FOLIATIONS GENERATED BY DIFFERENTIALS OF ABELIAN TYPE

KOCHETKOV YU. YU.

ABSTRACT. The geometry of foliations, generated by some differentials of Abelian type is considered. The case when all fibers are closed is studied.

Keywords: foliations, abelian differentials.

Consider the following differential

$$\eta = \frac{\sqrt{z}\, dz}{\sqrt{(z-A)(z-B)(z-C)}},$$

where A, B and C — are pairwise different nonzero complex numbers. Differential η defines two foliations in $\mathbb{C}$: F_r and F_i. Fibers of F_i are integral curves of the differential equation $\operatorname{Im} \eta = 0$, and fibers of F_r are integral curves of the differential equation $\operatorname{Re} \eta = 0$. We will study the foliation F_r.

1. General case

If $|z| \gg 1$, then

$$\frac{\sqrt{z}}{\sqrt{(z-A)(z-B)(z-C)}} = \frac{1}{z} + \mathrm{O}(|z|^{-2}).$$

Therefore along F_r-fiber we have $\arg(dz) = \pm\pi/2 + \arg(z) + \mathrm{O}(|z|^{-1})$ and along F_i-fiber we have $\arg(dz) = \arg(z) + \mathrm{O}(|z|^{-1})$. Hence a F_r-fiber that pass through the point z_0, $|z_0| \gg 1$, is almost a circle with the center at the origin, and a F_i-fiber that pass through the same point is almost a radial line.

Lemma 1. *Let $|z_0| \gg 1$, then F_r-fiber that pass through z_0 is a closed loop.*

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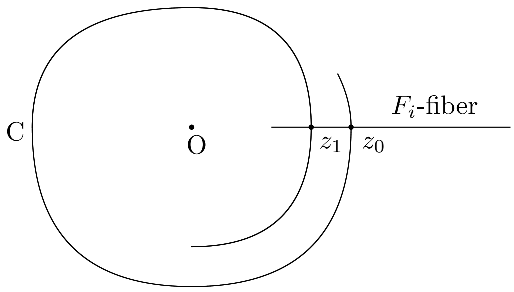
Proof. Let C be our F_r-fiber and let us assume that C doesn't return to z_0 after one rotation around the origin (Picture 1).



Picture 1

Let D be F_i-fiber that pass through z_0 and let z_1 be the first intersection point of D and C when we move along D from z_0 to the origin. Let us consider a closed contour L that consists of the part of the fiber C from z_0 to z_1 and of the part the fiber D from z_1 to z_0. Branch points of the differential η are inside L, hence $\int_L \eta = 2\pi i$. But the integral along the C-part of L is imaginary (by the definition of a F_r-fiber) and the integral along the D-part is real and nonzero. □

Definition 1. A closed F_r-fiber L such that branch points A, B, C and 0 are inside L will be called a O-fiber.

Lemma 2. *Denote by G the the union of all O-fibers. Then G is an open set.*

Proof. Let L be an O-fiber and z_0 belong to a small neighborhood of L. Consider the F_r-fiber that pass through z_0. If this fiber is not closed, that the reasoning of the proof of Lemma 1 is valid here. □

Corollary. *The boundary C_0 of the set G is a closed F_r-fiber that contains a branch point of η.*

Lemma 3. *The origin belongs to C_0.*

Proof. It is enough to demonstrate that only one arc of an F_r-fiber goes out from points A, B and C (three arcs of F_r-fibers go out from 0). Indeed, let α be an argument of the tangent vector to the F_r-trajectory at the point A. Then

$$\frac{\alpha}{2} + \frac{\arg(A)}{2} - \frac{\arg(A-B)}{2} - \frac{\arg(A-C)}{2} = \pm\frac{\pi}{2}.$$

Hence,

$$\alpha = \pm\pi + \arg(A-B) + \arg(A-C) - \arg(A).$$

If φ is an argument of the tangent vector to the F_r-trajectory at point 0, then

$$\frac{3\varphi}{2} - \frac{3\pi}{2} - \frac{\arg(ABC)}{2} = \frac{\pi}{2} + \pi n.$$

Hence,

$$\varphi = \frac{\arg(ABC)}{3} + \frac{2\pi n}{3}.$$

Therefore, the closed fiber C_0 necessary contains the origin 0 and point A (or B or C) belongs to C_0 only in the case, when C_0 contains an arc, that connects 0 and A. □

Now we can describe the F_r-foliation.

