NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Andrei S. Akhremenko, Alexander P. Petrov,
Egor A. Yureskul

CYCLICALLY BALANCED GROWTH PATHS IN A MODEL OF ECONOMIC GROWTH WITH ENDOGENOUS POLICY SWITCHING

BASIC RESEARCH PROGRAM
WORKING PAPERS

SERIES: ECONOMICS
WP BRP 109/EC/2015

Andrei S. Akhremenko[1], *Alexander P. Petrov*[2], *Egor A. Yureskul*[3]

CYCLICALLY BALANCED GROWTH PATHS IN A MODEL OF ECONOMIC GROWTH WITH ENDOGENOUS POLICY SWITCHING[4]

This paper deals with a model of economic growth, which we expand to include endogenous policy switching based on retrospective voting. It is shown that under certain conditions the solution has a special form that we call a cyclically balanced growth path. This type of solution is an analogue to balanced growth paths, which often occur in growth models with constant policies. Cyclically balanced growth paths are investigated analytically, and the growth rate over the cycle has been found. Results of numerical experiments are also provided and possible empirical applications of the model are outlined.

Keywords: differential equations; dynamical model; policy space; democracy; autocracy; economic growth; efficiency; public capital; growth path; retrospective voting

JEL Classification: C61, H54, O41

---

[1] National Research University Higher School of Economics, Laboratory for Qualitative and Quantitative Analysis of Political Regimes, Director; E-mail: aakhremenko@hse.ru

[2] National Research University Higher School of Economics, Laboratory for Qualitative and Quantitative Analysis of Political Regimes, Senior researcher; E-mail: appetrov@hse.ru

[3] National Research University Higher School of Economics, Laboratory for Qualitative and Quantitative Analysis of Political Regimes, Assistant researcher; E-mail: eyureskul@hse.ru

[4] This Paper is an output of a research project implemented by the Laboratory for Qualitative and Quantitative Analysis of Political Regimes within HSE's Annual Thematic Plan for basic and applied research in 2015. Project title: "The Impact of Feedback Loops in a State-Society System on Resource Redistribution and Economic Growth Prospects in Democratic, Autocratic and Hybrid Regimes".

1. Introduction

Policy change occurs in a discrete manner: tax rates, electoral rules, tariffs, trade regulations change discontinuously at certain moments in time. Discrete policy changes are essential to political life *per se* (though they are more natural in democracies, where policy switching is built into the basic mechanism, namely the change of power through elections). However, political science and political economy usually do not focus on policy change. One possible reason for this may be that the mainstream formal paradigm of studying policy choice issues—game theory—seeks equilibrium policy solutions which maximize individual utility or common social benefits ([Aiyagari et al. 2002], [Bassetto 2014], [Battaglini and Coate 2007]). As a consequence, rational choice models focus on a single (optimal or not) policy ([Andre et al. 2010], [Hassler et al. 2005]). This approach ignores the real-life variety of both actual policies and opinions on policy issues. Moreover, this approach yields policies chosen "once and for all" (even in dynamic environments [Battaglini et al. 2012], [Devereux and Wen 1998]). Sometimes policy changes are allowed in a probabilistic manner; in this case they are usually linked to the chances of the incumbent losing office ([Angelopoulos and Economides 2008], [Malley et al. 2007]). These exogenously defined probabilities affect the equilibrium solution, but they do not produce actual policy change. Consequently, the real-life phenomenon of a social system with regular policy switching stays out of focus.

For this reason, in the present work we eschew the game-theory approach in favour of a dynamic model with endogenously-defined policy switching. The switching mechanism is based on economic retrospective voting [Anderson 2007], [Healy and Malhotra 2013], if private capital has declined between elections, the public votes for the challenger, which leads to policy change. Otherwise, the public supports the incumbent, and the policy remains the same.

More formally, the model's main expressions are dynamic discrete time equations formulated for economic variables (such as private capital, state budget). The equations contain the governing parameter (tax rate) which is used as a proxy for the economic policy of the party currently in office. Elections take place at set time intervals corresponding to electoral cycles (each interval is considered a single time period). Two parties vie for office, with the results being defined by economic retrospective voting: the incumbent party loses if the results of its policy have led to diminished private capital in the latest electoral cycle. The incumbent losing the elections brings about a change in policy, e.g. if the system was governed by Party 1 and the private capital has decreased, then at the next interval of time it will be governed by Party 2. However, if private capital has increased (or remained the same), then there is no change in power, and the system remains governed by Party 1.

We focus on regimes with cyclically changing parties in office. Let such regimes be denoted e.g. 122122122…. The latter notation means that in the first time interval the office was held by Party 1, then by Party 2 at the second and third time intervals, again by the Party 1 at the forth time interval, and so on. The cyclical switching of power may be accompanied either by non-negative economic growth over the cycle or by decline.

We show that in this dynamical system with endogenous switching, an analogue to a balanced growth path appears. A balanced growth path is a solution which has all of its components increase or diminish proportionally over any time interval, either exponentially (in continuous-time models) or geometrically (in discrete-time models). Such solutions exist in many models of economic growth with constant policies.

In our model with endogenous policy switching all components of the solution increase (or decrease) in the same proportion over the political cycle (not over any interval of time). We refer to these solutions as cyclically balanced growth paths.

Let us also note that depending on the starting point and parameter values, the system may have solutions of a different type, such as regimes with no policy change after a transition period, e.g. 12222222222…

Another important issue which the current model allows us to touch upon is the difference between short-term and long-term policy effects. Given the economic retrospective behaviour of voters, the public might vote for short-term benefits rather than long-term ones. A long-term growth policy might lead to short-term economic decline, with the party who carries out such a policy losing office after its first term in power. This in turn may lead to the public temporarily supporting long-term decline (but short-term rise) policy, with the party that proposes long-term decline policy remaining in power for additional terms.

The model under consideration is close to the model presented in [Akhremenko and Petrov 2014], but instead of the Cobb-Douglas production function we use the Leontief production function. The main advantage of the Leontief form is that it opens more opportunities for analytical research. Therefore, in contrast with the previous work, the solutions are studied analytically as well as quantitatively.

The rest of the paper is organized as follows. Section 2 is an overview of the model itself. Section 3 contains an analysis of the model given a constant tax rate, i.e. there are no elections (we call it an autocracy), which allows us to define long-term growth and decline policies. Section 4 explains how a party carrying out a long-term growth policy may lose office as a result of retrospective economic voting, while a party with a long-term decline policy may stay in office in the short-term. Section 5 considers cyclically balanced growth paths.

2. Model

The model uses the Leontief production function $Y(t) = A\min(K(t)/b, G(t))$, where $Y(t)$ is aggregate output, $K(t)$ is private capital, $G(t)$ is public capital, b is a technologically determined constant, $A>0$ is a time-invariant total factor productivity parameter. Two points are of note considering our production function choice.

The first point refers to the production factor choice. Traditionally, two-factor production functions use labour and capital. Historically, the choice of public and private capital as production factors in economic growth models seems to have been made only by [Arrow and Kurz 1969]. However, in recent literature public capital is becoming an increasingly popular subject of research focus ([Arslanalp et al. 2010], [Aschauer 2000], [Gupta et al. 2014], [Pritchett 2000]). This may be due to growing interest in the interaction between state policy (including tax policy) and economic development.

The second point considers the production function specification. Econometric research focuses mainly on Cobb-Douglas functions, while theoretical works tend to use CES functions that generalize many other production functions and, given a set of parameters, can be reduced to, for example, Cobb-Douglas, Leontief, linear functions. Leontief production function as a standalone form is used more rarely. Our choice of Leontief PF is due to a technical consideration: while the solutions that are the focus of the current work (cyclically-balanced growth paths) may numerically be observed using other production functions, the Leontief function is better suited for analytical solutions. Note that as both production factors are kinds of capital and Leontief function is uniform ($F(\lambda K, \lambda G) = \lambda F(K, G)$), the model is somewhat similar to AK model [Aghion and Howitt 2008] in the sense that both have constant returns to capital.

We will now proceed to construction of the model.

