ON ENUMERATION OF TREE-ROOTED PLANAR CUBIC MAPS

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Abstract. We consider planar cubic maps, i.e. connected cubic graphs embedded into plane, with marked spanning tree and marked directed edge (not in this tree). The number of such objects with $2n$ vertices is $C_{2n} \cdot C_{n+1}$, where C_k is Catalan number.

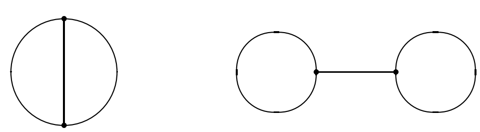
1. Introduction

Plane triangulation is a planar map, where the perimeter of each face is three. The corresponding dual graph is *cubic*, i.e. the degree of each vertex is three. A plane triangulation will be called *proper*, if each edge is incident to exactly *two* faces. Otherwise it will be called *improper*.

Example 1.1.

proper triangulation   improper triangulation

The corresponding dual graphs are presented below:



A connected graph with marked directed edge will be called *edge-rooted*. Proper edge-rooted triangulations where enumerated by Tutte in the work [8]: the number T_n of proper planar triangulations with $2n$ faces and marked directed edge is

$$T_n = \frac{2\,(4n-3)!}{n!\,(3n-1)!}.$$

A combinatorial proof of Tutte formula see in [6] (see also [1]).

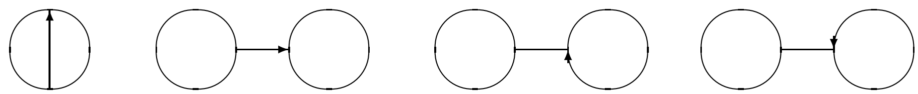
Let F_n be the number of planar edge-rooted cubic graphs with $2n$ vertices, i.e. the number of planar edge-rooted triangulations (proper and improper) with $2n$ faces. Let us define numbers f_n, $n \geqslant -1$, in the following way:

- $f_{-1} = 1/2$;
- $f_0 = 2$;
- $f_n = (3n+2)F_n$, $n > 0$.

In [3] a recurrent relation for numbers f_n was proposed:

$$f_n = \frac{4(3n+2)}{n+1} \sum_{\substack{i \geqslant -1,\, j \geqslant -1 \\ i+j=n-2}} f(i)f(j). \tag{1}$$

Example 1.2. From (1) it follows that $F_1 = 4$. Indeed, there are four ways to choose a root edge in a planar cubic map with two vertices:



Also we have that $F_2 = 32$. Indeed, there are six cubic maps with 4 vertices (and 6 edges):

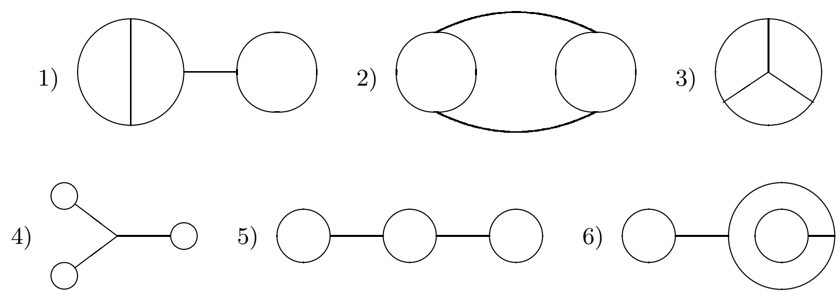


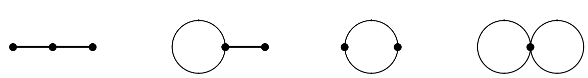
Figure 1

Group of automorphisms of the first map is trivial, of the second has order 4, of the third has order 12, of the forth has order 3, of the fifth and the sixth has order 2. Thus, there are 12 ways to choose a root edge in the first map, 3 — in the second, 1 — in the third, 4 — in the forth, 6 — in the fifth and the sixth. All this gives us 32 edge-rooted maps.

However, this formula does not seem to have a geometrical/combinatorial explanation.

