Journal of Applied Mathematics and Mechanics 74 (2010) 611–616

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Optimal singular and chattering modes in the problem of controlling the vibrations of a string with clamped ends☆

L.A. Manita

Moscow, Russia

ARTICLE INFO

Article history:
Received 1 November 2009

ABSTRACT

The problem of minimizing the root mean square deviation of a uniform string with clamped ends from an equilibrium position is investigated. It is assumed that the initial conditions are specified and the ends of the string are clamped. The Fourier method is used, which enables the control problem with a partial differential equation to be reduced to a control problem with a denumerable system of ordinary differential equations. For the optimal control problem in the l_2 space obtained, it is proved that the optimal synthesis contains singular trajectories and chattering trajectories. For the initial problem of the optimal control of the vibrations of a string it is also proved that there is a unique solution for which the optimal control has a denumerable number of switchings in a finite time interval.

The problem of controlling the vibrations of a string was considered previously[1–3] in the class of generalized solutions, and a method of determining the optimal control was presented for which the string transfers from an arbitrary specified position to an arbitrary specified final position with a minimum value of the boundary energy integral. Here the control is either a displacement of one of the ends of the string,[1] or an elastic force acting on one of the ends,[2] or a combined control is considered, namely, an elastic force, applied to the left end of the string and a displacement of the right end.[3] The problem of the transfer of the string from a rest position to a specified fixed position in a specified time, when the displacement of one of the ends of the string serves as the control was considered in Ref. [4]; it was shown that a sequence of finite-dimensional approximations of the problem (based on the Fourier representation) can be reduced in the norm of the $L_2[0,T]$ space to the solution of the initial problem. Using a modified method of moments, the problem of the control of the motion of an elastic system (described by a linear hyperbolic equation) by a concentrated force-type boundary action was investigated in Ref. [5]. A detailed review of the results for problems of controlling elastic systems can be found in Ref. [6].

In this paper it is shown that, for the problem of minimizing the root mean square deviation of a uniform string with clamped ends from an equilibrium position, the optimal control is a chattering control, i.e., it has a denumerable number of switchings in a finite time interval.

1. Formulation of the problem

Consider the equation of small transverse vibrations of a tensed string of length l and constant linear density ρ

$$u_{tt}(t,x) - a^2 u_{xx}(t,x) = g(t,x); \quad a^2 = K/\rho \tag{1.1}$$

where $u(t, x)$ and $g(x, t)$ are the displacement of the string and the external force density (per unit mass of the string) at the instant of time t at the point x, and K is the tension in the string. We will assume that the ends of the string are clamped:

$$u\big|_{x=0} = u\big|_{x=l} = 0, \quad t > 0 \tag{1.2}$$

and that the initial position and velocity of the string are fixed:

$$u\big|_{t=0} = \alpha(x), \quad u_t\big|_{t=0} = \beta(x), \quad x \in [0,l] \tag{1.3}$$

☆ *Prikl. Mat. Mekh.* Vol. 74, No. 5, pp. 857–863, 2010.

E-mail address: 1manita@rambler.ru

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.11.011

We will consider the external force density in the form

$$g(t,x) = q(t)f(x)$$

where $f(x)$ is a certain specified function, and $q(t)$ is the control function that obeys the constraints

$$-1 \le q(t) \le 1 \tag{1.4}$$

The control $q(t)$ is chosen so as to minimize the root mean square deviation of the string from the equilibrium position:

$$\int_0^\infty \int_0^l u^2(t,x)dxdt \to \inf \tag{1.5}$$

To solve problem (1.1)–(1.5) of the optimal control of the vibrations of a string, we will use Fourier's method, which is successful in solving many problems, for example, in problems of controlling a Timoshenko beam.[7] In other words, we will seek a solution $u(t, x)$ in the form of a series in eigenfunctions of the elliptic operator $L=-d^2/dx^2$ with Fourier coefficients that depend on the variable t. We will show below that the series obtained converges and gives a unique solution of the problem (a solution almost everywhere). The Fourier coefficients turn out to be solutions of a certain optimal control problem with a denumerable system of ordinary differential equations – the problem of stabilizing a denumerable number of oscillators under the action of a single force of limited modulus. We will prove that, for initial conditions from a certain neighbourhood of the origin of coordinates, a solution of the problem exists and is unique, and the optimal trajectories contain parts of a singular trajectory, which is connected with a non-singular control with an infinite number of switchings.