The output *Y(t)* is taxed at a fixed rate τ, tax revenue fills the state budget *T(t),* so $T(t) = \tau(t)Y(t)$. The whole budget *T* is invested into public capital, adjusted for investment efficiency γ (which may be empirically measured as the Public Investment Management Index (PIMI) γ [Pritchett 2000]). Taking into consideration the depreciation rate δ, public capital dynamics take the form $G(t+1) = (1-\delta)G(t) + \gamma T(t)$.

Private capital at t+1 is the remnant of the previous period's output after taxes have been applied: $K(t+1) = (1-\tau(t))Y(t)$.

Thus, the discrete time model is expressed as:

$$Y(t) = A\min\left(K(t)/b, G(t)\right) \tag{1}$$

$$T(t) = \tau(t)Y(t) \tag{2}$$

$$K(t+1) = \left(1-\tau(t)\right)Y(t) \tag{4}$$

$$G(t+1) = (1-\delta)G(t) + \gamma T(t). \tag{5}$$

Here τ is the switching parameter. For every *t* either $\tau(t) = \tau_1$ or $\tau(t) = \tau_2$, depending on which party is currently in office. Initially Party 1 (that is Party τ_1) is in office:

$$\tau(0) = \tau_1. \tag{6}$$

Retrospective economic voting occurs at $t = 1, 2, 3, ..$ and is expressed as:

$$\tau(t) = \tau(t-1), \text{ if } K(t) \geq K(t-1) \tag{7}$$

$$\tau(t) = \tau_1 + \tau_2 - \tau(t-1), \text{ if } K(t) < K(t-1). \tag{8}$$

Reducing (1)–(5) to a system of two equations gives us:

$$K(t+1) = \left(1-\tau(t)\right)A\min\left(K(t)/b, G(t)\right) \tag{9}$$

$$G(t+1) = (1-\delta)G(t) + \gamma\tau(t)A\min\left(K(t)/b, G(t)\right). \tag{10}$$

This is the model that is subject to analysis.

The next section is auxiliary, devoted to systems with no endogenous switching (following the [Akhremenko and Petrov 2014] terminology, autocratic regimes), i.e. the system (9)–(10) with constant τ. This will allow us to introduce certain notions concerning party policies within the system.

In Section 5, the policies defined will be used to analyse two-party systems in which cyclically balanced growth paths may occur.

3. Long-run growth policies and long-run decline policies

The current section considers a model with no switching of policies. We will demonstrate that the policy τ may be defined as long-run growth or long-run decline depending on the long-term consequences of its application to the system.

In addition, we base the definitions of the high-tax policy and the low-tax policy on the following consideration. In (1)–(5) public capital is financed through tax collection, therefore in the long-run, high τ leads to $G(t) > K(t)/b$, thus the production function (1) is reduced to $Y(t) = AK(t)/b$. Conversely, a sufficiently low τ leads to $G(t) < K(t)/b$ and the production function (1) taking the form $Y(t) = AG(t)$. Therefore, the high tax policy leads to an abundance of public capital and deficiency of private capital in the long run (i.e. at high *t*); vice versa for the low-tax policy.

Note that this holds true for any production function form, yet with a Leontief function this effect is the most prominent.

Consider system (9)–(10) with constant τ. Balanced growth path is defined as follows:

$$\begin{pmatrix} K(t+1) \\ G(t+1) \end{pmatrix} = (1+a) \begin{pmatrix} K(t) \\ G(t) \end{pmatrix}. \tag{11}$$

If $a > 0$, we say that the system is growing in the long-run and refer to solution (11) and the solution's parameter τ as a long-term growth regime and long-term growth policy, respectively. If $a < 0$, the system has a long-run decline.

We also call it long-run decline if both components of the solution (not necessarily taking the form (11)) tend to zero ($\lim_{t \to \infty} K(t) = 0, \lim_{t \to \infty} G(t) = 0$) for any positive starting point $K(0) > 0, G(0) > 0$.

Let us represent the results for a constant policy system (9)–(10) in the form of two theorems.

Theorem 1.

Let $A > 0, b > 0, 0 \le \gamma \le 1, 0 \le \delta \le 1$, *and let* τ *be constant.*

Then

(i) if $A < b - b\delta$*, then for any policy* $\tau \in [0;1]$ *the system (9)–(10) has a long-run decline and has no solutions of the form (11).*

(ii) if $b - b\delta \le A < b + \delta/\gamma$*, then for any policy* $\tau \in [0;1]$ *the system has long-run decline. Moreover, for* $0 \le \tau < 1 - (1-\delta)b/A$*, the system (9)–(10) has a solution of the form (11).*

(iii) if $A > b + \delta/\gamma$*, then:*

if $\dfrac{\delta}{\gamma A} < \tau < 1 - \dfrac{b}{A}$*, the system (9)–(10) has a long-run growth and has a solution of the form (11);*

if $0 \le \tau < \dfrac{\delta}{\gamma A}$*, or* $1 - \dfrac{b}{A} < \tau < 1 - \dfrac{b}{A}(1-\delta)$*, the system (9)–(10) has a long-run decline and has a solution of the form (11),*

if $1 - (1-\delta)b/A \le \tau \le 1$*, the system (9)–(10) has a long-run decline and has no solutions of the form (11).*

Case (iii) is of the most interest here. More detailed results for its outcomes are given by the following theorem.

Theorem 2.

Let $b > 0, 0 \le \gamma \le 1, 0 \le \delta \le 1$, τ *is constant, and let* $A > b + \delta/\gamma$*. Then*

if $\dfrac{1-(1-\delta)b/A}{1+\gamma b} < \tau < 1 - (1-\delta)b/A$*, the system (9)–(10) has a solution of the form (11) such that* $K(t) < bG(t)$*;*

if $0 \le \tau < \dfrac{1-(1-\delta)b/A}{1+\gamma b}$*, the system (9)–(10) has a solution of the form (11) such that* $K(t) > bG(t)$.

Proving Theorems 1 and 2 is quite straightforward, based on direct substitution of (11) into the equations (9)–(10). We omit the proof due to triviality. Theorem 2 allows us to define the high-tax and low-tax policies. Let τ be a low-tax policy if

$$0 \le \tau < \frac{1-(1-\delta)b/A}{1+\gamma b}. \tag{12}$$

And a high-tax policy if

$$\frac{1-(1-\delta)b/A}{1+\gamma b} < \tau \le 1. \tag{13}$$

Theorems 1 and 2 are illustrated in Fig.1.

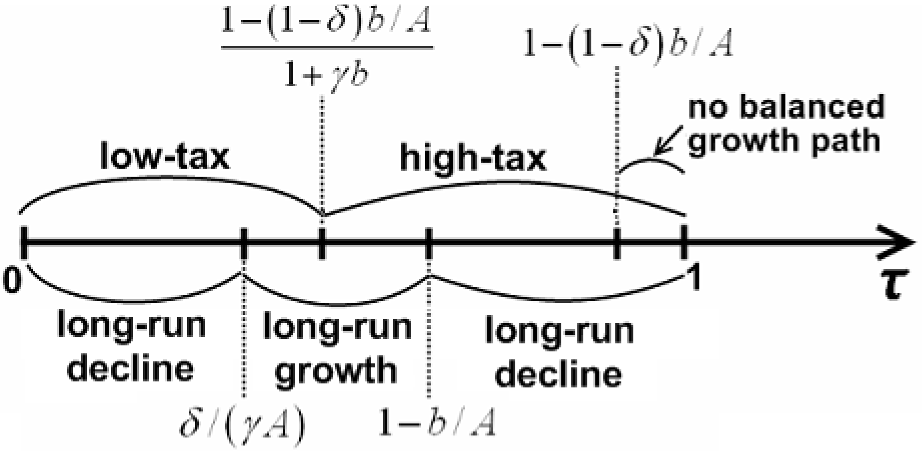

Fig.1. Policy space for $A > b + \delta/\gamma$

4. Long-run and short-run dynamics

The previous section was concerned with long-term effects of various policies. However, the short-term effects may drastically differ from the long-term ones, for example with certain starting conditions a long-run growth policy may lead to a short-term decline, and vice-versa. Therefore, with economic retrospective voting a party conducting a long-run growth policy may lose office after its first term in power without actually seeing the positive results of its policy. In contrast, a long-run decline party may stay in office for several terms.