In [5] a nice formula was proposed for the number tree-rooted planar maps, i.e. edge-rooted planar maps with distinguished spanning tree: the number of such maps with n edges is $C_n \cdot C_{n+1}$, where C_k is k-th Catalan number. An elegant proof of this formula see in [2].

Example 1.3. There are four planar maps with two edges:



1 2 3 4

- There is one way to choose a spanning tree in the first map and two ways to choose a directed edge.
- There is one way to choose a spanning tree in the second map and four ways to choose a directed edge.
- There is one way to choose a spanning tree in the third map and two ways to choose a directed edge.

- There is no spanning trees in the forth map and two ways to choose a directed edge.

Thus we have $10 = C_2 \cdot C_3$ tree rooted planar maps with two edges.

We will study tree-rooted cubic maps with additional property: a root edge *does not* belong to the spanning tree.

Theorem. *The number of such tree-rooted cubic maps with $2n$ vertices is $C_{2n} \cdot C_{n+1}$, where C_k is k-th Catalan number.*

2. The main construction: from map to curve

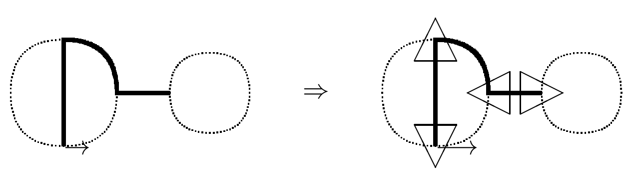
Definition 2.1. By tree-rooted plane cubic map we will understand a cubic graph imbedded into plane (sphere) with

- marked spanning tree;
- marked directed edge that *does not* belong to the spanning tree.

Let G be a tree-rooted pane cubic map with $2n$ vertices. We draw triangles, one triangle for each vertex, in such way that:

- triangles are disjoint;
- each vertex is inside the corresponding triangle;
- each side of triangle intersect one outgoing edge of corresponding vertex.

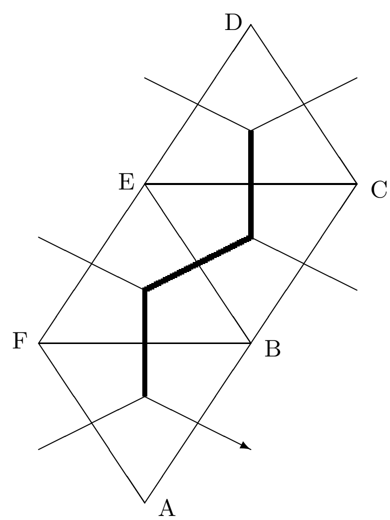
Example 2.1.



Thick lines above mark spanning tree and an arrow indicates the direction of the root edge.

Two triangles will be called adjacent, if the corresponding vertices are adjacent and the edge, that connects them, belongs to the spanning tree. The sides of adjacent triangles that intersect this edge also will be called adjacent. We construct a polygon P by glewing adjacent triangles by adjacent sides. This polygon has $2n + 2$ sides and is divided into $2n$ triangles. Each edge of the cubic map, that does not belong to the spanning tree, intersects two sides of P and we will say that these sides constitute a pair. Polygon P has a marked side: the marked edge of the cubic map intersects it in direction from inside P to outside.

Continuation of Example.



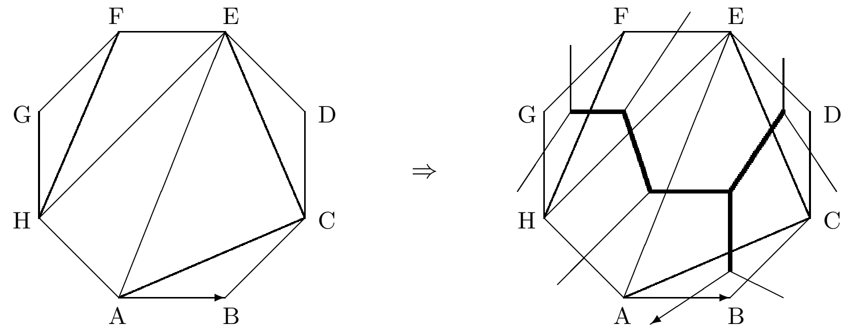
Here EF and FA, AB and BC, CD and DE are pairs and AB is the marked edge. If we identify sides that are in pairs (i.e. EF with FA, AB with BC and CD with DE), then we will obtain a triangulated genus 0 curve.

