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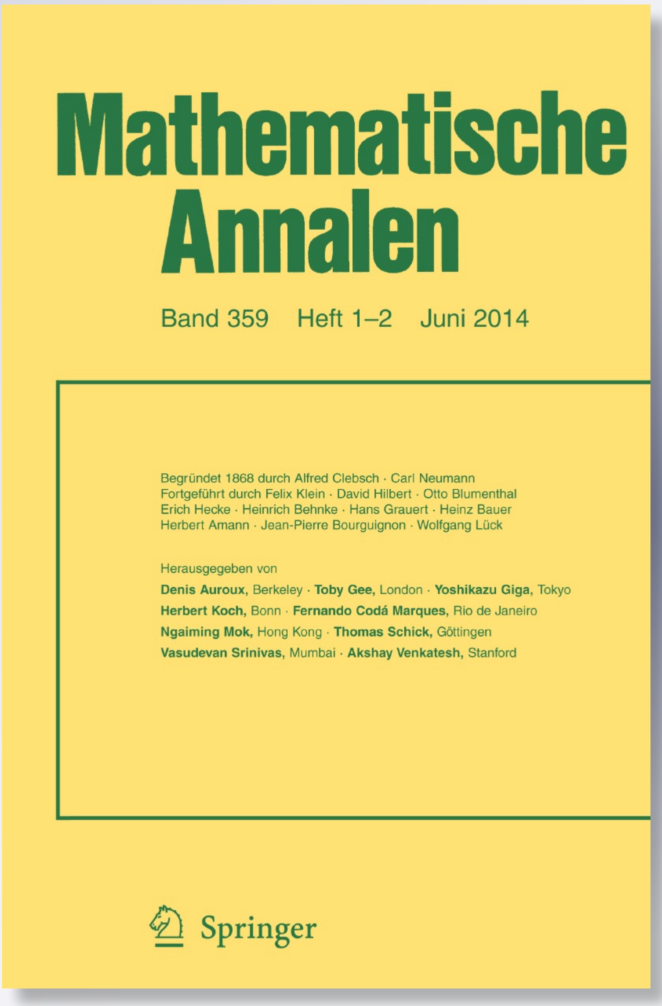
Mathematische Annalen

ISSN 0025-5831
Volume 359
Combined 1-2

Math. Ann. (2014) 359:45-59
DOI 10.1007/s00208-013-0985-3



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Weyl modules and q-Whittaker functions

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Received: 14 December 2012 / Revised: 14 October 2013 / Published online: 6 December 2013

Abstract Let G be a semi-simple simply connected group over $\mathbb{C}$. Following Gerasimov et al. (Comm Math Phys 294:97–119, 2010) we use the q-Toda integrable system obtained by quantum group version of the Kostant–Whittaker reduction (cf. Etingof in Am Math Soc Trans Ser 2:9–25, 1999, Sevostyanov in Commun Math Phys 204:1–16, 1999) to define the notion of q-Whittaker functions $\Psi_{\check{\lambda}}(q,z)$. This is a family of invariant polynomials on the maximal torus $T \subset G$ (here $z \in T$) depending on a dominant weight $\check{\lambda}$ of G whose coefficients are rational functions in a variable $q \in \mathbb{C}^*$. For a conjecturally the same (but a priori different) definition of the q-Toda system these functions were studied by Ion (Duke Math J 116:299–318, 2003) and by Cherednik (Int Math Res Notices 20:3793–3842, 2009) [we shall denote the q-Whittaker functions from Cherednik (Int Math Res Notices 20:3793–3842, 2009) by $\Psi'_{\check{\lambda}}(q,z)$]. For $G = SL(N)$ these functions were extensively studied in Gerasimov et al. (Comm Math Phys 294:97–119, 2010; Comm Math Phys 294:121–143, 2010; Lett Math Phys 97:1–24, 2010). We show that when G is simply laced, the function $\hat{\Psi}_{\check{\lambda}}(q,z) = \Psi_{\check{\lambda}}(q,z) \cdot \prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check{\lambda}\rangle}(1-q^r)$ (here I denotes the set of vertices of the Dynkin diagram of G) is equal to the character of a certain finite-dimensional $G[[t]] \rtimes \mathbb{C}^*$-module $D(\check{\lambda})$ (the Demazure module). When G is not simply laced a twisted version of the above statement holds. This result is known for $\Psi_{\check{\lambda}}$ replaced by

To the memory of Andrei Zelevinsky who taught us the beauty of symmetric functions.

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$\Psi'_{\check\lambda}$ (cf. Sanderson in J Algebraic Combin 11:269–275, 2000 and Ion in Duke Math J 116:299–318, 2003); however our proofs are algebro-geometric [and rely on our previous work (Braverman, Finkelberg in Semi-infinite Schubert varieties and quantum K-theory of flag manifolds, arXiv/1111.2266, 2011)] and thus they are completely different from Sanderson (J Algebraic Combin 11:269–275, 2000) and Ion (Duke Math J 116:299–318, 2003) [in particular, we give an apparently new algebro-geometric interpretation of the modules $D(\check\lambda)$].

Mathematics Subject Classification (2000) 19E08

1 Introduction

1.1 The q-Whittaker functions

Let G be a semi-simple, simply connected group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$; we choose a pair of opposite Borel subgroups B, B_- of G with unipotent radicals U, U_-; the intersection $B\cap B_-$ is a maximal torus T of G. It will be convenient for us to denote the weight lattice of T by $\check\Lambda$ and the coweight lattice by Λ. In this paper we study certain invariant polynomials $\Psi_{\check\lambda}(q,z)$ on T (the invariance is with respect to the Weyl group W of G). Here $z\in T$, $q\in\mathbb{C}^*$ and $\check\lambda : T\to\mathbb{C}^*$ is a dominant weight of G. The function $\Psi_{\check\lambda}(q,z)$ is a polynomial function of z with coefficients which are rational functions of q (in fact, later were are going to work with a certain modification $\hat\Psi_{\check\lambda}(q,z)$ of $\Psi_{\check\lambda}(q,z)$ which will be polynomial in q).

The definition of $\Psi_{\check\lambda}(q,z)$ is as follows. Let $\check G$ denote the Langlands dual group of G with its maximal torus $\check T$. In [7,21] the authors define (by adapting the so called Kostant–Whittaker reduction to the case of quantum groups) a homomorphism $\mathcal{M}:\mathbb{C}[T]^W\to \mathrm{End}_{\mathbb{C}(q)}\mathbb{C}(q)[\check T]$ called the quantum difference Toda integrable system associated with $\check G$. For each $f\in\mathbb{C}[T]^W$ the operator $\mathcal{M}_f := \mathcal{M}(f)$ is indeed a difference operator: it is a $\mathbb{C}(q)$-linear combination of shift operators $\mathbf{T}_{\check\beta}$ where $\check\beta\in\check\Lambda$ and

$$\mathbf{T}_{\check\beta}(F(x)) = F(q^{\check\beta}x).$$

Remark In principle the constructions of [7,21] depend on a choice of orientation of the Dynkin diagram of $\check G$; however one can deduce from the main result of [9] that the resulting homomorphism is independent of this choice.

In particular, the above operators can be restricted to operators acting in the space of functions on the lattice $\check\Lambda$ by means of the embedding $\check\Lambda\hookrightarrow\check T$ sending every $\check\lambda$ to $q^{\check\lambda}$. For any $f\in\mathbb{C}[T]^W$ we shall denote the corresponding operator by $\mathcal{M}^{\mathrm{lat}}_f$. The following conjecture should probably be not very difficult; however, at the moment we don't know how to prove it:

Conjecture 1.1 *1. There exists a unique collection of $\mathbb{C}(q)$-valued polynomials $\Psi_{\check\lambda}(q,z)$ on T satisfying the following properties:*

(a) $\Psi_{\check{\lambda}}(q,z) = 0$ *if* $\check{\lambda}$ *is not dominant.*
(b) $\Psi_0(q,z) = 1$.
(c) Let us consider all the functions $\Psi_{\check{\lambda}}(q,z)$ *as one function* $\Psi(q,z) : \check{\Lambda} \to \mathbb{C}(q)$ *depending on* $z \in T$. *Then for every* $f \in \mathbb{C}[T]^W$ *we have*

$$\mathcal{M}_f^{\mathrm{lat}}(\Psi(q,z)) = f(z)\Psi(q,z).$$

2. *The polynomials* $\Psi_{\check{\lambda}}(q,z)$ *are* W*-invariant.*

Of course, the second statement follows from the "uniqueness" part of the first.