Fig. 2a shows a vector field for (9)–(10) in the case of a long-run growth policy. If a starting point $(G(0), K(0))$ is situated in Area 1, then $K(1) < K(0)$, which leads to the party losing office at $t = 1$. Fig. 2b shows a vector field in the case of a long-run decline policy. Area 2 denotes initial conditions under which the ineffective party remains in office for one additional term, that is, until the long-term consequences of the party's policy begin to outweigh its short-term benefits.

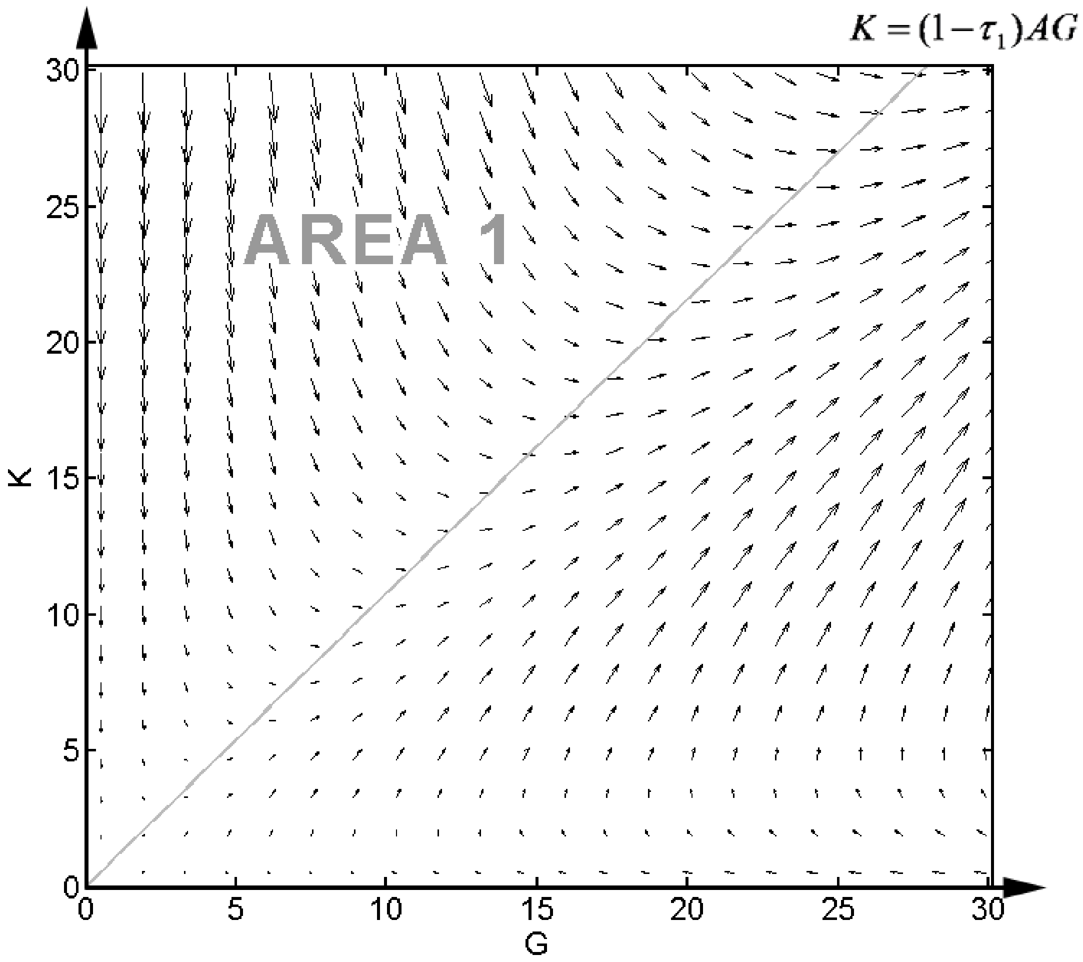

Fig. 2a. Vector field for $A = 1.8; b = 0.5; \gamma = 0.8; \delta = 0.2; \tau_1 = 0.4$

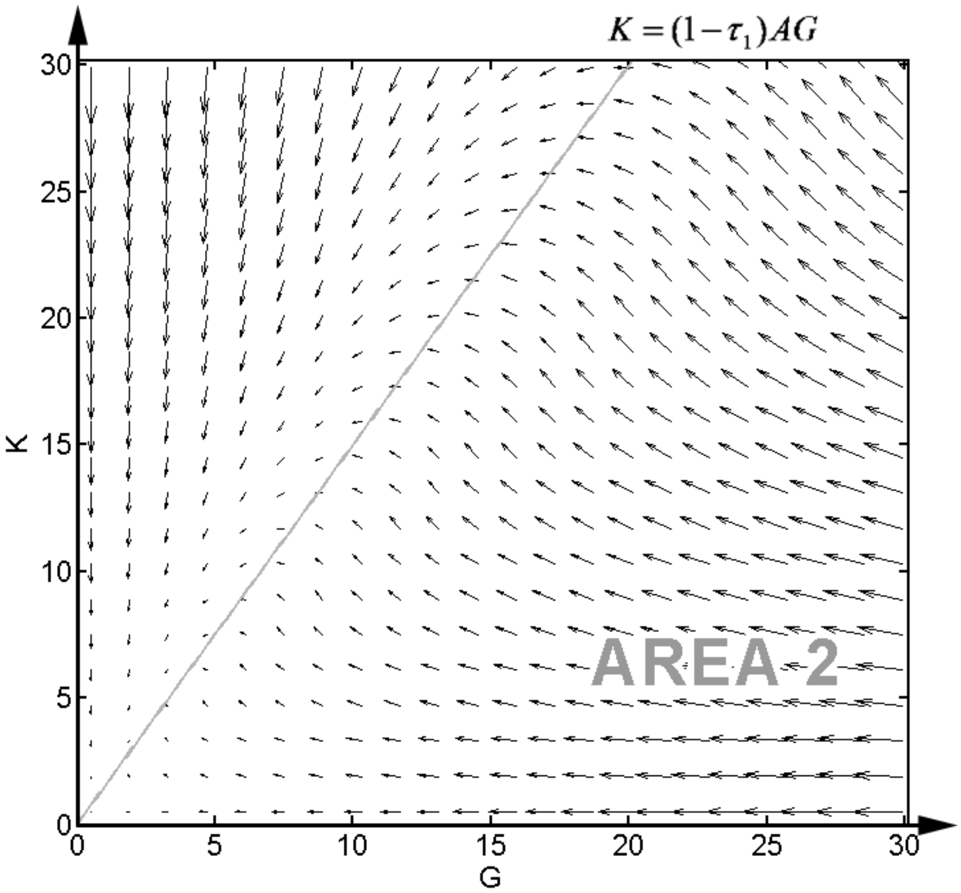

Fig. 2b Vector field for $A = 2.5; b = 1; \gamma = 0.4; \delta = 0.8; \tau_1 = 0.4$

5. Cyclically balanced growth paths

In what follows, we assume that $A > b + \delta / \gamma$, i.e. long-run growth policies exist. Sections 3 and 4 were auxiliary and considered a system with no switching of policies, i.e. governed by a single policy chosen "once and for all". We now proceed to analyse systems with endogenous switching of policies. Let us consider several cases.

5.1 Two low-tax policies

The current section considers (7)–(10) in the case of two low-tax policies, with Party 1's policy being long-run growth and Party 2's policy being long-run decline. Therefore,

$$\frac{\delta}{\gamma A} < \tau_1 < \frac{1 - (1-\delta) b / A}{1 + \gamma b}, \quad \tau_2 < \frac{\delta}{\gamma A}. \tag{14}$$

Let us first prove an auxiliary Lemma.

Lemma 1. *Let* $b>0\,,0\le\gamma\le 1,0\le\delta\le 1,A>b+\delta/\gamma$, *and*

$$0\le\tau(t)<\frac{1-(1-\delta)b/A}{1+\gamma b}\ \textit{for any}\ t=0,1,2,\ldots\ . \tag{15}$$

Then for any $K(0)>0,G(0)>0$ *the solution for (9)–(10) will lead to* $K(t)>bG(t)$ *starting from a certain* t.

