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Minimal Basis of the Symmetry Algebra for Three-Frequency Resonance

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Abstract. An explicit description of a finite minimal basis of generators is given for the algebra of symmetries of a generic quantum three-frequency resonance oscillator.

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INTRODUCTION

In quantum mechanics, when studying states localized near a stable equilibrium, an important role is played by systems with Hamiltonians of the form

$$\text{“oscillator”} + \text{“perturbation”}. \tag{1}$$

In particular, these systems were studied from the viewpoint of normal forms (see, e.g., [1–4]) and from the viewpoint of general semiclassical approximation theory (see [6–9]). After the quantum averaging of system (1), the perturbation turns out to commute with the leading term, i.e., with the oscillator. In other words, the averaged perturbation becomes an element of the symmetry algebra of the oscillator.

If the frequencies of the oscillator are not in resonance, then the symmetry algebra is trivial and commutative. The complicated behavior of the dynamics and spectrum happens under a resonance. In this case, the symmetry algebra is noncommutative.

The simplest example of resonant oscillator is the isotropic one, which has equal frequencies for all the n degrees of freedom. The symmetry algebra here is the simple Lie algebra $\mathrm{su}(n)$. In particular, the two-dimensional case $n = 2$ (the Schwinger model) was studied in [10–14], and the three-dimensional case ($n = 3$) in [15–17].

In the anisotropic resonant case, for a generic set of commensurable frequencies, the symmetry algebra is no longer a Lie algebra. It is described by finitely many nonlinear permutation relations. This fact was discovered in [18–20]. In this generic case, a nontrivial problems arises: describe a finite basis of generators of the symmetry algebra. For a generic n-dimensional resonance, the construction of the basis is unknown. At present, there is only an existence proof for a finite basis and an upper bound for the number of its elements, see [19].

In the present paper, the problem posed in [19] is solved for the first nontrivial case $n = 3$. For a generic three-frequency resonance oscillator, we give a complete description of the finite minimal basis of generators of the algebra of its symmetries. This description was announced in [21].

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1. SYMMETRIES OF A RESONANT OSCILLATOR

The quantum oscillator with frequencies $\omega_1, \dots, \omega_n$ is described by the Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{j=1}^{n} \big(-\hbar^2 \partial^2/\partial q_j^2 + \omega_j^2 q_j^2 - \hbar\omega_j \big). \tag{2}$$

This operator acts with respect to the variables $q_1, \dots, q_n$ in the space $L^2(\mathbb{R}^n)$. Assume that the parameter $\hbar$ and all frequencies ω_j are positive.

As was shown in [19], studying the symmetry algebra of the operator $\hat{H}$ (i.e., the algebra of operators in $L^2(\mathbb{R}^n)$ commuting with $\hat{H}$) can be reduced to studying the cases for which the following resonance condition is satisfied:

$$\textit{all the frequencies } \omega_j \textit{ are integer and pairwise coprime.} \tag{3}$$

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Proposition 1. *In the case of resonance* (3)*, the algebra of symmetries of the oscillator* (2) *is generated by the operators*

$$\hat{z}^{*l}\hat{z}^{m}. \tag{4}$$

Here the operators $\hat{z} = (\hat{z}_1, \dots, \hat{z}_n)$ *are defined by the formula* $\hat{z}_j = \big(\hbar\partial/\partial q_j + \omega_j q_j\big)/\sqrt{2\omega_j}$*, the operators* $\hat{z}^* = (\hat{z}_1^*, \dots, \hat{z}_n^*)$ *are adjoint to* $\hat{z}$*, and the vectors* $l = (l_1, \dots, l_n)$ *and* $m = (m_1, \dots, m_n)$ *satisfy the conditions*

$$\sum_{j=1}^{n} \omega_j (l_j - m_j) = 0, \qquad l_j, m_j \in \mathbb{Z}_+. \tag{5}$$

Note that there are infinitely many generators of type (4), (5). They are not independent. Some of them can be represented as polynomials in other generators. Therefore, there is a problem of extracting smallest subsets from the large set of generators (4), (5).

Definition 1. By a *basis* we mean a minimal subset of the set of generators (4), (5) still generating the entire algebra of symmetries.

In the two-frequency case (for $n = 2$), the problem of extracting the minimal basis of generators can be solved easily (see [19]).

Proposition 2. *For* $n = 2$*, a basis of the set of generators* (4), (5) *is formed by the four operators* $\hat{S}_1 = \hat{z}_1^*\hat{z}_1$*,* $\hat{S}_2 = \hat{z}_2^*\hat{z}_2$*,* $\hat{A} = \hat{z}_2^{*\omega_1}\hat{z}_1^{\omega_2}$*, and* $\hat{A}^* = \hat{z}_1^{*\omega_2}\hat{z}_2^{\omega_1}$*.*

In the case of multifrequency resonance ($n \geqslant 3$), the problem of extracting the basis is much more difficult. The following fact was proved in [19].

Proposition 3. *For any* $n \in \mathbb{N}$*, there exists a finite basis of generators* (4)*,* (5)*.*

In the present paper, the problem of explicit description of the basis is solved in the three-frequency case ($n = 3$).

2. BASIS AND MINIMAL VECTORS

Each generator (4) is uniquely determined by the pair of vectors $l, m \in \mathbb{Z}_+^n$ satisfying condition (5). Therefore, the problem of extracting the finite basis of generators (4), (5) can be reformulated in the language of vectors.

To this end, we introduce the notion of minimal vector. Choose a frequency vector $\omega \in \mathbb{N}^n$. A vector $\rho \in \mathbb{Z}^n$ with integer Cartesian coordinates ρ_j is referred to as a *resonance vector* if it is orthogonal to the frequency vector, i.e.,

$$\sum_{j=1}^{n} \omega_j \rho_j = 0. \tag{6}$$

The set of all resonance vectors is called the *resonance lattice* and denoted by $\mathcal{R} = \mathcal{R}[\omega]$.

Since all the frequencies ω_j are positive, it follows that the Cartesian coordinates of the resonance vector cannot be of the same sign. We say that a resonance vector $\rho \in \mathcal{R}$ belongs to the *normal sublattice* $\mathcal{R}^{j_1,\dots,j_k} \subset \mathcal{R}$ if its Cartesian coordinates with indices $j_1, \dots, j_k$ are nonnegative and the other coordinates are nonpositive. Therefore, since k can take any values between 1 and $n-1$, the total number of normal sublattices is $\sum_{k=1}^{n-1} C_n^k = 2^n - 2$.

For example, for $n = 2$, there are only two normal sublattices, $\mathcal{R}^1$ and $\mathcal{R}^2$; for $n = 3$, there are six normal sublattices, namely, $\mathcal{R}^1$, $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{R}^{12}$, $\mathcal{R}^{23}$, and $\mathcal{R}^{31}$.

The union of all normal sublattices gives the entire resonance lattice $\mathcal{R}$.

If all Cartesian coordinates of the resonance vector ρ are nonzero, then it belongs to only one normal sublattice. Such a vector ρ is called an *internal* vector of the given sublattice. If at least one of the coordinates of the resonance vector ρ is zero, then it belongs to the intersection of at least two normal sublattices. Such a vector is called a *face* vector of these sublattices.

For example, for $j_1 \neq j_2$, the intersection $\mathcal{R}^{j_1} \cap \mathcal{R}^{j_2}$ consists of the zero vector only, and the intersection $\mathcal{R}^{j_1} \cap \mathcal{R}^{j_1,j_2}$ consists of all resonance vectors whose coordinate with index j_1 is nonnegative, the coordinate with index j_2 is zero, and the other coordinates are nonpositive.

Note that each normal sublattice is an additive semigroup.

Definition 2. A nonzero resonance vector is said to be *minimal* if it is not the sum of any two nonzero vectors in a normal sublattice.

Denote the set of all minimal vectors by $\mathcal{M} = \mathcal{M}[\omega]$.

Proposition 4 [19, 21]**.** *Any resonance vector ρ can be decomposed into the sum of minimal vectors with nonnegative coefficients,*

$$\rho = \sum_{\varkappa\in\mathcal{M}_\rho} n^\rho_\varkappa\,\varkappa, \qquad n^\rho_\varkappa \in \mathbb{Z}_+, \quad \mathcal{M}_\rho \subset \mathcal{M}. \tag{7}$$

Here $\mathcal{M}_\rho$ stands for the set of minimal vectors in the intersection of all normal sublattices to which the vector ρ belongs.

