Concepts of Stability in Discrete Optimization Involving Generalized Addition Operations

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The paper addresses the tolerance approach to the sensitivity analysis of optimal solutions to the nonlinear optimization problem of the form

$$\bigoplus_{y\in S} C(y)\to \min \quad \text{over} \quad S\in\mathcal{S},$$

where $\mathcal{S}$ is a collection of nonempty subsets of a finite set X such that $\cup\mathcal{S}=X$ and $\cap\mathcal{S}=\varnothing$, C is a cost (or weight) function from X into $\mathbb{R}^+=[0,\infty)$ or $(0,\infty)$, and $\oplus$ is a continuous, associative, commutative, nondecreasing and unbounded binary operation of generalized addition on $\mathbb{R}^+$, called an A-operation. We evaluate and present sharp estimates for upper and lower bounds of costs of elements from X, for which an optimal solution to the above problem remains stable. These bounds present new results in the sensitivity analysis as well as extend most known results in a unified way. We define an invariant of the optimization problem—the tolerance function, which is independent of optimal solutions, and establish its basic properties, among which we mention a characterization of the set of all optimal solutions, the uniqueness of optimal solutions and extremal values of the tolerance function on an optimal solution.

Key words: optimization problem; A-operation; nonlinear objective function; optimal solution; stability interval; tolerance function; uniqueness
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1. Introduction. The purpose of this paper is to introduce and study certain concepts of stability of optimal solutions to the following nonlinear problem of discrete optimization:

$$f_C(S)\equiv f(C)(S)=\bigoplus_{y\in S} C(y)\to \min, \qquad S\in\mathcal{S}, \tag{1.1}$$

where $\mathcal{S}$ is a collection of nonempty subsets (called trajectories) of a finite set X of cardinality $|X|\ge 2$ such that $\cup\mathcal{S}=X$ and $\cap\mathcal{S}=\varnothing$ and $C:X\to\mathbb{R}^+$ is a given cost (or weight) function of elements from X with $\mathbb{R}^+=[0,\infty)$ or $(0,\infty)$. The objective function $f_C:\mathcal{S}\to\mathbb{R}^+$ in (1.1) is given by means of an operation $\oplus$ on the set $\mathbb{R}^+$, called an A-operation, which generalizes simultaneously the addition operation on $\mathbb{R}^+=[0,\infty)$ and the operation of multiplication on $\mathbb{R}^+=(0,\infty)$. More specifically, we assume that the operation $(u,v)\mapsto u\oplus v$ from $\mathbb{R}^+\times\mathbb{R}^+$ into $\mathbb{R}^+$ is associative, commutative, nondecreasing in each variable, unbounded in the sense that $u\oplus v\to\infty$ as $u\to\infty$ for all $v\in\mathbb{R}^+$, and continuous as a function of two real variables. Simple examples of A-operations are the usual addition operation $u\oplus v=u+v$, the operation of taking the maximum $u\oplus v=\max\{u,v\}$ and the usual operation of multiplication $u\oplus v=u\cdot v$.

In this paper we adopt the tolerance approach to the sensitivity analysis of optimal solutions to problem (1.1): given an optimal solution $S^*\in\mathcal{S}$ to problem (1.1) and an element $x\in X$, we are interested to what extent the cost $C(x)$ can be changed (the other costs remaining unperturbed) so that S^* is still an optimal solution. In other words, this can be expressed as how tolerant is the optimal solution S^* with respect to a change of the single cost $C(x)$. The tolerance approach has been applied in the literature for a variety of combinatorial optimization problems: shortest path and network flow problems [12, 24, 26], the traveling salesman problem [13, 17, 25, 27], the matrix coefficients in linear programming problems [23], the bottleneck problems [1, 6, 7, 9, 10, 22, 25]. We refer to [6, 11, 28] for a comprehensive literature on the sensitivity (or post-solution) analysis.

Up to now almost all works in the sensitivity analysis, including the ones referred to above, are concerned with two (most popular and natural) operations $\oplus$ in (1.1), namely, the addition $+$ or max, and so, the corresponding objective function f_C is either linear $f_C(S)=\sum_{y\in S} C(y)$ or bottleneck $f_C(S)=\max_{y\in S} C(y)$, respectively. One may intuitively feel the difference between the two operations: all costs contribute to the linear objective function equally well, while only the largest costs contribute to the bottleneck objective function. A more relevant explanation (in accordance with the theory of A-operations developed in Section 2) is that, given $v\geq 0$, the function $\Phi(u)=u+v$ is strictly increasing (and so, the operation $+$ is termed to be strict), while the function $\Psi(u)=\max\{u,v\}$ is only nondecreasing on $[0,\infty)$ (and so, the operation max is not strict). In calculating the upper and lower bounds for costs of elements from X, for which an optimal solution to problem (1.1) remains stable, one encounters the (generally, tacit) necessity to take the inverse(s) of the operation under consideration (in our case $+$ or max, cf. references above), which results in inverting functions Φ and Ψ as above. Since Φ is strictly increasing (and continuous), its inverse, the subtraction, behaves well and causes no problem. However, the usual inverse of the nondecreasing function Ψ does not exist, and so, one has to deal with its right and/or left inverse [15], which is why optimization problems with bottleneck objective functions are more complicated. Technically speaking, problems (1.1) with nonstrict A-operations $\oplus$ tend to exhibit certain pathological, hysteretic-like properties involving retardness (cf. [14]).

Given a generic A-operation $\oplus$ on $\mathbb{R}^+$, we introduce two inverses of $\oplus$, called the upper subtraction $\overline{\ominus}$ and the lower subtraction $\underline{\ominus}$. It turns out that the two subtractions replace the ordinary subtraction (in the case of $+$), and they coincide on the common part of their domains iff (= if and only if) the A-operation $\oplus$ is strict. By means of the upper (lower) subtraction we determine the upper (lower, respectively) stability interval of costs for an optimal solution to problem (1.1). In the case of strict A-operations $\oplus$ the induced upper and lower subtractions are translation invariant with respect to $\oplus$, which permits us to introduce upper tolerances for elements in an optimal solution to (1.1) and lower tolerances for elements outside the optimal solution and establish precise formulas for their evaluation. They are exact counterparts for all strict A-operations of the corresponding formulae from [17] established in the simplest case of $+$. The tolerance function for problem (1.1) is defined on X via any optimal solution S^* to (1.1) as follows: its value at $x\in S^*$ is equal to the upper tolerance of x and its value at $y\in X\setminus S^*$ is equal to the lower tolerance of y. We prove that the tolerance function is independent of a particular optimal solution to (1.1), and so, it is an invariant of the problem (1.1) itself. Also, we show that the tolerance function is useful in characterizing the whole set of optimal solutions and, in particular, their uniqueness: the optimal solution to (1.1) is unique iff the tolerance function never vanishes.

It is to be noted that the case of nonstrict A-operations $\oplus$ is more intricate and is yet to be studied in detail. Following the tradition, we elaborate only on the case of nonstrict A-operation of addition $\oplus=\max$. Also, we do not touch upon the computational complexity of the sensitivity analysis in our approach, which is a separate problem in itself (concerning the latter see [1, 8, 16, 18, 21, 22]).

The paper is organized as follows. In Section 2 we develop a theory of A-operations and their two subtractions, upper and lower, to be applied throughout the paper. In Section 3 we formulate a general optimization problem for objective functions generated by A-operations. In Section 4 we determine upper stability intervals and in Section 5—lower stability intervals of costs, for which an optimal solution preserves its optimality. In Section 6 for strict A-operations we define an invariant of the optimization problem—the tolerance function, and establish its basic properties.

2. A-operations on $\mathbb{R}^+$ and their inverses. Throughout the paper $\mathbb{R}^+$ denotes the set $[0,\infty)$ of all nonnegative real numbers or the set $(0,\infty)$ of all positive real numbers.

The aim of this section is to introduce an operation of generalized addition or generalized multiplication on $\mathbb{R}^+$, called an A-operation, and to gather its properties for future reference.

2.1. Definition. A continuous function $A:[0,\infty)\times[0,\infty)\to[0,\infty)$ is said to be an A-*operation* on $\mathbb{R}^+$ if A maps $\mathbb{R}^+\times\mathbb{R}^+$ into $\mathbb{R}^+$ and, given $u,v,w\in[0,\infty)$, the following four conditions (axioms) are satisfied:

(A.1) $A(u,A(v,w))=A(A(u,v),w)$ (associativity of A);
(A.2) $A(u,v)=A(v,u)$ (commutativity of A);
(A.3) $u<v$ implies $A(u,w)\le A(v,w)$ (monotonicity of A);
(A.4) $A(u,v)\to\infty$ as $u\to\infty$ for all $v\in\mathbb{R}^+$ (unboundedness of A).

If, instead of (A.3), the function A satisfies

(A.3$_\mathrm{s}$) given $u,v\in[0,\infty)$ and $w\in\mathbb{R}^+$, $u<v$ implies $A(u,w)<A(v,w)$,

then the A-operation A is said to be *strict* on $\mathbb{R}^+$.

We denote by $\mathcal{A}(\mathbb{R}^+)$ the set of all A-operations on $\mathbb{R}^+$ and by $\mathcal{A}_\mathrm{s}(\mathbb{R}^+)$—the set of those operations $A\in\mathcal{A}(\mathbb{R}^+)$, which are strict on $\mathbb{R}^+$.