Proof. The proof consists of two parts. First we prove that if $K(t)\ge bG(t)$ for a certain *t*, then $K(t+1)>bG(t+1)$. Then we prove that if $K(0)<bG(0)$, then $K(t)>bG(t)$ for some *t*.

Part one. Let $K(t)\ge bG(t)$. Then equations (9)–(10) take the form

$$K(t+1)=(1-\tau(t))AG(t),\ G(t+1)=(1-\delta)G(t)+\gamma\tau(t)AG(t),$$

so

$$\frac{G(t+1)}{K(t+1)}=\frac{(1-\delta)+\gamma\tau(t)A}{(1-\tau(t))A}. \tag{16}$$

Using inequality (15), we get from (16):

$$\frac{G(t+1)}{K(t+1)}<\frac{(1-\delta)+\gamma A\dfrac{1-(1-\delta)b/A}{1+\gamma b}}{\left(1-\dfrac{1-(1-\delta)b/A}{1+\gamma b}\right)A}.$$

This can be reduced to $K(t+1)>bG(t+1)$.

Part two. Let $K(0)<bG(0)$. It follows that there is some constant $b_1<b$ such that

$$K(0)/G(0)<b_1. \tag{17}$$

Let us prove that $K(t) > bG(t)$ for some *t*. Equations (9)–(10) for $t = 0$ yield:

$$K(1) = (1-\tau(0))AK(0)/b, \quad G(1) = (1-\delta)G(0) + \gamma\tau(0)AK(0)/b$$

so

$$\frac{G(1)}{K(1)} = \frac{(1-\delta)}{(1-\tau(0))A}\frac{bG(0)}{K(0)} + \frac{\gamma\tau(0)}{1-\tau(0)}. \tag{18}$$

Using the inequality (15), from (18) after simplifications we get:

$$\frac{G(1)}{K(1)} < \frac{G(0)}{K(0)}\frac{1}{\gamma + (1-\delta)/A}\left\{\frac{(1+\gamma b)(1-\delta)}{A} + \frac{K(0)}{G(0)}\frac{\gamma}{b}\left[1 - \frac{(1-\delta)b}{A}\right]\right\}. \tag{19}$$

Substituting (17) into (19), after simplifications we get:

$$\frac{G(1)}{K(1)} < \frac{G(0)}{K(0)}\left\{1 - \gamma\left(1 - \frac{b_1}{b}\right)\left[1 - \frac{(1-\delta)b}{A}\right]\right\}.$$

Let us denote the expression on the right-hand part of above inequality as $\frac{G(0)}{K(0)}(1-C)$. It can be easily seen that $0 < C < 1$. If $\frac{G(1)}{K(1)} < \frac{1}{b}$, then $K(t) > bG(t)$ starting $t = 1$. Otherwise, if $\frac{G(1)}{K(1)} \geq \frac{1}{b}$, it is easily shown that, similarly,

$$\frac{G(2)}{K(2)} < \frac{G(1)}{K(1)}(1-C) < \frac{G(0)}{K(0)}(1-C)^2.$$

As the sequence $\frac{G(0)}{K(0)}(1-C)^t$, $t=1,2,3,..$ forms a convergent geometric progression, then, from a certain point in time we have $\frac{G(0)}{K(0)}(1-C)^t<\frac{1}{b}$, i.e. $K(t)>bG(t)$.

This completes the proof of Lemma 1.

The economic meaning of Lemma 1 shows that after a certain point in time a low-tax policy (i.e. if the inequality (15) holds true) leads to private capital being the abundant production factor and public capital being the deficient production factor.

Let us proceed to system (9)–(10) with endogenous switching (7)–(8) that corresponds to economic retrospective voting. According to Lemma 1, if the inequalities (14) hold true, then at a certain point in time we have $K(t)>bG(t)$. Let us denote this point as t_0. Consider the case where the incumbent at t_0 carries out a long-run growth policy, i.e. $\tau(t_0)=\tau_1$ (the case where $\tau(t_0)=\tau_2$ can be considered similarly).

Then equations (9)–(10) for $t=t_0$ take the form:

$$K(t_0+1)=(1-\tau_1)AG(t_0) \tag{20}$$

$$G(t_0+1)=\left[(1-\delta)+\gamma\tau_1 A\right]G(t_0). \tag{21}$$

Voting rule (7)–(8) takes the form

$$\tau(t_0+1)=\tau_1,\ \text{ if }\ (1-\tau_1)AG(t_0)\geq K(t_0) \tag{22}$$

$$\tau(t_0+1)=\tau_2,\ \text{ if }\ (1-\tau_1)AG(t_0)<K(t_0). \tag{23}$$

If $(1-\tau_1)AG(t_0)\geq K(t_0)$, then $K(t_0)<K(t_0+1)<K(t_0+2)<\ldots$, and Party τ_1 remains in office forever. So there is no switching in this case, and the solution here is a balanced growth path of the form (11) (not a cyclically balanced growth path).

Consider now the case:

$$(1-\tau_1)AG(t_0)<K(t_0) \tag{24}$$

in which Party τ_2 takes office at $t = t_0 + 1$. It follows from Lemma 1 that $K(t_0+1) > bG(t_0+1)$, so

$$K(t_0+2) = (1-\tau_2)AG(t_0+1) \tag{25}$$

$$G(t_0+2) = \left[(1-\delta)+\gamma\tau_2 A\right]G(t_0+1). \tag{26}$$

In order to get the election outcome at $t = t_0 + 2$, let us calculate the growth of private capital:

$$\begin{aligned} K(t_0+2) - K(t_0+1) &= (1-\tau_2)A\left[(1-\delta)+\gamma\tau_1 A\right]G(t_0) - (1-\tau_1)AG(t_0) = \\ &= \left[(1-\tau_2)\left[(1-\delta)+\gamma\tau_1 A\right] - (1-\tau_1)\right]AG(t_0) \end{aligned}.$$

It follows from (14) that $1-\tau_2 > 1-\tau_1$ and $(1-\delta)+\gamma\tau_1 A > 1$, therefore $K(t_0+2) > K(t_0+1)$ and the Party τ_2 remains in office.

For the next point in time we have:

$$K(t_0+3) = (1-\tau_2)AG(t_0+2) \tag{27}$$

$$G(t_0+3) = \left[(1-\delta)+\gamma\tau_2 A\right]G(t_0+2) \tag{28}$$

therefore $K(t_0+3)/K(t_0+2) = G(t_0+2)/G(t_0+1) = (1-\delta)+\gamma\tau_2 A < 1$, leading to Party τ_2 losing office. Therefore, at the next moment the system is governed by τ_1:

$$K(t_0+4) = (1-\tau_1)AG(t_0+3) \tag{29}$$

$$G(t_0+4) = \left[(1-\delta)+\gamma\tau_1 A\right]G(t_0+3) \tag{30}$$

For the change in private capital we have:

$$\begin{aligned} K(t_0+4) - K(t_0+3) &= (1-\tau_1)A\left[(1-\delta)+\gamma\tau_2 A\right]G(t_0+2) - (1-\tau_2)AG(t_0+2) = \\ &= \left[(1-\tau_1)\left[(1-\delta)+\gamma\tau_2 A\right] - (1-\tau_2)\right]AG(t_0+2) \end{aligned}.$$

Since $1-\tau_1 < 1-\tau_2$ and $(1-\delta)+\gamma\tau_2 A < 1$, then $K(t_0+4)-K(t_0+3)<0$, leading to the Party τ_1 losing office.

Thus, we get a political cycle 1221221… . It is easily shown from (20)–(21) and (25)–(30) that the growth rate over the cycle is given by

$$\frac{K(t+3)}{K(t)} = \frac{G(t+3)}{G(t)} = \left[(1-\delta)+\gamma\tau_1 A\right]\left[(1-\delta)+\gamma\tau_2 A\right]^2 \tag{31}$$

for any integer $t \ge t_0$. If (31) is greater than 1, it means economic growth, otherwise it means economic decline.

This sub-section's results may be summarized by Theorem 3.

Theorem 3.

