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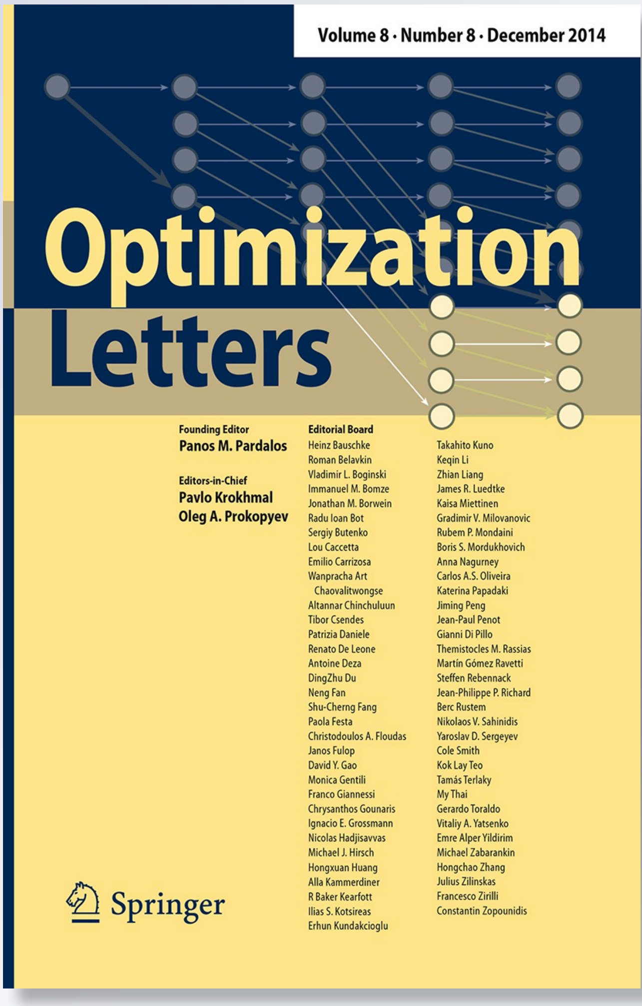

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ORIGINAL PAPER

The coloring problem for classes with two small obstructions

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Abstract The coloring problem is studied in the paper for graph classes defined by two small forbidden induced subgraphs. We prove some sufficient conditions for effective solvability of the problem in such classes. As their corollary we determine the computational complexity for all sets of two connected forbidden induced subgraphs with at most five vertices except 13 explicitly enumerated cases.

Keywords Vertex coloring · Computational complexity · Polynomial-time algorithm

1 Introduction

The coloring problem is one of classical problems on graphs. Its formulation is as follows. A *coloring* is an arbitrary mapping of colors to vertices of some graph. A graph coloring is called *proper* if any neighbors are colored in different colors. The *chromatic number of a graph* G (denoted by $\chi(G)$) is the minimal number of colors in proper colorings of G. The *coloring problem* for a given graph and a number k is to determine whether its chromatic number is at most k. The *vertex k-colorability problem* is to verify whether vertices of a given graph can be colored with at most k colors. The *edge k-colorability problem* is formulated by analogy.

A graph H is called an *induced subgraph* of G if H is obtained from G by deletion of vertices. The *induced subgraph relation* is denoted by $\subseteq_i$. In other words, $H \subseteq_i G$ if H is an induced subgraph of G. A *class* is a set of simple unlabeled graphs. A class of graphs is called *hereditary* if it is closed under deletion of vertices. It is well known

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that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$ in this case. If a hereditary class can be defined by a finite set of forbidden induced subgraphs, then it is called *finitely defined*. For a finitely defined class $\mathcal{X} = Free(\mathcal{S})$ and a graph G the problem whether G belongs to $\mathcal{X}$ is decided in polynomial time (e.g., by determining in G an induced copy of a graph in S).

