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Hall Quantum Hamiltonians and Electric 2D-Curvature

M. V. Karasev

Moscow Institute for Electronics and Mathematics
at National Research University HSE,
Moscow 109028, Russia
E-mail: karasev@miem.edu.ru

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Abstract. For the Dirac 2D-operator in a constant magnetic field with perturbing electric potential, we derive Hamiltonians describing the quantum quasiparticles (Larmor vortices) at Landau levels. The discrete spectrum of this Hall-effect quantum Hamiltonian can be computed to all orders of the semiclassical approximation by a deformed Planck-type quantization condition on the 2D-plane; the standard magnetic (symplectic) form on the plane is corrected by an "electric curvature" determined via derivatives of the electric field. The electric curvature does not depend on the magnitude of the electric field and vanishes for homogeneous fields (i.e., for the canonical Hall effect). This curvature can be treated as an effective magnetic charge of the inhomogeneous Hall 2D-nanosystem.

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1. INTRODUCTION

Microscopic intrinsic sources, like hydrogen-type centers, quantum dots, etc., can create highly inhomogeneous electric fields acting on free electrons in 2D-materials. In the presence of a magnetic field, these electrons form Larmor vortices (quasiparticles) whose motion, i.e., the Hall current, is controlled by the electric field. On the level of the classical mechanical picture, the current lines are perpendicular to the electric field directions. If this field is inhomogeneous and has maxima, minima, saddle points, etc., then the Hall current lines can be closed (circular). On the level of quantum mechanics, the states of such quasiparticles are described by Hamiltonians generated by the electric field. The phase space for these Hamiltonians is the surface of the 2D-material considered as a quantum surface. The circular behavior of the Hall current lines means that these Hamiltonians have a discrete spectrum.

The quantum description of "Hall phenomena" of this kind has been developed for a long time (see, e.g., [1–6]) and, in some cases, the approximate quantum Hamiltonian of a quasiparticle was computed [3, 7–10].

In the present paper, we describe a general scheme of computing the Hall-effect quantum Hamiltonians. We consider qraphene-type systems which are described by an effective Dirac equation [11–15]; however, the same approach certainly works for Schrödinger equations, say, for the bilayer graphene and for other 2D-systems.

We represent the quantum Hamiltonians of the quasiparticle exactly and show how to compute their spectra for all orders of the semiclassical approximation.

As the first semiclassical correction, we find an interesting addition to the usual magnetic differential form on the surface of the 2D-material. This additional form represents something like electric curvature in the Hall phenomena. It contributes to the spectrum by deforming the usual magnetic flux in the Planck-type quantization condition.

It is important to note that, in the case of nanosystems living just on the border of the quantum and classical worlds, the knowledge of high-order semiclassical terms becomes crucial, since the effective semiclassical parameter $\hbar$ ("the Planck constant") for these nanosystems is not too small.

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For example, for $\hbar \sim 1/3$, one certainly needs to know terms of order $\hbar^3 \sim 4 \cdot 10^{-2}$ and to compare them with possible "exponential" terms of order $e^{-1/\hbar} \sim 5 \cdot 10^{-2}$. The semiclassical picture in this nanoarea has to be clarified in much more detail than that usually needed in quantum mechanical problems. The objects which appear in the higher-order semiclassical terms are invariants and play a crucial role in nanophysics.

The computation of the Hall effect Hamiltonians on the surface with an inhomogeneous electric field represents a good example in which a careful and advanced semiclassical analysis is required and new interesting objects arise.

Finally, note that here we consider here only the simplest flat model, ignoring local nontrivial geometry and possible inhomogeneity of the magnetic field (or the lattice strain) [16–24]. A more general situation will be considered in another publication.

2. DIRAC-TYPE HAMILTONIAN

Consider the 2D-system described (in dimensionless quantities) by the Dirac-type Hamiltonian

$$\hat{H} = \gamma \cdot \hat{p} + \varepsilon\hbar V(q), \tag{2.1}$$

where the coordinates $q = (q^1, q^2)$ and the momenta $\hat{p} = (\hat{p}_1, \hat{p}_2)$, as well as the γ-matrices, satisfy the relations

$$[\gamma^j, \gamma^k]_+ = 2\delta^{jk}, [\gamma^j, \hat{p}_k] = [\gamma^j, q^k] = 0, [\hat{p}_1, \hat{p}_2] = i\hbar, [\hat{p}_1, q^1] = [\hat{p}_2, q^2] = -i\hbar, [\hat{p}_1, q^2] = [\hat{p}_2, q^1] = [q^1, q^2] = 0. \tag{2.2}$$

Here the effective "Planck constant" is given by $\hbar = l^2/L^2$, where l stands for the magnetic length (the scale of the Larmor vortex) and L characterizes the scale of the electric field inhomogeneity. The additional parameter ε in (2.1) controls the scale of the electric strength, and usually $\varepsilon \sim 10^{-1} \div 10$.