Let $b>0, 0\le\gamma\le 1, 0\le\delta\le 1,\ A>b+\delta/\gamma,\ K(t_0)>0,\ G(t_0)>0,\ \tau(t_0)=\tau_1$, *and let inequalities (14) hold true. Then,*

if $(1-\tau_1)AG(t_0)\ge K(t_0)$, *then system (7)–(10) has a solution of the form (11) for* $t\ge t_0$,

if $(1-\tau_1)AG(t_0)< K(t_0)$, *then the solution to (7)–(10) satisfies the inequalities (31) for* $t\ge t_0$, *and the parties in office alternate in the form 1221221…*

The results of numerical experiments illustrating Theorem 3 are shown in Figs. 3. Fig. 3a shows the trajectory of a long-run growing system (i.e. $K(t+3)/K(t)>1$), while Fig. 3b show the trajectory of a long-run declining system.

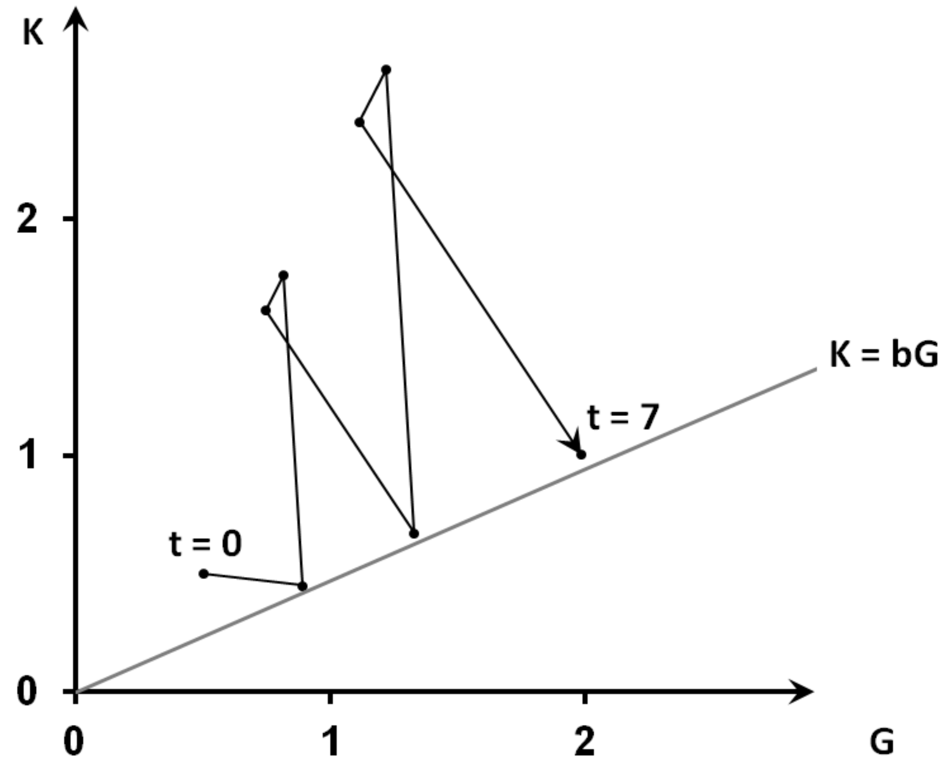

Fig. 3a The trajectory for $A = 2; b = 0.5; \gamma = 0.8; \delta = 0.1$; $\tau_1 = 0.55$ (long-run growth, low-tax); $\tau_2 = 0.01$ (long-run decline, low-tax); $G(0) = 0.5$; $K(0)=0.5$

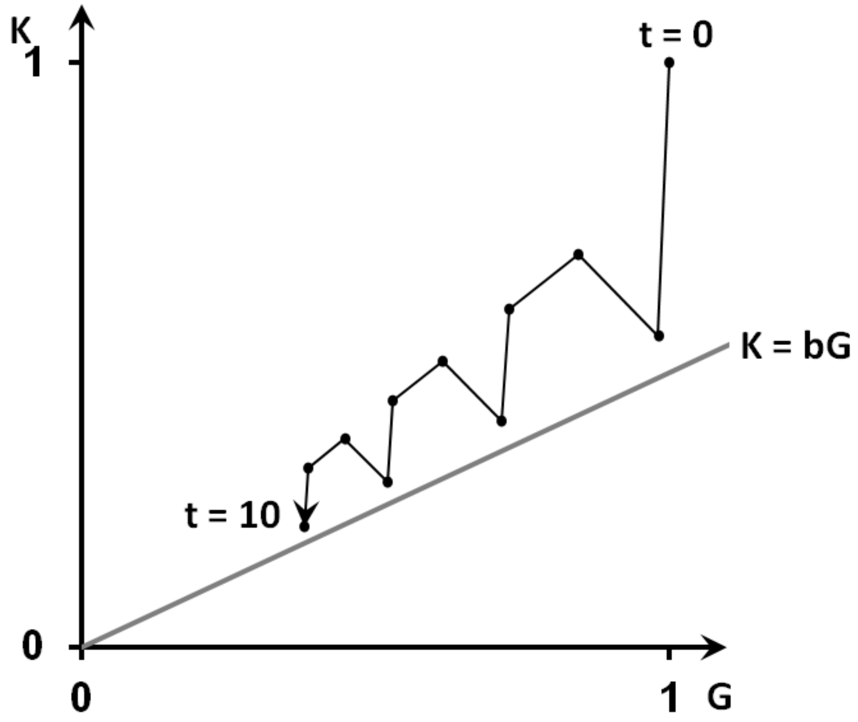

Fig. 3b The trajectory for $A = 0.76; b = 0.5; \gamma = 0.8; \delta = 0.2$; $\tau_1 = 0.33$ (long-run growth, low-tax); $\tau_2 = 0.291$ (long-run decline, low-tax); $G(0) = 1$; $K(0)=1$

5.2 Long-run growth high-tax policy and long-run decline low-tax policy

The current section considers model (7)–(10) for rivalry between the long-run growth high-tax policy of Party 1, and the long-run decline low-tax policy of Party 2. Thus:

$$\frac{1-(1-\delta)b/A}{1+\gamma b}<\tau_1<1-\frac{b}{A}, \quad \tau_2<\frac{\delta}{\gamma A} \tag{32}$$

Lemma 2.

Let $b>0, 0\le\gamma\le1, 0\le\delta\le1, A>b+\delta/\gamma$, *and*

$$\tau(t)>\frac{1-(1-\delta)b/A}{1+\gamma b}, \; K(t)>0, G(t)>0 \tag{33}$$

for some t. Then the solution of (9)–(10) satisfies $K(t+1)<bG(t+1)$.

Proof. Let us consider two cases: $K(t)\le bG(t)$ and $K(t)>bG(t)$, and show that each leads to $K(t+1)<bG(t+1)$.

Suppose that $K(t)\le bG(t)$. Then system (9),(10) takes the form

$$K(t+1)=(1-\tau(t))AK(t)/b \tag{34}$$

$$G(t+1)=(1-\delta)G(t)+\gamma\tau(t)AK(t)/b \tag{35}$$

Thus, we have in this case

$$\frac{G(t+1)}{K(t+1)}=\frac{(1-\delta)G(t)+\gamma\tau(t)AK(t)/b}{(1-\tau(t))AK(t)/b}\ge\frac{(1-\delta)+\gamma\tau(t)A}{(1-\tau(t))A}.$$

Using (33), we get

$$\frac{G(t+1)}{K(t+1)}>\frac{(1-\delta)(1+\gamma b)+\gamma A(1-(1-\delta)b/A)}{A(\gamma b+(1-\delta)b/A)}=\frac{1}{b}.$$

Now suppose that $K(t) > bG(t)$. In this case system (9)–(10) takes the form

$$K(t+1) = (1-\tau(t))AG(t) \tag{36}$$

$$G(t+1) = (1-\delta)G(t) + \gamma\tau(t)AG(t). \tag{37}$$

Hence,

$$\frac{G(t+1)}{K(t+1)} = \frac{(1-\delta)+\gamma\tau(t)A}{(1-\tau(t))A} > \frac{1}{b}$$

This completes the proof of Lemma 2.

The economic meaning of Lemma 2 is that a high-tax policy moves the system into the private capital deficiency area within a single time period. As a reminder, Lemma 1 stated that a low-tax policy moved the system into the public capital deficiency area within a finite number of time periods, for which an upper estimate can be easily obtained.

Lemma 3.