Note that the decomposition (7) is not unique in general.

By Proposition 4, one can reduce the problem of extracting a finite basis of generators (4), (5) to the problem of describing the set $\mathcal{M}$ of minimal vectors.

Introduce the following auxiliary notation. For each resonance vector ρ, define the vectors $\rho^+, \rho^- \in \mathbb{Z}^n_+$ by the rule

$$\rho^+_j \overset{\text{def}}{=} \begin{cases} \rho_j, & \rho_j \geqslant 0, \\ 0, & \rho_j \leqslant 0, \end{cases} \qquad \rho^-_j \overset{\text{def}}{=} \begin{cases} 0, & \rho_j \geqslant 0, \\ -\rho_j, & \rho_j \leqslant 0 \end{cases} \quad (j = 1, \dots, n). \tag{8}$$

Theorem 1. *Let the resonance condition* (3) *be satisfied. Then the operators*

$$\hat{S}_j \overset{\text{def}}{=} \hat{z}^*_j \hat{z}_j \quad (j = 1, \dots, n), \qquad \hat{A}_\rho \overset{\text{def}}{=} \hat{z}^{*\rho^+} \hat{z}^{\rho^-} \quad (\rho \in \mathcal{M}) \tag{9}$$

form a basis of generators of the symmetry algebra of the n-frequency oscillator (2)*. Here $\mathcal{M}$ stands for the set of all minimal vectors.*

To prove Theorem 1, we need the following lemma.

Lemma 1. *Assume that the resonance vectors $\varkappa$ and θ belong to the same normal sublattice. Then* (a) $(\varkappa + \theta)^\pm = \varkappa^\pm + \theta^\pm$, (b) $\varkappa^+_j \theta^-_j \quad (j = 1, \dots, n)$.

Proof of Theorem 1. Note first that the operators $\hat{S}_j$ and $\hat{A}_\rho$ can be represented in the form (4), (5). In representation (4), the operator $\hat{S}_j$ corresponds to the vectors $l = m = \Delta^{(j)}$, where $\Delta^{(j)}$ stands for the vector whose jth coordinate is equal to 1 and the other coordinates are zero. The operator $\hat{A}_\rho$ in representation (4) corresponds to the vectors $l = \rho^+$ and $m = \rho^-$; these vectors satisfy conditions (5), because $\rho^+ - \rho^- = \rho$ is a resonance vector (see (6)).

Let us now show that any generator $\hat{z}^{*l}\hat{z}^m$ (4), (5) can be represented as a polynomial in operators (9). First, consider the case in which the vectors l and m (5) satisfy the additional condition

$$l_j m_j = 0 \qquad (j = 1, \dots, n). \tag{10}$$

Then it follows from (5), (6), and (8) that $\rho \overset{\text{def}}{=} l - m$ is a resonance vector and that $\rho^+ = l$ and $\rho^- = m$. By Proposition 4, ρ can be decomposed into a linear combination $\rho = \sum_{\varkappa\in\mathcal{M}_\rho} n^\rho_\varkappa\,\varkappa$ of minimal vectors $\varkappa$ in a subset $\mathcal{M}_\rho$ belonging to the normal sublattice. Here the coefficients $n^\rho_\varkappa$ are nonnegative. Therefore, Lemma 1 can be applied to the vectors $n^\rho_\varkappa\,\varkappa$, which gives

$$l = \rho^+ = \sum_{\varkappa\in\mathcal{M}_\rho} n^\rho_\varkappa\,\varkappa^+, \qquad m = \rho^- = \sum_{\varkappa\in\mathcal{M}_\rho} n^\rho_\varkappa\,\varkappa^-, \qquad \tilde{\varkappa}^+_j\,\tilde{\tilde{\varkappa}}^-_j = 0 \quad (j = 1, \dots, n;\ \tilde{\varkappa}, \tilde{\tilde{\varkappa}} \in \mathcal{M}_\rho).$$

Hence, for the generator (4), (5), (10), we have the representation

$$\hat{z}^{*l}\hat{z}^m = \prod_{\varkappa\in\mathcal{M}_{l-m}} (\hat{z}^{*\varkappa^+})^{n^{l-m}_\varkappa} \cdot \prod_{\varkappa\in\mathcal{M}_{l-m}} (\hat{z}^{\varkappa^-})^{n^{l-m}_\varkappa}$$

in which any two multipliers commute, $[\hat{z}^{*\tilde{\varkappa}^+}, \hat{z}^{*\tilde{\tilde{\varkappa}}^+}] = [\hat{z}^{*\tilde{\varkappa}^+}, \hat{z}^{\tilde{\tilde{\varkappa}}^-}] = [\hat{z}^{\tilde{\varkappa}^-}, \hat{z}^{\tilde{\tilde{\varkappa}}^-}] = 0$ ($\tilde{\varkappa}, \tilde{\tilde{\varkappa}} \in \mathcal{M}_{l-m}$). Transposing these multipliers, we obtain the following expression for the generator (4), (5), (10) in terms of the generators (9):

$$\hat{z}^{*l}\hat{z}^m = \prod_{\varkappa\in\mathcal{M}_{l-m}} (\hat{z}^{*\varkappa^+}\hat{z}^{\tilde{\varkappa}^-})^{n^{l-m}_\varkappa} = \prod_{\varkappa\in\mathcal{M}_{l-m}} (\hat{A}_\varkappa)^{n^{l-m}_\varkappa}.$$

Assume now that the vectors l and m satisfying (5) do not satisfy the additional condition (10), i.e., for some j, we have $l_j m_j \neq 0$. Then $l_j \geqslant 1$ and $m_j \geqslant 1$. In this case, the vectors $l' \stackrel{\text{def}}{=} l - \Delta^{(j)}$ and $m' \stackrel{\text{def}}{=} m - \Delta^{(j)}$ also satisfy conditions (5). Therefore,

$$\hat{z}^{*l} \hat{z}^m = \hat{z}^{*l'} \hat{z}^{m'} (\hat{S}_j - \hbar m'_j), \tag{11}$$

which expresses the generator $\hat{z}^{*l} \hat{z}^m$ in terms of the generators $\hat{S}_j$ and $\hat{z}^{*l'} \hat{z}^{m'}$. (Here we have used the commutation relation $\hat{S}_j \hat{z}_k = \hat{z}_k(\hat{S}_j - \hbar\, \delta_{jk})$.) Applying formula (11) several times, we can express $\hat{z}^{*l} \hat{z}^m$ in terms of $\hat{S}_1, \dots, \hat{S}_n$ and $\hat{z}^{*\tilde{l}} \hat{z}^{\tilde{m}}$, where $\tilde{l}$ and $\tilde{m}$ satisfy conditions (5), (10). Hence, any generator (4), (5) can be represented as a polynomial in the generators (9).

It remains to show that no generator of the form (9) can be represented as a polynomial in the other generators in (9). The operators (9) can be represented in the form

$$\hat{S}_j = s_j(\overset{2}{\hat{z}^*}, \overset{1}{\hat{z}}), \qquad \hat{A}_\rho = a_\rho(\overset{2}{\hat{z}^*}, \overset{1}{\hat{z}}),$$

where the symbols $s_j(\overline{z}, z) = \overline{z}_j z_j$ and $a_\rho(\overline{z}, z) = \overline{z}^{\rho^+} z^{\rho^-}$ are the polynomials in the variables $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and the complex conjugate variables $\overline{z}$ Hence, it suffices to prove our statement for symbols only. Moreover, since the symbols s_j and a_ρ are monomials, the assumption that one of the generators (9) can be represented as a polynomial in the other generators (9) implies that the symbol of this generator is the product of the other symbols raised to certain powers. This follows from the fact that the noncommutative product of polynomials in z and $\overline{z}$ is equal to their usual commutative product plus a polynomial of lower degree. For example, if $\hat{A}_\rho$ can be polynomially expressed in terms of $\hat{S}_j$ $(j = 1, \dots, n)$ and $\hat{A}_\sigma$ $(\sigma \in \mathcal{M}, \sigma \neq \rho)$, then

$$a_\rho(\overline{z}, z) = \prod_{j=1}^{n} s_j(\overline{z}, z)^{k_j} \cdot \prod_{\sigma \in \mathcal{M},\, \sigma \neq \rho} a_\sigma(\overline{z}, z)^{n_\sigma},$$