The coloring problem for G-free graphs is polynomial-time solvable if $G \subseteq_i P_4$ or $G \subseteq_i P_3 \oplus K_1$ and it is NP-complete in all other cases [1]. A study of forbidden pairs was also initialized in the paper. When we forbid two induced subgraphs, the situation becomes more difficult than in the monogenic case. Here only partial results are known [2–6]. The next statement is a survey of such achievements [5].

Theorem 1 *Let G_1 and G_2 be two fixed graphs. The coloring problem is NP-complete for $Free(\{G_1, G_2\})$ if:*

- $C_p \subseteq_i G_1$ *for some* $p \geq 3$ *and* $C_q \subseteq_i G_2$ *for some* $q \geq 3$
- $K_{1,3} \subseteq_i G_1$ *and* $K_{1,3} \subseteq_i G_2$
- $K_{1,3} \subseteq_i G_1$ *and either* $K_4 \subseteq_i G_2$ *or* $K_4 - e \subseteq_i G_2$ *(or vice versa)*
- $K_{1,3} \subseteq_i G_1$ *and* $C_p \subseteq_i G_2$ *for some* $p \geq 4$ *(or vice versa)*
- G_1 *and* G_2 *contain a spanning subgraph of* $2K_2$ *as an induced subgraph*
- $C_3 \subseteq_i G_1$ *and* $K_{1,p} \subseteq_i G_2$ *for some* $p \geq 5$ *(or vice versa)*
- $C_3 \subseteq_i G_1$ *and* $P_{164} \subseteq_i G_2$ *(or vice versa)*
- $C_p \subseteq_i G_1$ *for* $p \geq 5$ *and* G_2 *contains a spanning subgraph of* $2K_2$ *as an induced subgraph (or vice versa)*
- *either* $C_p \oplus K_1 \subseteq_i G_1$ *for* $p \in \{3, 4\}$ *or* $\overline{C_q} \subseteq_i G_1$ *for* $q \geq 6$ *and* G_2 *contains a spanning subgraph of* $2K_2$ *as an induced subgraph (or vice versa)*

It is polynomial-time solvable for $Free(\{G_1, G_2\})$ if:

- G_1 *and* G_2 *are induced subgraphs of* P_4 *or* $P_3 \oplus K_1$
- $G_1 \subseteq_i K_{1,3}$ *and* $G_2 \subseteq_i C_3 \oplus K_1$ *(or vice versa)*
- $G_1 \subseteq_i paw$ *and* $G_2 \neq K_{1,5}$ *is a forest with at most six vertices (or vice versa)*
- $G_1 \subseteq_i paw$ *and either* $G_2 \subseteq_i pK_2$ *or* $G_2 \subseteq_i P_5 \oplus pK_1$ *for some* $p \geq 1$ *(or vice versa)*
- $G_1 \subseteq_i K_p$ *for* $p \geq 3$ *and either* $G_2 \subseteq_i qK_2$ *or* $G_2 \subseteq_i P_5 \oplus qK_1$ *for some* $q \geq 1$ *(or vice versa)*
- $G_1 \subseteq_i gem$ *and either* $G_2 \subseteq_i P_4 \oplus K_1$ *or* $G_2 \subseteq_i P_5$ *(or vice versa)*
- $G_1 \subseteq_i \overline{P}_5$ *and either* $G_2 \subseteq_i P_4 \oplus K_1$ *or* $G_2 \subseteq_i 2K_2$ *(or vice versa)*

In the present article we prove some sufficient conditions for NP-completeness and polynomial-time solvability of the coloring problem for $\{G_1, G_2\}$-free graphs. They add new information about its complexity for some cases that Theorem 1 does not cover. For instance, the problem is NP-complete for $\{K_{1,4}, bull\}$-free graphs, but it is polynomial-time solvable for $Free(\{K_{1,3}, P_5\})$, $Free(\{K_{1,3}, hammer\})$, $Free(\{P_5, C_4\})$. The complexity was earlier open for these four cases. As a corollary of the conditions we determine the complexity for all sets $\{G_1, G_2\}$ of connected graphs with at most five vertices except 13 listed cases.