Some remarks about the literature are necessary here. First of all, Conjecture 1.1 is easy for $G = SL(N)$. In this case, the functions $\Psi_{\check{\lambda}}(q,z)$ are extensively studied in [13–15]. Second, for general G there exists another definition of the q-Toda system using double affine Hecke algebras, studied for example in [5]. Since it is not clear to us how to prove that the definition of q-Toda from [5] and the definition of [7,21] are the same, we shall denote the operators from [5] by $\mathcal{M}'_f$. It is easy to see that $\mathcal{M}_f = \mathcal{M}'_f$ for $G = SL(N)$.[1] Similarly we shall denote by $(\mathcal{M}_f^{\mathrm{lat}})'$ their "lattice" version. Then it is shown in [5] that the existence part of Conjecture 1.1 holds for any G if the operators $\mathcal{M}_f^{\mathrm{lat}}$ are replaced by $(\mathcal{M}_f^{\mathrm{lat}})'$. We shall denote the corresponding polynomials by $\Psi'_{\check{\lambda}}(q,z)$.

The main result of this paper will imply the following:

Theorem 1.2 *1. There exists a collection of* W*-invariant polynomials* $\Psi_{\check{\lambda}}(q,z)$ *on* T *with coefficients in* $\mathbb{C}(q)$ *satisfying (a), (b) and (c) above.*

2. *Let* $\hat{\Psi}_{\check{\lambda}}(q,z) = \Psi_{\check{\lambda}}(q,z) \cdot \prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check{\lambda}\rangle}(1-q^r)$. *Then* $\hat{\Psi}_{\check{\lambda}}(q,z)$ *is a polynomial function on* $\mathbb{A}^1 \times T$.

We are going to construct the above polynomials explicitly by algebro-geometric means. Thus we prove the existence part of Conjecture 1.1.

We shall usually refer to the polynomials $\Psi_{\check{\lambda}}$ and $\hat{\Psi}_{\check{\lambda}}$ as q-Whittaker functions (following [13–15]). It is not difficult to see that

$$\lim_{q\to 0}\Psi_{\check{\lambda}} = \lim_{q\to 0}\hat{\Psi}_{\check{\lambda}} = \chi\big(L(\check{\lambda})\big)$$

where $\chi(L(\lambda))$ stands for the character of the irreducible representation $L(\check{\lambda})$ of G with highest weight $\check{\lambda}$.

The main purpose of this paper is to give several (algebro-geometric and representation-theoretic) interpretations of the functions $\Psi_{\check{\lambda}}$ and $\hat{\Psi}_{\check{\lambda}}$; as a byproduct we shall show that $\hat{\Psi}_{\check{\lambda}}(q,z)$ is *positive*, i.e. it is a linear combination of the functions $\chi(L(\check{\mu}))$ with coefficients in $\mathbb{Z}_{\geq 0}[q]$ (this also implies that $\Psi_{\check{\lambda}}$ is a linear combination of the $\chi(L(\check{\mu}))$'s with coefficients in $\mathbb{Z}_{\geq 0}[[q]]$). All of our results are known for the polynomials $\Psi'_{\check{\lambda}}$ (and thus, in particular, we can show that $\Psi_{\check{\lambda}} = \Psi'_{\check{\lambda}}$) due to [5,18,20] but our proofs are totally different from [5,18,20].

[1] In fact, as we are going to explain later, the results of this paper together with the results of [18] imply that $\mathcal{M}_f = \mathcal{M}'_f$ for any G, but we would like to have a more direct proof of this fact.

1.2 Weyl modules

Recall the notion of Weyl $\mathfrak{g}[\mathsf{t}]$-module $\mathcal{W}(\check{\lambda})$ for dominant $\check{\lambda} \in \Lambda_+^\vee$, see e.g. [3]. It is the maximal G-integrable $\mathfrak{g}[\mathsf{t}]$-quotient module of $\mathrm{Ind}_{\mathfrak{u}[\mathsf{t}]\oplus\mathfrak{t}}^{\mathfrak{g}[\mathsf{t}]}\mathbb{C}_{\check{\lambda}}$ where $\mathfrak{u} \subset \mathfrak{g}$ is the nilpotent radical of a Borel subalgebra, containing $\mathfrak{t}$. There is also a natural notion of *dual Weyl module* $\mathcal{W}(\check{\lambda})^\vee$ (one has to replace the induction by coinduction and "quotient module" by "submodule"). Both $\mathcal{W}(\check{\lambda})$ and $\mathcal{W}(\check{\lambda})^\vee$ are endowed with a natural action of $\mathbb{C}^*$ by "loop rotation". When restricted to $G \times \mathbb{C}^*$ the module $\mathcal{W}(\check{\lambda})$ becomes a direct sum of finite-dimensional representations and the character $\chi(\mathcal{W}(\check{\lambda}))$ makes sense; moreover it is a linear combination of $\chi(L(\check{\mu}))$'s with coefficients in $\mathbb{Z}_{\geq 0}[[q]]$. Also we have $\chi(\mathcal{W}(\check{\lambda})) = \chi(\mathcal{W}(\check{\lambda})^\vee)$.

Let $\mathbb{A}^{\check{\lambda}}$ denote the space of all formal linear combinations $\sum \gamma_i x_i$ where $x_i \in \mathbb{A}^1$ and γ_i are dominant weights of G such that $\sum \gamma_i = \check{\lambda}$. The character of $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ with respect to the natural action of $\mathbb{C}^*$ is equal to $\prod_{i\in I}\prod_{r=1}^{\langle\alpha_i,\check{\lambda}\rangle}(1-q^r)$. According to [3] there exists an action of $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ on $\mathcal{W}(\check{\lambda})$ such that

1. This action commutes with $G[\mathsf{t}] \rtimes \mathbb{C}^*$;
2. $\mathcal{W}(\check{\lambda})$ is finitely generated and free over $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$.

Let $D(\check{\lambda})$ be the fiber of $\mathcal{W}(\check{\lambda})$ at $\check{\lambda} \cdot 0 \in \mathbb{A}^{\check{\lambda}}$. This module is called a Demazure module (for reasons explained in [4,12]). This is a finite-dimensional $G[\mathsf{t}] \rtimes \mathbb{C}^*$-module (in fact, it is easy to see that the action of $G[\mathsf{t}]$ on $D(\check{\lambda})$ extends to an action of $G[[\mathsf{t}]]$). We are going to prove the following

Theorem 1.3 *Assume that G is simply laced. Then*

1.
$$\chi\left(\mathcal{W}(\check{\lambda})\right) = \Psi_{\check{\lambda}}(q,z) \tag{1.1}$$

2.
$$\chi\left(D(\check{\lambda})\right) = \hat{\Psi}_{\check{\lambda}}(q,z). \tag{1.2}$$

In particular, $\hat{\Psi}_{\check{\lambda}}(q,z)$ is positive in the sense discussed above.

When G is not simply laced, the above result is still true, if one replaces $G[[\mathsf{t}]]$ by some twisted (in the sense of Kac-Moody groups) version of it; we shall not give the details here (cf. Sect. 1.4 for a discussion of the non-simply laced case).

Theorem 1.3(2) is proved in [18] for $\hat{\Psi}'_{\check{\lambda}}$ instead of $\hat{\Psi}_{\check{\lambda}}$.[2] Thus Theorem 1.3 together with [18] imply the following:

Corollary 1.4 *Assume that G is simply laced. Then we have $\hat{\Psi}'_{\check{\lambda}} = \hat{\Psi}_{\check{\lambda}}$. Hence for any $f \in \mathbb{C}[T]^W$ we have $\mathcal{M}_f = \mathcal{M}'_f$.*

[2] It is important to emphasize that the definition of Demazure modules used in this paper (as fibers of Weyl modules) is not obviously equivalent to the standard definition used in [18]; however, the equivalence of the two definitions is proved in [4] in type A, and in [12] in general.