The second term in (2.1) is regarded as a perturbation: either ε is small and $\hbar \lesssim 1$ or ε could not be small, while the semiclassical parameter $\hbar$ is small enough, $\varepsilon\hbar \ll 1$.

The leading term $\gamma \cdot \hat{p}$ in (2.1) represents the graphene Hamiltonian itself. Its spectrum is well known. Our purpose is to study the perturbed Hamiltonian (2.1) in the first approximation with respect to the perturbation parameter $\varepsilon\hbar$ and to reduce it to 1D-effective Hamiltonians. Another objective is to derive the asymptotical expansion of the corresponding effective Hamiltonians of a quasiparticle on the quantum plane for all orders of $\hbar$ and to present asymptotic formulas for its spectrum.

We first introduce complex γ-matrices and complex coordinate-momenta, namely,

$$\gamma^\pm = \gamma^1 \pm i\gamma^2, \qquad z^\pm = \frac{1}{\sqrt{2}}(q^1 \pm iq^2), \qquad \Pi_\pm = \frac{1}{\sqrt{2}}(\hat{p}_1 \mp i\hat{p}_2).$$

Note that [24]

$$[\gamma^+, \gamma^-]_+ = 4, \qquad [\gamma^\pm, \gamma^0] = \pm 2\gamma^\pm, \qquad \gamma^-\gamma^+ = 2(1+\gamma^0), \qquad (\gamma^+)^2 = (\gamma^-)^2 = 0,$$

where

$$\gamma^0 \overset{def}{=\!=} \frac{i}{2}[\gamma^1, \gamma^2], \qquad (\gamma^0)^2 = 1. \tag{2.3}$$

Then

$$\gamma \cdot \hat{p} = \frac{1}{\sqrt{2}}(\gamma^+\Pi_+ + \gamma^-\Pi_-), \tag{2.4}$$

where

$$[\Pi_-, \Pi_+] = \hbar, \qquad [\Pi_\pm, z^\pm] = -i\hbar, \qquad [\Pi_\pm, z^\mp] = 0. \tag{2.5}$$

Squaring (2.4), we see that

$$\frac{1}{2}(\gamma\cdot\hat{p})^2 = \Pi_+\Pi_- + \frac{\hbar}{2} + \frac{\hbar}{2}\gamma^0 \overset{def}{=\!=} A, \qquad [A, \Pi_\pm\gamma^\pm] = 0, \qquad [\Pi_+\Pi_-, \Pi_\pm] = [A, \Pi_\pm] = \pm\hbar\Pi_+. \quad (2.6)$$

Squaring (2.1), we have

$$\frac{1}{2}\hat{H}^2 = \frac{1}{2}(\gamma\cdot\hat{p})^2 + \frac{\varepsilon\hbar}{2}((\gamma\cdot\hat{p})\cdot V + V\cdot(\gamma\cdot\hat{p})) + \frac{1}{2}\varepsilon^2\hbar^2 V^2 = A + \frac{\varepsilon\hbar}{\sqrt{2}}(\sqrt{A}\cdot V + V\cdot\sqrt{A}) + O(\varepsilon^2\hbar^2). \quad (2.7)$$

Assume now that we have found an invertible operator U such that

$$U\cdot(A + \frac{\varepsilon\hbar}{\sqrt{2}}(\sqrt{A}\cdot V + V\cdot\sqrt{A}))\cdot U^{-1} = A + \frac{\varepsilon\hbar}{\sqrt{2}}(\sqrt{A}\cdot \underline{V} + \underline{V}\cdot\sqrt{A}) + O(\varepsilon^2\hbar^2), \quad (2.8)$$

where an operator $\underline{V}$ commutes with A,

$$[A, \underline{V}] = 0. \quad (2.9)$$

Then (2.8) yields $(1/2)(U\cdot\hat{H}\cdot U^{-1})^2 = A + \sqrt{2}\varepsilon\hbar\sqrt{A}\cdot\underline{V} + O(\varepsilon^2\hbar^2)$, and thus

$$U\cdot\hat{H}\cdot U^{-1} = \sqrt{2A} + \varepsilon\hbar\underline{V} + O(\varepsilon^2\hbar^2). \quad (2.10)$$

In the next section we explain how to construct the operator U and compute an operator $\underline{V}$.