2 Notation

As usual, P_n, C_n, K_n, O_n and $K_{p,q}$ stand respectively for the simple path with n vertices, the chordless cycle with n vertices, the complete graph with n vertices, the empty graph with n vertices and the complete bipartite graph with p vertices in the first part and q vertices in the second. The graph $K_n - e$ is obtained by deleting an arbitrary edge in K_n. The graph *paw* is obtained from $K_{1,3}$ by adding a new edge incident to its vertices of degree two. The graphs $fork, gem, hammer, bull, butterfly$ have the vertex set $\{1, 2, 3, 4, 5\}$. The edge set for *fork* is $\{(1, 2), (1, 3), (1, 4), (4, 5)\}$, for *gem* is $\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (3, 4), (4, 5)\}$, for *hammer* is $\{(1, 2), (1, 3), (2, 3), (1, 4), (4, 5)\}$, for *bull* is $\{(1, 2), (1, 3), (2, 3), (1, 4), (2, 5)\}$, for *butterfly* is $\{(1, 2), (1, 3), (2, 3), (1, 4), (1, 5), (4, 5)\}$.

The *complement graph* of G (denoted by $\overline{G}$) is a graph on the same set of vertices and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The sum $G_1 \oplus G_2$ is the disjoint union of G_1 and G_2. The disjoint union of k copies of a graph G is denoted by kG. For a graph G and a set $V' \subseteq V(G)$ the formula $G \backslash V'$ denotes the subgraph of G obtained by deleting all vertices in V'.

3 Boundary graph classes

The notion of a boundary graph class is a helpful tool for the analysis of the computational complexity of graph problems in the family of hereditary graph classes. This notion was originally introduced by Alekseev for the independent set problem [7]. It was applied for the dominating set problem later [8]. A study of boundary graph classes for some graph problems was extended in the paper of Alekseev et al. [9], where the notion was formulated in its most general form. Let us give the necessary definitions.

Let Π be an NP-complete graph problem. A hereditary graph class is called Π-*easy* if Π is polynomial-time solvable for its graphs. If the problem Π is NP-complete for graphs in a hereditary class, then this class is called Π-hard. A class of graphs is said to be Π-limit if this class is the limit of an infinite monotonically decreasing sequence of Π-hard classes. In other words, $\mathcal{X}$ is Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_1 \supseteq \ldots$ of Π-hard classes, such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. A minimal under inclusion Π-limit class is called Π-boundary.

The following theorem certifies the significance of the boundary class notion.

Theorem 2 [7] *A finitely defined class containing a Π-boundary class is Π-hard. If it does not contain Π-boundary classes, then it is Π-easy* ($unless\, P = NP$).

The theorem shows that knowledge of all Π-boundary classes leads to a complete classification of finitely defined graph classes with respect to the complexity of Π. Two concrete classes of graphs are known to be boundary for several graph problems. First of them is $\mathcal{S}$. It is constituted by all forests with at most three pendant vertices in each connected component. The second one is $\mathcal{T}$, which is a set of the line graphs of graphs in $\mathcal{S}$. The paper [9] is a good survey about graph problems for which either $\mathcal{S}$ or $\mathcal{T}$ is boundary.

Some classes are known to be limit and boundary for the coloring problem. The set of all forests (denoted by $\mathcal{F}$) and the set of line graphs of forests with degrees at most three (denoted by $\mathcal{T}'$) are limit classes for it [10]. The set $co(\mathcal{T}) = \{G : \overline{G} \in \mathcal{T}\}$ is a boundary class for the problem [11]. The set of boundary graph classes for the coloring problem is continuum [12,18]. Some continuum sets of boundary classes for the vertex k-colorability and the edge k-colorability problems are known for any fixed $k \geq 3$ [11,13].