3. The main construction: from curve to map

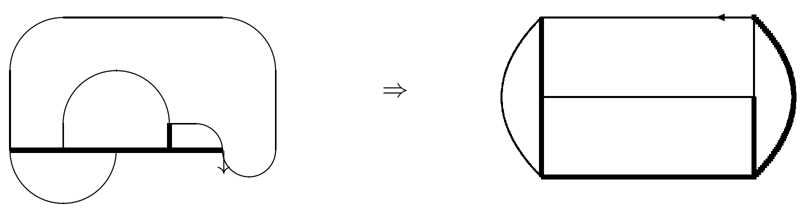
Let P be a $2n$-gon with marked side M and triangulated by non-intersecting diagonals into $2n-2$ triangles. Sides of P are divided into pairs in such way, that the identification of sides in each pair gives us a genus 0 curve. We will construct a plane tree-rooted cubic map with root edge (not in the spanning tree) in the following way.

- We put a vertex v_i inside each triangle $\triangle_i$ and connect vertices in adjacent triangles — the spanning tree is constructed.
- Let sides L and L' be in pair. L and L' are sides of triangles $\triangle_i$ and $\triangle_j$, respectively (these triangles may coincide). We draw an arc that connect v_i and v_j in the following way: going from v_i the arc intersects L. Its next part lies in the exterior of P and connects L and L'. After intersecting L' the arc goes to v_j.
- An arc, that intersects M will be the root edge. At intersection point it is directed from inside P to outside.

Example 3.1.



Here sides AB and FG, BC and CD, DE and EF, GH and HA constitute pairs and AB is the marked side. Thus, we must connect the arc that intersects AB with the arc that intersects FG, the arc that intersects BC with the arc that intersects CD, the arc that intersects DE with the arc that intersects EF and the arc that intersects GH with the arc that intersects HA. An arrow in the arc that intersects AB indicates the direction of the root edge of the cubic graph. The cubic graph itself and its "simplification" are presented in the figure below.



Lemma 3.1. *We can draw above mentioned arcs in such way, that they do not intersect in the exterior of P.*

Proof. Let us connect midpoints of all sides in pairs by segments inside P. As the identification of sides in pairs generates a genus zero curve, then these segments do not intersect. The polygon P is embedded into sphere, so we can interchange its interior and exterior domains. □

4. Main statement

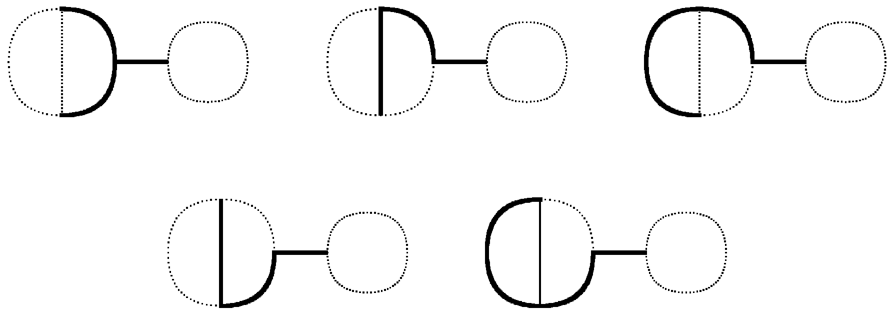
Theorem 4.1. *The number of tree-rooted cubic maps with $2n$ vertices and a marked edge, that does not belong to the spanning tree, is $C_{2n} \cdot C_{n+1}$, where C_k is k-th Catalan number.*

Proof. Our theorem follows from two statements.

(1) A convex n-gon with a marked side can be divided into triangles by non-intersecting diagonals in C_{n-2} ways [7].
(2) There are C_n ways to define a pairwise identification of sides of a convex $2n$-gon with a marked side to obtain a genus 0 curve [4].