2. Reduction to an optimal-control problem in l_2 space

In the domain of definition

$$D_L = \left\{ \upsilon \in C^2[0,l] : \upsilon(0) = \upsilon(l) = 0 \right\}$$

we will consider the positive self-adjoint operator $L=-d^2/dx^2$, which possesses a complete system of eigenfunctions $\{h_j(x)\}_{j=1}^{\infty}$, orthonormalized in the $L_2(0, l)$ space, and a corresponding system of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$:

$$h_j(x) = \sqrt{2/l}\sin\left(\sqrt{\lambda_j}x\right), \quad \lambda_j = (\pi j/l)^2$$

We will assume that the functions $\alpha(x),\beta(x), f(x)\in L_2(0,l)$. We expand the solution $u(t, x)$ of Eq. (1.1) and the function $f(x)$ in the system of eigenfunctions of the operator L (everywhere henceforth summation is carried out from $j=1$ to $j=\infty$):

$$u(t,x) = \sum s_j(t)h_j(x), \quad s_j(t) = \left(u,h_j\right)_{L_2(0,l)}, \quad f(x) = \sum C_j h_j(x), \quad C_j = \left(f,h_j\right)_{L_2(0,l)} \tag{2.1}$$

The cost functional (1.5) then takes the form

$$\int_0^\infty \sum s_j^2(t)dt \to \inf \tag{2.2}$$

Substituting expansions (2.1) into Eq. (1.1), we obtain

$$\sum\left(\ddot{s}_j(t) + \omega_j^2 s_j(t) - C_j q(t)\right)h_j(x) = 0, \quad \omega_j = a\sqrt{\lambda_j} = \pi j a/l$$

Hence, by virtue of the orthogonality of the system of eigenfunctions it follows that the Fourier coefficients $s_j(t)$ satisfy the following denumerable system of ordinary differential equations

$$\ddot{s}_j(t) + \omega_j^2 s_j(t) = C_j q(t), \quad j = 1,2,\ldots \tag{2.3}$$

and the problem consists of obtaining the Fourier coefficients $s_j(t)$.

We expand the functions $\alpha(x)$ and $\beta(x)$ in Fourier series in the system $\{h_j(x)\}_{j=1}^{\infty}$.

We obtain

$$\alpha(x) = \sum \alpha_j h_j(x), \alpha_j = \left(\alpha,h_j\right)_{L_2(0,l)}, \quad \beta(x) = \sum \beta_j h_j(x), \beta_j = \left(\beta,h_j\right)_{L_2(0,l)}$$

The initial conditions for the functions $s_j(t)$ take the form

$$s_j(0) = \alpha_j, \quad \dot{s}_j(0) = \beta_j \tag{2.4}$$

3. Optimal control with a denumerable system of oscillators

Thus, to obtain the functions $s_j(t)$ we consider the optimal control problem (2.2)–(2.4). The control $q(t)$ is a scalar measurable function that satisfies constraints (1.4). The following conditions are imposed on the problem parameters: $c_j \neq 0$ for all j, and the vectors $\alpha=(\alpha_1, \alpha_2,\ldots)$, $\beta=(\beta_1, \beta_2,\ldots)$ and $C=(C_1, C_2,\ldots)$ from the space l_2.