2.2. In extending an $A\in\mathcal{A}(\mathbb{R}^+)$ to any finite number of terms it is convenient to set $u\oplus v\equiv u\oplus_A v=A(u,v)$ for all $u,v\in[0,\infty)$ and, given $u_1,\dots,u_n\in[0,\infty)$ with an $n\in\mathbb{N}$, we put

$$u\oplus\varnothing=u,\qquad \oplus_{i=1}^{1}u_i=u_1,\qquad \oplus_{i=1}^{2}u_i=u_1\oplus u_2, \tag{2.1}$$

and, inductively,

$$\oplus_{i=1}^{n}u_i=A\big(\oplus_{i=1}^{n-1}u_i,u_n\big)=\big(\oplus_{i=1}^{n-1}u_i\big)\oplus u_n\quad\text{if}\quad n\ge 3. \tag{2.2}$$

We will write $A(u,v)$ or $u\oplus v$ indifferently as well as $A=\oplus\in\mathcal{A}(\mathbb{R}^+)$. Now, conditions (A.1)–(A.4) can be rewritten more commonly as

$$u\oplus(v\oplus w)=(u\oplus v)\oplus w,\quad u\oplus v=v\oplus u,\quad u<v\implies u\oplus w\le v\oplus w,$$
$$\text{and } u\oplus v\to\infty \text{ as } u\to\infty \text{ for all } v\in\mathbb{R}^+\text{, respectively.}$$

2.3. Generalized addition and multiplication. Of particular importance on $\mathbb{R}^+=[0,\infty)$ are A-operations $\oplus$, generalizing the usual addition, which, along with axioms (A.1)–(A.3), satisfy the condition

(A.5) $0\oplus v=v$ for all $v\in[0,\infty)$.

Note that the zero 0 plays the role of the *neutral* element with respect to $\oplus$, axiom (A.4) is redundant in this case (in fact, (A.3), (A.2) and (A.5) imply $u\oplus v\ge u\oplus 0=u$) and $u\oplus v\ge\max\{u,v\}$ for all $u,v\in[0,\infty)$. A-operations on $\mathbb{R}^+$, satisfying (A.5), are called F-operations in [20] (see also [2], [3], [19, Section 3]).

The case $\mathbb{R}^+=(0,\infty)$ is more appropriate for A-operations $\oplus$, generalizing the usual multiplication, i.e., satisfying (A.1)–(A.4) and

(A.6) $0\oplus v=0$ for all $v\in(0,\infty)$.

Also, we assume the existence of the *neutral* element (the unit with respect to $\oplus$) $\boldsymbol{e}\in\mathbb{R}^+$ such that $\boldsymbol{e}\oplus v=v$ for all $v\in[0,\infty)$.

2.4. Convention. In what follows A-operations $A=\oplus$ on $\mathbb{R}^+=[0,\infty)$ or $(0,\infty)$ will be treated in a unified way, however, on $[0,\infty)$ they are assumed to satisfy axioms (A.1)–(A.3) and (A.5) and are called A-*operations of addition* and on $(0,\infty)$—axioms (A.1)–(A.4) and (A.6) and are called A-*operations of multiplication*.

2.5. Examples of A-operations. Let $u, v \in [0, \infty)$ and $p \in (0, \infty)$. The following are examples of A-operations of addition on $\mathbb{R}^+ = [0, \infty)$ (cf. [3], [19, Chapter 3]):

(a$_1$) $A_1(u, v) = u + v$ (the usual addition operation);

(a$_2$) $A_2(u, v) = (u^p + v^p)^{1/p}$;

(a$_3$) $A_3(u, v) = \max\{u, v\} = \frac{1}{2}(u + v + |u - v|)$;

(a$_4$) $A_4(u, v) = \frac{1}{p}\log(e^{pu} + e^{pv} - 1)$;

(a$_5$) $A_5(u, v) = u + v + puv$;

(a$_6$) $A_6(u, v) = G(\max\{u, v\}, u + v)$, where $G(a, b)$ is defined for $0 \le a \le b$ by: $G(a, b) = b$ if $b < 1$, $G(a, b) = 1$ if $a < 1$ and $b \ge 1$, and $G(a, b) = a$ if $a \ge 1$.

Examples of A-operations of multiplication on $\mathbb{R}^+ = (0, \infty)$ are as follows:

(a$_7$) $A_7(u, v) = uv$ (the usual operation of multiplication) with the unit $\boldsymbol{e} = 1$;

(a$_8$) $A_8(u, v) = puv$, the neutral element being $\boldsymbol{e} = 1/p$;

(a$_9$) $A_9(u, v) = \frac{1}{p}\log\bigl(1 + (e^{pu} - 1)(e^{pv} - 1)\bigr)$ with the unit $\boldsymbol{e} = (\log 2)/p$;

(a$_{10}$) $A_{10}(u, v) = \frac{1}{p}\bigl(\exp[\log(1 + pu) \cdot \log(1 + pv)] - 1\bigr)$, the unit being $\boldsymbol{e} = (e - 1)/p$.

That all these ten operations are indeed A-operations on $\mathbb{R}^+$ (satisfying convention 2.4) will be more clear from Section 2.7.

Operations A_1, A_2, A_4 and A_5 are strict on $[0, \infty)$, operations A_7, A_8, A_9 and A_{10} are strict on $(0, \infty)$, while operations $A_3 = \max$ and A_6 are *not* strict on $[0, \infty)$.

For several elements $u_1, \ldots, u_n \in [0, \infty)$ operations A_1 through A_{10} extend in the way exposed in (2.2) and Sections 2.7 and 3.7 (cf. expression for $\widehat{\bigoplus}_{i=1}^{n} u_i$); see also Examples 3.4.

2.6. Two properties. Two simple properties of A-operations on $\mathbb{R}^+$ are straightforward: if $\oplus \in \mathcal{A}(\mathbb{R}^+)$ and $u, u_1, v, v_1 \in [0, \infty)$, then

$$u \le u_1 \text{ and } v \le v_1 \text{ imply } u \oplus v \le u_1 \oplus v_1,$$

and if, in addition, $\oplus$ is strict on $\mathbb{R}^+$, then, given $w \in \mathbb{R}^+$,

$$u \oplus w \le v \oplus w \text{ implies } u \le v \text{ (cancellation law)}, \tag{2.3}$$

which, in particular, gives: if $u \oplus w = v \oplus w$, then $u = v$.

2.7. Generating A-operations. Following [3] or [19, Section 3], here we introduce an equivalence relation on the set $\mathcal{A}(\mathbb{R}^+)$.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function vanishing at zero (only) and such that $\varphi(u) \to \infty$ as $u \to \infty$ (such functions are said to be *φ-functions*, cf. [3], [19]). Given $A \in \mathcal{A}(\mathbb{R}^+)$ and $u, v \in \mathbb{R}^+$, we set

$$(E_\varphi(A))(u, v) = \varphi^{-1}\bigl(A(\varphi(u), \varphi(v))\bigr),$$

where $\varphi^{-1} : [0, \infty) \to [0, \infty)$ is the inverse function of φ. Clearly, $E_\varphi(A) \in \mathcal{A}(\mathbb{R}^+)$, and so, E_φ maps $\mathcal{A}(\mathbb{R}^+)$ into itself.

If id_X denotes the identity map of a set X (i.e., $\mathrm{id}_X(x) = x$ for all $x \in X$) and $\varphi_0 = \mathrm{id}_{[0,\infty)}$, then $E_{\varphi_0} = \mathrm{id}_{\mathcal{A}(\mathbb{R}^+)}$. Also, given two φ-functions φ and ψ, we find $E_{\varphi\circ\psi} = E_\varphi \circ E_\psi$, where $\circ$ designates the usual composition of maps. It follows that the relation $\sim$ on $\mathcal{A}(\mathbb{R}^+)$ defined for $A, B \in \mathcal{A}(\mathbb{R}^+)$ by

$$B \sim A \text{ iff there exists a } \varphi\text{-function } \varphi \text{ such that } B = E_\varphi(A),$$

is an equivalence relation on $\mathcal{A}(\mathbb{R}^+)$. The equivalence class $[A]$ of an A-operation A under $\sim$ is given by $[A] = \{E_\varphi(A) : \varphi \text{ is a } \varphi\text{-function}\}$. It is to be noted that if $A \in \mathcal{A}_{\mathrm{s}}(\mathbb{R}^+)$, then $[A] \subset \mathcal{A}_{\mathrm{s}}(\mathbb{R}^+)$.

Now we turn back to Examples 2.5. We have:

$$\begin{aligned}
&A_2 = E_\varphi(A_1) \text{ and } A_7 = E_\varphi(A_7) \text{ with } \varphi(u) = u^p,\\
&A_4 = E_\psi(A_1) \text{ and } A_9 = E_\psi(A_7) \text{ with } \psi(u) = e^{pu} - 1,\\
&A_5 = E_\chi(A_1) \text{ and } A_{10} = E_\chi(A_7) \text{ with } \chi(u) = \log(1+pu),
\end{aligned}$$

and $A_8 = E_\xi(A_7)$ with $\xi(u) = pu$, where $p > 0$ and $u \in [0,\infty)$. Thus, $A_1 \sim A_2 \sim A_4 \sim A_5$ and $A_7 \sim A_8 \sim A_9 \sim A_{10}$, while A_1, A_3, A_6 and A_7 are not mutually equivalent under $\sim$. Note also that the equivalence class of $A_3 = \max$ is $[A_3] = \{A_3\}$.

If $\boldsymbol{e} \in \mathbb{R}^+$ is the neutral element with respect to the A-operation $\oplus$ and $\widehat{\oplus} = E_\varphi(\oplus)$ for a φ-function φ, then $\widehat{\boldsymbol{e}} = \varphi^{-1}(\boldsymbol{e})$ is the neutral element with respect to the A-operation $\widehat{\oplus}$: in fact, given $u \geq 0$, we find

$$\widehat{\boldsymbol{e}}\,\widehat{\oplus}\,u = \varphi^{-1}\big(\varphi(\widehat{\boldsymbol{e}}) \oplus \varphi(u)\big) = \varphi^{-1}\big(\boldsymbol{e} \oplus \varphi(u)\big) = \varphi^{-1}(\varphi(u)) = u.$$

2.8. Upper and lower subtractions. Here we treat two inverses of an A-operation $\oplus$, also called generalized subtractions (or divisions). Having an equation of the form $u \oplus v = w$ (or inequality $u \oplus v \leq w$) with $u, v, w \in \mathbb{R}^+$, we are going to write $u = w \ominus v$ (or $u \leq w \ominus v$), so that we would get $(w \ominus v) \oplus v = w$ (or $(w \ominus v) \oplus v \leq w$, respectively). We will achieve this by introducing two "operations" of upper and lower subtractions on $\mathbb{R}^+$, $\overline{\ominus}$ and $\underline{\ominus}$, as follows.