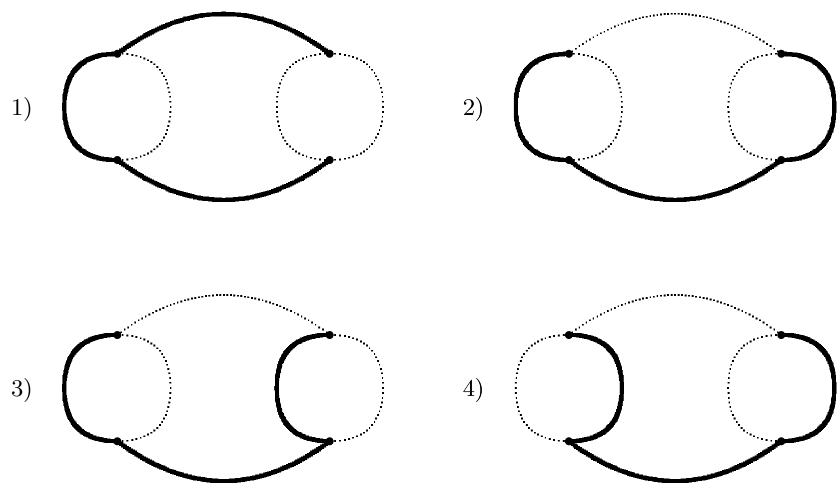
□

Example 4.1. According to theorem, we have $C_4 \cdot C_3 = 70$ tree-rooted cubic maps with 4 vertices. In what follows a map with a marked spanning tree will be called *t-map*. The first cubic map in Figure 1 generates six t-maps.



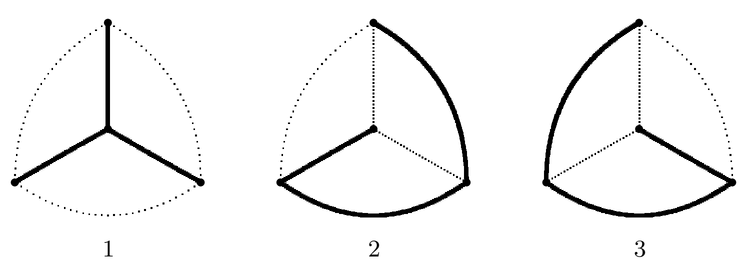
In each case we have six ways to choose a marked edge, that does not belong to the tree. Thus, the first map generates 30 tree-rooted maps.

The second cubic map in Figure 1 generates four t-maps.



The first two t-maps have trivial groups of automorphisms. Thus, they generate six tree-rooted cubic maps each. But the group of automorphisms of the third and the of forth t-maps has order two. Thus, they generate three tree-rooted cubic maps each and the second cubic map generates 18 tree-rooted maps.

The third cubic map in Figure 1 generates three t-maps.

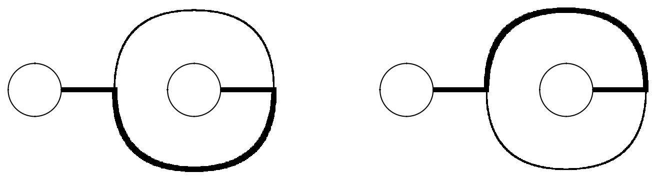


The group of automorphisms of the first of them has order 3 and of the second and the third — order 2. Thus they generate $2 + 3 + 3 = 8$ tree-rooted cubic maps.

The forth cubic map in Figure 1 generates one t-map with order three group of automorphisms. Thus, it generates 2 tree-rooted cubic maps.

The fifth cubic map in Figure 1 also generates one t-map with trivial group of automorphisms. Thus, it generates 6 tree-rooted cubic maps.

The sixth cubic map in Figure 1 generates two t-maps



with order two group of automorphisms each. Thus, they generate $3 + 3 = 6$ tree-rooted cubic maps.

So, we have

$$30 + 18 + 8 + 2 + 6 + 6 = 70$$

tree-rooted cubic maps, as expected.

References

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