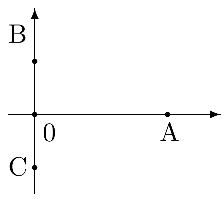
Theorem 1. *The F_r-foliation has the following properties:*

- *there exists a closed fiber C_0 that contains the origin 0;*
- *fibers outside C_0 are closed and at infinity are almost circles with the center at the origin;*
- *branch points A, B and C are inside C_0 and in an exceptional case these points (all or one) belong to C_0.*

Thus the behavior of F_r-fibers can be nontrivial only inside C_0.

2. Example

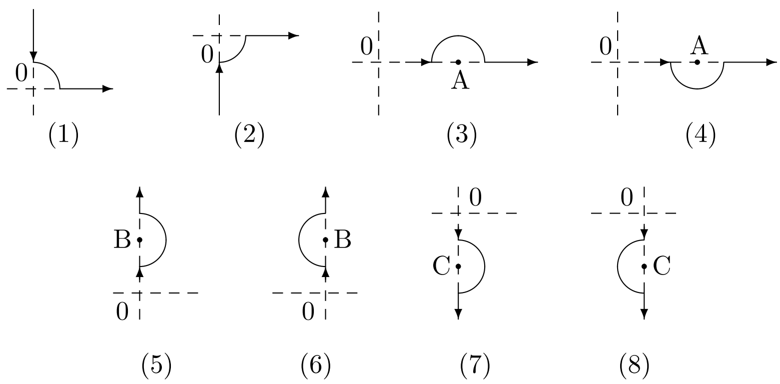
In this section we will consider in detail the case, when the point A belongs OX axis, $A \in (0, +\infty)$, points B and C belong to OY axis, $B \in (0, +i\infty)$, $C \in (0, -i\infty)$, and $|B| > |C|$ (Picture 2).



Picture 2

We will study the behavior of F_r-fibers inside C_0. At first let us note, that when a point z moves along a coordinate axis from 0 to A, from 0 to B, from 0 to C, from A to $+\infty$, from B to $+i\infty$ or from C to $-i\infty$ then the sign of $\mathrm{Re}\,\eta$ doesn't change. Let us consider now circuits of branch points.

Lemma 4. *Consider the following circuits of branch points (Picture 3):*



Picture 3

The sign of $\mathrm{Re}\,\eta$ *can be changed after the turn in the following way:*

1. *the sign of* $\mathrm{Re}\,\eta$ *changes to the opposite;*
2. *the sign of* $\mathrm{Re}\,\eta$ *doesn't change;*
3. *the sign of* $\mathrm{Re}\,\eta$ *doesn't change;*
4. *the sign of* $\mathrm{Re}\,\eta$ *changes to the opposite;*
5. *the sign of* $\mathrm{Re}\,\eta$ *doesn't change;*

(6) *the sign of* $\operatorname{Re}\eta$ *changes to the opposite;*
(7) *the sign of* $\operatorname{Re}\eta$ *doesn't change;*
(8) *the sign of* $\operatorname{Re}\eta$ *changes to the opposite.*

Proof. We will give the proof of the first case (other cases can be proved analogously). Let $z \in (0, B)$ and $v = (z - A)(z - B)(z - C)$, then

$$\frac{\pi}{2} < \arg(v) < \pi, \quad \pi < \arg(v^{-1}) < \frac{3\pi}{2}, \quad -\frac{\pi}{2} < \arg(zv^{-1}) < 0.$$

Let us consider the branch of $\sqrt{zv^{-1}}$, where

$$-\frac{\pi}{4} < \arg(\sqrt{zv^{-1}}) < 0.$$

As we move from B to 0 then

$$-\frac{3\pi}{4} < \arg(\eta) < -\frac{\pi}{2} \Rightarrow \operatorname{Re}\eta < 0.$$