Suppose the A-operation $A = \oplus \in \mathcal{A}(\mathbb{R}^+)$ is given.

We define the domain $D(\overline{\ominus})$ of the *upper subtraction* $\overline{\ominus}$ for $\oplus$ by

$$D(\overline{\ominus}) = \big\{(w,v) \in \mathbb{R}^+ \times \mathbb{R}^+ : \exists\, u_0 \in \mathbb{R}^+ \text{ such that } u_0 \oplus v \leq w\big\},$$

and, given $(w,v) \in D(\overline{\ominus})$, the value $w \overline{\ominus} v \in \mathbb{R}^+$ is defined by

$$w \overline{\ominus} v = \max\{u \in \mathbb{R}^+ : u \oplus v \leq w\} = \max\{u \in \mathbb{R}^+ : u \oplus v = w\}. \tag{2.4}$$

The *lower subtraction* $\underline{\ominus}$ for $\oplus$ is defined for all $(w,v) \in \mathbb{R}^+ \times \mathbb{R}^+$ by

$$w \underline{\ominus} v = \min\{u \in \mathbb{R}^+ : u \oplus v \geq w\} \in \mathbb{R}^+. \tag{2.5}$$

Definitions (2.4) and (2.5) are similar to taking the right or left inverse of a not necessarily strictly increasing function on $[0,\infty)$ (cf. [15]), depending on a parameter.

Our primary aim now is to verify that these definitions are correct.

2.9. Lemma. *The subtractions $\overline{\ominus}$ and $\underline{\ominus}$ are well defined.*

Proof. 1. Let us show that definition (2.4) is correct. First, we note that the domain $D(\overline{\ominus})$ is nonempty. In fact, given $u, v \in \mathbb{R}^+$, setting $u_0 = u$ and $w = u \oplus v$, we find $u_0 \oplus v = w$, and so, $(u \oplus v, v) = (w, v) \in D(\overline{\ominus})$.

Now, let $(w,v) \in D(\overline{\ominus})$, and so, there exists $u_0 \in \mathbb{R}^+$ such that $u_0 \oplus v \leq w$. Defining the set $\overline{U} = \{u \in \mathbb{R}^+ : u \oplus v \leq w\}$, we find $\overline{U} \neq \varnothing$ (since $u_0 \in \overline{U}$), $\overline{U}$ is bounded from below (because $\overline{U} \subset \mathbb{R}^+$) and bounded from above (by axiom (A.4)) and $\overline{U}$ is closed in $\mathbb{R}$ (by the continuity of the function $[u \mapsto u \oplus v] : \mathbb{R}^+ \to \mathbb{R}^+$), and so, $\overline{U}$ is compact in $\mathbb{R}$. We set $\overline{u} = \sup \overline{U} = \max \overline{U} \in \mathbb{R}$ and assert that $\overline{u} \in \mathbb{R}^+$ and $\overline{u} \oplus v = w$. In fact, given $u \in \overline{U}$, we have $u \leq \overline{u}$ and, since $u_0 \in \overline{U}$, inequality $u_0 \leq \overline{u}$ implies $\overline{u} \in \mathbb{R}^+$. Since $\overline{u} = \max \overline{U}$, we have $\overline{u} \oplus v \leq w$. If we assume that $\overline{u} \oplus v < w$, then we note that, by (A.4), there exists $\widehat{u} \in \mathbb{R}^+$ such that $w < \widehat{u} \oplus v$ and, moreover, by (A.3), $\overline{u} < \widehat{u}$; now, by the intermediate value theorem, there exists $\widetilde{u} \in \mathbb{R}^+$ with $\overline{u} < \widetilde{u} < \widehat{u}$ such that $\widetilde{u} \oplus v = w$, and so, $\widetilde{u} \in \overline{U}$ and $\max \overline{U} = \overline{u} < \widetilde{u}$, which is a contradiction. It follows that $\overline{u} \oplus v = w$ and, hence, $w \overline{\ominus} v = \overline{u}$.

2. Let us show that the quantity (2.5) is well defined. Given $(w,v) \in \mathbb{R}^+ \times \mathbb{R}^+$, we define the set $\underline{U} = \{u \in \mathbb{R}^+ : u \oplus v \geq w\}$. Then $\underline{U} \neq \varnothing$ (by virtue of (A.4)) and $\underline{U}$ is bounded from below (since

$\underline{U} \subset \mathbb{R}^+$), and so, $\underline{u} = \inf \underline{U} \in [0,\infty) = \{0\} \cup \mathbb{R}^+$. If $\mathbb{R}^+ = [0,\infty)$, then we note that, by the continuity of the function $u \mapsto u \oplus v$, the set $\underline{U}$ is closed in $[0,\infty)$, and so, $\underline{u} = \min \underline{U}$ and $\underline{u} \in [0,\infty) = \mathbb{R}^+$. In the case when $\mathbb{R}^+ = (0,\infty)$, we have

$$u \oplus v \geq w \quad \text{for all} \quad u > \underline{u}, \tag{2.6}$$

and, by virtue of (A.6), $u \oplus v \to 0 \oplus v = 0$ as $u \to 0$, and so, there exists $u_1 > 0$, depending on $w > 0$, such that $u_1 \oplus v < w$. It follows from (2.6) that $\underline{u} \geq u_1$, i.e., $\underline{u} > 0$. Passing to the limit as $u \to \underline{u}$ in (2.6), by the continuity of $u \mapsto u \oplus v$, we get $\underline{u} \oplus v \geq w$, and so, $\underline{u} \in \underline{U}$ and $\underline{u} = \min \underline{U} \in (0,\infty) = \mathbb{R}^+$. Thus, $w \underline{\ominus} v = \underline{u}$. □

2.10. Remark. We have shown in step 2 in the above proof that if $\mathbb{R}^+ = (0,\infty)$, then (A.6) implies $\inf \underline{U} = \min \underline{U}$ for all $w > 0$. It is to be noted that the converse implication holds as well. In fact, given $\varepsilon > 0$, we set $u_\varepsilon = \inf \underline{U} = \min \underline{U}$, where $\underline{U} = \{u \in (0,\infty) : u \oplus v \geq \varepsilon\}$. We assert that $u \oplus v < \varepsilon$ for all $0 < u < u_\varepsilon$ (and so, $0 \oplus v = \lim_{u \to 0} u \oplus v = 0$ implying (A.6)): indeed, if we assume that $u \oplus v \geq \varepsilon$, then $u \in \underline{U}$, and so, $u \geq \min \underline{U} = u_\varepsilon$, which contradicts the inequality $u < u_\varepsilon$.

Note that the value $w \overline{\ominus} v$, as opposed to $w \underline{\ominus} v$, may *not* be defined for all $w, v \in \mathbb{R}^+$. A comparison of the two subtractions $\overline{\ominus}$ and $\underline{\ominus}$ for a $\oplus \in \mathcal{A}(\mathbb{R}^+)$ is given in the following

2.11. Lemma. (a) *If* $(w,v) \in D(\overline{\ominus})$, *then* $w \underline{\ominus} v \leq w \overline{\ominus} v$.

(b) *If* $\oplus$ *is an* A-*operation of addition on* $\mathbb{R}^+ = [0,\infty)$, *then* $(w,v) \in D(\overline{\ominus})$ *iff* $0 \leq v \leq w$, *and inequalities* $v \geq w \geq 0$ *imply* $w \underline{\ominus} v = 0$.

(c) *If* $\oplus$ *is an* A-*operation of multiplication on* $\mathbb{R}^+ = (0,\infty)$, *then* $(w,v) \in D(\overline{\ominus})$ *for all* $w, v > 0$.

Proof. (a) is a straightforward consequence of (2.4) and (2.5).

(b) Given $(w,v) \in D(\overline{\ominus})$, there exists $u_0 \geq 0$ such that $u_0 \oplus v \leq w$, and so, by virtue of (A.5) and (A.3), $v = 0 \oplus v \leq u_0 \oplus v \leq w$. On the other hand, if $v \leq w$, then, by (A.5), $0 \oplus v = v \leq w$, and so, $(w,v) \in D(\overline{\ominus})$.

Suppose $v \geq w$. By (A.3) and (A.5), $u \oplus v \geq u \oplus w \geq 0 \oplus w = w$ for all $u \geq 0$, and so, $\underline{U} = \{u \geq 0 : u \oplus v \geq w\} = [0,\infty)$ and $w \underline{\ominus} v = \min \underline{U} = 0$.

(c) Given $w, v > 0$, by virtue of (A.6), $u \oplus v \to 0$ as $u \to 0$, and so, there exists $u_0 > 0$ such that $u_0 \oplus v < w$ implying $(w,v) \in D(\overline{\ominus})$. □

2.12. Examples of upper and lower subtractions. Most examples in this paper will be demonstrated for the three basic (representatives of equivalence classes of) A-operations A_2, A_3 and A_7 from Section 2.5, i.e., $u \oplus v = (u^p + v^p)^{1/p}$ with $p > 0$, $u \oplus v = \max\{u,v\}$ and $u \oplus v = uv$, where $u, v \geq 0$, also abbreviated as *p-sum, max* and *product* operations, respectively.

Let $\oplus \in \mathcal{A}(\mathbb{R}^+)$ and $\overline{\ominus}$ and $\underline{\ominus}$ be the upper and lower subtractions for $\oplus$.