As was mentioned earlier we would like to have a more direct proof of this result (independent of the results of [18] and this paper). We would also like to emphasize that our proof of Theorem 1.3 is geometric (in fact it follows easily from the main result of [2]) and thus it is quite different from the proof in [18]. Also, Corollary 1.4 is wrong if G is not simply laced, cf. Sect. 1.4.

1.3 Geometric interpretation and spaces of (quasi-)maps

To prove Theorem 1.3 it is clearly enough to prove (1.1). This will be done by interpreting both the LHS and the RHS in terms of algebraic geometry.

Let us first do it for the LHS. The quotient $G[[\mathsf{t}]]/T \cdot U_{-}[[\mathsf{t}]]$ can naturally be regarded as a scheme over $\mathbb{C}$. Any weight $\check{\lambda}$ defines a $G[[\mathsf{t}]] \rtimes \mathbb{C}^{*}$-equivariant line bundle on this scheme in the standard way. We shall prove

Theorem 1.5 *There is a natural isomorphism* $\Gamma(G[[\mathsf{t}]]/T \cdot U_{-}[[\mathsf{t}]], \mathcal{O}(\check{\lambda})) \simeq \mathcal{W}(\check{\lambda})^{\vee}$. *Similarly,* $\Gamma(G[[\mathsf{t}]]/B_{-}[[\mathsf{t}]], \mathcal{O}(\check{\lambda})) \simeq D(\check{\lambda})^{\vee}$.

Remark Theorem 1.5 is not difficult; it can be thought of as an analog of Borel-Weil-Bott theorem for $G[[\mathsf{t}]]$. Let us also stress, that while the dual Weyl module $\mathcal{W}(\check{\lambda})^{\vee}$ has a natural action of $G[[\mathsf{t}]]$, the Weyl module $\mathcal{W}(\check{\lambda})$ itself only has an action of $G[\mathsf{t}]$.

On the other hand, there is a well known connection between the quotient $G[[\mathsf{t}]]/T \cdot U_{-}[[\mathsf{t}]]$ and the space of based maps $\mathbb{P}^1 \to G/B$. Moreover, in [2] we have given a construction of the universal eigen-function of the operators $\mathcal{M}_f$ via the geometry of the above spaces of maps. Using this construction, we can obtain (1.1) from Theorem 1.5 by a (simple) sequence of formal manipulations. Technically, in order to perform this we shall need to consider a compactification of the space of maps by the corresponding space of quasi-maps.

1.4 The case of non-simply laced G

Formally, the above results do not hold when G is not simply laced. However, it is easy to adjust all the results to the non-simply laced case following Section 7 of [2]; in particular, in the non-simply laced case the functions $\Psi_{\check{\lambda}}$ and $\hat{\Psi}_{\check{\lambda}}$ should be interpreted as the characters of global (resp. local) Weyl modules for the distinguished maximal parahoric subalgebra in a certain twisted affine algebra corresponding to $\mathfrak{g}$ (cf. Section 7 of [2] for more detail). The relevant theory of Weyl modules and their relation to Demazure modules in the twisted case is developed in [11]. On the other hand, the character of *nontwisted* local Weyl modules are identified with $\hat{\Psi}'_{\check{\lambda}}$ in [19].

1.5 Plan of the paper

This paper is organized as follows. In Sect. 2 we discuss certain line bundles on the space of (quasi-)maps and relate those to sections of a line bundle on $G[[\mathsf{t}]]/T \cdot U_{-}[[\mathsf{t}]]$. Section 3 is devoted to the proof of certain cohomology vanishing on the space of quasimaps. In Sect. 4 we give an interpretation of $\Psi_{\check\lambda}$ via quasi-maps. Finally in Sect. 5 we give a proof of Theorem 1.3.

2 Quasimaps' scheme

We follow the notations of [2], unless specified otherwise.

2.1 Ind-scheme $\mathfrak{Q}$

Given $\beta \geq \alpha \in \Lambda_+$ (the cone of positive integral combinations of the simple coroots) we consider the closed embedding $\varphi_{\alpha,\beta}: \mathcal{QM}^\alpha_{\mathfrak{g}} \hookrightarrow \mathcal{QM}^\beta_{\mathfrak{g}}$ adding the defect $(\beta-\alpha)\cdot 0$ at the point $0 \in \mathbf{C}$. We denote by $\mathfrak{Q}$ the direct limit of this system.

Recall that $V_{\check\omega_i},\ i \in I$, are the fundamental $\mathfrak{g}$-modules, and $\mathcal{QM}^\alpha_{\mathfrak{g}}$ is equipped with a closed embedding $\psi_\alpha: \mathcal{QM}^\alpha_{\mathfrak{g}} \hookrightarrow \prod_{i\in I} \mathbb{P}\Gamma(\mathbf{C}, V_{\check\omega_i} \otimes \mathcal{O}(\langle\alpha, \check\omega_i\rangle))$. Given a $\mathfrak{g}$-weight $\check\lambda = \sum_{i\in I} d_i\check\omega_i \in \Lambda^\vee$ we define a line bundle $\mathcal{O}(\check\lambda)^\alpha$ on $\mathcal{QM}^\alpha_{\mathfrak{g}}$ as $\psi^*_\alpha \bigotimes_{i\in I} \mathcal{O}(d_i)$. Note that if $\check\lambda$ is dominant, i.e. $d_i \geq 0\ \forall i$, then $\mathcal{O}(\check\lambda)^\alpha$ is the inverse image of $\mathcal{O}(1)$ on $\mathbb{P}\Gamma(\mathbf{C}, V_{\check\lambda} \otimes \mathcal{O}(\langle\alpha, \check\lambda\rangle))$ under the natural morphism $\mathcal{QM}^\alpha_{\mathfrak{g}} \to \mathbb{P}\Gamma(\mathbf{C}, V_{\check\lambda} \otimes \mathcal{O}(\langle\alpha, \check\lambda\rangle))$. Clearly, $\varphi^*_{\alpha,\beta}\mathcal{O}(\check\lambda)^\beta \simeq \mathcal{O}(\check\lambda)^\alpha$. The resulting line bundle on the ind-scheme $\mathfrak{Q}$ is denoted $\mathcal{O}(\check\lambda)$.

2.2 Infinite type scheme $\mathbf{Q}$

We denote $\mathbb{C}[[t^{-1}]]$ by R, and $\mathbb{C}((t^{-1}))$ by F. Recall that $R_n = R/(t^{-n})$. We denote the projection $R \twoheadrightarrow R_n$ by p_n. The $\mathbb{C}$-points of the infinite type scheme $\overline{G/U_-}(R)$ are the collections of vectors $v_{\check\lambda} \in V_{\check\lambda} \otimes R,\ \check\lambda \in \Lambda^\vee_+$ (dominant $\mathfrak{g}$-weights), satisfying the Plücker equations. We denote by $\widehat{\mathbf{Q}} \subset \overline{G/U_-}(R)$ the open subscheme formed by all the maps Spec $R \to \overline{G/U_-}$ whose restriction to the generic point of Spec R lands into $G/U_- \subset \overline{G/U_-}(R)$. It is equipped with a free action of the Cartan torus $T: \ h(v_{\check\lambda}) = \check\lambda(h)v_{\check\lambda}$. The quotient scheme $\mathbf{Q} = \widehat{\mathbf{Q}}/T$ is a closed subscheme in $\prod_{i\in I} \mathbb{P}(V_{\check\omega_i} \otimes R)$. Any weight $\check\lambda \in \Lambda^\vee$ gives rise to a line bundle $\mathcal{O}(\check\lambda)$ on $\mathbf{Q}$.