Let $b > 0, 0 \le \gamma \le 1, 0 \le \delta \le 1, A > b + \delta/\gamma$, *and*

$$\tau(t) < 1 - \frac{b}{A},\ K(t) > 0, G(t) > 0,\ K(t) \le bG(t)$$

for some t. *Then the solution of (9)–(10) satisfies* $K(t+1) > K(t)$.

Proof. The system (9)–(10) takes the form

$$K(t+1) = (1-\tau(t))AK(t)/b$$

$$G(t+1) = (1-\delta)G(t) + \gamma\tau(t)AK(t)/b.$$

We have

$$\frac{K(t+1)}{K(t)} = (1-\tau(t))A/b > \left(1-\left(1-\frac{b}{A}\right)\right)A/b = 1$$

This concludes the proof.

Let us consider the cyclical switching of parties in office. Let the Party τ_1 be in office at a moment t_0, (i.e. $\tau(t_0)=\tau_1$). It follows from Lemmas 2–3 that

if $K(t_0)\le bG(t_0)$, then the Party τ_1 remains in office for all $t>t_0$

if $K(t_0)>bG(t_0)$, then $K(t_0+1)<bG(t_0+1)$.

Therefore, Party τ_1 can lose the elections at t_0+1 only if $K(t_0)>bG(t_0)$. Let us consider the system in this case from this moment on. Thus, we have from (9)–(10):

$$K(t_0+1)=(1-\tau_1)AG(t_0) \tag{38}$$

$$G(t_0+1)=(1-\delta)G(t_0)+\gamma\tau_1 AG(t_0) \tag{39}$$

$$K(t_0)>bG(t_0). \tag{40}$$

The application of Lemma 2 yields $K(t_0+1)<bG(t_0+1)$.

Let us consider the elections at t_0+1. if $K(t_0+1)\ge K(t_0)$, then, according to Lemma 3, Party 1 remains in office for all $t>t_0$. Thus, cyclical switching is only possible if $K(t_0+1)<K(t_0)$, i.e.

$$(1-\tau_1)AG(t_0)<K(t_0). \tag{41}$$

Combining (40) and (41), we obtain the necessary condition for the cyclically balanced growth path:

$$0<\frac{G(t_0)}{K(t_0)}<\min\left(\frac{1}{(1-\tau_1)A},\frac{1}{b}\right)=\frac{1}{(1-\tau_1)A}. \tag{42}$$

Provided (42), we have $\tau(t_0+1)=\tau_2$. Therefore, equations (9)–(10) for $t=t_0+1$ yield:

$$K(t_0+2)=(1-\tau_2)AK(t_0+1)/b \tag{43}$$

$$G(t_0+2)=(1-\delta)G(t_0+1)+\gamma\tau_2 AK(t_0+1)/b\,. \tag{44}$$

It follows from Lemma 3 that $K(t_0+2)>K(t_0+1)$ and Party τ_2 remains in office for the next term.

After some simplifications it can be proved that

$$\frac{G(t_0+2)}{K(t_0+2)}=\frac{(1-\delta)b}{(1-\tau_2)A}\frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A}+\frac{\gamma\tau_2}{1-\tau_2}<\frac{1}{b}.$$

Therefore, equations (9),(10) for $t=t_0+2$ take the form

$$K(t_0+3)=(1-\tau_2)AG(t_0+2) \tag{45}$$

$$G(t_0+3)=(1-\delta)G(t_0+2)+\gamma\tau_2 AG(t_0+2). \tag{46}$$

It can be easily shown from (32) that $K(t_0+3)>bG(t_0+3)$.

Let us calculate the election outcome at $t=t_0+3$. It follows from (43)–(45) that

$$\frac{K(t_0+3)}{K(t_0+2)}=\frac{G(t_0+2)}{K(t_0+1)/b}=\frac{(1-\delta)bG(t_0+1)}{K(t_0+1)}+\gamma\tau_2 A. \tag{47}$$

Depending on $K(t_0+3)/K(t_0+2)$, two distinct cases may emerge.

Case 1: Short cycle 1221221...

Let $K(t_0+3)/K(t_0+2)<1$. This means that $\frac{(1-\delta)bG(t_0+1)}{K(t_0+1)}+\gamma\tau_2 A<1$, therefore,

$$\frac{G(t_0+1)}{K(t_0+1)}<\frac{1-\gamma\tau_2 A}{(1-\delta)b}. \tag{48}$$

On the other hand, from (38)–(39) we have:

$$\frac{G(t_0+1)}{K(t_0+1)}=\frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A}. \tag{49}$$

From (48)–(49), we get the necessary condition for the short cycle :

$$\frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A}<\frac{1-\gamma\tau_2 A}{(1-\delta)b}. \tag{50}$$

Provided (50), Party τ_1 takes office at $t=t_0+3$.

It can be easily seen that at $t=t_0+3$ the system falls into the same area as at $t=t_0$ (cf. (42)), namely

$$0<\frac{G(t_0+3)}{K(t_0+3)}=\frac{(1-\delta)+\gamma\tau_2 A}{(1-\tau_2)A}<\frac{1}{(1-\tau_1)A}.$$

However, $G(t_0+3)/K(t_0+3)$ and $G(t_0)/K(t_0)$ are not necessarily equal, as $G(t_0)/K(t_0)$ bears the imprint of the initial conditions. In other words, $t=t_0$ is the last point of the transitional process from the initial conditions to a cyclically balanced growth path, while $t=t_0+1$ is the starting point of this path.

Equations (9)–(10) for $t=t_0+3$ take the form:

$$K(t_0+4)=(1-\tau_1)AG(t_0+3) \tag{51}$$

$$G(t_0+4)=(1-\delta)G(t_0+3)+\gamma\tau_1 AG(t_0+3). \tag{52}$$

We have:

$$\frac{G(t_0+4)}{K(t_0+4)}=\frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A}=\frac{G(t_0+1)}{K(t_0+1)}. \tag{53}$$

Thus, the first loop of the cyclically balanced growth path occurs from $t = t_0 + 1$ to $t = t_0 + 4$. The growth over the cycle can be calculated from (38)–(39), (44), (46), (49), and (52). It is given by:

$$\frac{G(t+3)}{G(t)} = \frac{K(t+3)}{K(t)} =$$
$$= \left[(1-\delta)+\gamma\tau_1 A\right]\left[(1-\delta)+\gamma\tau_2 A\right]\left[(1-\delta)+\frac{\gamma\tau_2 A}{b}\frac{(1-\tau_1)A}{(1-\delta)+\gamma\tau_1 A}\right] \tag{54}$$

(here $t \geq t_0 + 1$).

If (54) is greater than 1, we have a cyclically balanced path with growth (Fig. 4a), otherwise we have a path with decline (Fig. 4b)

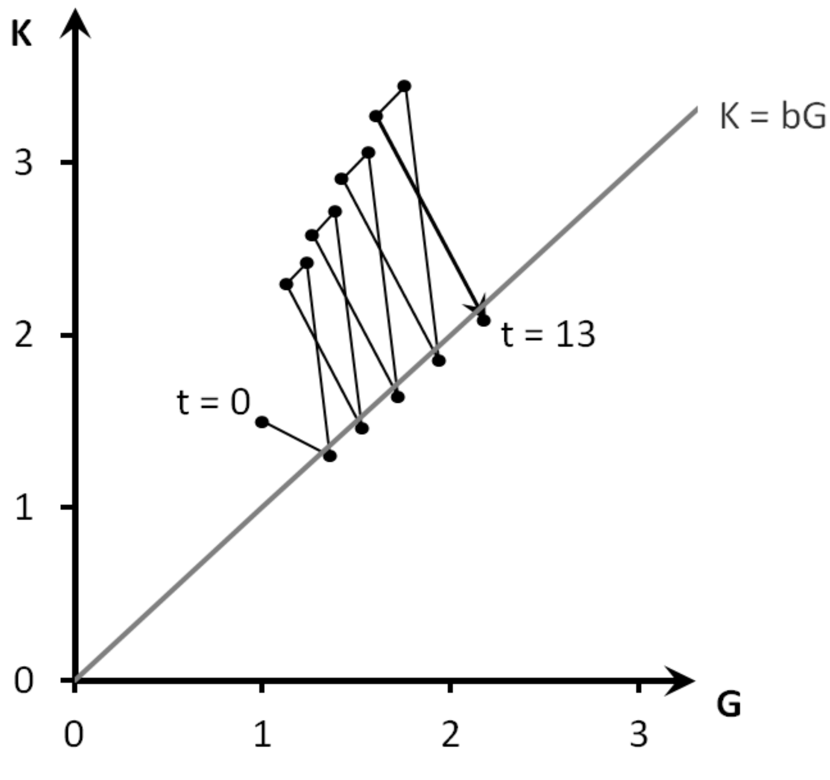

Fig 4a. The trajectory for $A = 2; b = 1; \gamma = 0.8; \delta = 0.2$; $\tau_1 = 0.36$ (long-run growth high-tax); $\tau_2 = 0.07$ (long-run decline low-tax); $G(0) = 1$; $K(0)=1.5$