(a) Suppose $u \oplus v = (u^p + v^p)^{1/p}$ on $\mathbb{R}^+ = [0,\infty)$ with $p > 0$. We have:

$$(w,v) \in D(\overline{\ominus}) \text{ iff } 0 \leq v \leq w, \text{ and } w \overline{\ominus} v = (w^p - v^p)^{1/p};$$

$$\text{given } w, v \geq 0, \quad w \underline{\ominus} v = \begin{cases} (w^p - v^p)^{1/p} & \text{if } v < w, \\ 0 & \text{if } v \geq w. \end{cases}$$

Note that $w \overline{\ominus} v = w \underline{\ominus} v$ for all $0 \leq v \leq w$.

(b) Let $u \oplus v = \max\{u,v\}$ on $\mathbb{R}^+ = [0,\infty)$. Then

$$(w,v) \in D(\overline{\ominus}) \text{ iff } 0 \leq v \leq w, \text{ and } w \overline{\ominus} v = w;$$

$$\text{given } w, v \geq 0, \quad w \underline{\ominus} v = \begin{cases} w & \text{if } v < w, \\ 0 & \text{if } v \geq w. \end{cases}$$

In fact, if $v \le w$, then $\overline{U} = \{u \ge 0 : \max\{u,v\} \le w\} = [0,w]$, and so, by (2.4), $w \overline{\ominus} v = \max \overline{U} = w$. By virtue of Lemma 2.11(b), $v \ge w$ implies $w \underline{\ominus} v = 0$, and if $v < w$, then $u \in \underline{U}$ iff $\max\{u,v\} \ge w$ iff $u \ge w$, and so, $\underline{U} = [w,\infty)$ and $w \underline{\ominus} v = \min \underline{U} = w$.

Note that $w \overline{\ominus} v = w = w \underline{\ominus} v$ for all $0 \le v < w$ and $0 \overline{\ominus} 0 = 0 = 0 \underline{\ominus} 0$, while if $w > 0$, then we have $w \underline{\ominus} w = 0 < w = w \overline{\ominus} w$.

(c) Suppose $u \oplus v = uv$ on $\mathbb{R}^+ = (0,\infty)$. It follows from Lemma 2.11(c) that $D(\overline{\ominus}) = (0,\infty) \times (0,\infty)$ and $w \overline{\ominus} v = w \underline{\ominus} v = w/v$ is the usual operation of division for all $w, v > 0$.

Now we establish basic (in)equalities related to the upper and lower subtractions $\overline{\ominus}$ and $\underline{\ominus}$ for an A-operation $\oplus$.

2.13. Lemma. *Given $\oplus \in \mathcal{A}(\mathbb{R}^+)$, we have*:

$$(w \overline{\ominus} v) \oplus v = w \qquad \text{for all } (w,v) \in D(\overline{\ominus}); \tag{2.7}$$

$$w \le (w \underline{\ominus} v) \oplus v \qquad \text{for all } w, v \in \mathbb{R}^+; \tag{2.8}$$

$$(w \oplus v) \underline{\ominus} v \le w \le (w \oplus v) \overline{\ominus} v \qquad \text{for all } w, v \in \mathbb{R}^+. \tag{2.9}$$

Also, the following criterion holds:

$$\oplus \text{ is strict on } \mathbb{R}^+ \text{ iff } w \overline{\ominus} v = w \underline{\ominus} v \text{ for all } (w,v) \in D(\overline{\ominus}). \tag{2.10}$$

Proof. 1. Equality (2.7) is the characterizing property of the quantity $w \overline{\ominus} v$, which follows immediately from (2.4).

2. Inequality (2.8) is the characterizing property of $w \underline{\ominus} v$, which is a consequence of (2.5) (cf. also Remark 2.14(a)).

3. The left-hand side inequality in (2.9) follows from (2.5):

$$(w \oplus v) \underline{\ominus} v = \min\{u \in \mathbb{R}^+ : u \oplus v \ge w \oplus v\} \le w.$$

Setting $u_0 = w$, we find $u_0 \oplus v = w \oplus v$, and so, $(w \oplus v, v) \in D(\overline{\ominus})$, and the right-hand side inequality in (2.9) follows from (2.4):

$$(w \oplus v) \overline{\ominus} v = \max\{u \in \mathbb{R}^+ : u \oplus v = w \oplus v\} \ge w.$$

4. Let us prove (2.10). Let $\oplus$ be strict. Given $(w,v) \in D(\overline{\ominus})$, by virtue of Lemma 2.11(a), we have to show only that $w \underline{\ominus} v \ge w \overline{\ominus} v$. In fact, we assert that $u \ge w \overline{\ominus} v$ for all $u \in \underline{U} = \{u \in \mathbb{R}^+ : u \oplus v \ge w\}$, for, otherwise, if $u < w \overline{\ominus} v$, then (A.3$_\mathrm{s}$) and (2.7) imply $u \oplus v < (w \overline{\ominus} v) \oplus v = w$, which is a contradiction. Thus, $w \underline{\ominus} v = \min \underline{U} \ge w \overline{\ominus} v$.

Suppose the equality on the right in (2.10) holds, and let $u > v \ge 0$ and $w \in \mathbb{R}^+$. It follows from (A.3) that $u \oplus w \ge v \oplus w$. If we assume that $u \oplus w = v \oplus w$, then, by (2.4) and the left-hand side inequality in (2.9), we get

$$u \le (v \oplus w) \overline{\ominus} w = (v \oplus w) \underline{\ominus} w \le v,$$

which contradicts to $u > v$. Thus, $u \oplus w > v \oplus w$, and (A.3$_\mathrm{s}$) follows. □

2.14. Remarks. (a) Strict inequality may hold in (2.8) even for strict A-operations: if $\oplus$ is as in Lemma 2.11(b) and $v > w \ge 0$, then

$$w < v = 0 \oplus v = (w \underline{\ominus} v) \oplus v.$$

Also, one cannot replace the inequality $\ge w$ in (2.5) by the equality $= w$: in fact, if $\oplus = \max$, then taking into account Example 2.12(b), we have, for $v > w \ge 0$,

$$w \underline{\ominus} v = \min\{u \ge 0 : \max\{u,v\} \ge w\} = 0,$$

whereas $\{u \geq 0 : \max\{u,v\} = w\} = \varnothing$.

(b) If $\oplus$ is strict on $\mathbb{R}^+$, then, by (2.10), equalities hold in (2.9). However, if $\oplus$ is not strict, then inequalities in (2.9) may be strict. By virtue of Example 2.12(b), for $\oplus = \max$ we have: if $v = w > 0$, then

$$(w \oplus v) \underline{\ominus} v = (w \oplus w) \underline{\ominus} w = \max\{w,w\} \underline{\ominus} w = w \underline{\ominus} w = 0 < w,$$

and if $v > w \geq 0$, then

$$(w \oplus v) \overline{\ominus} v = \max\{w,v\} \overline{\ominus} v = v \overline{\ominus} v = v > w.$$

In this respect we note that, by Example 2.12(b), $w \overline{\ominus} v = w$ for all $0 \leq v \leq w$, and so, equality (2.7) is of the form

$$(w \overline{\ominus} v) \oplus v = w \oplus v = \max\{w,v\} = w.$$

The following lemma shows that the functions $w \mapsto w \overline{\ominus} v$ and $w \mapsto w \underline{\ominus} v$ are nondecreasing (in the first variable).

2.15. Lemma. *Let $\oplus \in \mathcal{A}(\mathbb{R}^+)$, $w_1, w_2, v \in \mathbb{R}^+$ and $w_1 \leq w_2$. We have:*
(a) *if $(w_1, v) \in D(\overline{\ominus})$, then $(w_2, v) \in D(\overline{\ominus})$ and $w_1 \overline{\ominus} v \leq w_2 \overline{\ominus} v$;*
(b) *$w_1 \underline{\ominus} v \leq w_2 \underline{\ominus} v$.*

Proof. (a) Condition $(w_1, v) \in D(\overline{\ominus})$ implies the existence of $u_0 \in \mathbb{R}^+$ such that $u_0 \oplus v \leq w_1$, which gives $u_0 \oplus v \leq w_2$, and so, $(w_2, v) \in D(\overline{\ominus})$. Setting $u_1 = w_1 \overline{\ominus} v$, by virtue of (2.7), we find

$$u_1 \oplus v = (w_1 \overline{\ominus} v) \oplus v = w_1 \leq w_2,$$

and so, (2.4) yields $u_1 \leq w_2 \overline{\ominus} v$.

(b) If $u_2 = w_2 \underline{\ominus} v$, then it follows from (2.8) that

$$w_1 \leq w_2 \leq (w_2 \underline{\ominus} v) \oplus v = u_2 \oplus v,$$

whence (2.5) implies $w_1 \underline{\ominus} v \leq u_2$. □

Several more inequalities will be needed in the sequel. If u, v and w are real numbers, then for the usual operations of addition $+$ and subtraction $-$ we have: $u \leq v$ implies $w - v \leq w - u$, and $w - (v - u) = (w + u) - v$. In the next two lemmas we study to what extent these two properties carry over to general A-operations $\oplus \in \mathcal{A}(\mathbb{R}^+)$ and their upper and lower subtractions $\overline{\ominus}$ and $\underline{\ominus}$.

2.16. Lemma. *Suppose $u, v, w \in \mathbb{R}^+$ and $u \leq v$. Then we have:*
(a) *if $(w, v) \in D(\overline{\ominus})$, then $(w, u) \in D(\overline{\ominus})$ and $w \overline{\ominus} v \leq w \overline{\ominus} u$;*
(b) *$w \underline{\ominus} v \leq w \underline{\ominus} u$.*

Proof. (a) By virtue of (A.3) and (2.7), we get

$$(w \overline{\ominus} v) \oplus u \leq (w \overline{\ominus} v) \oplus v = w,$$

and so, $(w, u) \in D(\overline{\ominus})$, and the definition (2.4) of $w \overline{\ominus} u$ (i.e., the maximality of $w \overline{\ominus} u$) implies the desired inequality in (a).