2.3 The embedding $\mathfrak{Q} \hookrightarrow \mathbf{Q}$

We fix a coordinate t on $\mathbf{C}$ such that $t(0) = 0,\ t(\infty) = \infty$. For $\alpha \in \Lambda_+$ we define a T-torsor $\widehat{\mathcal{QM}^\alpha_{\mathfrak{g}}} \xrightarrow{p} \mathcal{QM}^\alpha_{\mathfrak{g}}$ as follows. The $\mathbb{C}$-points of $\widehat{\mathcal{QM}^\alpha_{\mathfrak{g}}}$ are the collections $(v_{\check\lambda} \in \mathcal{L}_{\check\lambda} \subset V_{\check\lambda} \otimes \mathcal{O}_{\mathbf{C}}),\ \check\lambda \in \Lambda^\vee_+$, such that

(a) $(\mathcal{L}_{\check\lambda} \subset V_{\check\lambda} \otimes \mathcal{O}_{\mathbf{C}})_{\check\lambda\in\Lambda^\vee_+} \in \mathcal{QM}^\alpha_{\mathfrak{g}}$; (b) $v_{\check\lambda} \in \Gamma(\mathbf{C} - 0, \mathcal{L}_{\check\lambda})$ are the nonvanishing sections satisfying the Plücker equations.

The projection p forgets the sections $v_{\check\lambda}$. The action of T on $\widehat{\mathcal{QM}^\alpha_{\mathfrak{g}}}$ is defined as follows: $h(v_{\check\lambda} \in \mathcal{L}_{\check\lambda}) = (\check\lambda(h)v_{\check\lambda} \in \mathcal{L}_{\check\lambda})$.

Taking a formal expansion of $v_{\check{\lambda}}$ at $\infty \in \mathbf{C}$ we obtain a closed embedding $s_\alpha : \widehat{\mathcal{QM}_{\mathfrak{g}}^\alpha} \hookrightarrow \widehat{\mathbf{Q}}$. Clearly, s_α is T-equivariant, and gives rise to the same named closed embedding $s_\alpha : \mathcal{QM}_{\mathfrak{g}}^\alpha \hookrightarrow \mathbf{Q}$. Evidently, for $\beta \geq \alpha$ we have $s_\alpha = s_\beta \circ \varphi_{\alpha,\beta}$. Hence we obtain the closed embedding $s : \mathfrak{Q} \hookrightarrow \mathbf{Q}$. The restriction of the line bundle $\mathcal{O}(\check{\lambda})$ on $\mathbf{Q}$ to $\mathfrak{Q}$ coincides with the line bundle $\mathcal{O}(\check{\lambda})$ on $\mathfrak{Q}$.

2.4 Open subschemes $\mathfrak{Q}_\infty \subset \mathfrak{Q}$ and $\mathbf{Q}_\infty \subset \mathbf{Q}$

We define an open subscheme $\overset{\circ}{\mathcal{QM}}{}_{\mathfrak{g}}^\alpha \subset \mathcal{QM}_{\mathfrak{g}}^\alpha$ formed by all the quasimaps without defect at $\infty \in \mathbf{C}$. Clearly, $\varphi_{\alpha,\beta}(\overset{\circ}{\mathcal{QM}}{}_{\mathfrak{g}}^\alpha) \subset \overset{\circ}{\mathcal{QM}}{}_{\mathfrak{g}}^\beta$. The direct limit of this system is denoted by $\mathfrak{Q}_\infty$; it is an open sub ind-scheme of $\mathfrak{Q}$.

Note that $s(\mathfrak{Q}_\infty) \subset G(R)/T \cdot U_-(R) \subset \mathbf{Q}$. We are going to denote the open subscheme $G(R)/T \cdot U_-(R) \subset \mathbf{Q}$ by $\mathbf{Q}_\infty$. For $n \geq 1$, we have a natural projection $p_n : \mathbf{Q}_\infty \to G/U_-(R_n)/T =: \mathbf{Q}_n$.

Lemma 1 *The restriction $\Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) \to \Gamma(\mathfrak{Q}_\infty, \mathcal{O}(\check{\lambda}))$ is an isomorphism for any $\check{\lambda} \in \Lambda^\vee$.*

Proof It suffices to prove that the restriction $\Gamma(\mathcal{QM}_{\mathfrak{g}}^\alpha, \mathcal{O}(\check{\lambda})) \to \Gamma(\overset{\circ}{\mathcal{QM}}{}_{\mathfrak{g}}^\alpha, \mathcal{O}(\check{\lambda}))$ is an isomorphism for any $\alpha \in \Lambda_+$. Since the complement of $\overset{\circ}{\mathcal{QM}}{}_{\mathfrak{g}}^\alpha$ in $\mathcal{QM}_{\mathfrak{g}}^\alpha$ has codimension 2, it suffices to know that $\mathcal{QM}_{\mathfrak{g}}^\alpha$ is normal. However, locally in the étale topology, $\mathcal{QM}_{\mathfrak{g}}^\alpha$ is isomorphic to the product of the Zastava space $Z_{\mathfrak{g}}^\alpha$ and the flag variety $\mathcal{B}_{\mathfrak{g}}$. Finally, the normality of $Z_{\mathfrak{g}}^\alpha$ is proved in [2, Corollary 2.10]. □

The following conjecture is not needed in this paper, but it might be useful for future purposes.

Conjecture 2.1 *The restriction $\Gamma(\mathbf{Q}, \mathcal{O}(\check{\lambda})) \to \Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))$ is an isomorphism for any $\check{\lambda} \in \Lambda^\vee$.*

Let us make a few remarks about Conjecture 2.1. As in the proof of Lemma 1, it suffices to know that the scheme $\mathbf{Q}$ is normal. According to [6,17], the formal completion of $\mathbf{Q}$ at a closed point $x \in \mathbf{Q}$ is isomorphic to the product of the formal completion of a certain $\mathcal{QM}_{\mathfrak{g}}^\alpha$ at a closed point $\phi \in \mathcal{QM}_{\mathfrak{g}}^\alpha$, and countably many copies of the formal disc. So the normality of the formal neighborhood of every closed point follows from the normality of $\mathcal{QM}_{\mathfrak{g}}^\alpha$. Unfortunately, since $\mathbf{Q}$ is not noetherian it does not imply the normality of $\mathbf{Q}$ itself.

The group $\mathbb{G}_m$ acts on $\mathfrak{Q}$ and $\mathbf{Q}$ by loop rotations, and the line bundles $\mathcal{O}(\check{\lambda})$ are $\mathbb{G}_m$-equivariant. Hence $\mathbb{G}_m$ acts on the global sections of these line bundles. We will denote by $\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) \subset \Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$ the subspace of $\mathbb{G}_m$-finite sections.

Theorem 2.2 *The restriction $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda})) \to \widetilde{\Gamma}(\mathfrak{Q}_\infty, \mathcal{O}(\check{\lambda})) = \widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$ is an isomorphism for any $\check{\lambda} \in \Lambda^\vee$.*

Proof The closed embedding $\varphi_{\alpha,\beta} : \mathcal{QM}_{\mathfrak{g}}^\alpha \hookrightarrow \mathcal{QM}_{\mathfrak{g}}^\beta$ lifts in an evident way to the same named closed embedding of T-torsors $\widehat{\mathcal{QM}_{\mathfrak{g}}^\alpha} \hookrightarrow \widehat{\mathcal{QM}_{\mathfrak{g}}^\beta}$. We denote the limit of this system by $\widehat{\mathfrak{Q}}$, a T-torsor over $\mathfrak{Q}$. The construction of Sect. 2.3 defines a T-equivariant closed embedding $s: \widehat{\mathfrak{Q}} \hookrightarrow \widehat{\mathbf{Q}}_\infty := G/U_-(R)$. We have to prove that the restriction $\mathbb{C}[\widehat{\mathbf{Q}}_\infty] \to \widetilde{\mathbb{C}}[\widehat{\mathfrak{Q}}_\infty] = \widetilde{\mathbb{C}}[\widehat{\mathfrak{Q}}]$ is an isomorphism. Here $\widetilde{\mathbb{C}}[\widehat{\mathfrak{Q}}_\infty]$ (resp. $\widetilde{\mathbb{C}}[\widehat{\mathfrak{Q}}]$) stands for the ring of $\mathbb{G}_m$-finite functions on $\widehat{\mathfrak{Q}}_\infty$ (resp. $\widehat{\mathfrak{Q}}$).