If $|z|$ is small, then $\arg(\sqrt{zv^{-1}}) \approx -\pi/4$. After the circuit of the origin O the argument of z decreases by $\pi/2$ and the value of $\arg(\sqrt{zv^{-1}})$ now is approximately $-\pi/2$. The vector v now is in the second quarter, i.e. $\pi/2 < \arg(v) < \pi$ and $-\pi < \arg(v^{-1}) < -\pi/2$. As z now is real positive, then

$$-\frac{\pi}{2} < \arg(\sqrt{zv^{-1}}) < -\frac{\pi}{4}.$$

Hence, $\operatorname{Re}\eta > 0$. □

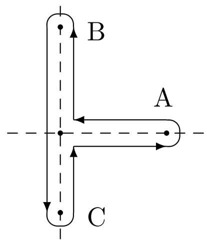
These rules allow us to formulate the following important statement.

Lemma 5. *Let*

$$a = \int_0^A |\operatorname{Re}\eta|, \quad b = \int_0^B |\operatorname{Re}\eta|, \quad c = \int_0^C |\operatorname{Re}\eta|,$$

then $b = a + c$.

Proof. Consider the contour L (Picture 4)



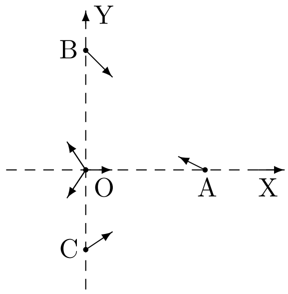
Picture 4

As L is closed and branch points are inside it, then $\int_L \eta = 2\pi i$ and $\int_L \operatorname{Re}\eta = 0$. Let us consider the branch of η, where $\operatorname{Re}\eta > 0$, if $z \in (0, C)$. Then $\operatorname{Re}\eta > 0$, if $z \in (C, 0)$; $\operatorname{Re}\eta > 0$, if $z \in (0, A)$ (rule 2); $\operatorname{Re}\eta > 0$, if $z \in (A, 0)$; $\operatorname{Re}\eta < 0$, if $z \in (0, B)$ (rule 1); $\operatorname{Re}\eta < 0$, if $z \in (B, 0)$. Hence, $2c + 2a - 2b = 0$. □

Using the reasoning of the proof of Lemma 3 we can find directions of F_r-trajectories outgoing from branch points (Picture 5):

- vectors, tangent to F_r-trajectories outgoing from points B and C, are directed to the point A;

- let α be the argument of the vector, tangent to the F_r-trajectory outgoing from the point A, then $\arg(B-A) < \alpha < \pi$;
- arguments of vectors, tangent to F_r-trajectories outgoing from the origin, are equal to 0 and $\pm 2\pi/3$.



Picture 5

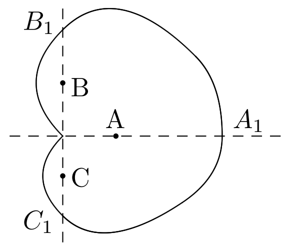
In what follows the axis $(0, +\infty)$ will be called A-axis, the axis $(0, +i\infty)$ — B-axis , and the axis $(0, -i\infty)$ — C-axis. We will define new coordinates at these three axes: if z belong to A-, B- or C-axis, then

$$||z|| = \int_0^z |\mathrm{Re}\, \eta|.$$

Thus $||A|| = a$, $||B|| = b$ and $||C|| = c$.

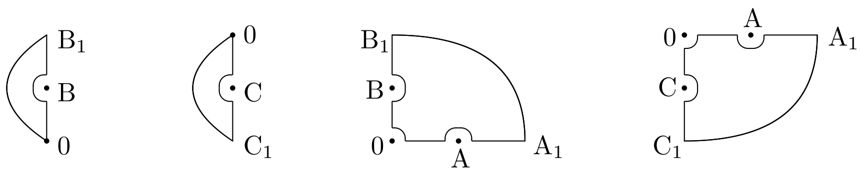
Now we can describe the closed contour C_0.