(b) It follows form (A.3) and (2.8) that

$$(w \underline{\ominus} u) \oplus v \geq (w \underline{\ominus} u) \oplus u \geq w,$$

and the minimality of $w \underline{\ominus} v$ from (2.5) gives the inequality in (b). □

2.17. Lemma. *Let* $\oplus\in\mathcal{A}(\mathbb{R}^+)$ *and* $u,v,w\in\mathbb{R}^+$. *We have*:
(a) *if* $(v,u)\in D(\overline{\ominus})$ *and* $(w,v\overline{\ominus}u)\in D(\overline{\ominus})$, *then* $(w\oplus u,v)\in D(\overline{\ominus})$ *and*

$$(w\oplus u)\underline{\ominus}v\le w\overline{\ominus}(v\overline{\ominus}u)\le(w\oplus u)\overline{\ominus}v;$$

(b) *if* $(v,u)\in D(\overline{\ominus})$, *then* $(w\oplus u)\underline{\ominus}v\le w\underline{\ominus}(v\overline{\ominus}u)$;
(c) *if* $(w,v\underline{\ominus}u)\in D(\overline{\ominus})$, *then* $(w\oplus u,v)\in D(\overline{\ominus})$ *and* $w\overline{\ominus}(v\underline{\ominus}u)\le(w\oplus u)\overline{\ominus}v$;
(d) $(w\oplus u)\underline{\ominus}[(v\underline{\ominus}u)\oplus u]\le w\underline{\ominus}(v\underline{\ominus}u)$.

Proof. (a) Setting $w_1=w\overline{\ominus}(v\overline{\ominus}u)$, by virtue of (2.7), we find

$$w=[w\overline{\ominus}(v\overline{\ominus}u)]\oplus(v\overline{\ominus}u)=w_1\oplus(v\overline{\ominus}u),$$

and so, once again (2.7) implies

$$w\oplus u=w_1\oplus(v\overline{\ominus}u)\oplus u=w_1\oplus v.$$

Consequently, $(w\oplus u,v)\in D(\overline{\ominus})$. Applying (2.9), we get

$$(w\oplus u)\underline{\ominus}v=(w_1\oplus v)\underline{\ominus}v\le w_1\le(w_1\oplus v)\overline{\ominus}v=(w\oplus u)\overline{\ominus}v.$$

(b) Set $w_2=w\underline{\ominus}(v\overline{\ominus}u)$. It follows from (2.8) that

$$w\le[w\underline{\ominus}(v\overline{\ominus}u)]\oplus(v\overline{\ominus}u)=w_2\oplus(v\overline{\ominus}u),$$

and so, by (A.3) and (2.7),

$$w\oplus u\le w_2\oplus(v\overline{\ominus}u)\oplus u=w_2\oplus v.$$

The desired inequality $(w\oplus u)\underline{\ominus}v\le w_2$ follows from definition (2.5).

(c) It follows from (2.7) that if $w_3=w\overline{\ominus}(v\underline{\ominus}u)$, then

$$w=[w\overline{\ominus}(v\underline{\ominus}u)]\oplus(v\underline{\ominus}u)=w_3\oplus(v\underline{\ominus}u),$$

and so, (2.8) implies

$$w\oplus u=w_3\oplus(v\underline{\ominus}u)\oplus u\ge w_3\oplus v.$$

Thus, $(w\oplus u,v)\in D(\overline{\ominus})$ and, by (2.4), $w_3\le(w\oplus u)\overline{\ominus}v$.

(d) Setting $w_4=w\underline{\ominus}(v\underline{\ominus}u)$ and applying (2.8), we have

$$w\le[w\underline{\ominus}(v\underline{\ominus}u)]\oplus(v\underline{\ominus}u)=w_4\oplus(v\underline{\ominus}u),$$

and so, by (A.3), $w\oplus u\le w_4\oplus(v\underline{\ominus}u)\oplus u$. Now, the desired inequality in (d) follows from definition (2.5). □

In the final lemma of this section we address the *translation invariance* of subtractions $\overline{\ominus}$ and $\underline{\ominus}$ with respect to the A-operation $\oplus$ that generates them.

2.18. Lemma. *Suppose* $\oplus$ *is an* A*-operation on* $\mathbb{R}^+$.
(a) *If* $u\in\mathbb{R}^+$ *and* $(w,v)\in D(\overline{\ominus})$, *then* $(u\oplus w,u\oplus v)\in D(\overline{\ominus})$ *and*

$$w\overline{\ominus}v\le(u\oplus w)\overline{\ominus}(u\oplus v). \tag{2.11}$$

In addition, if $\oplus$ *is strict, then we have the equality* $(u\oplus w)\overline{\ominus}(u\oplus v)=w\overline{\ominus}v$.

(b) *If* $u,v,w\in\mathbb{R}^+$, *then* $(u\oplus w)\underline{\ominus}(u\oplus v)\le w\underline{\ominus}v$. *If, in addition,* $\oplus$ *is strict, then we have the equality* $(u\oplus w)\underline{\ominus}(u\oplus v)=w\underline{\ominus}v$.

Proof. (a) Since $(w,v)\in D(\overline{\ominus})$, there exists $u_0\in\mathbb{R}^+$ such that $u_0\oplus v\le w$, and so, by (A.1)–(A.3), $u_0\oplus(u\oplus v)=u\oplus(u_0\oplus v)\le u\oplus w$, which implies that the pair $(u\oplus w,u\oplus v)$ is in the domain of $\overline{\ominus}$.

Taking into account (2.7) and (A.2), we get

$$(w\,\overline{\ominus}\,v)\oplus(u\oplus v)=[(w\,\overline{\ominus}\,v)\oplus v]\oplus u=w\oplus u=u\oplus w,$$

and so, inequality (2.11) is a consequence of definition (2.4).

Now, suppose $\oplus$ is strict. Setting $u_1=(u\oplus w)\,\overline{\ominus}(u\oplus v)$, by (2.7), we have

$$u_1\oplus u\oplus v=[(u\oplus w)\,\overline{\ominus}(u\oplus v)]\oplus(u\oplus v)=u\oplus w.$$

By virtue of (2.3), we cancel by u and get $u_1\oplus v=w$. Since, again by (2.7), $(w\,\overline{\ominus}\,v)\oplus v=w$, we find $u_1\oplus v=w=(w\,\overline{\ominus}\,v)\oplus v$, and so, cancelling by v, we arrive at $u_1=w\,\overline{\ominus}\,v$, which is the desired equality.

(b) Set $u_2=(u\oplus w)\,\underline{\ominus}(u\oplus v)$. It follows from (A.1)–(A.3) and (2.8) that

$$(w\,\underline{\ominus}\,v)\oplus(u\oplus v)=[(w\,\underline{\ominus}\,v)\oplus v]\oplus u\ge w\oplus u=u\oplus w,$$

and so, definition (2.5) implies $u_2=(u\oplus w)\,\underline{\ominus}(u\oplus v)\le w\,\underline{\ominus}\,v$.

Let $\oplus$ be strict. By virtue of (2.8), we get

$$u_2\oplus u\oplus v=[(u\oplus w)\,\underline{\ominus}(u\oplus v)]\oplus(u\oplus v)\ge u\oplus w,$$

and so, cancelling by u, we find $u_2\oplus v\ge w$, which, by virtue of definition (2.5), implies $w\,\underline{\ominus}\,v\le u_2$, and the desired equality readily follows. □

2.19. Remark. The inequalities in Lemma 2.18 may be strict if $\oplus$ is not necessarily strict. To see this, we set $\oplus=\max$ and take into account Example 2.12(b): given $0\le v<w<u$, we have

$$w\,\overline{\ominus}\,v=w<u=u\,\overline{\ominus}\,u=\max\{u,w\}\,\overline{\ominus}\max\{u,v\}=(u\oplus w)\,\overline{\ominus}(u\oplus v)$$

and

$$(u\oplus w)\,\underline{\ominus}(u\oplus v)=\max\{u,w\}\,\underline{\ominus}\max\{u,v\}=u\,\underline{\ominus}\,u=0<w=w\,\underline{\ominus}\,v.$$

3. Optimization problems. In this section we introduce notation, definitions and assumptions to be used throughout the paper.

3.1. Optimization space. Let X be a finite set of cardinality $|X|\ge 2$ (e.g., $X=\{1,2,\dots,n\}$ with $n\ge 2$), called the *ground set*, 2^X be the family of all subsets of X (i.e., the power set of X) and $\dot{2}^X=2^X\setminus\{\varnothing\}$. For instance, a ground set may be thought of as the collection of all edges of a graph (or arcs in the directed case). Given a nonempty set Y, we denote, as usual, by Y^X the set of all functions (maps) $g:X\to Y$ from X into Y.

A *set of trajectories* on X (or a canonical collection on X, cf. [5]) is a collection $\mathcal{S}\subset\dot{2}^X$ of subsets of X such that

$$\cup\mathcal{S}=X \quad \text{and} \quad \cap\mathcal{S}=\varnothing, \tag{3.1}$$

where $\cup\mathcal{S}$ is the union of $\mathcal{S}$ (= the set of all $x\in X$ such that $x\in S$ for some $S\in\mathcal{S}$) and $\cap\mathcal{S}$ is the intersection of $\mathcal{S}$ (= the set of all $x\in X$ such that $x\in S$ for all $S\in\mathcal{S}$). It follows immediately from (3.1) that there are at least two trajectories in $\mathcal{S}$, and so, $2\le|\mathcal{S}|<2^{|X|}$.