To this end we mimick the argument of [2, Section 2]. We choose a *regular* dominant $\mu \in \Lambda^+$, and consider the corresponding T-fixed point $t^\mu \in \mathrm{Gr}_G$. Its stabilizer St_μ in $G[t^{-1}]$ has the unipotent radical RadSt_μ, and the quotient $\mathrm{St}_\mu / \mathrm{RadSt}_\mu$ is canonically isomorphic to T. The quotient $G[t^{-1}]/\mathrm{St}_\mu$ is the $G[t^{-1}]$-orbit $\mathsf{W}_{G,\mu} \subset \mathrm{Gr}_G$ of t^μ (see [2, Section 2.4]), and the quotient $G[t^{-1}]/\mathrm{RadSt}_\mu$ is a T-torsor $\widehat{\mathsf{W}}_{G,\mu}$.

NB: The group denoted St_μ in [2, Section 2.6] is the intersection of our present St_μ with the first congruence subgroup $G_1 \subset G[t^{-1}]$.

In modular terms, $\mathsf{W}_{G,\mu}$ parametrizes the G-bundles on $\mathbf{C}$ of isomorphism type $W\mu$ equipped with a trivialization on $\mathbf{C} - 0$ (see [2, Proof of Theorem 2.8]). Such a bundle $\mathcal{F}_G$ possesses a canonical Harder-Narasimhan flag $HN(\mathcal{F}_G)$. Note that this flag is complete, i.e. it is a reduction to the Borel, since μ is regular. In particular, the fiber $\mathcal{F}_{G,\infty}$ of $\mathcal{F}_G$ at $\infty \in \mathbf{C}$ is equipped with a canonical reduction to the Borel. Now $\widehat{\mathsf{W}}_{G,\mu}$ parametrizes the data as above along with a further reduction of $\mathcal{F}_{G,\infty}$ to the unipotent radical of the Borel. □

In complete similarity with [2, Lemma 2.7] we have

Lemma 2 1. *Fix $n \geq 1$, and let $\mu \in \Lambda^+_{\mathrm{reg}}$ satisfy the following condition: $\langle \mu, \check{\alpha} \rangle \geq n$ for every positive root $\check{\alpha}$ of $\mathfrak{g}$. Then the image of RadSt_μ in $G[t^{-1}]/G_n = G(R_n)$ is equal to $U_-(R_n)$. In particular, we have a natural map $\pi_{\mu,n}: \widehat{\mathsf{W}}_{G,\mu} \to G(R_n)/U_-(R_n)$.*
2. *Under the assumption of (1), for every $k < n$, the map $\pi^*_{\mu,n}: \mathbb{C}[G(R_n)/U_-(R_n)] \to \mathbb{C}[\widehat{\mathsf{W}}_{G,\mu}]$ induces an isomorphism on functions of homogeneity degree k with respect to $\mathbb{G}_m$.*

We denote the intersection of $\mathsf{W}_{G,\mu} \subset \mathrm{Gr}_G$ with $\overline{\mathrm{Gr}}^\lambda_G$ by $\overline{\mathsf{W}}^\lambda_{G,\mu}$. We denote the preimage of $\overline{\mathsf{W}}^\lambda_{G,\mu} \subset \mathsf{W}_{G,\mu}$ in $\widehat{\mathsf{W}}_{G,\mu}$ by $\widehat{\mathsf{W}}^\lambda_{G,\mu}$. In complete similarity with [2, Theorem 2.8] we have

Lemma 3 1. *Let $\lambda \geq \mu \in \Lambda^+_{\mathrm{reg}}$, and let $\alpha = \lambda - \mu$. Then there exists a natural birational $T \times \mathbb{G}_m$-equivariant morphism $s^\lambda_\mu : \widehat{\mathsf{W}}^\lambda_{G,\mu} \to \widehat{\mathcal{QM}}^\alpha_\mathfrak{g}$ such that for any n satisfying the condition in Lemma 2(1), the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{\mathsf{W}}^\lambda_{G,\mu} & \xrightarrow{s^\lambda_\mu} & \widehat{\mathcal{QM}}^\alpha_\mathfrak{g} \\ \downarrow{\scriptstyle \pi_{\mu,n}} & & \downarrow{\scriptstyle p_n \circ s_\alpha} \\ G(R_n)/U_-(R_n) & \xrightarrow{\mathrm{id}} & G(R_n)/U_-(R_n) \end{array} \tag{2.1}$$

(s_α was constructed in Sect. 2.3).
2. *The map $(s^\lambda_\mu)^*: \mathbb{C}[\widehat{\mathcal{QM}}^\alpha_\mathfrak{g}] \to \mathbb{C}[\widehat{\mathsf{W}}^\lambda_{G,\mu}]$ induces an isomorphism on functions of degree $< n$ for any n satisfying the condition in Lemma 2(1).*

Now Theorem 2.2 immediately follows from Lemma 2 and Lemma 3.

Note that if one assumes Conjecture 2.1 then it follows that the restriction $\Gamma(\mathbf{Q}, \mathcal{O}(\check{\lambda})) \to \widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$ is an isomorphism for any $\check{\lambda} \in \Lambda^\vee$. (this follows immediately from Theorem 2.2, Lemma 1, and Conjecture 2.1).

3 Cohomology vanishing

3.1 From now on we assume that G is simply laced.

The group $\mathbb{G}_m$ acts on $\mathfrak{Q}$ and $\mathbf{Q}$ by loop rotations, and the line bundles $\mathcal{O}(\check{\lambda})$ are $\mathbb{G}_m$-equivariant. Hence $\mathbb{G}_m$ acts on the cohomology $H^n(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) := \varprojlim H^n(\mathfrak{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda}))$ of these line bundles. We will denote by $\widetilde{H}^n(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) \subset H^n(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$ the subspace of $\mathbb{G}_m$-finite classes.

Recall that $\alpha \mapsto \alpha^*$ stands for the natural (linear) isomorphism between the coroot lattice of $\mathfrak{g}$ and its root lattice, taking the simple coroots to the corresponding simple roots. Now Λ_+ contains a cofinal subsystem $\Lambda_+^{\check{\lambda}}$ formed by α such that $\alpha^* + \check{\lambda}$ is dominant.

Theorem 3.1 1. *For $n > 0$ and $\alpha \in \Lambda_+^{\check{\lambda}}$ we have $H^n(\mathfrak{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda})) = 0$.*
2. *For $n > 0$ and $\check{\lambda} \in \Lambda^\vee$ we have $\widetilde{H}^n(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) = 0$.*
3. *For $\check{\lambda} \notin \Lambda_+^\vee$ we have $\widetilde{H}^0(\mathfrak{Q}, \mathcal{O}(\check{\lambda})) = 0$.*

Proof (3) is clear, and (2) follows from (1). We prove (1).

According to [2, Proposition 5.1], $Z_{\mathfrak{g}}^{\alpha}$ is a Gorenstein variety with rational singularities. Since $\mathfrak{QM}_{\mathfrak{g}}^{\alpha}$ is, locally in étale topology, isomorphic to $Z_{\mathfrak{g}}^{\alpha} \times \mathcal{B}_{\mathfrak{g}}$, we conclude that $\mathfrak{QM}_{\mathfrak{g}}^{\alpha}$ is a Gorenstein variety with rational singularities as well. (It is here that we use the assumption that G is simply laced.) Let us denote the dualizing sheaf of $\mathfrak{QM}_{\mathfrak{g}}^{\alpha}$ by ω^{α}. □

Lemma 4 $\omega^{\alpha} \simeq \mathcal{O}(-\alpha^* - 2\check{\rho})$.