Lemma 6. *The F_r-trajectory, that constitute the contour C_0, leaves the origin with the argument $2\pi/3$. Then it intersects the B-axis at the point B_1, $||B_1|| = 2b$; then it intersects the A-axis at the point A_1, $||A_1|| = 2b$; then it intersects the C-axis at the point C_1, $||C_1|| = 2c$; and then it returns to the origin with the argument $\pi/3$ (Picture 6).*



Picture 6

Proof. It is enough to consider integrals of $\mathrm{Re}\, \eta$ by following contours (Picture 7)



Picture 7

and use rules of Lemma 4. □

Consider now a F_r-fiber inside C_0.

Lemma 7. *Let L be a F_r-fiber inside C_0, then*

(1) *if L intersects B-axis at point y from the second to the first quarter, then after that L intersects A-axis at point x, $||x|| = ||y||$;*
(2) *if L intersects B-axis at point y from the first to the second quarter, then, after half turn around B, L intersects B-axis at point y_1, $||y_1|| = 2b - ||y||$, i.e.*

$$\int_y^B |\mathrm{Re}\,\eta| = \int_B^{y_1} |\mathrm{Re}\,\eta|;$$

(3) *if L intersects C-axis at point z from the third to the forth quarter, then after that L intersects A-axis at point x, $||x|| = ||z|| + 2a$;*
(4) *if L intersects C-axis at point z from the forth to the third quarter, then, after half turn around C, L intersects C-axis at point z_1, $||z_1|| = 2c - ||z||$, i.e.*

$$\int_z^C |\mathrm{Re}\,\eta| = \int_C^{z_1} |\mathrm{Re}\,\eta|;$$

(5) *if L intersects A-axis at point x from the first to the forth quarter, then: a) if $||x|| > 2a$, the after that L intersects C-axis at point z, $||z|| = ||x|| - 2a$ (see 3); b) if $||x|| < 2a$, then, after half turn around A, L intersects A-axis at point x_1, $||x_1|| = 2a - ||x||$, i.e.*

$$\int_x^A |\mathrm{Re}\,\eta| = \int_A^{x_1} |\mathrm{Re}\,\eta|;$$

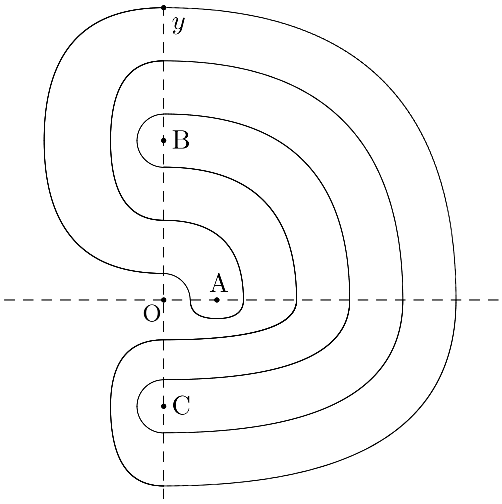
c) if $||x|| = 2a$, then L goes into the origin;
(6) *if L intersects A-axis at point x from the forth to the first quarter, the after that L intersects B-axis at point y, $||y|| = ||x||$ (see 1).*

Proof. The proof is analogous to the proof of Lemma 6. □

Example. Let $a = 1$, $c = 2$, then $b = 3$. Consider the F_r-fiber that pass through point y at B-axis, $||y|| = 11/2$, from the second to the first quarter (Picture 8).

$$\frac{11}{2}(B) \to \frac{11}{2}(A) \to \frac{7}{2}(C) \to \frac{1}{2}(C) \to \frac{5}{2}(A) \to$$
$$\to \frac{5}{2}(B) \to \frac{7}{2}(B) \to \frac{7}{2}(A) \to \frac{3}{2}(C) \to \frac{5}{2}(C) \to \frac{9}{2}(A) \to$$
$$\to \frac{9}{2}(B) \to \frac{3}{2}(B) \to \frac{3}{2}(A) \to \frac{1}{2}(A) \to \frac{1}{2}(B) \to \frac{11}{2}(B).$$

The fiber is closed.