where k_j and n_σ are some nonnegative integers. Equating the exponents of $\overline{z}$ and z on the left- and right-hand sides of this formula, we obtain the vector relations

$$\rho^\pm = \sum_{j=1}^{n} k_j \Delta^{(j)} + \sum_{\sigma \in \mathcal{M},\, \sigma \neq \rho} n_\sigma \sigma^\pm, \tag{12}$$

which imply that the minimal vector $\rho = \rho^+ - \rho^-$ can be decomposed into a linear combination $\rho = \sum_{\sigma \in \mathcal{M},\, \sigma \neq \rho} n_\sigma\, \sigma$ of nonzero vectors σ with nonnegative integer coefficients n_σ. By the definition of the minimal vector, this is possible only if this linear combination contains vectors $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ belonging to different normal sublattices ($n_{\tilde{\sigma}} \neq 0$ and $n_{\tilde{\tilde{\sigma}}} \neq 0$). There is at least one index j for which the jth coordinates of the vectors have opposite signs, i.e., $\tilde{\sigma}_j\, \tilde{\tilde{\sigma}}_j < 0$. Therefore, we have the inequality $(n_{\tilde{\sigma}}\, \tilde{\sigma} + n_{\tilde{\tilde{\sigma}}}\, \tilde{\tilde{\sigma}})_j^\pm < n_{\tilde{\sigma}}\, \tilde{\sigma}_j^\pm + n_{\tilde{\tilde{\sigma}}}\, \tilde{\tilde{\sigma}}_j^\pm$. Hence,

$$\rho_j^\pm = \Big(\sum_{\sigma \in \mathcal{M},\, \sigma \neq \rho} n_\sigma\, \sigma \Big)_j^\pm < \sum_{\sigma \in \mathcal{M},\, \sigma \neq \rho} n_\sigma\, \sigma_j^\pm.$$

However, this contradicts relations (12). Therefore, the generator $\hat{A}_\rho$ cannot be represented as a polynomial in the other generators (9). Obviously, $\hat{S}_j$ cannot be represented as polynomials in the other generators (9) either. Thus, the operators (9) satisfy the conditions for a basis.

Remark 1. By Theorem 1, the basis of generators (4), (5) is related to the set $\mathcal{M}$ of minimal vectors. Therefore, the basis of generators (4), (5) can naturally be referred to as *minimal basis.*

3. SET $\mathcal{M}$ OF MINIMAL VECTORS IN THE CASE OF $n = 3$

Thus, the problem in question is reduced to the problem of describing the set $\mathcal{M}$ of minimal vectors. We give the solution of the latter problem in the three-frequency case ($n = 3$). Let us first describe the set $\mathcal{R}$ of resonance vectors for $n = 3$. Consider the Diophantine equation

$$\mu\omega_1 + \nu\omega_2 + \omega_3 = 0 \tag{13}$$

for the unknown integers μ and ν.

Lemma 2. *Let ω_1, ω_2, and ω_3 be pairwise coprime positive integers. Then there is a unique solution of Eq.* (13) *such that*

$$0 \leqslant \nu \leqslant \omega_1 - 1. \tag{14}$$

In particular, if $\omega_1 = 1$, then this solution is given by $\mu = -\omega_3$ and $\nu = 0$. If $\omega_1 \geqslant 2$, then $\nu \geqslant 1$.

Proof. Consider the set of numbers of the form $\nu\omega_2 + \omega_3$, where $\nu \in \{0, 1, \dots, \omega_1 - 1\}$. Divide each of these numbers by ω_1 with the remainder, $\nu\omega_2 + \omega_3 = d\omega_1 + r$, $0 \leqslant r \leqslant \omega_1 - 1$. Note that, since ω_1 and ω_2 are coprime and inequality (14) holds, different values of the parameter ν are associated with different remainders r. Therefore, the number of values attained by r coincides with the number of values of ν, and hence is equal to ω_1. Therefore, r takes (only once) all the values in the set $\{0, 1, \dots, \omega_1 - 1\}$. In particular, for some ν (which is unique), the corresponding remainder is zero. For this value of ν, the number $\nu\omega_2 + \omega_3$ is divisible by ω_1, i.e., $\mu \stackrel{\text{def}}{=} -\dfrac{\nu\omega_2 + \omega_3}{\omega_1}$ is an integer. These numbers (μ, ν) form a pair giving a unique solution of Eq. (13) satisfying condition (14).

For each $k \in \mathbb{Z}_+$, introduce the notation

$$\nu^{(k)} = k\nu \ (\mathrm{mod}\, \omega_1), \qquad \mu^{(k)} = -\frac{k\omega_3 + \nu^{(k)}\omega_2}{\omega_1}, \tag{15}$$

where ν is the solution of the Diophantine equation (13) with condition (14). The numbers $\mu^{(k)}$ and $\nu^{(k)}$ are integer, and $0 \leqslant \nu^{(k)} \leqslant \omega_1 - 1$. If $k = 0$, then $\mu^{(0)} = \nu^{(0)} = 0$; for $k = 1$, we obtain $\mu^{(1)} = \mu$ and $\nu^{(1)} = \nu$.

Proposition 5. *In the three-frequency case, under condition* (3)*, the resonance lattice $\mathcal{R} = \mathcal{R}^{23} \cup \mathcal{R}^{31} \cup \mathcal{R}^{12} \cup \mathcal{R}^{1} \cup \mathcal{R}^{2} \cup \mathcal{R}^{3}$ has the following structure.*

The normal sublattice $\mathcal{R}^{23}$ consists of the resonance vectors

$$(\mu^{(k)} - l\omega_2, \nu^{(k)} + l\omega_1, k), \qquad k \in \mathbb{Z}_+, \quad l \in \mathbb{Z}_+. \tag{16}$$

The vectors of the other normal sublattices $\mathcal{R}^{31}$ and $\mathcal{R}^{12}$ can be obtained from the description of the vectors in $\mathcal{R}^{23}$ by cyclic permutation of the indices 1, 2, 3*. The vectors in the normal sublattice $\mathcal{R}^{j}$ have the form $(-\sigma)$, where σ is the resonance vector in the sublattice $\mathcal{R}^{kl}$, and k and l are the indices complementing the index j to the triple of the indices* 1, 2, 3*.*

Proof. Let $n = 3$. By definition, the normal sublattice $\mathcal{R}^{23}$ consists of the vectors with Cartesian coordinates (x, y, k), where $k \in \mathbb{Z}_+$ and (x, y) are integer solutions of the equation

$$x\omega_1 + y\omega_2 + k\omega_3 = 0 \tag{17}$$

under the conditions

$$x \leqslant 0, \qquad y \geqslant 0. \tag{18}$$

It follows from the second formula in (15) that $(x, y) = (\mu^{(k)}, \nu^{(k)})$ is a particular solution of the inhomogeneous equation (17). Since the frequencies ω_1 and ω_2 are coprime, the general solution of the corresponding homogeneous equation $\tilde{x}\omega_1 + \tilde{y}\omega_2 = 0$ has the form $(\tilde{x}, \tilde{y}) = (-l\omega_2, l\omega_1)$, where $l \in \mathbb{Z}$. Hence, the general solution of Eq. (17) is given by the formula $(x, y) = (\mu^{(k)} - l\omega_2, \nu^{(k)} + l\omega_1)$. It remains to note that conditions (18) are satisfied for $l \geqslant 0$. As a result, we see that the normal sublattice $\mathcal{R}^{23}$ consists of the vectors (16).

The other statements in Proposition 6 are obvious.

Let us now describe the set $\mathcal{M}$ of minimal vectors in the case of $n = 3$.

Theorem 2. *In the three-frequency case, under condition* (3)*, the minimal vectors in the resonance lattice $\mathcal{R} = \mathcal{R}^{23} \cup \mathcal{R}^{31} \cup \mathcal{R}^{12} \cup \mathcal{R}^{1} \cup \mathcal{R}^{2} \cup \mathcal{R}^{3}$ have the following structure.*

If $\omega_1 = 1$, then there are no internal minimal vectors in the sublattice $\mathcal{R}^{23}$.