Proof In case $G = \mathrm{SL}(N)$, the lemma is proved in [16, Theorem 3]. For arbitrary simply laced G we first prove that $\omega^{\alpha} \simeq \mathcal{O}(\check{\lambda})$ for some $\check{\lambda}$. It is enough to check this on the open subscheme $\mathring{\mathfrak{QM}}_{\mathfrak{g}}^{\alpha}$ since the complement is of codimension two. We have the morphism of evaluation at $\infty \in \mathbf{C}$: $\mathring{\mathfrak{QM}}_{\mathfrak{g}}^{\alpha} \xrightarrow{ev_\infty} \mathcal{B}_{\mathfrak{g}}$. It is a G-equivariant fibration with fibers isomorphic to $Z_{\mathfrak{g}}^{\alpha}$. Since the big cell $U \cdot e_- \subset \mathcal{B}_{\mathfrak{g}}$ is a free orbit of U, we have $ev_\infty^{-1}(U \cdot e_-) \simeq Z_{\mathfrak{g}}^{\alpha} \times U$. The canonical class of $Z_{\mathfrak{g}}^{\alpha}$ is trivial (see [2, Proof of Proposition 5.1]), hence the canonical class of $ev_\infty^{-1}(U \cdot e_-)$ is trivial as well. Thus ω^{α} has a nowhere vanishing section σ on $ev_\infty^{-1}(U \cdot e_-)$. Hence the class of ω^{α} on $\mathring{\mathfrak{QM}}_{\mathfrak{g}}^{\alpha}$ is a linear combination of the pullbacks under ev_∞ of the Schubert divisors on $\mathcal{B}_{\mathfrak{g}}$. The pullback of an irreducible Schubert divisor being $\mathcal{O}(\check{\omega}_i)$ we conclude that there exists $\check{\lambda}$ such that $\omega^{\alpha} \simeq \mathcal{O}(\check{\lambda})$.

It remains to check $\check{\lambda} = -\alpha^* - 2\check{\rho}$. We will do this on another open subscheme $\dot{\mathfrak{QM}}_{\mathfrak{g}}^{\alpha} \subset \mathfrak{QM}_{\mathfrak{g}}^{\alpha}$ with the complement of codimension two. Namely, $\dot{\mathfrak{QM}}_{\mathfrak{g}}^{\alpha}$ is the moduli space of quasimaps with defect at most a simple coroot (or no defect at all). Note that

$\overset{\bullet}{\mathfrak{QM}}{}^{\alpha}_{\mathfrak{g}}$ is smooth, and the Kontsevich resolution is an isomorphism over it. Let us fix a quasimap without defect $\phi\in\mathfrak{QM}^{\alpha-\alpha_i}_{\mathfrak{g}}$, and consider a curve $C_i^{\phi}\subset\overset{\bullet}{\mathfrak{QM}}{}^{\alpha}_{\mathfrak{g}}$ formed by all the quasimaps $\phi(\alpha_i\cdot c)$, $c\in\mathbf{C}$ (twisting ϕ by an arbitrary point of $\mathbf{C}$). It is easy to see that $\deg(\mathcal{O}(\check{\omega}_j)|_{C_i^{\phi}})=\delta_{ij}=\langle\alpha_i,\check{\omega}_j\rangle$. Hence it remains to check that $\deg(\omega^{\alpha}|_{C_i^{\phi}})=-\langle\alpha_i,\alpha^*+2\check{\rho}\rangle$. This is done in [10, Proposition 4.4]. Although [10] is formulated for $G=\mathrm{SL}(N)$, its proof goes through word for word for arbitrary simple G.

The lemma is proved. □

We are ready to finish the proof of the theorem. For $\alpha\in\Lambda_+^{\check{\lambda}}$ the line bundle $\mathcal{L}=\mathcal{O}(\check{\lambda})\otimes(\omega^{\alpha})^*$ on $\mathfrak{QM}^{\alpha}_{\mathfrak{g}}$ is very ample. We have to prove that $H^n(\mathfrak{QM}^{\alpha}_{\mathfrak{g}},\mathcal{O}(\check{\lambda}))=H^n(\mathfrak{QM}^{\alpha}_{\mathfrak{g}},\mathcal{L}\otimes\omega^{\alpha})=0$ for $n>0$. According to [2, Proposition 5.1], $\mathfrak{QM}^{\alpha}_{\mathfrak{g}}$ has rational singularities. Let $\pi:X\to\mathfrak{QM}^{\alpha}_{\mathfrak{g}}$ be a resolution of singularities. Then for the canonical line bundle ω_X of X we have $R\pi_*\omega_X=\omega^{\alpha}$. Hence $H^n(\mathfrak{QM}^{\alpha}_{\mathfrak{g}},\mathcal{L}\otimes\omega^{\alpha})=H^n(X,\pi^*\mathcal{L}\otimes\omega_X)=0$ (for $n>0$) by Kawamata-Viehweg vanishing since $\pi^*\mathcal{L}$ is nef and big.

This completes the proof of the theorem.

4 q-Whittaker functions

4.1 The character of $R\Gamma(\mathfrak{QM}^{\alpha}_{\mathfrak{g}},\mathcal{O}(\check{\lambda}))$

Recall [2, Introduction] that $\mathfrak{J}_{\alpha}(q,z)$ is the character of $T\times\mathbb{G}_m$-module $\mathbb{C}[Z^{\alpha}_{\mathfrak{g}}]$, a rational function on $T\times\mathbb{G}_m$. Let x_i stand for the character of the dual torus $\check{T}$ corresponding to the simple coroot α_i. For $\alpha\in\Lambda_+$ the corresponding character of $\check{T}$ is denoted by x^{α}. We consider the formal generating functions $J_{\mathfrak{g}}(q,z,x)=\sum_{\alpha\in\Lambda_+}x^{\alpha}\mathfrak{J}_{\alpha}$, and $\mathfrak{J}_{\mathfrak{g}}(q,z,x)=\prod_{i\in I}x_i^{\log(\check{\omega}_i)/\log q}J_{\mathfrak{g}}(q,z,x)$, cf. [1, Equation (18)].

According to [2, Corollary 1.6], the function $\mathfrak{J}_{\mathfrak{g}}(q,z,x)$ is an eigen-function of the quantum difference Toda integrable system associated with $\mathfrak{g}$. For example, if $G=\mathrm{SL}(N)$, the function $\mathfrak{J}_{\mathfrak{g}}(q,z,x)$ is an eigenfunction of the operator $\mathfrak{H}=T_1+T_2(1-x_1)+\cdots+T_N(1-x_{N-1})$, cf. [1, Equation (16)], where $T_k(F(q,z,x_1,\ldots,x_{N-1}))=F(q,z,x_1,\ldots,x_{k-2},q^{-1}x_{k-1},qx_k,x_{k+1},\ldots,x_{N-1})$.

Note that if we plug $x=q^{\check{\lambda}}$ into $J_{\mathfrak{g}}(q^{-1},z,x)$ or into $\mathfrak{J}_{\mathfrak{g}}(q^{-1},z,x)$, then for $\check{\lambda}\in\Lambda_+^{\vee}$ these formal series converge, and we have $\mathfrak{J}_{\mathfrak{g}}(q^{-1},z,q^{\check{\lambda}}):=\prod_{i\in I}(q^{\langle\alpha_i,\check{\lambda}\rangle})^{\log(\check{\omega}_i)/\log q}J_{\mathfrak{g}}(q^{-1},z,q^{\check{\lambda}})=z^{\check{\lambda}}J_{\mathfrak{g}}(q^{-1},z,q^{\check{\lambda}})$ (a formal Taylor series in q with coefficients in Laurent polynomials in z).