Picture 8

This example illustrates conditions of the closure of F_r-fibers inside C_0.

Theorem 2. *If a and c are commensurable, then all F_r-fibers inside C_0 are closed. Otherwise all F_r-fibers inside C_0 are not closed.*

Proof. Let L be a closed F_r-fiber (trajectory) inside C_0. Let us assume that branch points do not belong to L and that L intersects B-axis and y is the B-coordinate of the intersection point. Then L will intersect axes in the following order:

$$y(B) \to y(A) \to y - 2a(C) \to 2c + 2a - y(C) \to 2c + 4a - y(A) \to \\ \to 2c + 4a - y(B) \to y - 2a(B) \to y - 2a(A)$$

After that L will intersect A-axis from the first to the forth quarter in points $y-4a, y-6a, \ldots$. If $y-2ka > 0$, but $y-2(k+1)a < 0$, then L intersects A-axis at point $y - 2ka$, makes half turn around A, intersects A-axis at point $2(k+1)a - y$ from the forth to the first quarter, intersects B-axis at point $2(k + 1)a - y$ from the first to the second quarter, makes half turn around B and intersects B-axis at point $y + 2c - 2ka$ from the second to the first quarter.

In other words, L intersects B-axis from the first to the second quarter at points $2lc - 2ka - y$ and from the second to the first quarter at points $y + 2mc - 2na$. As L is closed, then either $y = y + 2mc - 2na$, or $y = 2lc - 2ka - y$. In the second case L gets into a branch point and returned to y along itself in contradiction to the assumption. Thus $y = y + 2mc - 2na$ and a and c are commensurable.

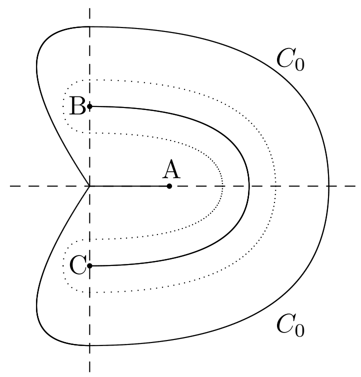
If a branch point belongs to a closed F_r-fiber, then this fiber connects two branch points. In this case we can put $y = b = a+c$, for example, and consider the following cases:

- $a + c + 2mc - 2na = a$ (the fiber came into A);
- $a + c + 2mc - 2na = c$ (the fiber came into C);
- $2lc - 2ka - a - c = 0$ (the fiber came into 0).

In all these case we have the commensurability of a and c.

Let now a and c be commensurable: $a = kc/l$, $b = (k+l)c/l$. Consider a F_r-fiber that intersects B-axis at point y. Computing as above all intersection points in c-units, we have that a F_r-fiber intersects B-axis at points $\pm y + s/l$, where denominator l is fixed and numerator s is bounded. This proves the closure of fibers in the case of the commensurability of a and c. □

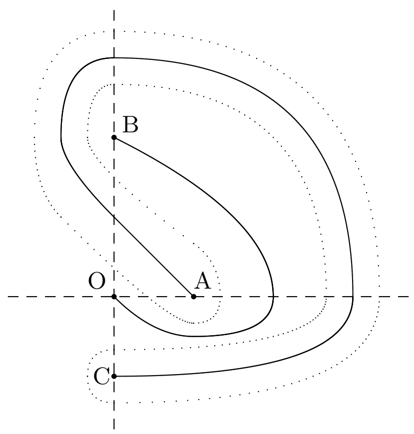
Example. $a = 0$, $b = c$. In this case $B = -C$ and fibers are of the form (Picture 9):



Picture 9

Here the dotted curve is a typical fiber inside C_0.

Example. $c = a$, $b = 2a$. In this case fibers are of the form (Picture 10):



Picture 10

Here the dotted curve is a typical fiber inside C_0.

If a and c are not commensurable, then a fiber inside C_0 is dense in the interior of C_0 and demonstrates a pseudo random behavior.

E-mail address: yuyuk@prov.ru