4 NP-completeness of the coloring problem for $\{K_{1,4}, bull\}$-free graphs

The results listed above on limit and boundary classes for the coloring problem together with Theorem 2 allow to prove NP-completeness of the problem for some finitely defined classes. Namely, if $\mathcal{Y}$ is a finite set of graphs and no one graph in $\mathcal{Y}$ belongs to a class in $\{\mathcal{F}, \mathcal{T}', co(\mathcal{T})\}$, then the problem is NP-complete for $Free(\mathcal{Y})$. But this idea cannot be applied to $Free(\{K_{1,4}, bull\})$, because $K_{1,4} \in \mathcal{F}$, $bull \in \mathcal{T}'$, $bull \in co(\mathcal{T})$. Nevertheless, the coloring problem is NP-complete for it. To show this we use the operation with a graph called the diamond implantation.

Let G be a graph and x be one of its nonpendant vertex. Applying the *diamond implantation* to x implies:

- an arbitrary splitting of the neighborhood of x into two nonempty parts A and B
- deletion of x and addition of new vertices y_1, y_2, y_3, y_4
- addition of all edges of the kind (y_1, a), $a \in A$ and of the kind (y_4, b), $b \in B$
- addition of the edges (y_1, y_2), (y_1, y_3), (y_2, y_3), (y_2, y_4), (y_3, y_4)

Clearly that for any graph and any its nonpendant vertex applying the diamond implantation preserves vertex 3-colorability. This property and the paper [14] give the key idea of the proof of Lemma 1.

Lemma 1 *The vertex 3-colorability problem* (*hence*, *thecoloringproblem*) *is NP-complete for the class* $Free(\{K_{1,4}, bull\})$.

Proof The vertex 3-colorability problem is known to be NP-complete for triangle-free graphs with degrees at most four [15]. Let us consider connected such a graph with at least two vertices. We will sequentially apply the described above operation to its vertices with edgeless (i.e., inducing the empty graph) neighborhoods. In other words, if H is a current graph, then it is applied to an arbitrary vertex of H that does not belong to any triangle. The sets A and B are arbitrarily formed with the condition $||A| - |B|| \leq 1$. The whole process is finite, because the number of its steps is no more than the quantity of vertices in the initial graph. Hence, the resultant graph is formed in polynomial time. It belongs to $Free(\{K_{1,4}, bull\})$, since (with respect to the definition of the diamond implantation) degrees of its vertices are at most four and any its vertex belongs to an induced copy of $K_4 - e$, whose two vertices of degree three have in the resultant graph the same degrees. Thus, the vertex 3-colorability problem for triangle-free graphs with degrees at most four is polynomially reduced to the same problem for graphs in $Free(\{K_{1,4}, bull\})$. Hence, it is NP-complete for $Free(\{K_{1,4}, bull\})$. □

5 Some structural results on graphs in some classes defined by small obstructions

For a hereditary class $\mathcal{X}$ and a number k the class $[\mathcal{X}]_k$ is a set of graphs, for which one can delete at most k vertices such that the result belongs to $\mathcal{X}$.

Lemma 2 *If some connected graph $G \in Free(\{K_{1,3}, P_5\})$ contains an induced cycle C of length at least four, then $G \in [Free(\{O_3\})]_5$.*

Proof Length of C is equal to either four or five. We will show first that C dominates all vertices of G. Assume that there is a vertex of G that does not belong to C and adjacent to no one vertex of the cycle. Then due to the connectivity of G there are vertices $x, y \in V(G)\backslash V(C)$, such that $(x, y) \in E(G)$, x is not adjacent to any vertex of C and y is adjacent to at least one vertex of C. Since $G \in Free(\{K_{1,3}\})$, then y is adjacent to exactly two vertices of C. The vertices x, y and some three consecutive vertices of the cycle (one of which is adjacent to y) induce the subgraph P_5. Thus, C dominates all vertices of G.

We will show that the graph $G\backslash V(C)$ does not contain three pairwise nonadjacent vertices. This fact implies the validity of Lemma 2. Assume that $G\backslash V(C)$ has a set V' of three pairwise nonadjacent vertices. Since G is $K_{1,3}$-free, then the intersection of the neighborhood of each vertex in V' with $V(C)$ is a set of at least two (three for $C = C_5$) consecutive vertices of C.