The following lemma is a reformulation of [16, Proposition 2]:

Lemma 5 *The class of $R\Gamma(\mathfrak{QM}^{\alpha}_{\mathfrak{g}},\mathcal{O}(\check{\lambda}))$ in $K_{T\times\mathbb{G}_m}(pt)$ equals*

$$\sum_{\substack{\gamma+\beta=\alpha\\ w\in W}} z^{w\check{\lambda}}q^{\langle\gamma,\check{\lambda}\rangle}\mathfrak{J}_{\gamma}\left(q^{-1},wz\right)\mathfrak{J}_{\beta}(q,wz)\prod_{\check{\alpha}\in\check{R}^+}\left(1-wz^{\check{\alpha}}\right)^{-1}.$$

Proof Let $\pi:\overline{M}_{0,0}(\mathbb{P}^1\times\mathcal{B}_{\mathfrak{g}},(1,\alpha))\to\mathfrak{QM}^{\alpha}_{\mathfrak{g}}$ (resp. $\varpi:M^{\alpha}_{\mathfrak{g}}\to Z^{\alpha}_{\mathfrak{g}}$) be the Kontsevich resolution, see e.g. [8, Appendix] (resp. [2, Proof of Proposition 5.1]).

Since the singularities of $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ (resp. $Z_{\mathfrak{g}}^{\alpha}$) are rational, we have $R\Gamma(\mathcal{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda})) = R\Gamma(\overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1,\alpha)), \pi^*\mathcal{O}(\check{\lambda}))$ (resp. $\mathbb{C}[Z_{\mathfrak{g}}^{\alpha}] = \mathbb{C}[M_{\mathfrak{g}}^{\alpha}]$). Hence we have to express the character of $R\Gamma(\mathcal{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda}))$ via the characters of $\mathbb{C}[M_{\mathfrak{g}}^{\beta}]$. This is done in [16, Proof of Proposition 2] via the Atiyah–Bott–Lefschetz localization to $T \times \mathbb{G}_m$-fixed points of $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1,\alpha))$. As usually, we have to add that [16] deals with $G = \mathrm{SL}(N)$, however, the proof goes through word for word for arbitrary semisimple G. □

4.2 The character of $\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$

By the proof of Theorem 2.2 and Lemma 3(2), the character $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda})))$ is the limit of the characters $\chi(R^0\Gamma(\mathcal{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda})))$ as $\alpha \to \infty$. By Theorem 3.1(1), as $\alpha \to \infty$, the limit of the characters $\chi(R^{>0}\Gamma(\mathcal{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda})))$ vanishes. Thus, the character $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda})))$ is the limit of the characters $\chi(R\Gamma(\mathcal{QM}_{\mathfrak{g}}^{\alpha}, \mathcal{O}(\check{\lambda})))$ as $\alpha \to \infty$. We define $\mathfrak{J}_{\infty}(q,z) := \lim_{\alpha\to\infty} \mathfrak{J}_{\alpha}(q,z)$ (it is easy to see that the latter limit exists).

Proposition 4.1

$$\chi\left(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))\right) = \sum_{w\in W} \mathfrak{J}_{\mathfrak{g}}\left(q^{-1}, wz, q^{\check{\lambda}}\right)\mathfrak{J}_{\infty}(q, wz) \prod_{\check{\alpha}\in\check{R}^+} \left(1 - wz^{\check{\alpha}}\right)^{-1}.$$

Proof As α goes to ∞, the formula of Lemma 5 goes to

$$\begin{aligned}
&\sum_{\substack{\gamma\in\Lambda_+\\ w\in W}} z^{w\check{\lambda}} q^{\langle\gamma,\check{\lambda}\rangle} \mathfrak{J}_{\gamma}\left(q^{-1}, wz\right) \mathfrak{J}_{\infty}(q, wz) \prod_{\check{\alpha}\in\check{R}^+} \left(1 - wz^{\check{\alpha}}\right)^{-1}\\
&= \sum_{w\in W} z^{w\check{\lambda}} J_{\mathfrak{g}}\left(q^{-1}, wz, q^{\check{\lambda}}\right) \mathfrak{J}_{\infty}(q, wz) \prod_{\check{\alpha}\in\check{R}^+} \left(1 - wz^{\check{\alpha}}\right)^{-1}\\
&= \sum_{w\in W} \mathfrak{J}_{\mathfrak{g}}\left(q^{-1}, wz, q^{\check{\lambda}}\right) \mathfrak{J}_{\infty}(q, wz) \prod_{\check{\alpha}\in\check{R}^+} \left(1 - wz^{\check{\alpha}}\right)^{-1}.
\end{aligned}$$

□

Corollary 4.2 *Let $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))) = \Psi_{\check{\lambda}}(q,z)$. Then the functions $\Psi_{\check{\lambda}}(q,z)$ satisfy all the conditions of Conjecture 1.1.*

Proof Part 2 of Conjecture 1.1 is obvious by construction. Also Conjecture 1.1(1b) is obvious. According to Theorem 3.1(2), $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))) = 0$ if $\check{\lambda} \notin \Lambda_+^{\vee}$, which proves Conjecture 1.1(1a).

Let us prove Conjecture 1.1(1c). The function $\mathfrak{J}_{\mathfrak{g}}(q^{-1}, wz, q^{\check{\lambda}})$ on the lattice $\Lambda^{\vee}$ is an eigenfunction of the quantum difference Toda restricted to the lattice. According to Proposition 4.1, $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda})))$ is a linear combination of the functions $\mathfrak{J}_{\mathfrak{g}}(q^{-1}, wz, q^{\check{\lambda}})$ with coefficients independent of $\check{\lambda}$. Hence $\Psi_{\check{\lambda}}(q,z)$ is an eigenfunction of the quantum difference Toda as well. □

5 Weyl modules

5.1 Recall that $R = \mathbb{C}[[t^{-1}]]$. We introduce a new variable $\mathsf{t} = t^{-1}$, so that $R = \mathbb{C}[[\mathsf{t}]]$. We set $\widetilde{R} := \mathbb{C}[\mathsf{t}] \subset R$. The proalgebraic group $G(R)$ acts naturally on the profinite dimensional vector space $\Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))$. The continuous dual $\Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))^\vee$ coincides with the graded dual $\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))^\vee$, and is equipped with a natural action of $G(\widetilde{R})$: $g \cdot v^*(v) := v^*(\tau g \cdot v)$. Here $g \mapsto \tau g$ is the Chevalley antiinvolution of G identical on T. The derivative of these actions gives rise to the actions of $\mathfrak{g}(R)$ and $\mathfrak{g}(\widetilde{R})$. According to Theorem 2.2, the $\mathfrak{g}(\widetilde{R})$-module $\Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))^\vee$ coincides with the graded dual $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))^\vee$.

We denote the preimage of the big cell $U \cdot e_- \subset \mathcal{B}_\mathfrak{g}$ in $G/U_- \to \mathcal{B}_\mathfrak{g}$ by $C \subset G/U_-$. We denote the open subscheme $C(R)/T \subset G(R)/T \cdot U_-(R) = \mathbf{Q}_\infty$ by $\overset{\circ}{\mathbf{Q}}$. We have the restriction morphism of $\mathfrak{g}(R)$-modules $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda})) \hookrightarrow \Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))$. Now $C(R)$ is a free orbit of $B(R) \subset G(R)$, and $\Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda})) = \mathrm{CoInd}_{\mathfrak{u}(R)\oplus\mathfrak{t}}^{\mathfrak{g}(R)} \mathbb{C}_{\check{\lambda}}$. The graded dual $\Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))^\vee = \mathrm{Ind}_{\mathfrak{u}(\widetilde{R})\oplus\mathfrak{t}}^{\mathfrak{g}(\widetilde{R})} \mathbb{C}_{\check{\lambda}}$.

Lemma 6 $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda})) \subset \Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))$ *is the maximal* G*-integrable* $\mathfrak{g}(R)$*-submodule. Equivalently,* $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))^\vee$ *is the maximal* G*-integrable* $\mathfrak{g}(\widetilde{R})$*-quotient module of* $\Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))^\vee$.