We introduce the following notation[8]

$$\tau_j(t) = \dot{s}_j(t)/\omega_j, \quad c_j = C_j/\omega_j, \quad a_j = \alpha_j, \quad b_j = \beta_j/\omega_j$$

Problem (2.2)–(2.4) then takes the form

$$\int_0^\infty \sum s_j^2(t)dt \to \inf \tag{3.1}$$

$$\dot{s}_j = \omega_j\tau_j, \quad \dot{\tau}_j = -\omega_j s_j + c_j q \tag{3.2}$$

$$s_j(0) = a_j, \quad \tau_j(0) = b_j, \quad j = 1,2,\ldots \tag{3.3}$$

$$-1 \le q(t) \le 1 \tag{3.4}$$

We define the following vectors

$$s(t) = (s_1(t), s_2(t), \ldots), \quad \tau(t) = (\tau_1(t), \tau_2(t), \ldots)$$

$$a = (a_1, a_2, \ldots), \quad b = (b_1, b_2, \ldots), \quad c = (c_1, c_2, \ldots)$$

It necessarily follows from the minimization (3.1) that

$$\lim_{T\to\infty} s(T) = \lim_{T\to\infty} \tau(T) = 0$$

The existence, uniqueness and continuous dependence on the initial data and the continuability over the whole numerical axis of the solutions of Cauchy problem (3.2)–(3.3) were obtained[8] from the explicit representation of the corresponding solutions.

For problem (3.1)–(3.4) it was shown in Ref. 8 that, for all initial data (a, b) from a certain open neighbourhood of the origin of the space $l_2 \times l_2$, the set of admissible trajectories is non-empty, namely, an admissible control exists which ensures an exponential decay in the trajectory of system (3.2), whence the finiteness of functional (3.1) follows. Hence we obtain the assertion that a solution of problem (3.1)–(3.4) exists for any initial data (a, b) from a certain open neighbourhood of the origin. The uniqueness of the optimal trajectory follows from the strict convexity of functional (3.1) in the solutions of system (3.2).

We will apply a formally generalized Pontryagin's maximum principle to problem (3.1)–(3.4). We desire the Pontryagin function

$$H(\psi_1, \psi_2, s, \tau, q) = \sum\left(\psi_{1j}\omega_j\tau_j - \psi_{2j}\omega_j s_j + \psi_{2j}c_j q - s_j^2/2\right)$$

Denote the vector $(\psi_{i1}, \psi_{i2}\ldots)(i = 1,2)$ by ψ_i.

Consider the following Hamiltonian system in the space $l_2 \times l_2 \times l_2 \times l_2$

$$\begin{aligned} &\dot{\psi}_{1j} = \psi_{2j}\omega_j + s_j, \quad \dot{s}_j = \omega_j\tau_j \\ &\dot{\psi}_{2j} = -\psi_{1j}\omega_j, \quad \dot{\tau}_j = -\omega_j s_j + c_j\hat{q}(t), \quad j = 1,2,\ldots \end{aligned} \tag{3.5}$$

where

$$\hat{q}(t) = \arg\max_{q\in[-1,1]} H = \operatorname{sgn} H_1(t), \quad H_1(t) = \sum \psi_{2j}(t)c_j \tag{3.6}$$

It was proved in Ref. 8 that for problem (3.1)–(3.4) Pontryagin's maximum principle is the necessary and sufficient condition of optimality. Thus we have the following assertions.

Lemma 1 ((the sufficient condition of optimality)). Suppose $(\psi_1(t), \psi_2(t), s(t), \tau(t))$ is an arbitrary solution of system (3.5)–(3.6) with the boundary conditions

$$s(0) = a, \quad \tau(0) = b, \quad s(T) = 0, \quad \tau(T) = 0$$

Then $(s(t), \tau(t))$ is a solution of problem (3.1)–(3.4).

Suppose $q^*(t)$ is the optimal control in problem (3.1)–(3.4) (it exists and is unique) and $(s^*(t), \tau^*(t))$ is the corresponding optimal trajectory.

Lemma 2 ((the necessary condition of optimality)). A non-trivial function $(\psi_1(\cdot), \psi_2(\cdot))$ exists with values in the space $l_2 \times l_2$, which satisfies the adjoint system of equations

$$\dot{\psi}_{1j} = \psi_{2j}\omega_j + s_j^*, \quad \dot{\psi}_{2j} = -\psi_{1j}\omega_j, \quad j = 1,2,\ldots$$

such that following maximum condition holds

$$\max_{-1\le q(t)\le 1}\left(\sum \psi_{2j}(t)c_j q(t)\right) = \sum \psi_{2j}(t)c_j q^*(t)$$

The optimal control is determined uniquely from the maximum condition, provided $H_1(t) \ne 0$.