If $\omega_1 \geqslant 2$, then all internal minimal vectors in the sublattice $\mathcal{R}^{23}$ are determined by the sequence

$$(\mu^{(k)}, \nu^{(k)}, k), \qquad k = 1, \dots, \omega_1 - 1, \tag{19}$$

and the vector with index k is preserved in the sequence (19) *only if $\nu^{(k)} < \nu^{(j)}$ for any j in $\{1, \dots, k-1\}$.*

The face minimal vectors in $\mathcal{R}^{23}$ have the form

$$(-\omega_3, 0, \omega_1) \in \mathcal{R}^{23} \cap \mathcal{R}^{3}, \qquad (-\omega_2, \omega_1, 0) \in \mathcal{R}^{23} \cap \mathcal{R}^{2}. \tag{20}$$

The minimal vectors in the normal sublattices $\mathcal{R}^{31}$ and $\mathcal{R}^{12}$ can be obtained from the above description of the vectors in $\mathcal{R}^{23}$ by cyclic permutation of the indices 1, 2, 3*.*

The minimal vectors in the normal sublattices $\mathcal{R}^j$ have the form $(-\sigma)$, where σ is the minimal vector in the sublattice in $\mathcal{R}^{kl}$ and k and l are the indices complementing the index j to the triple of the indices 1, 2, 3.

To prove Theorem 2, we need the following lemma.

Lemma 3. *Let ω_1 and ω_2 be coprime. Then, for any values of $j, k \in \mathbb{Z}_+$, only one of the following two conditions is possible for the numbers* (15): *either $\nu^{(j+k)} = \nu^{(j)} + \nu^{(k)}$ and $\mu^{(j+k)} = \mu^{(j)} + \mu^{(k)}$ or $\nu^{(j+k)} = \nu^{(j)} + \nu^{(k)} - \omega_1$ and $\mu^{(j+k)} = \mu^{(j)} + \mu^{(k)} + \omega_2$.*

Proof of Lemma 3. It follows from (15) that $\nu^{(j+k)} - \nu^{(j)} - \nu^{(k)}$ can be divided by ω_1. However, the numbers $\nu^{(j+k)}$, $\nu^{(j)}$, and $\nu^{(k)}$ satisfy the conditions $0 \leqslant \nu^{(j+k)} \leqslant \omega_1 - 1$, $0 \leqslant \nu^{(j)} \leqslant \omega_1 - 1$, and $0 \leqslant \nu^{(k)} \leqslant \omega_1 - 1$. Therefore, $2 - 2\omega_1 \leqslant \nu^{(j+k)} - \nu^{(j)} - \nu^{(k)} \leqslant \omega_1 - 1$. Only two numbers in the family $\{2 - 2\omega_1, 3 - 2\omega_1, \dots, \omega_1 - 1\}$ are divisible by ω_1. These are 0 and $-\omega_1$. Hence, either $\nu^{(j+k)} - \nu^{(j)} - \nu^{(k)} = 0$ or $\nu^{(j+k)} - \nu^{(j)} - \nu^{(k)} = -\omega_1$. In the first case, $\mu^{(j+k)} - \mu^{(j)} - \mu^{(k)} = 0$ by (15). In the other case, $\mu^{(j+k)} - \mu^{(j)} - \mu^{(k)} = \omega_2$.

Proof of Theorem 2. It is sufficient to describe the minimal vectors in the normal sublattice $\mathcal{R}^{23}$. Recall that the sublattice $\mathcal{R}^{23}$ consists of the vectors $(\mu^{(k)} - l\omega_2, \nu^{(k)} + l\omega_1, k)$, $k \in \mathbb{Z}_+$, $l \in \mathbb{Z}_+$ (see (16)). We are to extract the subset of minimal vectors from this set.

First, consider the vectors corresponding to the value $k = 0$, i.e., $\rho^{(l)} \overset{\text{def}}{=} (-l\omega_2, l\omega_2, 0)$, $l \in \mathbb{Z}_+$. These are the face vectors in the intersection $\mathcal{R}^{23} \cap \mathcal{R}^2$. For $l \geqslant 2$, the vector $\rho^{(l)}$ is not minimal, because it can be decomposed into the sum of two nonzero vectors from the sublattice $\mathcal{R}^{23}$, $\rho^{(l)} = \rho^{(l-1)} + \rho^{(1)}$. Further, for $l = 0$, we obtain the zero vector $\rho^{(0)}$. By Definition 2, this vector is also not minimal. Finally, for $l = 1$, we obtain the vector $\rho^{(1)} = (-\omega_2, \omega_1, 0)$. Let us show that this vector is minimal. Assume that $\rho^{(1)}$ can be represented as the sum $\rho^{(1)} = \tilde{\rho} + \tilde{\tilde{\rho}}$ of vectors $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ in the same normal sublattice. Then the coordinates with index 3 of these vectors must be zero. Hence, $\tilde{\rho} = (-\tilde{j}\omega_2, \tilde{j}\omega_1, 0)$ and $\tilde{\tilde{\rho}} = (-\tilde{\tilde{j}}\omega_2, \tilde{\tilde{j}}\omega_1, 0)$, where $\tilde{j}$ and $\tilde{\tilde{j}}$ stand for some integers such that $\tilde{j} + \tilde{\tilde{j}} = 1$. Since $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ belong to the same normal sublattice, $\tilde{j}$ and $\tilde{\tilde{j}}$ cannot have opposite signs. Hence, one of the numbers $\tilde{j}$ and $\tilde{\tilde{j}}$ vanishes, and therefore one of the vectors $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ is zero. Thus, we have proved that $\rho^{(1)}$ is minimal.

Now let $k \geqslant 1$. Note first that, for $l \geqslant 1$, the vector (16) is not minimal, because it can be decomposed into the sum of vectors in the normal sublattice $\mathcal{R}^{23}$, $(\mu^{(k)} - l\omega_2, \nu^{(k)} + l\omega_1, k) = (\mu^{(k)}, \nu^{(k)}, k) + (-l\omega_2, l\omega_1, 0)$. Therefore, we must look for the minimal vectors in the sublattice $\mathcal{R}^{23}$ only among the vectors $\sigma^{(k)} \overset{\text{def}}{=} (\mu^{(k)}, \nu^{(k)}, k)$, $k \in \mathbb{N}$.

Let us try to represent $\sigma^{(k)}$ as the sum of two nonzero vectors $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ in $\mathcal{R}^{23}$,

$$\sigma^{(k)} = \tilde{\sigma} + \tilde{\tilde{\sigma}}. \tag{21}$$

It is clear that the coordinates with index 3 of the vectors $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ are some nonnegative integers $\tilde{k}$ and $\tilde{\tilde{k}}$ such that $\tilde{k} + \tilde{\tilde{k}} = k$. Consider the following two possible cases separately.

(a) One of the numbers $\tilde{k}$ and $\tilde{\tilde{k}}$ is zero, and the other number is equal to k. To be definite, assume that $\tilde{k} = 0$ and $\tilde{\tilde{k}} = k$. Then $\tilde{\sigma} = (-m\omega_2, m\omega_1, 0)$ by Proposition 5, where $m \in \mathbb{Z}_+$. However, $\tilde{\sigma} = 0$ for $m = 0$ and, for $m \geqslant 1$, it follows from (21) that the coordinate with index 2 of the vector $\tilde{\tilde{\sigma}}$ is negative, $\tilde{\tilde{\sigma}}_2 = \sigma_2^{(k)} - \tilde{\sigma}_2 = \nu^{(k)} - m\omega_1 \leqslant \nu^{(k)} - \omega_1 < 0$. Hence $\tilde{\tilde{\sigma}} \notin \mathcal{R}^{23}$. Therefore, there is no decomposition (21) in which $\tilde{k}$ or $\tilde{\tilde{k}}$ vanishes.

We also note that, if $k = 1$, then k cannot be decomposed into the sum $k = \tilde{k} + \tilde{\tilde{k}}$, $\tilde{k} \geqslant 0$, $\tilde{\tilde{k}} \geqslant 0$, in any way which differs from that treated above, where $\tilde{k} = 0$ and $\tilde{\tilde{k}} = 1$ (or, conversely, $\tilde{k} = 1$ and $\tilde{\tilde{k}} = 0$). Therefore, the vector $\sigma^{(1)}$ is minimal.