A pair $(X,\mathcal{S})$ is called an *optimization space* if X is a ground set and $\mathcal{S}$ is a set of trajectories on X.

3.2. Operational procedures. If $(\mathbb{R}^+)^X$ denotes the set of all functions of the form $C: X\to\mathbb{R}^+$, we let $\mathcal{C}(X)$ be a subset of $(\mathbb{R}^+)^X$, called the set of all (admissible) *cost functions*, also representing distance, weight, time, etc. Given $C\in\mathcal{C}(X)$, to each element $x\in X$ a nonnegative number $C(x)$ is assigned uniquely, which is called the *cost* of x.

Since we are going to optimize (i.e., look for minima or maxima of) nonnegative functions on the set of trajectories $\mathcal{S}$, we denote by $\mathrm{Ob}(\mathcal{S})=(\mathbb{R}^+)^{\mathcal{S}}$ the family of all such functions, called *objective functions*.

A map of the form $f:\mathcal{C}(X)\to\mathrm{Ob}(\mathcal{S})$ is said to be an *operational procedure* on the optimization space $(X,\mathcal{S})$. In other words, to each cost function $C:X\to\mathbb{R}^+$ the operational procedure f assignes in a unique way the objective function of the form $f_C\equiv f(C):\mathcal{S}\to\mathbb{R}^+$. If the cost function C is fixed (somehow), notation $f_C(S)$ will be employed in place of $f(C)(S)$, where $S\in\mathcal{S}$.

Of particular importance for the developments to follow are operational procedures generated by A-operations on $\mathbb{R}^+$, which are most often encountered in practice.

Let $(X,\mathcal{S})$ be an optimization space, $C:X\to\mathbb{R}^+$—a cost function and $\oplus$—an A-operation on $\mathbb{R}^+$. Suppose the set function $F_C:2^X\to\mathbb{R}^+$ is given by

$$F_C(S)=\bigoplus_{y\in S}C(y) \quad \text{for all} \quad S\in\dot{2}^X, \quad \text{and} \quad F_C(\varnothing)=\varnothing, \tag{3.2}$$

where (cf. (2.2))

$$\bigoplus_{y\in S}C(y)=\bigoplus_{i=1}^{n}C(b(i)) \tag{3.3}$$

for a bijection $b:\{1,\dots,n\}\to S$ with $n=|S|$ (since $\oplus$ satisfies axioms (A.1) and (A.2), the right-hand side in (3.3) is independent of a bijection b chosen). By virtue of (A.1) and (A.2), F_C is a finitely additive *measure* on 2^X with respect to the A-operation $\oplus$, that is,

$$\text{if } S_1,S_2\in 2^X \text{ and } S_1\cap S_2=\varnothing, \text{ then } F_C(S_1\cup S_2)=F_C(S_1)\oplus F_C(S_2),$$

the term $F_C(\varnothing)=\varnothing$ being omitted (cf. (2.1)). The measure F_C will be called the *operational measure* corresponding to C and $\oplus$.

The operational procedure $f:\mathcal{C}(X)\to\mathrm{Ob}(\mathcal{S})$ on the optimization space $(X,\mathcal{S})$, *generated by the* A-*operation* $\oplus$, is given by

$$f_C(S)=F_C(S)=\bigoplus_{y\in S}C(y)\quad\text{for all}\quad S\in\mathcal{S}\quad\text{and}\quad C\in\mathcal{C}(X),\tag{3.4}$$

i.e., $f_C=F_C|_{\mathcal{S}}$ is the restriction of measure F_C to the set of trajectories $\mathcal{S}$ on X.

3.3. Remark. If $\oplus$ is an A-operation of addition, then the measure F_C can be expressed as $F_C(S)=\oplus_{x\in X}C(x)\delta_x(S)$, where $\delta_x:2^X\to\{0,1\}$ is the Dirac measure (or point mass) concentrated at $x\in X$, i.e., given $S\subset X$, one has $\delta_x(S)=1$ if $x\in S$, and $\delta_x(S)=0$ if $x\notin S$.

3.4. Examples of operational measures. Here we follow the pattern of Section 2.12. Let $C\in\mathcal{C}(X)$ and $S\in\dot{2}^X$.

(a) If $u\oplus v=(u^p+v^p)^{1/p}$ on $\mathbb{R}^+=[0,\infty)$ with $p>0$, then the p-sum operational measure is given by

$$F_C(S)=\left(\sum_{y\in S}C(y)^p\right)^{1/p}.$$

(b) If $u\oplus v=\max\{u,v\}$ on $\mathbb{R}^+=[0,\infty)$, then the max (or bottleneck) operational measure is of the form

$$F_C(S)=\max_{y\in S}C(y)=\max\big\{C(y):y\in S\big\}.$$

(c) If $u\oplus v=uv$ on $\mathbb{R}^+=(0,\infty)$, then the product operational measure is given by

$$F_C(S)=\prod_{y\in S}C(y)\quad\text{if}\quad C(y)>0\quad\text{for all}\quad y\in X.$$

3.5. Optimization problems. The triple $(X,\mathcal{S},f)$, where $(X,\mathcal{S})$ is an optimization space and f is an operational procedure on $(X,\mathcal{S})$, determines a (Discrete) Optimization Problem (OP, for short), which is formulated as follows: given a cost function $C:X\to\mathbb{R}^+$, *minimize or maximize the objective function* f_C *on* $\mathcal{S}$, that is, look for solutions to the following extremal problem:

$$f_C(S)\to\min\ (\text{or}\ \max),\qquad S\in\mathcal{S}.\tag{3.5}$$

The set of trajectories $\mathcal{S}$ in (3.5) plays the role of the set of all feasible (or admissible) solutions to the OP.

Throughout the paper we concentrate only on the minimization problem (3.5) with the objective function of the form (3.4), i.e., problem (1.1), where $\oplus$ is an A-operation on $\mathbb{R}^+$. Since the formulation of the problem (1.1) depends on X, $\mathcal{S}$, $\oplus$ and C, we will also refer to the problem (1.1) as $\mathrm{OP}(X,\mathcal{S},\oplus,C)$.

Examples of concrete OPs including combinatorial OPs are the well known traveling salesman problem, shortest path problem, assignment problem, Steiner problem, machine sequencing problem, min-cut problem, and many other problems on graphs, matroids, etc. (we refer to [5, 6, 9, 10, 16, 17, 23, 24, 25, 26]).

3.6. Optimal solutions. Given an $\mathrm{OP}(X,\mathcal{S},\oplus,C)$, we denote by

$$\mathcal{S}^*\equiv\mathcal{S}^*_C=\{S^*\in\mathcal{S}:f_C(S^*)\le f_C(S)\ \text{for all}\ S\in\mathcal{S}\}\tag{3.6}$$

the set of all *optimal solutions* to the OP (1.1). The collection $\mathcal{S}^*$ and its elements depend on the cost function C; however, if C is fixed (in a context), then it will be convenient (and brief) not to show the dependence $\mathcal{S}^*=\mathcal{S}^*(C)$ explicitly. Note that, since the set of trajectories $\mathcal{S}$ is finite, optimal solutions $S^*\in\mathcal{S}^*$ always exist, i.e., we have $\mathcal{S}^*\neq\varnothing$.

The minimal value of f_C on $\mathcal{S}$, called the *optimal value* of the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$, is determined uniquely and is independent of an optimal solution $S^*\in\mathcal{S}^*$, and it will be denoted by

$$f_C(\mathcal{S}^*)=\min_{S\in\mathcal{S}} f_C(S)=\min_{\mathcal{S}} f_C=f_C(S^*)\quad\text{for all}\quad S^*\in\mathcal{S}^*. \tag{3.7}$$

3.7. Equivalent OPs. Given an $\mathrm{OP}(X,\mathcal{S},\oplus,C)$ of the form (1.1) and a φ-function $\varphi:[0,\infty)\to[0,\infty)$ (cf. Section 2.7), we let $\widehat{\oplus}=E_\varphi(\oplus)$ (i.e., $u\widehat{\oplus}v=\varphi^{-1}\big(\varphi(u)\oplus\varphi(v)\big)$ for all $u,v\geq 0$) and $\widehat{C}=\varphi^{-1}\circ C$ (i.e., $\widehat{C}(y)=\varphi^{-1}(C(y))$ for all $y\in X$). We are going to show that the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$ is *equivalent* to the $\mathrm{OP}(X,\mathcal{S},\widehat{\oplus},\widehat{C})$ in the sense that their sets of optimal solutions, denoted here by $\mathcal{S}^*(\oplus)$ and $\mathcal{S}^*(\widehat{\oplus})$, respectively, coincide. In fact, first we note that if $u_1,\dots,u_n\geq 0$, then $\widehat{\oplus}_{i=1}^n u_i=\varphi^{-1}\big(\oplus_{i=1}^n\varphi(u_i)\big)$, $n\in\mathbb{N}$. It follows that, given $S\in\mathcal{S}$, for the corresponding objective functions $f_C^{\oplus}$ and $f_{\widehat{C}}^{\widehat{\oplus}}$ we have

$$\begin{aligned}f_C^{\oplus}(S)&=\bigoplus_{y\in S}C(y)=\bigoplus_{y\in S}\varphi\big(\varphi^{-1}(C(y))\big)=\bigoplus_{y\in S}\varphi\big(\widehat{C}(y)\big)=\\&=\varphi\bigg(\varphi^{-1}\Big(\bigoplus_{y\in S}\varphi\big(\widehat{C}(y)\big)\Big)\bigg)=\varphi\bigg(\widehat{\bigoplus_{y\in S}}\widehat{C}(y)\bigg)=\varphi\Big(f_{\widehat{C}}^{\widehat{\oplus}}(S)\Big).\end{aligned}$$

Since φ is strictly increasing, it follows from (3.6) that $\mathcal{S}^*(\oplus)=\mathcal{S}^*(\widehat{\oplus})$.