Let us consider the case $C = C_5$. No one vertex of V' can be adjacent to all vertices of C, since otherwise some vertex of C is adjacent to all vertices of V' (and G contains $K_{1,3}$ as an induced subgraph). One can assume that no one vertex in V' is adjacent to exactly four vertices of the cycle C, since in this case the graph G contains the induced cycle C_4 and the case $C = C_4$ will be considered later. Therefore, we can consider only the situation, where each vertex of V' is adjacent to three consecutive vertices of C and the corresponding sets of three consecutive vertices are distinct (otherwise G contains $K_{1,3}$ as an induced subgraph). Then, some two vertices of V' and some three vertices of C induce P_5. So, if $C = C_5$, then we have a contradiction.

Now we consider the case $C = C_4$. It is easy to verify that avoiding induced $K_{1,3}$ in G leads to only the following situations:

– one vertex of V' is adjacent to all vertices of C and the other two vertices of V' are adjacent to disjoint pairs of its consecutive vertices
– one vertex of V' is adjacent to two consecutive vertices of C and each of the other two vertices is adjacent to three consecutive vertices of C, they have two common neighbors in C and the first vertex has only one common neighbor in C with each of them
– each of two vertices of V' is adjacent to two consecutive vertices of C, the third one is adjacent to three consecutive vertices of C and any two vertices of V' have only one common neighbor in C

The graph G contains P_5 as an induced subgraph in all three cases. We come to a contradiction. Thus, the initial assumption was false. □

Lemma 3 *If some connected graph $G \in Free(\{K_{1,3}, hammer\})$ contains an induced cycle C_n $(n \geq 7)$, then G is isomorphic to C_n.*

Proof Assume opposite i.e., there is a vertex $x \in V(G)\backslash V(C_n)$. One can easily show that the vertex x is adjacent to at least one vertex of C_n. It is easy to verify that the set of $x's$ neighbors in C_n is constituted either by two, three or four consecutive vertices or by two pairs of consecutive vertices (otherwise $G \notin Free(\{K_{1,3}\})$). In both situations the graph G contains *hammer* as an induced subgraph. Hence, the assumption was false. □

Lemma 4 *If some connected graph $G \in Free(\{K_{1,3}, hammer\})$ contains C_6 as an induced subgraph, then $G\backslash V(C_6)$ is the disjoint union of at most three cliques.*

Proof Let us consider the set $V' = V(G)\backslash V(C_6)$. It is easy to verify that the intersection of the neighborhood of each vertex in V' with C_6 induces in G the subgraph $2K_2$. Let us consider now two arbitrary vertices in V'. If they are adjacent, then they have in C_6 the same sets of neighbors and if they are not adjacent, then the mentioned sets are distinct. This implies that V' does not contain four pairwise nonadjacent vertices. Thus, $G\backslash V(C_6)$ is the disjoint union of at most three cliques. □

Lemma 5 *For any connected graph $G \in Free(\{K_{1,3}, hammer\})$ at least one of the following properties is true:*

- *G is a simple cycle*
- *G contains the induced subgraph C_6*
- *G has a pendant vertex*
- *G belongs to the class $Free(\{P_5\})$*
- *G belongs to the class $[Free(\{O_3\})]_5$*

Proof Assume that $G \notin Free(\{P_5\})$. Let us consider an induced path P_n of G having the maximal length. Clearly, $n \geq 5$. Let us consider an arbitrary end of this path. One can assume that it is adjacent to some vertex $x \in V(G)\backslash V(P_n)$, otherwise G contains a pendant vertex. By the maximality of P_n the vertex x is adjacent to at least two vertices of the path. One can consider that x is adjacent to at least one interior vertex of P_n, otherwise G is a simple cycle (by Lemma 3) or it contains C_6 as an induced subgraph.