3. HALL QUANTUM HAMILTONIANS

One can seek a unitary operator U in the form

$$U = 1 + \frac{i\varepsilon}{\sqrt{2}}(\sqrt{A}\cdot C + C\cdot\sqrt{A}) + O(\varepsilon^2\hbar^2) \quad (3.1)$$

Then, by (2.8), the following equation holds for the self-adjoint operator C:

$$\frac{i}{\hbar}[A, C] = V - \underline{V}. \quad (3.2)$$

To solve this equation together with (2.9), note that the operator A has the spectrum $\hbar\mathbb{Z}_+$, i.e.,

$$\mathrm{Spectr}(A) = \{\hbar n \mid n = 0, 1, 2, \dots\}. \quad (3.3)$$

This follows from relations (2.6), (2.3) and from the fact that the operator $\Pi_- = -i\hbar\partial/\partial z^- - iz^+/2$ annihilates the vacuum state

$$|0\rangle = \exp(-z^+z^-/2\hbar). \quad (3.4)$$

By using (3.3), one can construct the solution of the homological equations (3.2) and (2.9) in a standard way [25], by an algebraic averaging, namely,

$$C = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{it}{\hbar}A}\cdot v(z^-|z^+)\cdot e^{-\frac{it}{\hbar}A}\, t\, dt, \qquad \underline{V} = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{it}{\hbar}A}\cdot v(z^-|z^+)\cdot e^{-\frac{it}{\hbar}A}\, dt. \quad (3.5)$$

Here we express the electric potential V in complex coordinates,

$$V(q) = v(z^-|z^+), \qquad z^\pm = \frac{1}{\sqrt{2}}(q^1 \pm iq^2). \quad (3.6)$$

The matrix part of the operator A (2.6) commutes with $\Pi_+\Pi_-$ and $z^\pm$. Therefore, one can replace the operator A by $\Pi_+\Pi_-$ in the transformation $e^{\frac{it}{\hbar}A}\cdot(\dots)\cdot e^{-\frac{it}{\hbar}A}$ entering (3.5).

Introduce also the operator[1]

$$\hat{z} \overset{def}{=\!=} z^+ - i\Pi_- \tag{3.7}$$

and take into account that

$$[\Pi_\pm, \hat{z}] = [\Pi_\pm, \hat{z}^*] = 0. \tag{3.8}$$

Then, in (3.5), we obtain

$$e^{\frac{it}{\hbar}A}\cdot v(z^-|z^+)\cdot e^{-\frac{it}{\hbar}A} = e^{\frac{it}{\hbar}\Pi_+\Pi_-}\cdot v(\overset{2}{\hat{z}^*} - i\overset{2}{\Pi_+} \mid \overset{1}{\hat{z}} + i\overset{1}{\Pi_-})\cdot e^{-\frac{it}{\hbar}\Pi_+\Pi_-} = v(\overset{2}{\hat{z}^*} - i\overset{2}{\Pi_+}(t) \mid \overset{1}{\hat{z}} + i\overset{1}{\Pi_-}(t)). \tag{3.9}$$

Here (following Maslov's notation in [26]) we use numbers above symbols to indicate the order of action of the operators. We also write

$$\Pi_\pm(t) \overset{def}{=\!=} e^{\frac{it}{\hbar}\Pi_+\Pi_-}\cdot\Pi_\pm\cdot e^{-\frac{it}{\hbar}\Pi_+\Pi_-}.$$

It can readily be seen from (2.6) that $\Pi_\pm(t) = e^{\pm it}\Pi_\pm$. Thus, by (3.9) we transform formulas (3.5) as follows:

$$C = \frac{1}{2\pi}\int_0^{2\pi} v(\overset{2}{\hat{z}^*} - ie^{it}\overset{2}{\Pi_+} \mid \overset{1}{\hat{z}} + ie^{-it}\overset{1}{\Pi_-})\,t\,dt, \qquad \underline{V} = \frac{1}{2\pi}\int_0^{2\pi} v(\overset{2}{\hat{z}^*} - ie^{it}\overset{2}{\Pi_+} \mid \overset{1}{\hat{z}} + ie^{-it}\overset{1}{\Pi_-})\,dt. \tag{3.10}$$