(b) $k \geqslant 2$ and both the numbers $\tilde{k}$ and $\tilde{\tilde{k}}$ are positive. Then, by Proposition 5, the resonance vectors $\tilde{\sigma}$ and $\tilde{\tilde{\sigma}}$ have the form $\tilde{\sigma} = (\mu^{(\tilde{k})} - \tilde{l}\omega_2, \nu^{(\tilde{k})} + \tilde{l}\omega_1, \tilde{k})$ and $\tilde{\tilde{\sigma}} = (\mu^{(\tilde{\tilde{k}})} - \tilde{\tilde{l}}\omega_2, \nu^{(\tilde{\tilde{k}})} + \tilde{\tilde{l}}\omega_1, \tilde{\tilde{k}})$, where $\tilde{l}$ and $\tilde{\tilde{l}}$ are some nonnegative numbers. On the other hand, it is easy to see that the numbers $\tilde{l}$ and $\tilde{\tilde{l}}$ are nonpositive. Indeed, it follows from relation (21) (for the coordinate with index 2) that $\tilde{l}\omega_1 \leqslant \nu^{(\tilde{k})} + \tilde{l}\omega_1 = \tilde{\sigma}_2 \leqslant \sigma_2^{(k)} = \nu^{(k)} \leqslant \omega_1 - 1$. This implies that $\tilde{l} \leqslant 0$. We can similarly prove that $\tilde{\tilde{l}} \leqslant 0$. Hence, $\tilde{l} = \tilde{\tilde{l}} = 0$.

Thus, if a decomposition (21) in case (b) exists, then it is of the form $(\mu^{(k)}, \nu^{(k)}, k) = (\mu^{(\tilde{k})}, \nu^{(\tilde{k})}, \tilde{k}) + (\mu^{(k-\tilde{k})}, \nu^{(k-\tilde{k})}, k - \tilde{k})$, where $1 \leqslant \tilde{k} \leqslant k-1$. By Lemma 3, for each $\tilde{k}$, $1 \leqslant \tilde{k} \leqslant k-1$, one of the following two cases is possible: either $\nu^{(k)} = \nu^{(\tilde{k})} + \nu^{(k-\tilde{k})}$ and $\mu^{(k)} = \mu^{(\tilde{k})} + \mu^{(k-\tilde{k})}$ or $\nu^{(k)} = \nu^{(\tilde{k})} + \nu^{(k-\tilde{k})} - \omega_1$ and $\mu^{(k)} = \mu^{(\tilde{k})} + \mu^{(k-\tilde{k})} + \omega_2$. Hence, for $k \geqslant 2$, the vector $\sigma^{(k)}$ can be represented as the sum (21) of nonzero vectors in a normal sublattice if and only if the relation $\nu^{(k)} = \nu^{(\tilde{k})} + \nu^{(k-\tilde{k})}$ holds for some $\tilde{k} \in \{1, 2, \dots, k-1\}$. For the vector $\sigma^{(k)}$ to be minimal, it is necessary and sufficient that $\nu^{(k)} = \nu^{(\tilde{k})} + \nu^{(k-\tilde{k})} - \omega_1$ for any $\tilde{k} \in \{1, 2, \dots, k-1\}$.

It remains to note that the relation $\nu^{(k)} = \nu^{(\tilde{k})} + \nu^{(k-\tilde{k})} - \omega_1$ is equivalent to the inequality $\nu^{(k)} < \nu^{(\tilde{k})}$. Hence, for $k \geqslant 2$, the vector $\sigma^{(k)}$ is minimal if and only if $\nu^{(k)} < \nu^{(\tilde{k})}$ for any $\tilde{k} \in \{1, 2, \dots, k-1\}$.

In conclusion, we note that $\nu^{(\omega_1)} = 0$. Therefore, if $k \geqslant \omega_1 + 1$, then $\nu^{(k)} \geqslant 0 = \nu^{(\tilde{k})}$ for $\tilde{k} = \omega_1$. Hence, for $k \geqslant \omega_1 + 1$, the vector $\sigma^{(k)}$ is not minimal.

However, if $k = \omega_1$, then, for the vector $\sigma^{(k)} = \sigma^{(\omega_1)} = (-\omega_3, 0, \omega_1)$, the minimality condition $0 < \nu^{(\tilde{k})}$ is satisfied for any $\tilde{k} \in \{1, 2, \dots, \omega_1 - 1\}$. Indeed, since the frequencies ω_1 and ω_3 are coprime, the numbers ν and ω_1 are also coprime by relation (13). Hence, $\tilde{k}\nu$ is divisible by ω_1 only if $\tilde{k}$ is a multiple of ω_1. Hence, for any $\tilde{k} \in \{1, 2, \dots, \omega_1 - 1\}$, the number $\nu^{\tilde{k}} = \tilde{k}\nu \pmod{\omega_1}$ is greater than zero. Therefore, the vector $\sigma^{(\omega_1)}$ is minimal. The vector $\sigma^{(\omega_1)}$ is a face vector. It belongs to the intersection $\mathcal{R}^{23} \cap \mathcal{R}^3$.

For $1 \leqslant k \leqslant \omega_1 - 1$, all the three coordinates of the vector $\sigma^{(k)}$ are nonzero. Hence, $\sigma^{(k)}$ is an internal vector of the sublattice $\mathcal{R}^{23}$.

Corollary 1. *In the three-frequency case* $(n = 3)$, *the number* $|\mathcal{M}|$ *of minimal resonance vectors has the upper bound* $|\mathcal{M}| \leqslant 2\sum_{j=1}^{3} \omega_j$.

Proof. The number of internal minimal vectors in the normal sublattice $\mathcal{R}^{23}$ coincides with the number of internal minimal vectors in $\mathcal{R}^1$ and, according to (19), does not exceed $\omega_1 - 1$. Similar estimates hold for the sublattices $\mathcal{R}^{31}$ (and for $\mathcal{R}^2$) and $\mathcal{R}^{12}$ (and for $\mathcal{R}^3$); in this case, the frequency ω_1 is replaced by ω_2 or ω_3. Thus, the total number of internal minimal vectors does not exceed $2[(\omega_1 - 1) + (\omega_2 - 1) + (\omega_3 - 1)] = 2\sum_{j=1}^{3} \omega_j - 6$, and the total number of face minimal vectors (see (20)) is equal to 6.

4. QUANTUM RESONANCE ALGEBRA

Following [21], let us describe the resonance algebra of the quantum n-frequency oscillator (2) in the case of positive frequencies satisfying the resonance condition (3). Introduce the following notation. For each pair of vectors $\alpha, \beta \in \mathbb{Z}^n$, define the vector $[\alpha|\beta]$ with the Cartesian coordinates by the formula $[\alpha|\beta]_j \stackrel{\text{def}}{=} \min\{\alpha_j^-, \beta_j^+\} - \min\{\beta_j^-, \alpha_j^+\}$ $(j = 1, \dots, n)$. Here the notation (8) was used.

For any $a \in \mathbb{R}$ and $m \in \mathbb{Z}$, write

$$(a)_m \stackrel{\text{def}}{=} \begin{cases} (a + \hbar) \dots (a + \hbar m) & \text{for} \quad m \geqslant 1, \\ 1 & \text{for} \quad m = 0, \\ a(a - \hbar) \dots (a - \hbar(|m| - 1)) & \text{for} \quad m \leqslant -1. \end{cases} \tag{22}$$

For the vectors $s \in \mathbb{R}^n$ and $\rho \in \mathbb{Z}^n$, set $(s)_\rho \stackrel{\text{def}}{=} (s_1)_{\rho_1} \cdots (s_n)_{\rho_n}$, where each of the multipliers is given by (22). Further, for a pair of vectors $\rho, \sigma \in \mathbb{Z}^n$, we define the polynomial $g_{\rho,\sigma}$ on $\mathbb{R}^n$ by the formula $g_{\rho,\sigma}(s) \stackrel{\text{def}}{=} (s - \hbar\rho)_{[\sigma|\rho]}$, $s \in \mathbb{R}^n$.

Theorem 3.

(a) *The algebra of symmetries of the quantum n-frequency oscillator $\hat{H}$ (2) in the case of positive frequencies satisfying the resonance condition* (3) *is generated by the operators $\hat{S} = (\hat{S}_1, \dots, \hat{S}_n)$ and $\hat{A}_\rho$ ($\rho \in \mathcal{M}$) defined in* (9)*. Here $\mathcal{M}$ is the set of minimal resonance vectors in $\mathbb{Z}^n$.*

(b) *The operators $\hat{S}_j$ and $\hat{A}_\rho$ satisfy the following quantum constraints and commutation relations.*

Quantum constraints of Hermitian type:

$$\hat{S}_j^* = \hat{S}_j, \qquad \hat{A}_\rho^* = \hat{A}_{-\rho} \tag{23}$$

for any $j \in \{1, \dots, n\}$ and any $\rho \in \mathcal{M}$.