Proof Note that $\mathbf{Q}_\infty$ is the G-saturation of $\overset{\circ}{\mathbf{Q}}$. Let $v \in \Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))$ lie in a finite-dimensional $\mathfrak{g}$-submodule $V \subset \Gamma(\overset{\circ}{\mathbf{Q}}, \mathcal{O}(\check{\lambda}))$. The action of $\mathfrak{g}$ on V integrates to the action of G. Let us view v as a $\check{\lambda}$-covariant function on $C(R)$. We have to check that v is the restriction of a $\check{\lambda}$-covariant function $\hat{v}$ on $G/U_-(R)$ to $C(R)$. Given a point $y \in G/U_-(R)$ we can find $g \in G$ such that $g(y) \in C(R)$. Then we define $\hat{v}(y) := u(gy)$ where we view $u := gv \in V$ as a $\check{\lambda}$-covariant function on $C(R)$. Clearly, this is well defined, i.e. independent of a choice of g. □

Recall the notion of Weyl $\mathfrak{g}(\widetilde{R})$-module $\mathcal{W}(\check{\lambda})$ for dominant $\check{\lambda} \in \Lambda_+^\vee$, see e.g. [3]. It is the maximal G-integrable $\mathfrak{g}(\widetilde{R})$-quotient module of $\mathrm{Ind}_{\mathfrak{u}(\widetilde{R})\oplus\mathfrak{t}}^{\mathfrak{g}(\widetilde{R})} \mathbb{C}_{\check{\lambda}}$ [3]. Thus Lemma 6 implies the first part of Theorem 1.3.

On the other hand, taking into account Theorem 2.2 we also get

Proposition 5.1 *For* $\check{\lambda} \in \Lambda_+^\vee$*, we have a natural isomorphism of* $\mathfrak{g}(\widetilde{R})$*-modules* $\Gamma(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))^\vee \simeq \mathcal{W}(\check{\lambda})$.

Combining this with Corollary 4.2 we get the following

Corollary 5.2 $\chi(\mathcal{W}(\check{\lambda})) = \Psi_{\check{\lambda}}(q, z)$.

This is actually the statement of Theorem 1.3(1). To prove Theorem 1.3(2) let us recall that the Demazure module $D(\check{\lambda})$ is a certain $\mathfrak{g}(\widetilde{R})$-submodule of an irreducible integrable level one representation of $\mathfrak{g}_{\mathrm{aff}}$, see e.g. [12, 2.2]. In addition, according to [3,12] there exists an action of $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ on $\mathcal{W}(\check{\lambda})$ such that

1. This action commutes with $G(R) \rtimes \mathbb{C}^*$.
2. $\mathcal{W}(\check{\lambda})$ is finitely generated and free over $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$.
3. The fiber of $\mathcal{W}(\check{\lambda})$ at $\check{\lambda} \cdot 0$ is isomorphic to $D(\check{\lambda})$.

Thus we get the following corollary, which is actually the statement of Theorem 1.3(2) (as was mentioned in the introduction it was proved in [15] for $G = \mathrm{SL}(N)$):

Corollary 5.3 *The product* $\chi(\widetilde{\Gamma}(\mathfrak{Q}, \mathcal{O}(\check{\lambda}))) \cdot \prod_{i \in I} \prod_{r=1}^{\langle \alpha_i, \check{\lambda} \rangle} (1 - q^r) = \hat{\Psi}_{\check{\lambda}}(q, z)$ *is equal to the character of the (finite dimensional) Demazure module* $D(\check{\lambda})$*. In particular, it is a finite linear combination of* $\chi(L(\check{\mu}))$*'s with coefficients in* $\mathbb{Z}_{\geq 0}[q]$.

5.1 Geometric interpretation of the $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$-action

We conclude the paper by giving an interpretation of the $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$-action on $\mathcal{W}(\check{\lambda})$ in terms of Theorem 1.3(1). This will enable us to prove the second assertion of Theorem 1.5. It would be nice to prove that this action is free directly by geometric means (without referring to [12]).

Let $T(R)_1$ denote the first congruence subgroup in $T(R)$ (i.e. the kernel of the natural map $T(R) \to T$). Let $\mathfrak{t}(R)_1$ denote its (abelian) Lie algebra (i.e. the kernel of the natural map $\mathfrak{t}(R) \to \mathfrak{t}$). We denote by $\mathfrak{t}(\widetilde{R})_1 \subset \mathfrak{t}(R)_1$ the corresponding subspace (consisting of all mappings $\mathbb{A}^1 \to \mathfrak{t}$ which are equal to 0 at 0). Then for every $\check{\lambda} \in \Lambda_+^\vee$ there exists a natural epimorphism $\pi_{\check{\lambda}} : U(\mathfrak{t}(R)_1) = \mathrm{Sym}\, \mathfrak{t}(R)_1) \to \mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ defined by the following formula:

$$\pi_{\check{\lambda}}(h\mathfrak{t}^n)\left(\sum_i \gamma_i x_i\right) = \sum_i \langle h, \gamma_i \rangle x_i^n .$$

Here $h \in \mathfrak{t}$ and $\sum_i \gamma_i x_i \in \mathbb{A}^{\check{\lambda}}$.

Clearly, the group $T(R)_1$ acts (on the right) on the scheme $\mathbf{Q}_\infty = G(R)/T \cdot U_-(R)$. Hence we get a natural action of $\mathrm{Sym}(\mathfrak{t}(R)_1)$ on $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))$ for every $\check{\lambda} \in \Lambda^\vee$. The following result is easy to prove; we leave the details to the reader:

Proposition 5.4 1. *The above action of* $\mathrm{Sym}(\mathfrak{t}(\widetilde{R})_1)$ *on* $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))$ *factors through* $\pi_{\check{\lambda}}$.
2. *The resulting action of* $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ *on* $\Gamma(\mathbf{Q}_\infty, \mathcal{O}(\check{\lambda}))^\vee = \mathcal{W}(\check{\lambda})$ *coincides with the action considered in* [3,12].

From Proposition 5.4 we immediately get the following

Corollary 5.5 *We have* $\Gamma(G(R)/B_-(R), \mathcal{O}(\check{\lambda})) \simeq D(\check{\lambda})^\vee$ *(this is the second assertion of Theorem 1.5).*

Proof It follows from Proposition 5.4 and from the fact that $D(\check{\lambda})$ is the fiber of $\mathcal{W}(\check{\lambda})$ over $\check{\lambda} \cdot 0 \in \mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ that $D(\check{\lambda})^\vee$ is isomorphic to the invariants of $\mathfrak{t}(\widetilde{R})$ on $\mathcal{W}(\check{\lambda})^\vee$. Since $\mathfrak{t}(\widetilde{R})_1$ is dense in $\mathfrak{t}(R)_1$, it follows that

$$\left(\mathcal{W}(\check{\lambda})^\vee\right)^{\mathfrak{t}(\widetilde{R})_1} = \left(\mathcal{W}(\check{\lambda})^\vee\right)^{\mathfrak{t}(R)_1}.$$

From Proposition 5.1 we get

$$(\mathcal{W}(\check{\lambda})^\vee)^{\mathfrak{t}(R)_1} = \Gamma\Big(G(R)/T \cdot U_-(R), \mathcal{O}(\check{\lambda})\Big)^{\mathfrak{t}(R)_1} = \Gamma\Big(G(R)/B_-(R), \mathcal{O}(\check{\lambda})\Big).$$

□

Acknowledgments This paper emerged as a result of numerous conversations held between the first author and Borodin in IHES in June 2011. In particular, A. Borodin has introduced the first author to the notion of q-Whittaker functions and brought to his attention the papers [13–15]. The first author wishes also to thank IHES staff for exceptionally pleasant working conditions and hospitality. We are grateful to B. Feigin and S. Loktev for their numerous patient explanations about Weyl modules. The observation that $\Gamma(G[[\mathsf{t}]]/T \cdot U_-[[\mathsf{t}]], \mathcal{O}(\check{\lambda}))$ is a dual Weyl module is essentially due to them. We would like to thank K. Schwede and other mathoverflowers for their explanations about Kodaira vanishing. The first author is also deeply indebted to Patrick Clark for his patient explanations about Macdonald polynomials and to Etingof and Cherednik for explanations about the q-Toda system. M.F. was partially supported by the RFBR grants 12-01-00944, 12-01-33101, 13-01-12401/13, the National Research University Higher School of Economics' Academic Fund award No.12-09-0062 and the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. This study comprises research findings from the "Representation Theory in Geometry and in Mathematical Physics" carried out within The National Research University Higher School of Economics' Academic Fund Program in 2012, grant No 12-05-0014.

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