We shall consider the second formula in (3.10) only; the first one can be treated similarly. One can represent (3.10) in the form

$$\underline{V} = \underline{v}(\overset{2}{\hat{z}^*}|\overset{1}{\hat{z}}), \tag{3.11}$$

where the operator-valued function $\underline{v}$ on the complex plane is defined by

$$\underline{v}(\overline{z}|z) \overset{def}{=\!=} \frac{1}{2\pi}\int_0^{2\pi} \exp(-ie^{it}\overset{2}{\Pi_+}\cdot\overline{\partial} + ie^{-it}\overset{1}{\Pi_-}\cdot\partial)v(\overline{z}|z)\,dt. \tag{3.12}$$

Here $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$, as usual. Note that $\partial\overline{\partial} = (1/2)\Delta = (1/2)\left(\partial^2/\partial q_1^2 + \partial^2/\partial q_2^2\right)$ and $z = (1/\sqrt{2})(q^1 + iq^2)$.

Expanding the exponential function in (3.12) in the power series gives

$$\underline{v}(\overline{z}|z) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}(\overset{2}{\Pi_+}\overset{1}{\Pi_-})^k(\partial\overline{\partial})^k v(\overline{z}|z). \tag{3.13}$$

Relations (2.6) readily imply that $(\overset{2}{\Pi_+}\overset{1}{\Pi_-})^k = (\Pi_+\Pi_-)(\Pi_+\Pi_- - \hbar)\dots(\Pi_+\Pi_- - (k-1)\hbar)$, and therefore formula (3.13) reads

$$\begin{aligned}\underline{v}(\overline{z}|z) &= \left[1 + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\frac{\Pi_+\Pi_-}{\hbar}\left(\frac{\Pi_+\Pi_-}{\hbar} - 1\right)\dots\left(\frac{\Pi_+\Pi_-}{\hbar} - (k-1)\right)\right]v(\overline{z}|z)\\ &= {}_1F_1\left(-\frac{\Pi_+\Pi_-}{\hbar}, 1; -\hbar\partial\overline{\partial}\right)v(\overline{z}|z).\end{aligned}$$

[1] This is the first approximation to the so-called "guiding center" operator describing the quantum coordinate of the Larmor vortex center.

Here we use the notation ${}_1F_1$ for the hypergeometric function,

$$ {}_1F_1(a,b;x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{x^n}{n!}, \qquad (a)_n \overset{def}{=\!=} a(a+1)\dots(a+n-1). $$

Since the spectrum of $\Pi_+\Pi_-$ coincides with $\hbar\mathbb{Z}_+$, i.e., $\mathrm{Spectr}(\Pi_+\Pi_-) = \{\hbar N \mid N = 0,1,2,\dots\}$, one can write

$$ \underline{v}(\overline{z}|z) = \sum_{N=0}^{\infty} \underline{v}_N(\overline{z}|z) I_N, \tag{3.14} $$

where I_N stands for the projection to the Nth eigensubspace of $\Pi_+\Pi_-$ (the Nth "Landau level") and

$$ \underline{v}_N \overset{def}{=\!=} {}_1F_1(-N,1;-\hbar\partial\overline{\partial})v = \sum_{k=0}^{N} \frac{N!}{(k!)^2(N-k)!} (\hbar\partial\overline{\partial})^k v. \tag{3.15} $$

Finally, by (3.11) and (3.14), we obtain

$$ \underline{V} = \sum_{N=0}^{\infty} \underline{v}_N(\overset{2}{\overline{z}^*}|\overset{1}{\hat{z}}) I_N, \tag{3.16} $$

where the scalar functions $\underline{v}_N$ on the complex plane are given by the hypergeometric series (3.15) via the electric potential $v(\overline{z}|z)$, and the operator $\hat{z}$ is given by (3.7).

Note that, in view of (3.8), the operator $\hat{z}$, as well as its conjugate $\hat{z}^*$, leave each eigenspace of $\Pi_+\Pi_-$ invariant (i.e., the projections I_N are invariant). It can also readily be seen that

$$ [\hat{z}^*, \hat{z}] = \hbar. \tag{3.17} $$

Thus, the operator $\hat{z}$ can be referred to as the quantum complex "coordinate" on the plane of the 2D-material (e.g., on the surface of graphene).

Formulas (2.10), (2.6), and (3.16) imply the following theorem.