We will assume that an interval (t_1, t_2) exists such that

$$H_1(t) \equiv 0, \quad \forall t \in (t_1, t_2)$$

We will differentiate the identity $H_1(t) \equiv 0$, by virtue of system (3.5), until a control appears with non-zero coefficient. We will then assume that all the series obtained converge. We will introduce the following notation

$$H_2(t) = -\sum c_j \psi_{1j}(t)\omega_j, \quad H_3(t) = -\sum c_j \omega_j(\psi_{2j}(t)\omega_j + s_j(t))$$
$$H_4(t) = -\sum c_j \omega_j^2(-\psi_{1j}(t)\omega_j + \tau_j(t))$$

We will have

$$\frac{d}{dt}H_1(t) = H_2(t), \quad \frac{d^2}{dt^2}H_1(t) = \frac{d}{dt}H_2(t) = H_3(t),$$
$$\frac{d^3}{dt^3}H_1(t) = \frac{d}{dt}H_3(t) = H_4(t)$$
$$\frac{d^4}{dt^4}H_1(t) = \frac{d}{dt}H_4(t) = \sum c_j \omega_j^3(\psi_{2j}\omega_j + 2s_j) - q\sum c_j^2 \omega_j^2 \tag{3.7}$$

It follows from relations (3.7) that, in the interval (t_1, t_2),

$$H_1(t) = H_2(t) = H_3(t) = H_4(t) = 0 \tag{3.8}$$

We will determine the extremal (i.e., the solution of system (3.5)–(3.6)), which lies on the surface (3.8) as a singular extremal. The control on the singular extremal is then found from the last equation of (3.7) and is equal to

$$q^0(t) = \sum c_j \omega_j^3(\psi_{2j}\omega_j + 2s_j) \Big/ \sum c_j^2 \omega_j^2 \tag{3.9}$$

The following is proved for problem (3.1)–(3.4). For the initial data (a, b) from a sufficiently small neighbourhood of the origin of coordinates, the optimal trajectory reaches the surface filled with the singular extremals in a finite time with an infinite number of control switchings, and then the optimal trajectory remains on the singular surface. Namely, the following assertion holds.

Theorem 1 ((see Ref. 8).). Suppose $c_j \neq \forall j$ and the vector $\left(c_1\omega_1^4,\ c_2\omega_2^4,\ c_3\omega_3^4, \ldots\right) \in l_2$. We will assume that positive constants δ and D exist such that

$$|\omega_{j+1}| - |\omega_j| \geq \delta, \quad |\omega_j| \leq Bj, \quad j = 1, 2, \ldots$$

Then, a neighbourhood of the origin of the space (s, τ) exists such that, for all the initial data (α, β) from this neighbourhood, the following assertions hold:

1) an optimal solution of problem (3.1)–(3.4) exists and is unique;
2) in the space $(s, \tau, \Psi_1, \Psi_2)$ there is a surface Σ of codimensionality 4, given by Eqs (3.8), filled with the singular extremals of problem (3.1)–(3.4), the control on which is defined by formula (3.9);
3) for all the initial data, not belonging to the projection of the singular surface Σ onto the space (s, τ), the optimal trajectories reach Σ in a finite time with an infinite number of control switchings.

4. The generalized solution of the problem of controlling the vibrations of a string and its properties

We proved in Section 3 that the optimal solution $(s^*(t), q^*(t))$ of problem (3.1)–(3.4) exists and is unique, and the optimal synthesis contains singular trajectories and chattering trajectories.