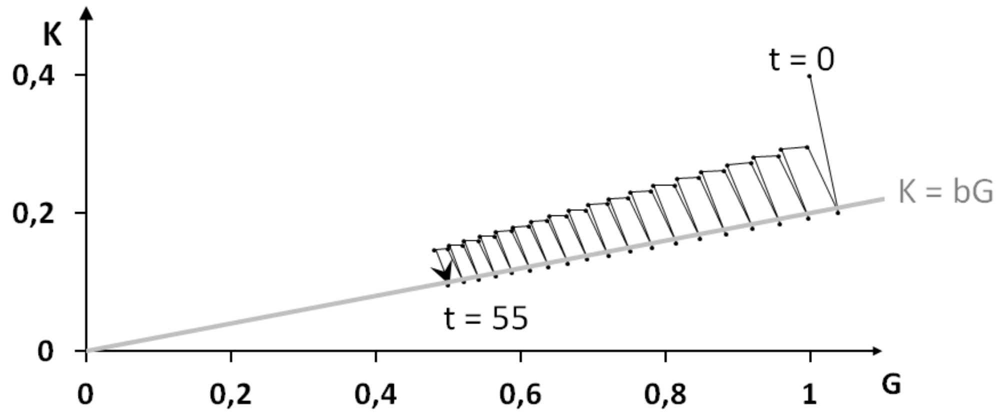

Fig. 4b. The trajectory for $A=0,5; b=0,2; \gamma=0.8; \delta=0.2$; $\tau_1=0.599$ (long-run growth high-tax); $\tau_2=0.41$ (long-run decline low-tax); G(0) = 1; K(0)=0.4

Case 2: Long cycle: 122212221...

Let $K(t_0+3)/K(t_0+2)\geq 1$ in (47). This means that (cf. (50)),

$$\frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A}\geq\frac{1-\gamma\tau_2 A}{(1-\delta)b}. \tag{55}$$

Provided (55), Party τ_2 remains in office at $t=t_0+3$. Therefore,

$$K(t_0+4)=(1-\tau_2)AG(t_0+3) \tag{56}$$

$$G(t_0+4)=(1-\delta)G(t_0+3)+\gamma\tau_2 AG(t_0+3). \tag{57}$$

To determine the election outcome at $t=t_0+4$, let us consider:

$$\frac{K(t_0+4)}{K(t_0+3)}=\frac{G(t_0+3)}{G(t_0+2)}=(1-\delta)+\gamma\tau_2 A. \tag{58}$$

Since $\tau_2<\delta/(\gamma A)$, then $K(t_0+4)<K(t_0+3)$. Therefore, Party τ_2 loses office at $t=t_0+4$. At this point of time:

$$0 < \frac{G(t_0+4)}{K(t_0+4)} < \frac{1}{(1-\tau_1)A} < \frac{1}{b}.$$

So, equations (9)–(10) for $t = t_0 + 4$ take the form:

$$K(t_0+5) = (1-\tau_1)AG(t_0+4) \tag{59}$$

$$G(t_0+5) = (1-\delta)G(t_0+4) + \gamma\tau_1 AG(t_0+4). \tag{60}$$

We have:

$$\frac{G(t_0+5)}{K(t_0+5)} = \frac{(1-\delta)+\gamma\tau_1 A}{(1-\tau_1)A} = \frac{G(t_0+1)}{K(t_0+1)}. \tag{61}$$

Thus, the first loop of the cyclically balanced growth path occurs from $t = t_0 + 1$ to $t = t_0 + 5$. The growth over the cycle can be calculated from (38)–(39), (44), (46), (57), and (60). It is given by

$$\begin{gathered} \frac{G(t+4)}{G(t)} = \frac{K(t+4)}{K(t)} = \\ = \left[(1-\delta)+\gamma\tau_1 A\right]\left[(1-\delta)+\gamma\tau_2 A\right]^2 \left[(1-\delta) + \frac{\gamma\tau_2 A}{b}\frac{(1-\tau_1)A}{(1-\delta)+\gamma\tau_1 A}\right] \end{gathered} \tag{62}$$

(here $t \geq t_0 + 1$).

If (62) is greater than 1, we have a cyclically balanced path with growth (Fig. 5a), otherwise we have a path with decline (Fig. 5b)

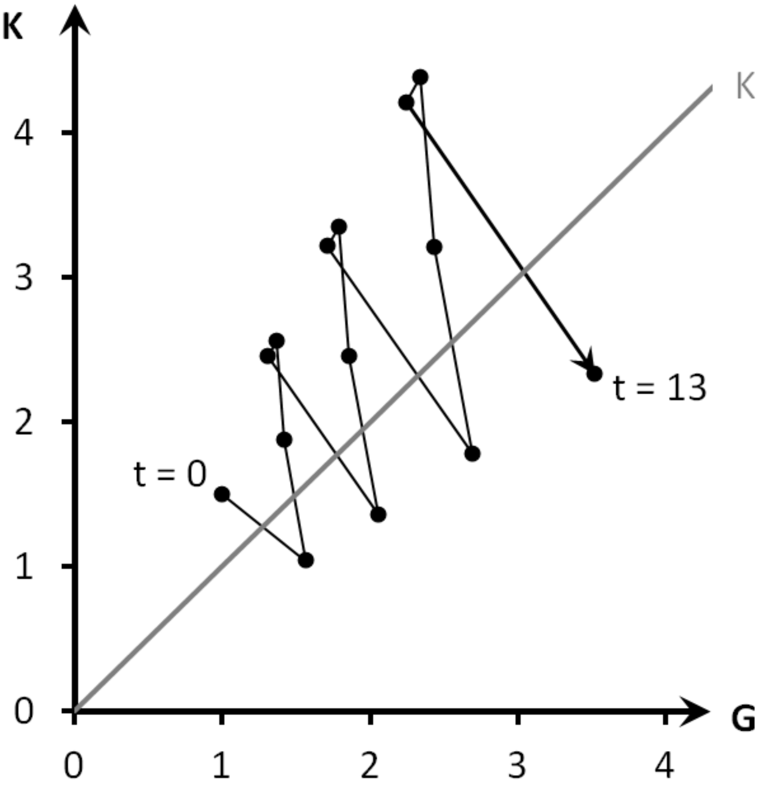

Fig 5a. The trajectory for $A=2; b=1; \gamma=0.8; \delta=0.2$; $\tau_1=0.48$ (long-run growth high-tax); $\tau_2=0.1$ (long-run decline low-tax); $G(0)$ = 1; $K(0)$=1.5