Theorem 3.1. *The Dirac-type operator* $\hat{H}$ (2.1) *is unitary equivalent up to* $O(\varepsilon^2\hbar^2)$ *to the direct sum of operators*

$$ (\pm\sqrt{\hbar}\sqrt{2N+1+\gamma^0} + \varepsilon\hbar\underline{v}_N(\overset{2}{\hat{z}^*}|\overset{1}{\hat{z}}))I_N, \qquad N = 0,1,\dots $$

Here I_N *is the projection to the* N*th "Landau level" and* γ^0 *is the matrix* (2.3) *with the spectrum* $\{\pm 1\}$. *The Hall quantum Hamiltonians* $\underline{v}_N(\overset{2}{\hat{z}^*}|\overset{1}{\hat{z}})$ *are given by the quantum coordinates* $\hat{z}$ *and* $\hat{z}^*$ *satisfying relation* (3.17) *and by the functions* $\underline{v}_N$ (3.15) *generated by the electric potential* $v = v(\overline{z}|z)$.

Note that, for the lowest ($N = 0$) Landau level, one has $\underline{v}_N = v$ by (3.15), i.e., in this case, the Hall effect Hamiltonian coincides with the electric potential itself. This result was known in some specific cases (see, e.g., [3, 5, 8–10, 27, 28]).

4. SPECTRAL ASYMPTOTICS

The Hall quantum Hamiltonians $\hat{v}_N = v_N(\overset{2}{\hat{z}^*}|\overset{1}{\hat{z}})$ were derived above without referring to the semiclassical approximation; the scheme works well for both small and not small $\hbar$.

The discrete spectrum of the operator $\hat{v}_N$ can be obtained by a method analogous to that developed by P. Argyres [29].

Theorem 4.1. *Suppose that the spectrum of $\hat{v}_N$ is discrete, say, the function v_N has a minimum and grows at infinity as a polynomial in all directions on the plane. Then the eigenvalues $E = E_{N,m}$ of the operator $\hat{v}_N$ can be computed by resolving the quantization condition*

$$(2\pi\hbar)^{-1} \int \rho_N^E \, d\overline{z} \, dz = m + 1/2, \tag{4.1}$$

where

$$\rho_N^E(\overline{z}|z) \stackrel{def}{=\!=} \theta(E - b_N)1, \qquad b_N \stackrel{def}{=\!=} v_N(\overset{2}{\overline{z}}|z - \hbar\overset{1}{\partial}). \tag{4.2}$$

Here θ is the usual Heaviside step function and 1 is the function identically equal to one.

In reality, to use this theorem, one needs to assume that $\hbar$ is small. In this case, the distribution ρ_N^E (4.2) can be calculated explicitly for any order of $\hbar$,

$$\rho_N^E \simeq \theta\left(E - v_N - \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hbar^l \partial^l v_N \cdot \overline{\partial}^l\right) 1 = \theta(E - v_N) - \frac{\hbar}{2} |\partial v_N|^2 \theta''(E - v_N) + O(\hbar^2)$$

(here, for simplicity, we wrote out only the first two terms of the asymptotic expansion). Then (4.1) yields

$$\frac{1}{2\pi\hbar} \int_{v_N \leqslant E} d\overline{z} \, dz - \frac{1}{8\pi} \frac{\partial^2}{\partial E^2} \int_{v_N \leqslant E} |\nabla v_N|^2 \, d\overline{z} \, dz + O(\hbar) = m + \frac{1}{2}. \tag{4.3}$$

Here we took into account that $\partial v_N = (1/\sqrt{2}) \left(\partial/\partial q^1 - i\partial/\partial q^2\right) v_N$ and $|\partial v_N|^2 = (1/2)|\nabla v_N|^2$.

The second integral in (4.3) can be transformed as follows:

$$\frac{\partial^2}{\partial E^2} \int_{v_N \leqslant E} |\nabla v_N|^2 \, d\overline{z} \, dz = \langle \Delta v_N, \delta(E - v_N) \rangle = \int_{v_N = E} \Delta v_N \, dt,$$

where t is the time coordinate in the Hamiltonian system corresponding to v_N and to the symplectic form $dq^2 \wedge dq^1$ on the plane. The form dt, at any energy level $v_N = \text{const}$, can be chosen as

$$dt = \left.\frac{\nabla v_N \times dq}{|\nabla v_N|^2}\right|_{v_N = \text{const}},$$

where the cross $\times$ stands for the skew product with respect to the symplectic structure $dq^2 \wedge dq^1$. Thus, Equation (4.3) reads

$$\frac{1}{2\pi\hbar} \int_{v_N \leqslant E} \left(1 - \frac{\hbar}{4} \operatorname{div}\left(\frac{\Delta v_N}{|\nabla v_N|^2} \nabla v_N\right) + O(\hbar^2)\right) dq^2 \wedge dq^1 = m + \frac{1}{2}. \tag{4.4}$$