Quantum constraints of commutative type:

$$\prod_\rho (\hat{A}_\rho)^{k_\rho} = \prod_\sigma (\hat{A}_\sigma)^{l_\sigma} \tag{24}$$

for any families of minimal vectors ρ and σ in the same normal sublattice and for the numbers $k_\rho, l_\sigma \in \mathbb{N}$ such that

$$\sum_\rho k_\rho \rho = \sum_\sigma l_\sigma \sigma. \tag{25}$$

Quantum constraints of noncommutative type: *if the minimal vectors ρ and σ do not belong to the same normal sublattice and $\rho \neq -\sigma$, then*

$$\hat{A}_\rho \hat{A}_\sigma = g_{\rho,\sigma}(\hat{S}) \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\hat{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}, \tag{26}$$

where $n_\varkappa^{\rho+\sigma}$ are the coefficients of decomposition (7) *of the vector $\rho + \sigma$ in minimal vectors from $\mathcal{M}_{\rho+\sigma}$.*

Commutation relations:

$$[\hat{S}_j, \hat{S}_k] = 0, \qquad [\hat{S}_j, \hat{A}_\rho] = \hbar \rho_j \hat{A}_\rho, \qquad [\hat{A}_{-\rho}, \hat{A}_\rho] = \hbar F_{-\rho,\rho}(\hat{S}) \tag{27}$$

for any $j, k \in \{1, \dots, n\}$ and $\rho \in \mathcal{M}$, where the polynomials $F_{\rho,\sigma}$ are given by the formula

$$F_{\rho,\sigma} \stackrel{\text{def}}{=} (g_{\rho,\sigma} - g_{\sigma,\rho})/\hbar. \tag{28}$$

(c) *Relations* (23), (24), (26), (27) *have the Casimir element $\sum_{j=1}^n \omega_j \hat{S}_j$. In representation* (9), *this element coincides with the Hamiltonian* (2) *of the oscillator, $\sum_{j=1}^n \omega_j \hat{S}_j = \hat{H}$.*

Remark 2. Relation of the form (25) is said to be *reducible* if the coefficients k_ρ and l_σ can be represented as the sum $k_\rho = k'_\rho + k''_\rho$, $l_\sigma = l'_\sigma + l''_\sigma$, where $k'_\rho, k''_\rho, l'_\sigma, l''_\sigma \in \mathbb{Z}_+$ and $(\sum_\rho k'_\rho) \cdot (\sum_\rho k''_\rho) \neq 0$ are such that $\sum_\rho k'_\rho \rho = \sum_\sigma l'_\sigma \sigma$. In the other cases, relation (25) is said to be *irreducible*.

Note that the number of irreducible relations (25), and hence the number of constraints of commutative type (24), is finite (see [21]).

Remark 3. The product $\prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\hat{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}$ on the right-hand side of (26) is well defined, because, due to constraints of commutative type, it does not depend on the choice of decomposition (7) of the vector $\rho + \sigma$ in minimal vectors.

Remark 4. In fact, the set of constraints of noncommutative type (26) consists of the commutation relations

$$[\hat{A}_\rho, \hat{A}_\sigma] = \hbar F_{\rho,\sigma}(\hat{S}) \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\hat{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}},$$

where $F_{\rho,\sigma}$ is defined in (28), and the anticommutation relations

$$[\hat{A}_\rho, \hat{A}_\sigma]_+ = \big(g_{\rho,\sigma}(S) + g_{\sigma,\rho}(S)\big) \prod_{\varkappa \in \mathcal{M}_{\rho+\sigma}} (\hat{A}_\varkappa)^{n_\varkappa^{\rho+\sigma}}. \tag{26$'$}$$

We call relations (26$'$) the *actual constraints of noncommutative type.*

The set of constraints of commutative type (24) also contains commutation relations. These are the constraints (24) corresponding to the vector equalities $\rho + \sigma = \sigma + \rho$ for the vectors $\rho, \sigma \in \mathcal{M}$ belonging to the same normal sublattice. Eliminating the commutation relations from the set of constraints (24), we obtain the set of *actual constraints of noncommutative type.* The actual constraints in this set need not be independent. More precisely, the number of independent actual constraints is equal to $|\mathcal{M}| - n + 1$, where $|\mathcal{M}|$ is the number of minimal vectors.

Definition 3. An algebra with involution generated by the elements $\hat{S}_j$ ($j = 1, \ldots, n$) and $\hat{A}_\rho$ ($\rho \in \mathcal{M}$) satisfying the relations (23), (24), (26), (27) is referred to as a *resonance algebra*.

An abstract resonance algebra (not necessarily represented by the realization (9)) was considered in [21], where the properties of the structure functions $g_{\rho,\sigma}$ of this algebra were studied, the commutation relations following from (23), (24), (26), (27) were obtained, the irreducible representations of the resonance algebra in Hilbert spaces of polynomials were constructed, and the constructions of the reproducing measure and coherent states were presented.

Since, for $n = 3$, the description of the set $\mathcal{M}$ was found in Section 3, we can explicitly present actual constraints and the commutation relations determining the resonance algebra. Here we consider only two examples.

Example 1. Let $n = 3$ and $\omega = (1, 2, 3)$. Then there are 10 minimal resonance vectors $\mathcal{M} = \{\pm\alpha, \pm\beta, \pm\gamma, \pm\delta, \pm\varepsilon\}$, where $\alpha \stackrel{\text{def}}{=} (-3, 0, 1)$, $\beta \stackrel{\text{def}}{=} (2, -1, 0)$, $\gamma \stackrel{\text{def}}{=} (0, 3, -2)$, $\delta \stackrel{\text{def}}{=} (1, -2, 1)$, and $\varepsilon \stackrel{\text{def}}{=} (1, 1, -1)$.

In the sublattice $\mathcal{R}^{23}$, there are no internal minimal vectors. The face minimal vectors in $\mathcal{R}^{23}$ are $\alpha \in \mathcal{R}^{23} \cap \mathcal{R}^3$ and $-\beta \in \mathcal{R}^{23} \cap \mathcal{R}^2$.

The sublattice $\mathcal{R}^{31}$ contains one internal minimal vector δ and two face vectors: $\beta \in \mathcal{R}^{31} \cap \mathcal{R}^1$ and $-\gamma \in \mathcal{R}^{31} \cap \mathcal{R}^3$. They satisfy the relation $\beta + (-\gamma) = 2\delta$.

The sublattice $\mathcal{R}^{12}$ also contains one internal minimal vector ε and two face vectors: $\gamma \in \mathcal{R}^{12} \cap \mathcal{R}^2$ and $-\alpha \in \mathcal{R}^{12} \cap \mathcal{R}^1$ with the relation $\gamma + (-\alpha) = 3\varepsilon$.

The minimal vectors in the sublattice $\mathcal{R}^1$ are $-\alpha$ and $-\beta$, in the sublattice $\mathcal{R}^2$, these are the vectors $-\delta$, $-\beta$, and γ, and in the sublattice $\mathcal{R}^3$, these are the vectors $-\varepsilon$, $-\gamma$, and α.