If we fix the control $q(t)=q^*(t)$, the components of the vector $s^*(t)$ can be written in the form

$$s_j^*(t) = \alpha_j \cos\left(a\sqrt{\lambda_j}t\right) + \frac{1}{a\sqrt{\lambda_j}}\beta_j \sin\left(a\sqrt{\lambda_j}t\right) + I_j(t), \quad j = 1, \ldots$$

$$I_j(t) = \frac{1}{a\sqrt{\lambda_j}} \int_0^t C_j q^*(\tau) \sin\left(a\sqrt{\lambda_j}(t-\tau)\right) d\tau \tag{4.1}$$

After substitution into the first series (2.1) we obtain

$$u^*(t,x) = \sum\left(\alpha_j \cos\left(a\sqrt{\lambda_j}t\right) + \frac{1}{a\sqrt{\lambda_j}}\beta_j \sin\left(a\sqrt{\lambda_j}t\right) + I_j(t)\right) h_j(x) \tag{4.2}$$

Series (4.2) formally satisfies Eq. (1.1), boundary conditions (1.2) and initial conditions (1.3). We will further show that, when certain constraints on these problems are satisfied, this series gives a generalized solution of problem (1.1)–(1.3).

We will put Q_T=(0, l) × (0, T), where T > 0. We will define the space $H^k(Q_T) = W_2{}^k(Q_T)$ as a Sobolev space of functions from the $L_2(Q_T)$ space, such that all their generalized derivatives up to order k belong to the $L_2(Q_T)$ space. We will denote by $H_0{}^k(Q_T)$ the space $W^k_{2,0}(Q_T)$, which is a supplement of the space $C_0^\infty(Q_T)$ with respect to the norm $H^k(Q_T)$.

Definition 1. The function $u \in H^1(Q_T)$ is called a generalized solution in Q_T of problem (1.1)–(1.3), if it satisfies boundary conditions (1.2), initial conditions (1.3) and the identity

$$\int_{Q_T}\left(a^2u_x\upsilon_x - u_t\upsilon_t\right)dxdt = \int_{Q_T} g\upsilon dxdt + \int_0^l \beta(x)\,\upsilon(0,x)dx$$

for all $\upsilon \in H^1(Q_T)$, for which

$$\upsilon|_{x=0} = \upsilon|_{x=l} = 0, \quad \upsilon|_{t=T} = 0$$

Definition 2. The function $u \in H^2(Q_T)$ is called the solution, almost everywhere, of the initial-boundary-volume problem (1.1)–(1.3), if it satisfies Eq. (1.1) in Q_T (for almost all $(t, x) \in Q_T$, and satisfies boundary conditions (1.2) and initial conditions (1.3).

We will formulate the main result for problem (1.1)–(1.5).

Theorem 2. Suppose the functions $\alpha \in H_0^2(0, l)$, $\beta \in H_0^1(0, l)$, the function $f(x)$ satisfies the conditions

$$f \in KC^3[0,l], \quad f(0) = f_{xx}(0) = 0, \quad f(l) = f_{xx}(l) = 0 \tag{4.3}$$

and, moreover, all the coefficients of the Fourier expansion of the function $f(x)$ in the system $\{h_j(x)\}_{j=1}^\infty$ are non-zero. Positive constants γ_1 and γ_2 exist such that if

$$\|\alpha\|_{L_2(0,l)} < \gamma_1, \quad \|\beta\|_{L_2(0,l)} < \gamma_2$$

then a) the optimal solution $u^*(t, x)$ of problem (1.1)–(1.5) exists and is unique, $u^*(t,x) \in H^2(Q_T)$, $\forall T > 0$ and b) the optimal trajectories are chatterings trajectories, i.e., the control on the optimal trajectory has an infinite number of switchings in a finite time interval.

Proof. We will check that the conditions of Theorem 1 are satisfied. Since the function $f(x)$ satisfies conditions (4.3), then, for its Fourier coefficients C_j in the expansion in the system of functions

$$h_j(x) = \sqrt{2/l}\sin\left(\sqrt{\lambda_j}x\right), \quad j = 1,2,...$$

the following estimates are true[9]

$$|C_j| \le \varepsilon_j j^{-3}, \quad \varepsilon_j > 0, \quad j = 1,2,..., \quad \sum\varepsilon_j^2 < \infty$$

We then have

$$c_j\omega_j^4 = C_j\omega_j^3 = C_j(\pi ja/l)^3 = \theta C_j j^3; \quad \theta = (\pi a/l)^3$$