Theorem 4.2. *The semiclassical asymptotics of the eigenvalues $E = E_{N,m}$ for the Hall quantum Hamiltonians $\hat{v}_N$ can be derived from the quantization condition* (4.4). *This can be done explicitly for all orders of $\hbar$.*

The function v_N in these theorems is given by (3.15), i.e.,

$$\underline{v}_N = \sum_{k=0}^{N} \frac{N!}{(k!)^2 (N-k)!} \left(\frac{\hbar}{2}\right)^k \Delta^k V, \tag{4.5}$$

where V is the original electric potential from (2.1). If the number N of the Landau level is not high, then (4.5) can be regarded as the asymptotic expansion, $\underline{v}_N = V + (\hbar N/2)\Delta V + O(\hbar^2)$.

Corollary 4.1. *For low Landau levels, the eigenvalues $E = E_{N,m}$ of the Hall quantum Hamiltonians can be derived (for all orders of $\hbar$) from the quantization condition*

$$\frac{1}{2\pi}\int_{V\leqslant E}\frac{dq^2\wedge dq^1}{\hbar}-\frac{1}{4\pi}\left(N+\frac{1}{2}\right)\int_{V\leqslant E}\operatorname{div}\left(\frac{\operatorname{div}\mathcal{E}}{|\mathcal{E}|^2}\mathcal{E}\right)dq^2\wedge dq^1+O(\hbar)=m+\frac{1}{2}, \tag{4.6}$$

where V stands for the perturbing potential in (2.1) and $\mathcal{E} = \nabla V$ is the electric field.

The first summand in (4.6) represents the magnetic flux through the area $\{V \leqslant E\}$. The second summand depends on the derivatives of the electric field $\mathcal{E}$; this "electric curvature" correction to the magnetic flux exists even at the lowest Landau level ($N = 0$); it does not depend on the magnitudes of the electric and magnetic fields, and it vanishes for a homogeneous electric field (i.e., for the canonical Hall effect).

It is useful to note that the divergence of the electric field $\operatorname{div}\mathcal{E}$ determining the electric curvature

$$K_{\text{electric}} = \operatorname{div}\left(\frac{\operatorname{div}\mathcal{E}}{|\mathcal{E}|^2}\mathcal{E}\right) \tag{4.7}$$

does not vanish in general since the vector field $\mathcal{E}$ is only a projection to the surface of the actual physical electric field whose 3D-divergence is zero indeed, by the Maxwell equation. Thus, the electric curvature (4.7) is, in fact, controlled by the third transversal component $\mathcal{E}_3$ of the physical electric field (which is "invisible" from the viewpoint of the 2D-world).

The electric curvature contributes to Planck's type quantization rule (4.6) at the same semiclassical order as the Kramers–Keller–Maslov correction $\frac{1}{2}$. As a result, the magnetic fluxes through the areas enveloped by the Hall circular current lines need not be not half-integer; these are numbers of generic kind which depend on the values of the electric curvature. Note that, in order to be written in actual physical units, the curvature expression in (4.6) must be multiplied by the magnetic flux quantum.

Thus, we see that the two-dimensionality and the presence of the transversal electric "frequency" induce an effective magnetic charge of the Hall nanosystem. When the input magnetic and electric fields are both generated by internal sources (like the lattice strain and the deformation of electronic shells [19–22]), this effective magnetic charge may be treated as an internal feature of the 2D-system embedded in the 3D-space. Possibly this charge is responsible for some magnetic properties of graphene and other 2D-materials.

Remark 4.1. The asymptotics of the eigenvalues $\lambda = \lambda_{N,m}$ of the original Hamiltonian (2.1) is derived from Theorems 3.1, 4.1, and 4.2 and from the quantization conditions (4.4) or (4.6) as follows:

$$\lambda_{N,m} = \pm\sqrt{\hbar(2N+1\pm 1)} + \varepsilon\hbar E_{N,m} + O(\varepsilon^2\hbar^2), \tag{4.8}$$

where ± 1 are the eigenvalues of γ^0 (2.3).

Although we explicitly present here only a couple of first terms, all asymptotic expansions can be made with arbitrary accuracy with respect to ε and $\hbar$ using the above approach.

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