Let $n > 5$. To avoid induced $K_{1,3}$ the vertex x must be adjacent to three or four consecutive vertices of P_n or to both its ends or to three vertices of P_n that induce the subgraph $K_2 \oplus K_1$ in G or to four vertices inducing $2K_2$. The graph G contains *hammer* as an induced subgraph in all these situations.

Let $n = 5$ now. One can assume that the graph $G\backslash V(P_5)$ has three pairwise nonadjacent vertices (otherwise $G \in [Free(\{O_3\})]_5$). It is easy to check that any of these three vertices must be adjacent to either three central vertices of P_5 or to all its vertices, except central or to the first, the third and the fourth vertices of P_5 (counting from some of the P_5's ends) or to the first and the last its vertices. The graph G contains C_6 as an induced subgraph in the last case. Hence, we can consider that no one among the three vertices is adjacent to only the ends of P_5. If one of the three vertices is adjacent to the first, the third and the fourth vertices of P_5 and other of these vertices is adjacent to the second, the third and the fifth ones, then G contains induced C_6. Therefore, one can assume that there are no such two vertices. Either the second or the fourth vertex of P_5 is adjacent to the three vertices and, hence, G is not $K_{1,3}$-free. Thus, the initial assumption was false. □

6 On formulae connecting the chromatic numbers of a graph and of its induced subgraphs

The following statement is obvious.

Lemma 6 *If G is a connected graph with at least three vertices and a pendant vertex v, then $\chi(G\backslash\{v\}) = \chi(G)$.*

Lemma 7 *Let G be a connected graph in $Free(\{P_5, C_4\})$ that contains an induced copy of C_5. Let V_1 be the set its vertices that are adjacent to all vertices of C_5, V_2 be the set of vertices in G that have three neighbors in C_5, G_1 and G_2 be the subgraphs of G, induced by $V(G)\backslash(V_1 \cup V_2 \cup V(C_5))$ and $V_1 \cup V_2 \cup V(C_5)$ correspondingly. Then, G_2 is O_3-free and the relation $\chi(G) = \max(\chi(G_1), \chi(G_2))$ holds.*

Proof Any vertex outside C_5 that is adjacent to at least one vertex of the cycle must be adjacent to either all vertices of the cycle or to some three consecutive its vertices. It is easy to verify taking into account that G is $\{P_5, C_4\}$-free. Therefore, any such a vertex belongs to either V_1 or V_2. Each vertex in V_2 has no neighbors outside $V(C_5) \cup V_1 \cup V_2$ (since $G \in Free(\{P_5\})$). Any vertex in V_1 is adjacent to every vertex in $V_1 \cup V_2 \cup V(C_5)$ except itself (if $V_1 \cup V_2$ contains a pair of nonadjacent vertices and one of them is in V_1, then the vertices and two their common nonadjacent neighbors in C_5 induce a copy of C_4). Assume that G_2 contains three pairwise nonadjacent vertices. No one of them belongs to V_1, since any such a vertex is adjacent to each vertex of G_2 except itself. If two of the mentioned three vertices belong to V_2, then the union of their neighborhoods must contain $V(C_5)$ (otherwise G is not $\{P_5, C_4\}$-free). Hence, one (and exactly one) of the three vertices belongs to $V(C_5)$. The neighborhoods of the two remaining vertices do not cover C_5. We have a contradiction and G_2 is O_3-free.