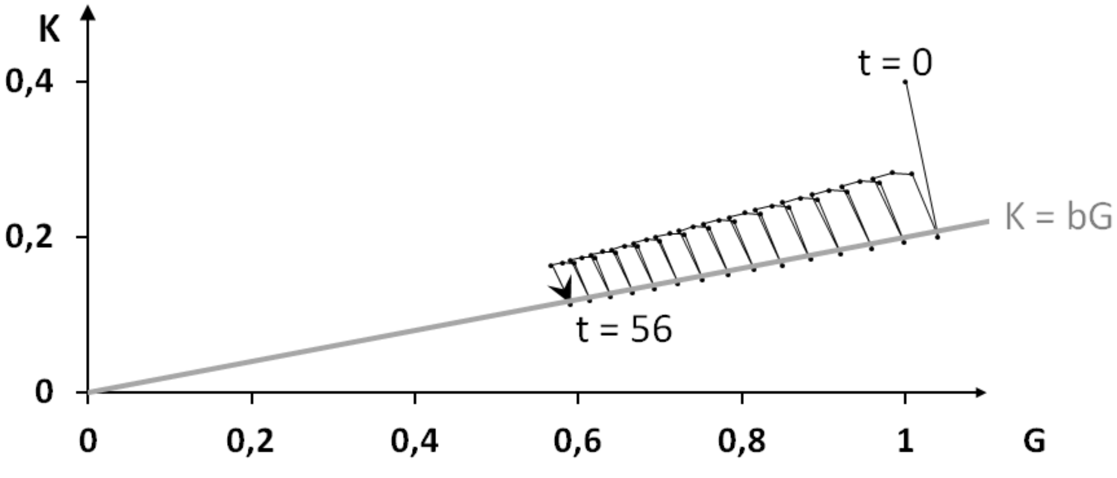

Fig 5b. The trajectory for $A=0{,}5; b=0{,}2; \gamma=0.8; \delta=0.2$; $\tau_1=0.599$ (long-run growth high-tax); $\tau_2=0.44$ (long-run decline low-tax); $G(0)$ = 1; $K(0)$=0.5

This sub-section may be summarized by the following theorem.

Theorem 4.

Let $b > 0, 0 \le \gamma \le 1, 0 \le \delta \le 1,\quad A > b + \delta/\gamma,\quad K(t_0) > 0,\quad G(t_0) > 0,$ $0 < G(t_0)/K(t_0) < 1/\left[(1-\tau_1)A\right]$, $\tau(t_0) = \tau_1$, *and let inequalities (32) be satisfied. Then*

if (50) is satisfied, then at $t \ge t_0$ *the solution to (7)–(10) satisfies (54), leading to cyclical switching for parties in office in the form 1221221…,*

if (55) is satisfied, then at $t \ge t_0$ *the solution to (7)–(10) satisfies (62), leading to cyclical switching for parties in office in the form 122212221….*

5.3 Certain other cases

It can be easily shown that if Party 1 carries out a long-run growth high-tax policy and Party 2 carries out a long-run decline high-tax policy, that is,

$$\frac{\delta}{\gamma A} < \tau_1 < 1 - \frac{b}{A}, \quad \tau_2 > 1 - \frac{b}{A} \tag{63}$$

then the system has no cyclically balanced growth path. This is due to Party 1 lowering taxes after coming into office, which in turn leads to private capital growth and allows Party 1 to remain in office as an incumbent during the next election. During subsequent elections the Party 1 remains in office due to economic growth.

The cases of both parties carrying out long-run growth or long-run decline policies are less interesting from a practical standpoint and are thus omitted. Lastly, borderline cases where $\tau = \delta/(\gamma A)$ or $\tau = 1 - b/A$ are also omitted because of their insipidness.

6. Conclusions

In this paper, we have introduced solutions which we call cyclically balanced growth paths. They generalize well-known balanced growth paths to systems with endogenous switching. Several cases have been considered, the conditions for existence of these solutions are found for each case, and growth rates over the cycle are found.

The potential applicability of the formal results within a wide range of contemporary problems at the crossroads of economics and political science. To name a few, "political business cycles" [Alesina et al. 1997], [Franzese 2002], [Golden and Min 2013], or new institutional studies, the search for formal rules to find optimal policy [Morelli 2004], [Persson 2007]. A particularly promising application of our results to the new developments in political science and economics is a novel approach to the study of populist politics [Acemoglu et al. 2013], [Postel 2007]. This is due to the ability of the model to distinguish between the long-term and short-term effects of various policies. The choice in favour of populist policies may be explicitly analysed as “long-term sacrifice for a short-term gain”. This analytical work of in this area is not yet complete. The issues mentioned above illustrate possible paths for future research.

References:

1. Acemoglu, D., Egorov G., Sonin, K.. (2013). A Political Theory of Populism, Quarterly Journal of Economics 128 (2):, 771-805
2. Aghion, Ph., Howitt, P. (2008). The Economics of Growth. MIT Press.
3. Aiyagari, R., Marcet, A., Sargent, T., Seppala, J. (2002). Optimal Taxation without State-Contingent Debt, - Journal of Political Economy, 110: 1220-1254.
4. Akhremenko A., Petrov A. (2014). Efficiency, Policy Selection, and Growth in Democracy and Autocracy: A Formal Dynamic Model. - NRU Higher School of Economics Political Series Working Paper
5. Alesina, A., Roubini, N. and Cohen, G. (1997). Political Cycles and the Macroeconomy. Cambridge, MA: MIT Press
6. Anderson, Ch. (2007). The end of Economic Voting? Contingency Dilemmas and the Limits of Democratic Accountability. - Annual Review of Political Science. Vol.10: 271-296
7. André, F., Cardenete, A., Romero C. (2010). Designing Public Policies: An Approach Based on Multi-Criteria Analysis and Computable General Equilibrium Modeling. – Springer.
8. Angelopoulos, K., and Economides, G. (2008). Fiscal Policy, Rent Seeking and Growth under Electoral Uncertainty. - Canadian Journal of Economics, Volume 41, Issue 4, pages 1375–1405.
9. Arrow K, Kurz M. Optimal Public Investment Policy and Controllability with Fixed Private Savings Ratio. JOURNAL OF ECONOMIC THEORY 1, 141-177 (1969)
10. Arslanalp, S., Bornhorst, F., Gupta S., Sze, E. (2010). Public Capital and Growth. - IMF Working Paper WP/10/175.
11. Aschauer, D.A. (2000). Public Capital and Economic Growth: Issues of Quantity, Finance, and Efficiency, - Economic Development and Cultural Change, Vol. 48(2).
12. Bassetto, M. (2014). Optimal Fiscal Policy with Heterogeneous Agents, - Quantitative Economics, 5: 675–704.

13. Battaglini M., Nunnari S., Palfrey T. (2012). Legislative Bargaining and the Dynamics of Public Investment, - American Political Science Review, Vol. 106, No. 2, P. 407-429.
14. Battaglini, M., Coate, S. (2007). Inefficiency in Legislative Policy-Making: A Dynamic Analysis, - American Economic Review, 97(1): 118-149.
15. Devereux, M., Wen, J-F. (1998). Political instability, capital taxation and growth, - European Economic Review, 42, 1635—1651.
16. Franzese, R. (2002). Electoral and Partisan Cycles in Economic Policies and Outcomes. Annu. Rev. Polit. Sci. 5:369–421
17. Golden, M., Min, B. (2013). Distributive Politics Around the World. Annu. Rev. Polit. Sci., 16:73–99
18. Gupta S., Kangur, A., Papageorgiou, Ch., Wane, A. (2014). Efficiency-Adjusted Public Capital and Growth. – World Development. Vol. 57:. P. 164-178.
19. Hassler, J., Krusell, P., Storesletten, K., Zilibotti, F. (2005). The Dynamics of Government. - Journal of Monetary Economics. 52(7): 1331-1358
20. Healy, A., Malhotra N. (2013). Retrospective Voting Reconsidered. – Annual Review of Political Science, Vol.16: 285 – 306.
21. Malley, J., Philippopoulos, A., Woitek, U. (2007). Electoral Uncertainty, Fiscal Policy and Macroeconomic Fluctuations, - Journal of Economic Dynamics and Control, 31, pp. 1051–1080
22. Morelli, R. (2004). Party Formation and Policy Outcomes under Different Electoral Systems. Review of Economic Studies, 71: 829–853.
23. Persson, T., Roland G., Tabellini, G. (2007). Electoral Rules and Government Spending in Parliamentary Democracies. Quarterly Journal of Political Science, 2: 155-188
24. Postel, Ch. (2007). The Populist Vision. Oxford: Oxford University Press
25. Pritchett, L. (2000. The tyranny of concepts: CUDIE (Cumulated, Depreciated, Investment Effort) is not capital. Journal of Economic Growth, 5, 361–384)

Andrei S. Akhremenko
National Research University Higher School of Economics, Laboratory for Qualitative and Quantitative Analysis of Political Regimes, Director; E-mail: aakhremenko@hse.ru