The resonance algebra is determined by the constraints (23) of Hermitian type, by the following *actual constraints of commutative and noncommutative type*:

$$\hat{A}_\beta \hat{A}_{-\gamma} = (\hat{A}_\delta)^2, \qquad \hat{A}_\gamma \hat{A}_{-\alpha} = (\hat{A}_\varepsilon)^2,$$

$$[\hat{A}_\alpha, \hat{A}_\beta]_+ = 2(\hat{S}_1^2 + 2\hbar\hat{S}_1 + 3\hbar^2)\hat{A}_{-\varepsilon}, \qquad [\hat{A}_\alpha, \hat{A}_\gamma]_+ = 2(\hat{S}_3 + \hbar)\hat{A}_{-\beta}\hat{A}_{-\delta},$$

$$[\hat{A}_\alpha, \hat{A}_\delta]_+ = (2\hat{S}_1 + 3\hbar)(\hat{A}_{-\varepsilon})^2 \qquad [\hat{A}_\alpha, \hat{A}_{-\delta}]_+ = (2\hat{S}_3 + \hbar)(\hat{A}_{-\beta})^2,$$

$$[\hat{A}_\alpha, \hat{A}_\varepsilon]_+ = (2\hat{S}_1\hat{S}_3 + \hbar\hat{S}_1 + 3\hbar\hat{S}_3)\hat{A}_{-\beta}, \qquad [\hat{A}_\beta, \hat{A}_\gamma]_+ = (2\hat{S}_2 - \hbar)(\hat{A}_\varepsilon)^2,$$

$$[\hat{A}_\beta, \hat{A}_{-\delta}]_+ = 2(\hat{S}_1\hat{S}_2 - \hbar^2)\hat{A}_\varepsilon, \qquad [\hat{A}_\beta, \hat{A}_\varepsilon]_+ = (2\hat{S}_2 + \hbar)\hat{A}_{-\alpha},$$

$$[\hat{A}_\beta, \hat{A}_{-\varepsilon}]_+ = 2\hat{S}_1\hat{A}_\delta, \qquad [\hat{A}_\gamma, \hat{A}_{-\varepsilon}]_+ = (2\hat{S}_2\hat{S}_3 + 2\hbar\hat{S}_2 - \hbar\hat{S}_3 - 4\hbar^2)\hat{A}_{-\delta},$$

$$[\hat{A}_\gamma, \hat{A}_\delta]_+ = 2(\hat{S}_2^2\hat{S}_3 + \hbar\hat{S}_2^2 - 3\hbar^2\hat{S}_2 + 2\hbar^2\hat{S}_3 + 2\hbar^3)\hat{A}_\varepsilon,$$

$$[\hat{A}_\delta, \hat{A}_\varepsilon]_+ = (2\hat{S}_2\hat{S}_3 + \hbar\hat{S}_2 + 2\hbar\hat{S}_3)\hat{A}_\beta, \qquad [\hat{A}_\delta, \hat{A}_{-\varepsilon}]_+ = (2\hat{S}_1 + \hbar)\hat{A}_{-\gamma},$$

and by the following *commutation relations*: $[\hat{A}_\alpha, \hat{A}_{-\beta}] = 0$, $[\hat{A}_\alpha, \hat{A}_{-\gamma}] = 0$,

$$[\hat{A}_\alpha, \hat{A}_{-\varepsilon}] = 0, \quad [\hat{A}_\beta, \hat{A}_{-\gamma}] = 0, \quad [\hat{A}_\beta, \hat{A}_\delta] = 0, \quad [\hat{A}_\gamma, \hat{A}_{-\delta}] = 0, \quad [\hat{A}_\gamma, \hat{A}_\varepsilon] = 0,$$

$$[\hat{A}_\alpha, \hat{A}_\beta] = 6\hbar(\hat{S}_1 + \hbar)\hat{A}_{-\varepsilon}, \qquad [\hat{A}_\alpha, \hat{A}_\gamma] = -2\hbar\hat{A}_{-\beta}\hat{A}_{-\delta}, \qquad [\hat{A}_\alpha, \hat{A}_\delta] = 3\hbar(\hat{A}_{-\varepsilon})^2,$$

$$[\hat{A}_\alpha, \hat{A}_{-\delta}] = -\hbar(\hat{A}_{-\beta})^2, \qquad [\hat{A}_\alpha, \hat{A}_\varepsilon] = \hbar(3\hat{S}_3 - \hat{S}_1)\hat{A}_{-\beta}, \qquad [\hat{A}_\beta, \hat{A}_\gamma] = 3\hbar(\hat{A}_\varepsilon)^2,$$

$$[\hat{A}_\beta, \hat{A}_{-\delta}] = 2\hbar(\hat{S}_1 - \hat{S}_2)\hat{A}_\varepsilon, \qquad [\hat{A}_\beta, \hat{A}_\varepsilon] = \hbar\hat{A}_{-\alpha}, \qquad [\hat{A}_\beta, \hat{A}_{-\varepsilon}] = -2\hbar\hat{A}_\delta,$$

$$[\hat{A}_\gamma, \hat{A}_{-\varepsilon}] = \hbar(2\hat{S}_2 - 3\hat{S}_3 - 4\hbar)\hat{A}_{-\delta}, \qquad [\hat{A}_\gamma, \hat{A}_\delta] = 2\hbar(\hat{S}_2^2 - 3\hat{S}_2\hat{S}_3 - 3\hbar\hat{S}_2 + 2\hbar^2)\hat{A}_\varepsilon,$$

$$[\hat{A}_\delta, \hat{A}_\varepsilon] = \hbar(2\hat{S}_3 - \hat{S}_2)\hat{A}_\beta, \qquad [\hat{A}_\delta, \hat{A}_{-\varepsilon}] = -\hbar\hat{A}_{-\gamma},$$

$$[\hat{S}_1, \hat{S}_2] = 0, \qquad [\hat{S}_2, \hat{S}_3] = 0, \qquad [\hat{S}_3, \hat{S}_1] = 0,$$

$$[\hat{S}_1, \hat{A}_\alpha] = -3\hbar\hat{A}_\alpha, \quad [\hat{S}_1, \hat{A}_\beta] = 2\hbar\hat{A}_\beta, \quad [\hat{S}_1, \hat{A}_\delta] = \hbar\hat{A}_\delta, \quad [\hat{S}_1, \hat{A}_\gamma] = 0, \quad [\hat{S}_1, \hat{A}_\varepsilon] = \hbar\hat{A}_\varepsilon,$$

$$[\hat{S}_2, \hat{A}_\alpha] = 0, \quad [\hat{S}_2, \hat{A}_\beta] = -\hbar\hat{A}_\beta, \quad [\hat{S}_2, \hat{A}_\gamma] = 3\hbar\hat{A}_\gamma, \quad [\hat{S}_2, \hat{A}_\delta] = -2\hbar\hat{A}_\delta, \quad [\hat{S}_2, \hat{A}_\varepsilon] = \hbar\hat{A}_\varepsilon,$$

$$[\hat{S}_3, \hat{A}_\alpha] = \hbar\hat{A}_\alpha, \quad [\hat{S}_3, \hat{A}_\beta] = 0, \quad [\hat{S}_3, \hat{A}_\gamma] = -2\hbar\hat{A}_\gamma, \quad [\hat{S}_3, \hat{A}_\delta] = \hbar\hat{A}_\delta, \quad [\hat{S}_3, \hat{A}_\varepsilon] = -\hbar\hat{A}_\varepsilon,$$

$[\hat{A}_{-\alpha}, \hat{A}_{\alpha}] = \hbar(\hat{S}_1^3 - 9\hat{S}_1^2\hat{S}_3) - 3\hbar^2(\hat{S}_1^2 + 3\hat{S}_1\hat{S}_3) + 2\hbar^3(\hat{S}_1 - 3\hat{S}_3),$
$[\hat{A}_{-\beta}, \hat{A}_{\beta}] = \hbar(4\hat{S}_1\hat{S}_2 - \hat{S}_1^2) + \hbar^2(\hat{S}_1 + 2\hat{S}_2),$
$[\hat{A}_{-\gamma}, \hat{A}_{\gamma}] = \hbar(9\hat{S}_2^2\hat{S}_3^2 - 4\hat{S}_2^3\hat{S}_3) + \hbar^2(9\hat{S}_2\hat{S}_3^2 + 3\hat{S}_2^2\hat{S}_3 - 2\hat{S}_2^2) + \hbar^3(6\hat{S}_2^2 + 6\hat{S}_3^2 - 17\hat{S}_2\hat{S}_3) - 2\hbar^4(2\hat{S}_2 + 3\hat{S}_3),$
$[\hat{A}_{-\delta}, \hat{A}_{\delta}] = \hbar(\hat{S}_1\hat{S}_2^2 + \hat{S}_2^2\hat{S}_3 - 4\hat{S}_1\hat{S}_2\hat{S}_3) + \hbar^2(\hat{S}_2^2 - \hat{S}_1\hat{S}_2 - \hat{S}_2\hat{S}_3 - 2\hat{S}_1\hat{S}_3) - \hbar^3\hat{S}_2,$
$[\hat{A}_{-\varepsilon}, \hat{A}_{\varepsilon}] = \hbar(\hat{S}_1\hat{S}_3 + \hat{S}_2\hat{S}_3 - \hat{S}_1\hat{S}_2) + \hbar^2\hat{S}_3.$

Thus, in this example, the number of generators is 13 ($\hat{A}_\alpha, \dots, \hat{A}_\varepsilon$ with their conjugates and $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$), and the number of independent actual constraints is 8. The quantum manifold determined by these constraints is of dimension 5.