Hence it follows that

$$\sum\left(c_j\omega_j^4\right)^2 \le \theta^2\sum\varepsilon_j^2 < \infty, \quad \text{т.е.} \quad \left(c_1\omega_1^4, c_2\omega_2^4, c_3\omega_3^4, ...\right) \in l_2$$

Moreover

$$|\omega_{j+1}| - |\omega_j| \ge \delta, \quad |\omega_j| \le Bj, \quad j = 1,2,...; \quad \delta = \pi a/l, \quad B = \pi a/l$$

Hence, for problem (3.1)–(3.4) Theorem 1 holds, i.e., the optimal control $q^*(t)$ has an infinite number of switchings in a finite time interval.

We set up the series

$$\sum s_j^*(t)h_j(x) \tag{4.4}$$

where $s_j^*(t)(j = 1, 2, \ldots)$ is the optimal solution of problem (3.1)–(3.4).

When the conditions of the theorem, imposed on the functions f, α and β, are satisfied, it follows[10,11] that the function $u^*(t, x)$, defined by series (4.4), is a unique generalized solution of problem (1.1)–(1.3), where $u^*(t, x) \in H^2(Q_T)$. Consequently,[10] $u^*(t, x)$ is the solution, almost everywhere, of problem (1.1)–(1.3), i.e., the function $u^*(t, x)$ satisfies in Q_T (for any T > 0) Eq. (1.1) (for almost all $(t, x) \in Q_T$), boundary conditions (1.2) and initial conditions (1.3).

In addition, we have

$$\int_0^\infty\sum\left(s_j^*(t)\right)^2dt = \int_0^\infty\int_0^l(u^*(t,x))^2dxdt = \lim_{T\to\infty}\int_0^T\int_0^l(u^*(t,x))^2dxdt$$

Hence, if the function $s^*(t) = \left(s_1^*(t),\ s_2^*(t), \ldots\right)$ yields the minimum of functional (2.2), then the function $u^*(t, x)$, defined by series (4.4), yields a minimum of functional (1.5), i.e., $u^*(t, x)$ is a solution of problem (1.1)–(1.5).

Acknowledgement

This research was supported financially by the Russian Foundation for Basic Research (08-01-00474-a).

References

1. Il'in VA, Moiseyev YeI. Optimization of boundary control by displacement at one end of a string with the other end free after an arbitrary fairly long time interval. *Diff Urav* 2007;**417**(1):12–7.
2. Il'in VA, Moiseyev YeI. Optimization of boundary control by an elastic force on one end of a string, based on finding the minimum of the integral of the elastic force raised to an arbitrary power p $\geq$ 1. *Dokl Ross Akad Nauk* 2007;**412**(6):732–5.
3. Il'in VA, Moiseyev YeI. Optimization of combined boundary control of the vibrations of a string by an elastic force at one end and displacement at the other end. *Dokl Ross Akad Nauk* 2005;**402**(5):590–5.
4. Vasil'ev FP, Kurzhanskii MA, Razgulin AV. Fourier's method for solving a problem of controlling the vibrations of a string. *Vestnik MGU* 1993;**2**:3–8.
5. Akulenko LD. Steering of an elastic system to a specified state by a boundary force. *Prikl Mat Mekh* 1981;**45**(6):1095–103.
6. Znamenskaya LN. *Control of Elastic Vibrations*. Moscow: Fizmatlit; 2004.
7. Zelikin MI, Manita LA. Optimal chattering modes in the problem of the control of a Timoshenko beam. *Prikl Mat Mekh* 2006;**70**(2):295–304.
8. Borisov VF, Zelikin MI, Manita LA. Extremals with accumulation of switchings in an infinite-dimensional space. *Advances in science and Technology*, vol. 58. Moscow: VINITI; 2008, 3–55.
9. Kudryavtsev LD. *A Course in Mathematical Analysis. Vol. 3*. Moscow: Vyssh Shkola; 1989.
10. Mikhailov VP. *Partial Differential Equations*. Moscow: Nauka; 1976.
11. Ladyzhenskaya OA. *Boundary-Value Problems of Mathematical Physics*. Moscow: Nauka; 1973.

Translated by R.C.G.