4. Upper stability intervals.

4.1. In the Sensitivity Analysis one is interested in numerical characteristics of elements x from the ground set X, which express the degree of invariance of an optimal solution to the OP (1.1) with respect to a change of the single cost $C(x)$. The following two notions serve this purpose and are adopted in the literature ([4]–[10], [17]–[18], [24]–[26]). By the *upper tolerance* $u_{S^*}(x)$ (*lower tolerance* $\ell_{S^*}(x)$) of $x\in X$ with respect to $S^*\in\mathcal{S}^*$ one means the *maximum increase* (*maximum decrease*, respectively) of the cost $C(x)$ only, so that the optimal solution S^* to the original OP (1.1) remains an optimal solution to the "perturbed" OP (1.1), in which the costs $C(y)$ are unchanged if $y\neq x$ and the cost $C(x)$ of x is increased (decreased, respectively) as compared to $C(x)$. These two notions will be studied in detail in the framework of general A-operations in this and the next sections.

Let the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$ of the form (1.1) be given and $\mathcal{S}^*$ be the set of all its optimal solutions (cf. (3.6)).

4.2. Perturbed objective functions. Given $x\in X$, we perturb the cost function $C\in\mathcal{C}(X)$ at its value $C(x)$ by defining the *perturbed cost function* $C_{x,\gamma}:X\to\mathbb{R}^+$ with $\gamma\in\mathbb{R}^+$ as follows: $C_{x,\gamma}(y)=C(y)$ if $y\in X$ and $y\neq x$, and $C_{x,\gamma}(x)=\gamma$. Since $C_{x,\gamma}\in\mathcal{C}(X)$, we let $f(C_{x,\gamma})\equiv f_{C_{x,\gamma}}$ be the objective function (3.4) corresponding to the cost function $C_{x,\gamma}$, called the *perturbed objective function* (as compared to f_C), and so, it is of the form

$$f(C_{x,\gamma})(S)=\bigoplus_{y\in S}C_{x,\gamma}(y)\quad\text{for all}\quad S\in\mathcal{S}. \tag{4.1}$$

In order to (properly) define upper and lower tolerances of $x\in X$ with respect to an optimal solution $S^*\in\mathcal{S}^*$ to problem (1.1), we ought to determine (further) restrictions on $\gamma\in\mathbb{R}^+$, under which

$$S^*\in\mathcal{S}^*\quad\text{implies}\quad f(C_{x,\gamma})(S^*)=\min_{S\in\mathcal{S}} f(C_{x,\gamma})(S). \tag{4.2}$$

For this, let us express the perturbed objective function (4.1) in terms of the original cost function C, the initial objective function f_C and the operational measure F_C.

In order to do it, it will be helpful to introduce two ad hoc subcollections of the set of trajectories $\mathcal{S}$ by

$$\mathcal{S}_{-x} = \{S \in \mathcal{S} : x \notin S\} \quad \text{and} \quad \mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}, \qquad x \in X; \tag{4.3}$$

in other words, given $x \in X$ and $S \in \mathcal{S}$, we have: $x \notin S$ iff $S \in \mathcal{S}_{-x}$, and $x \in S$ iff $S \in \mathcal{S}_x$. By virtue of (3.1), both collections $\mathcal{S}_{-x}$ and $\mathcal{S}_x$ are nonempty, $\mathcal{S}_{-x} \cup \mathcal{S}_x = \mathcal{S}$ and $\mathcal{S}_{-x} \cap \mathcal{S}_x = \varnothing$ for all $x \in X$.

Now, given $S \in \mathcal{S}$, we have either $S \in \mathcal{S}_{-x}$ or $S \in \mathcal{S}_x$. If $S \in \mathcal{S}_{-x}$ (or $x \notin S$), then $C_{x,\gamma}(y) = C(y)$ for all $y \in S$, and so, (4.1) and (3.4) imply

$$f(C_{x,\gamma})(S) = \bigoplus_{y \in S} C(y) = F_C(S) = f_C(S).$$

If $S \in \mathcal{S}_x$ (or $x \in S$), then $C_{x,\gamma}(y) = C(y)$ if $y \in S$ and $y \neq x$, and $C_{x,\gamma}(x) = \gamma$, and so, (4.1), (3.2), (A.1) and (A.2) yield

$$f(C_{x,\gamma})(S) = \gamma \oplus \left(\bigoplus_{y \in S \setminus \{x\}} C(y) \right) = \gamma \oplus F_C(S \setminus \{x\}),$$

where the term $F_C(S \setminus \{x\})$ is omitted if $S = \{x\}$ (cf. (2.1)).

Thus, given $x \in X$ and $S \in \mathcal{S}$, the perturbed objective function $f(C_{x,\gamma})$ is expressed as

$$f(C_{x,\gamma})(S) = \begin{cases} f_C(S) & \text{if } S \in \mathcal{S}_{-x} \text{ (i.e., } x \notin S), \\ \gamma \oplus F_C(S \setminus \{x\}) & \text{if } S \in \mathcal{S}_x \quad \text{(i.e., } x \in S). \end{cases} \tag{4.4}$$

In particular, if $\gamma = C(x)$, then $C_{x,\gamma}(y) = C(y)$ for all $y \in X$, and so, (4.4) implies

$$f_C(S) = C(x) \oplus F_C(S \setminus \{x\}) \quad \text{if} \quad S \in \mathcal{S}_x. \tag{4.5}$$

Formula (4.4) is valid for all A-operations $\oplus$ on $\mathbb{R}^+$ and, particularly, as it will be seen later, it works well for all strict operations. However, for certain nonstrict A-operations, such as max, a different (more subtle) form of formula (4.4) is needed.

4.3. The perturbed objective function in the case $\oplus = \max$. Suppose that $\oplus = \max$ on $\mathbb{R}^+ = [0, \infty)$. Let us transform the lower part of formula (4.4) taking into account certain specific features of the A-operation max.

Only the case $S \in \mathcal{S}_x$ with $S \neq \{x\}$ is to be considered. By virtue of (4.5) and Example 3.4(b), we have

$$f_C(S) = \max\{C(x), F_C(S \setminus \{x\})\} \quad \text{with} \quad F_C(S \setminus \{x\}) = \max_{y \in S \setminus \{x\}} C(y), \tag{4.6}$$

and the second line of (4.4) can be rewritten as

$$f(C_{x,\gamma})(S) = \max\{\gamma, F_C(S \setminus \{x\})\}. \tag{4.7}$$

Assuming that $\gamma \geq C(x)$ and considering the two possibilities in (4.6), which are of the form

$$f_C(S) = C(x) \quad \text{or} \quad f_C(S) = F_C(S \setminus \{x\}), \tag{4.8}$$

we find from (4.7) that

$$f(C_{x,\gamma})(S) = \max\{\gamma, f_C(S)\} = \gamma \oplus f_C(S). \tag{4.9}$$

In fact, if $f_C(S)=C(x)$, then

$$F_C(S\setminus\{x\})=\max_{y\in S\setminus\{x\}}C(y)\le\max_{y\in S}C(y)=f_C(S)=C(x)\le\gamma,$$

and so, by (4.7), $f(C_{x,\gamma})(S)=\gamma$. Now, if $f_C(S)=F_C(S\setminus\{x\})$, then once again (4.7) implies $f(C_{x,\gamma})(S)=\max\{\gamma,f_C(S)\}$.

Assume that $0\le\gamma\le C(x)$ and consider the possibilities (4.8). If $f_C(S)=C(x)$, then, by (4.7), we have

$$f(C_{x,\gamma})(S)=\max\{f_C(S)+\gamma-C(x),F_C(S\setminus\{x\})\},\tag{4.10}$$

and if $f_C(S)=F_C(S\setminus\{x\})$, then

$$\gamma\le C(x)\le\max_{y\in S}C(y)=f_C(S)=F_C(S\setminus\{x\}),\tag{4.11}$$

and so, (4.7) implies $f(C_{x,\gamma})(S)=F_C(S\setminus\{x\})=f_C(S)$. It follows that equality (4.10) holds under both possibilities (4.8).

Thus, given $x\in X$ and $S\in\mathcal{S}$, taking into account (4.4), (4.9) and (4.10), we have the following alternative expression for the perturbed objective function in the case $\oplus=\max$:

$$f(C_{x,\gamma})(S)=\begin{cases}f_C(S) & \text{if } S\in\mathcal{S}_{-x},\\ \max\{\gamma,f_C(S)\} & \text{if } S\in\mathcal{S}_x \text{ and } \gamma\ge C(x),\\ \max\bigl\{f_C(S)+\gamma-C(x),F_C(S\setminus\{x\})\bigr\} & \text{if } S\in\mathcal{S}_x \text{ and } 0\le\gamma\le C(x),\end{cases}\tag{4.12}$$

where $f_C(S)=F_C(S)$ is as in Example 3.4(b) and $F_C(S\setminus\{x\})$ is given in (4.6).