Example 2. Let $n = 3$, and $\omega = (1, 1, N)$, where $N \in \{1, 2, 3\}$. Then the set of minimal resonance vectors is $\mathcal{M} = \{\pm\rho^{(0)}, \dots, \pm\rho^{(N)}, \pm\sigma\}$, where $\rho^{(k)} = (N - k, k, -1)$, $\sigma = (1, -1, 0)$.

For any N, there are no internal minimal vectors in the sublattice $\mathcal{R}^{23}$, and the face minimal vectors are $-\rho^{(0)} \in \mathcal{R}^{23} \cap \mathcal{R}^3$ and $-\sigma \in \mathcal{R}^{23} \cap \mathcal{R}^2$.

The sublattice $\mathcal{R}^{31}$ does not contain internal minimal vectors either, and the face minimal vectors are $\sigma \in \mathcal{R}^{31} \cap \mathcal{R}^1$ and $-\rho^{(N)} \in \mathcal{R}^{31} \cap \mathcal{R}^3$.

In the sublattice $\mathcal{R}^{12}$, there are $N + 1$ minimal vectors: $\rho^{(0)}, \dots, \rho^{(N)}$. The face vectors are $\rho^{(0)} \in \mathcal{R}^{12} \cap \mathcal{R}^1$ and $\rho^{(N)} \in \mathcal{R}^{12} \cap \mathcal{R}^2$. The number of internal minimal vectors depends on N. If $N = 1$, then there are no internal minimal vectors. If $N \geqslant 2$, then there are $N - 1$ internal minimal vectors: $\rho^{(1)}, \dots, \rho^{(N-1)}$. For $N = 2$, the minimal vectors in the sublattice $\mathcal{R}^{12}$ satisfy only one irreducible relation,

$$\rho^{(0)} + \rho^{(2)} = 2\rho^{(1)}. \tag{29}$$

For $N = 3$, in addition to (29), there are four irreducible relations $2\rho^{(0)} + \rho^{(3)} = 3\rho^{(1)}$, $\rho^{(0)} + 2\rho^{(3)} = 3\rho^{(2)}$, $\rho^{(1)} + \rho^{(3)} = 2\rho^{(2)}$, and $\rho^{(0)} + \rho^{(3)} = \rho^{(1)} + \rho^{(2)}$.

The minimal vectors in the sublattice $\mathcal{R}^1$ are $\rho^{(0)}$ and σ, in the sublattice $\mathcal{R}^2$, the vectors $-\sigma$ and $\rho^{(N)}$, and in the sublattice $\mathcal{R}^3$, the vectors $-\rho^{(0)}, \dots, -\rho^{(N)}$.

The minimal vectors in different sublattices satisfy the relations $\rho^{(k)} + \sigma = \rho^{(k-1)}$ $(k = 1, \dots, N)$, $-\rho^{(k)} + \sigma = -\rho^{(k+1)}$ $(k = 0, \dots, N-1)$, $-\rho^{(l)} + \rho^{(k)} = (l - k)\sigma$ $(k = 0, \dots, N-1, l = k+1, \dots, N)$.

The resonance algebra is determined by the constraints (23) of Hermitian type and the following relations.

Actual constraints of commutative type:

— for $N = 1$, there are no such constraints;

— for $N = 2$, $\hat{A}_{\rho^{(0)}}\hat{A}_{\rho^{(2)}} = (\hat{A}_{\rho^{(1)}})^2$;

— for $N = 3$, $\hat{A}_{\rho^{(0)}}\hat{A}_{\rho^{(2)}} = (\hat{A}_{\rho^{(1)}})^2$, $(\hat{A}_{\rho^{(0)}})^2\hat{A}_{\rho^{(3)}} = (\hat{A}_{\rho^{(1)}})^3$, $\hat{A}_{\rho^{(0)}}(\hat{A}_{\rho^{(3)}})^2 = (\hat{A}_{\rho^{(2)}})^3$, $\hat{A}_{\rho^{(1)}}\hat{A}_{\rho^{(3)}} = (\hat{A}_{\rho^{(2)}})^2$, $\hat{A}_{\rho^{(0)}}\hat{A}_{\rho^{(3)}} = \hat{A}_{\rho^{(1)}}\hat{A}_{\rho^{(2)}}$.

Actual constraints of noncommutative type:

$$[\hat{A}_\sigma, \hat{A}_{\rho^{(k)}}]_+ = (2\hat{S}_2 - \hbar(k-2))\hat{A}_{\rho^{(k-1)}} \qquad (k = 1, \dots, N),$$
$$[\hat{A}_\sigma, \hat{A}_{-\rho^{(k)}}]_+ = (2\hat{S}_1 + \hbar(N-k))\hat{A}_{-\rho^{(k+1)}} \qquad (k = 0, \dots, N-1),$$
$$[\hat{A}_{-\rho^{(l)}}, \hat{A}_{\rho^{(k)}}]_+ = \Big((\hat{S}_1)_{N-l}\,(\hat{S}_2 + \hbar l)_{-k}\,\hat{S}_3 + (\hat{S}_1 - \hbar(l-k))_{l-N}\,(\hat{S}_2)_{-k}\,(\hat{S}_3 + \hbar)\Big)(\hat{A}_\sigma)^{l-k} \qquad (0 \leqslant k < l \leqslant N).$$

Commutation relations:

$$[\hat{A}_\sigma, \hat{A}_{\rho^{(0)}}] = 0, \quad [\hat{A}_\sigma, \hat{A}_{-\rho^{(N)}}] = 0, \quad [\hat{A}_{\rho^{(k)}}, \hat{A}_{\rho^{(l)}}] = 0 \quad (0 \leqslant k < l \leqslant N),$$
$$[\hat{A}_\sigma, \hat{A}_{\rho^{(k)}}] = \hbar k\hat{A}_{\rho^{(k-1)}} \qquad (k = 1, \dots, N),$$
$$[\hat{A}_\sigma, \hat{A}_{-\rho^{(k)}}] = -\hbar(N-k)\hat{A}_{-\rho^{(k+1)}} \qquad (k = 0, \dots, N-1),$$
$$[\hat{A}_{-\rho^{(l)}}, \hat{A}_{\rho^{(k)}}] = \Big((\hat{S}_1)_{N-l}\,(\hat{S}_2 + \hbar l)_{-k}\,\hat{S}_3 - (\hat{S}_1 - \hbar(l-k))_{l-N}\,(\hat{S}_2)_{-k}\,(\hat{S}_3 + \hbar)\Big)(\hat{A}_\sigma)^{l-k} \qquad (0 \leqslant k < l \leqslant N),$$

$$
\begin{aligned}
&[\hat A_{-\sigma},\hat A_\sigma]=\hbar(\hat S_2-\hat S_1), \qquad [\hat S_1,\hat S_2]=0, \qquad [\hat S_2,\hat S_3]=0, \qquad [\hat S_3,\hat S_1]=0,\\
&[\hat S_1,\hat A_\sigma]=\hbar\hat A_\sigma, \qquad [\hat S_1,\hat A_{\rho^{(l)}}]=\hbar(N-l)\hat A_{\rho^{(l)}},\\
&[\hat S_2,\hat A_\sigma]=-\hbar\hat A_\sigma, \qquad [\hat S_2,\hat A_{\rho^{(l)}}]=\hbar l\hat A_{\rho^{(l)}},\\
&[\hat S_3,\hat A_\sigma]=0, \qquad [\hat S_3,\hat A_{\rho^{(l)}}]=-\hbar\hat A_{\rho^{(l)}} \qquad (l=0,1,\dots,N).
\end{aligned}
$$

Thus, for $N = 1$, the number of generators of the resonance algebra is 9, and the number of independent constraints is 4; for $N = 2$, the number of generators is 11, and the number of independent constraints is 6; for $N = 3$, the number of generators is 13, and the number of independent constraints is 8. In all these cases, the quantum manifold determined by these constraints is of dimension 5.

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