cdot\|$. Given $x_0 \in M$, we denote by X the set of all functions $x: [a, b] \to M$ such that $x(a) = x_0$. Given $\lambda > 0$ and $x, y \in X$, we set

$$w_\lambda(x, y) = \sup_P \sum_{i=1}^{m} \varphi\left(\frac{\|x(t_i) + y(t_{i-1}) - x(t_{i-1}) - y(t_i)\|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1}),$$

where the supremum is taken over all partitions $P = \{t_i\}_{i=0}^{m}$ of $[a, b]$, i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$. Then ([3, 4]) the family $w = \{w_\lambda\}_{\lambda > 0}$ is a strict convex modular on X, and one can show that the (nonlinear) modular space X_w^* (around the constant function $x_0(t) \equiv x_0$, $t \in [a, b]$) is modular complete. Recall that a function $x: [a, b] \to M$ is in X_w^* if and only if $x(a) = x_0$ and there exists a constant $\lambda = \lambda(x) > 0$ such that

$$w_\lambda(x, x_0) = \sup_P \sum_{i=1}^{m} \varphi\left(\frac{\|x(t_i) - x(t_{i-1})\|}{\lambda(t_i - t_{i-1})}\right)(t_i - t_{i-1}) < \infty;$$

the value $w_\lambda(x, x_0)$ with $\lambda = 1$ is usually said to be the total φ-variation of x in the sense of F. Riesz, Yu. T. Medvedev and W. Orlicz [4].

Denote by $\mathrm{AC}([a, b]; M)$ the set of all absolutely continuous functions $x: [a, b] \to M$, by $\mathrm{L}^1([a, b]; M)$ — the set of all strongly measurable and Bochner summable functions $x: [a, b] \to M$ and by $\mathrm{L}^\varphi([a, b]; M)$— the Orlicz space of all strongly measurable functions x: $[a, b] \to M$ such that $\int_a^b \varphi\left(\frac{\|x(t)\|}{\lambda}\right) dt < \infty$ for some $\lambda > 0$.

The following criterion is well known [3, 4, 8]: given $x: [a, b] \to M$, we have: $X_w^* : x \in X_w^*$ if and only if $x \in \mathrm{AC}([a, b]; M)$, $x(a) = x_0$ and $w_\lambda(x, x_0) = \int_a^b \varphi\left(\frac{\|x'(t)\|}{\lambda}\right) dt < \infty$ for some number $\lambda = \lambda(x) > 0$, and so, the strong derivative x' (evaluated in the norm $\|\cdot\|$), which is defined almost everywhere on $[a, b]$, belongs to the space $L^\varphi([a, b]; M)$.

Theorem 3. *Let $f: [a, b] \times M \to M$ be a (Carathéodory-type) function satisfying the following two conditions*:

(C.1) *for each $x \in M$ the function $f(\cdot, x) = [t \mapsto f(t, x)]: [a, b] \to M$ is strongly measurable and $f(\cdot, y_0) \in \mathrm{L}^\varphi([a, b]; M)$ for some $y_0 \in M$;*

(C.2) *there exists a constant $L > 0$ such that $\|f(t, x) - f(t, y)\| \le L\|x - y\|$ for almost all $t \in [a, b]$ and all $x, y \in M$.*

Then the integral operator

$$(Tx)(t) = x_0 + \int_a^t f(s, x(s))ds, \quad x \in X_w^*, \quad t \in [a, b]$$

maps the modular space X_w^* into itself, and the following inequality holds:

$$w_{L(b-a)\lambda}(Tx, Ty) \le w_\lambda(x, y) \ \text{for all} \ \lambda > 0 \ \text{and} \ x, y \in X_w^*.$$

As a corollary, we note that, under the assumptions (C.1) and (C.2) from Theorem 3, for each $x_0 \in M$ and any interval $[a, b]$ such that $L(b - a) < 1$ Theorem 2 implies the existence of a fixed point $x \in X_w^*$ of the integral operator T, and so, the Cauchy problem $x'(t) = f(t, x(t))$ for almost all $t \in [a, b]$ and $x(a) = x_0$ admits a solution $x \in X_w^*$. This generalizes certain results on the existence of absolutely continuous solutions to the Caratheodory differential equations under the assumption that $f(\cdot, y_0) \in \mathrm{L}^1([a, b]; M)$ for some $y_0 \in M$ (see [9]).

Finally, it is to be noted that the modular w, defined at the beginning of this section, has been chosen on the basis that in the corresponding modular space X_w^* the modular convergence is not equivalent to the metric one.

Example. Set $\varphi(u) = e^u - 1$ if $u \ge 0$, $[a, b] = [0, 1]$, $M = \mathbb{R}$ and $x_0 = 0$. Define a sequence of functions $x_n \in X_w^*$, $n \in \mathbb{N}$, as follows: $x_n(t) = t - (t + \alpha_n)\log(t + \alpha_n) + \alpha_n\log\alpha_n$ for all $0 \le t \le 1$, where $\alpha_n = \frac{1}{n}$, and set $x(t) = t - t\log t$ if $0 < t \le 1$, and $x(0) = 0$. Then $x \in X_w^*$, x_n converges uniformly on $[0,1]$ to x as $n \to \infty$, $w_\lambda(x_n, x) \to 0$ as $n \to \infty$ only for $\lambda > 1$ (and so, $x_n \xrightarrow{w} x$), and at the same time $d_w^*(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.

ACKNOWLEDGMENTS

This study was carried out within "The National Research University Higher School of Economics' Academic Fund Program in 2010–2012, research grant no. 10-01-0071."

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