The inequality $\chi(G) \geq \max(\chi(G_1), \chi(G_2))$ is obvious. We will show that G can be colored with $\max(\chi(G_1), \chi(G_2))$ colors. Let c_1 and c_2 be optimal colorings of G_1 and G_2 correspondingly. If $\chi(G_1) \geq \chi(G_2)$, then c_1 has $\chi(G_1) - |V_1| \geq \chi(G_2) - |V_1| \geq 0$ colors that do not appear in V_1. Hence, c_1 can be extended to a proper coloring of G with $\chi(G_1)$ colors by coloring $G_2 \backslash V_1$ with $\chi(G_2) - |V_1|$ colors of the mentioned type. By the same reasons c_2 is extendable to a proper coloring of G with $\chi(G_2)$ colors when $\chi(G_2) \geq \chi(G_1)$. □

7 Some results on polynomial-time solvability of the coloring problem

Lemma 8 *Let $\mathcal{X}$ be an easy case for the coloring problem, the problem whether a graph belongs to $\mathcal{X}$ is polynomial-time solvable and for some fixed number p the inclusion $\mathcal{X} \subseteq Free(\{O_p\})$ $(p \geq 2)$ holds. Then, for any fixed q this problem is polynomial-time solvable in the class $[\mathcal{X}]_q$.*

Proof Let G be a graph in $[\mathcal{X}]_q$. Deleting some set V' $(|V'| \leq q)$ of its vertices leads to a graph in $\mathcal{X}$ (this set is determined in polynomial time by the exhaustive search algorithm). We will consider all partial proper colorings of G with at most $|V'|$ color classes, in which every vertex of V' is colored. Obviously, any such a coloring has at most $(p-1)q$ colored vertices (hence, all partial colorings are enumerated in polynomial time). For any considered partial coloring deleting all colored vertices leads to a graph in $\mathcal{X}$ and its chromatic number is computed in polynomial time. For every our partial coloring we will find the sum of the number of used colors and the chromatic number of the subgraph induced by the set of uncolored vertices. Any such a sum corresponds to the number of colors in a proper coloring of G (by optimal coloring the uncolored part with colors different from the used ones in the earlier colored part). Hence, any sum is at least $\chi(G)$. Every optimal coloring of G can be partitioned in two parts: vertices having a common color with a vertex in V' and the other vertices. The sum of the chromatic numbers of the subgraphs induced by the parts is equal to $\chi(G)$. Hence, extending some of the partial colorings leads to an optimal coloring of G. Therefore, minimal among the corresponding sums is equal to $\chi(G)$. Thus, $\chi(G)$ is computed in polynomial time. □

A graph is called *chordal* if it does not contain induced cycles with four or more vertices. The problem whether a graph is chordal is solved in polynomial time [16]. The coloring problem is known to be polynomial-time solvable for chordal graphs [16].

Lemma 9 *The classes* $Free(\{K_{1,3}, P_5\})$, $Free(\{K_{1,3}, hammer\})$, $Free(\{P_5, C_4\})$ *are easy for the coloring problem.*

Proof We will show that for every considered class the coloring problem is polynomially reduced to the same problem for chordal graphs. This fact implies the lemma. The problem is polynomial-time solvable in $Free(\{O_3\})$, since it is equivalent to the matching problem. The reduction for $\{K_{1,3}, P_5\}$-free graphs follows from this observation, Lemmas 2 and 8.

Let G be a graph in $Free(\{K_{1,3}, hammer\})$ containing the induced subgraph C_6. By Lemma 4, deleting vertices of this cycle leads to a chordal O_4-free graph. Hence, by Lemma 8, $\chi(G)$ is computed in polynomial time. Thus, by Lemmas 5 and 6 the coloring problem for the class is polynomially reduced to the same problem for graphs in $Free(\{K_{1,3}, P_5\}) \cup [Free(\{O_3\})]_5$. Hence, it is reduced to chordal graphs.

Let G be a connected graph in $Free(\{P_5, C_4\})$ that is not chordal. Hence, G contains an induced copy of C_5. The graphs G_1 and G_2 defined in the formulation of Lemma 7 are constructed in polynomial time. Moreover, G_2 is O_3-free and $|V(G)| - |V(G_1)| \geq 5$. Therefore, by Lemma 7 the considered problem for $\{P_5, C_4\}$-free graphs is also polynomially reduced to the same problem for chordal graphs. □

8 The main result and its corollaries

Remind that $\mathcal{F}$ is the class of forests, $\mathcal{T}'$ is the set of line graphs of forests with degrees at most three and $co(\mathcal{T})$ is constituted by complement graphs of line graphs of forests with at most three pendant vertices in each connected component.

The following theorem is the main result of the paper.

Theorem 3 *Let H_1 and H_2 be graphs. If there is a class $\mathcal{Y} \in \{\mathcal{F}, \mathcal{T}', co(\mathcal{T})\}$ with either $H_1, H_2 \notin \mathcal{Y}$ or $K_{1,4} \subseteq_i H_1$ and bull $\subseteq_i H_2 (or vice versa)$, then the coloring* *problem is NP-complete for $Free(\{H_1, H_2\})$. It is polynomial-time solvable in the class if at least one of the following properties holds:*

- $H_1 \subseteq_i P_4$ *or* $H_2 \subseteq_i P_4$
- $H_1 \subseteq_i P_5$ *or* $H_2 \subseteq_i K_5$ *(or vice versa)*
- $H_1 \subseteq_i P_5$ *or* $H_2 \subseteq_i gem$ *(or vice versa)*
- $H_1 \subseteq_i P_5$ *or* $H_2 \subseteq_i C_4$ *(or vice versa)*
- $H_1 \subseteq_i P_5$ *or* $H_2 \subseteq_i K_{1,3}$ *(or vice versa)*
- $H_1 \subseteq_i K_{1,4}$ *or* $H_2 \subseteq_i paw$ *(or vice versa)*
- $H_1 \subseteq_i fork$ *or* $H_2 \subseteq_i paw$ *(or vice versa)*
- $H_1 \subseteq_i K_{1,3}$ *or* $H_2 \subseteq_i hammer$ *(or vice versa)*

Proof The classes $\mathcal{F}, \mathcal{T}', co(\mathcal{T})$ are limit for the coloring problem. This fact, Theorem 2 and Lemma 1 imply the first part of the statement. The set of P_4-free graphs is well known to be an easy case for the coloring problem [17]. The classes $Free(\{P_5, gem\})$ and $Free(\{P_5, K_5\})$ are easy for the problem [2,5]. The same is true for $Free(\{fork, paw\})$ [5] and $Free(\{K_{1,4}, paw\})$ [1]. These facts and Lemma 9 imply the second part of the theorem. □

Both parts of Theorem 3 add new information about the complexity of the coloring problem for some classes. For example, its complexity status for the classes $Free(\{K_{1,3}, bull\})$, $Free(\{K_{1,3}, P_5\})$, $Free(\{K_{1,3}, hammer\})$, $Free(\{P_5, C_4\})$ was open.

Theorem 3 gives the following criterion.

Corollary 1 *Let H_1 and H_2 be connected graphs with at most four vertices. The coloring problem is polynomial-time solvable for $\{H_1, H_2\}$-free graphs if either $H_1 \subseteq_i P_4$ or $H_2 \subseteq_i P_4$ or $\{H_1, H_2\} = \{K_{1,3}, paw\}$ or $\{H_1, H_2\} = \{K_{1,3}, C_3\}$. It is NP-complete in all other cases.*

Theorem 3 can not be applied to some pairs of connected graphs with at most five vertices. If $\{H_1, H_2\}$ is such a set, then either H_1 or H_2 belongs to $\{K_{1,3}, fork, K_{1,4}, P_5\}$. This observation helps to enumerate all connected cases with at most five vertices that the theorem does not cover.

Corollary 2 *Theorem 3 does not give the complexity status of the coloring problem for the following sets of forbidden induced connected subgraphs (a number in the brackets shows the quantity of such kind sets):*

- $\{K_{1,3}, G\}$, *where* $G \in \{bull, butterfly\}$ (2)
- $\{fork, bull\}$ (1)
- $\{P_5, G\}$, *where* $G \notin \{K_5, gem\}$ *is an arbitrary connected five-vertex graph in* $co(\mathcal{T})$ (10)

Determining the complexity of the problem for any of the listed above 13 cases is a challenging research problem.

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