4.4. Unrestricted upper tolerances. In order to define the upper tolerance $u_{S^*}(x)$ of $x\in X$ with respect to an $S^*\in\mathcal{S}^*$ following the pattern exposed in Section 4.1, we have to increase the cost $C(x)$ to the value $\gamma\ge C(x)$ in such a way that the implication (4.2) holds, i.e., S^* is also an optimal solution to the perturbed $\mathrm{OP}(X,\mathcal{S},\oplus,C_{x,\gamma})$, which is the problem (1.1) with C replaced by $C_{x,\gamma}$ (cf. also (4.1)). It is to be noted that for certain elements $x\in X$ implication (4.2) holds automatically for all $\gamma\ge C(x)$; for instance, it is intuitively clear that $x\notin S^*$ are such elements, and so, the upper tolerance $u_{S^*}(x)$ for them may be thought of as infinite (unrestricted). This assertion is made precise in the following

4.5. Lemma. *If $S^*\in\mathcal{S}^*$ and $x\in X\setminus S^*$, then for all $\gamma\ge C(x)$ we have*

$$f(C_{x,\gamma})(S^*)\le f(C_{x,\gamma})(S)\quad\textit{for all}\quad S\in\mathcal{S}.\tag{4.13}$$

Proof. Let us fix $\gamma\ge C(x)$ arbitrarily. Since $S^*\in\mathcal{S}^*$, it follows from (3.6) that $S^*\in\mathcal{S}$ and $f_C(S^*)\le f_C(S)$ for all $S\in\mathcal{S}$. Assumption $x\notin S^*$ is equivalent to $S^*\in\mathcal{S}_{-x}$, and so, (4.4) implies

$$f(C_{x,\gamma})(S^*)=f_C(S^*).$$

Now, given $S\in\mathcal{S}$, we have either $S\in\mathcal{S}_{-x}$ or $S\in\mathcal{S}_x$. If $S\in\mathcal{S}_{-x}$, then taking into account (4.4), we get (for all $\gamma\in\mathbb{R}^+$)

$$f(C_{x,\gamma})(S^*)=f_C(S^*)\le f_C(S)=f(C_{x,\gamma})(S).$$

If $S\in\mathcal{S}_x$, then, by virtue of (4.5), the monotonicity of $\oplus$ (cf. axiom (A.3)) and (4.4), we find

$$\begin{aligned}f(C_{x,\gamma})(S^*)&=f_C(S^*)\le f_C(S)=C(x)\oplus F_C(S\setminus\{x\})\le\\&\le\gamma\oplus F_C(S\setminus\{x\})=f(C_{x,\gamma})(S),\end{aligned}$$

which was to be proved. □

Lemma 4.5 is valid for all A-operations $\oplus$ and even those satisfying only axioms (A.1)–(A.3). However, for lower tolerances, to be considered in Section 5, the situation is more subtle (cf. Theorem 5.4).

4.6. Upper stability intervals. Intuitively, for minimization problems (1.1) the cost $C(x)$ of an $x\in S^*$ cannot be increased indefinitely so that (4.2) holds. So, we are interested in finding (restrictions on $\oplus$ and) the largest closed interval of costs $[C(x),C^+_{S^*}(x)]$ with $C^+_{S^*}(x)\ge C(x)$, called the *upper stability interval*, such that the implication (4.2) is valid for all $\gamma\in[C(x),C^+_{S^*}(x)]$.

Given $S^*\in\mathcal{S}^*$ and $x\in S^*$, we set (cf. (4.3))

$$C^+_{S^*}(x)=\max\Gamma_{x,S^*}(\mathcal{S}_{-x}), \tag{4.14}$$

where, given a subcollection $\mathcal{S}_1\subset\mathcal{S}$ (usually, $\mathcal{S}_1=\mathcal{S}_{-x}$, $\mathcal{S}_x$ or $\mathcal{S}$),

$$\Gamma_{x,S^*}(\mathcal{S}_1)=\big\{\gamma\in\mathbb{R}^+: f(C_{x,\gamma})(S^*)\le f(C_{x,\gamma})(S) \text{ for all } S\in\mathcal{S}_1\big\}. \tag{4.15}$$

Note that, since $S^*\in\mathcal{S}^*$ and $C_{x,\gamma}(y)=C(y)$ for all $y\in X$ if $\gamma=C(x)$, it follows from (3.6) that $C(x)\in\Gamma_{x,S^*}(\mathcal{S}_1)$ (cf. also (4.22) below).

In order to evaluate (and estimate) the quantity (4.14) (see Theorem 4.7), it will be convenient to apply the notation of the form (3.7) for subcollections $\mathcal{S}_{-x}$ and $\mathcal{S}_x$ from (4.3):

$$f_C(\mathcal{S}^*_{-x})=\min_{S\in\mathcal{S}_{-x}} f_C(S) \quad\text{and}\quad f_C(\mathcal{S}^*_x)=\min_{S\in\mathcal{S}_x} f_C(S), \qquad x\in X \tag{4.16}$$

(to avoid ambiguities, we may explicitly set $\mathcal{S}^*_{-x}=(\mathcal{S}_{-x})^*\ne(\mathcal{S}^*)_{-x}$ and $\mathcal{S}^*_x=(\mathcal{S}_x)^*\ne(\mathcal{S}^*)_x$).

In the next theorem $\overline{\ominus}$ and $\underline{\ominus}$ denote the upper and lower subtractions for $\oplus$.

4.7. Theorem. *Given $S^*\in\mathcal{S}^*$ and $x\in S^*$, we have:*
(a) $C^+_{S^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,F_C(S^*\setminus\{x\})$;
(b) $C(x)\le C^+_1(x)\le C^+_{S^*}(x)\le C^+_2(x)$ *and* $C^+_1(x)\in\Gamma_{x,S^*}(\mathcal{S}_{-x})$*, where*

$$C^+_1(x) = f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,[f_C(S^*)\,\overline{\ominus}\,C(x)], \tag{4.17}$$
$$C^+_2(x) = f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,[f_C(S^*)\,\underline{\ominus}\,C(x)]; \tag{4.18}$$

(c) *if, in addition, $\oplus$ is strict on $\mathbb{R}^+$ or $\oplus=\max$ on $[0,\infty)$, then*

$$C^+_{S^*}(x)=C^+_1(x)=C^+_2(x)=[C(x)\oplus f_C(\mathcal{S}^*_{-x})]\,\overline{\ominus}\,f_C(S^*)=\max\Gamma_{x,S^*}(\mathcal{S}), \tag{4.19}$$

the implication (4.2) *holds for all $\gamma\in[C(x),C^+_{S^*}(x)]$ and, moreover, in the case $\oplus=\max$ we also have $C^+_{S^*}(x)=f_C(\mathcal{S}^*_{-x})$.*

Proof. (a) Since $x\in S^*$ iff $S^*\in\mathcal{S}_x$, it follows from (4.4) that, given $S\in\mathcal{S}_{-x}$ and $\gamma\in\mathbb{R}^+$,

$$f(C_{x,\gamma})(S^*)=\gamma\oplus F_C(S^*\setminus\{x\}) \quad\text{and}\quad f(C_{x,\gamma})(S)=f_C(S). \tag{4.20}$$

Therefore, (4.15) with $\mathcal{S}_1=\mathcal{S}_{-x}$ and (4.16) yield

$$\begin{aligned}\Gamma_{x,S^*}(\mathcal{S}_{-x})&=\big\{\gamma\in\mathbb{R}^+:\gamma\oplus F_C(S^*\setminus\{x\})\le f_C(S)\text{ for all } S\in\mathcal{S}_{-x}\big\}=\\&=\big\{\gamma\in\mathbb{R}^+:\gamma\oplus F_C(S^*\setminus\{x\})\le f_C(\mathcal{S}^*_{-x})\big\}.\end{aligned} \tag{4.21}$$

Since $S^*\in\mathcal{S}^*$ and $x\in S^*$, (4.5) and (3.6) imply

$$C(x)\oplus F_C(S^*\setminus\{x\})=f_C(S^*)\le f_C(S) \quad\text{for all}\quad S\in\mathcal{S}_{-x}, \tag{4.22}$$

and so,

$$C(x)\oplus F_C(S^*\setminus\{x\})\le f_C(\mathcal{S}^*_{-x}). \tag{4.23}$$

Hence, $C(x)\in\Gamma_{x,S^*}(\mathcal{S}_{-x})$ and the pair $(f_C(\mathcal{S}^*_{-x}),F_C(S^*\setminus\{x\}))$ belongs to the domain $D(\overline{\ominus})$ of the upper subtraction $\overline{\ominus}$ for $\oplus$. Definitions (4.14) and (2.4), (4.21) and (4.23) imply inequality $C^+_{S^*}(x)\ge C(x)$ and assertion (a).

(b) Taking into account the equality in (4.22) and applying inequalities (2.9) from Lemma 2.13 (with $w = F_C(S^*\setminus\{x\})$ and $v = C(x)$), we get

$$f_C(S^*)\underline{\ominus}C(x) \le F_C(S^*\setminus\{x\}) \le f_C(S^*)\overline{\ominus}C(x), \tag{4.24}$$

where, by (3.7), $f_C(S^*) = f_C(\mathcal{S}^*)$ is the optimal value of problem (1.1).

Now, we put $w = f_C(\mathcal{S}^*_{-x})$.

First, we set $u = f_C(S^*)\underline{\ominus}C(x)$ and $v = F_C(S^*\setminus\{x\})$. By (4.24), $u \le v$, and it follows from (4.23) that $(w,v)\in D(\overline{\ominus})$, and so, applying Lemma 2.16(a), we get $(w,u)\in D(\overline{\ominus})$ and $w\overline{\ominus}v \le w\overline{\ominus}u$. This inequality together with Theorem 4.7(a) and (4.18) gives $C^+_{S^*}(x) \le C^+_2(x)$.

Second, we set $u = F_C(S^*\setminus\{x\})$ and $v = f_C(S^*)\overline{\ominus}C(x)$. Then (4.24) implies $u\le v$, and equality (2.7) from Lemma 2.13 and (A.2) give

$$C(x)\oplus v = C(x)\oplus[f_C(S^*)\overline{\ominus}C(x)] = f_C(S^*) \le f_C(\mathcal{S}^*_{-x}) = w, \tag{4.25}$$

i.e., $(w,v)\in D(\overline{\ominus})$. Applying Lemma 2.16(a), we find $w\overline{\ominus}v \le w\overline{\ominus}u$, and so, by (4.17) and Theorem 4.7(a), $C^+_1(x) \le C^+_{S^*}(x)$.

In order to show that $C^+_1(x)\in\Gamma_{x,S^*}(\mathcal{S}_{-x})$, we apply the notation for u, v and w from the previous paragraph. By virtue of (4.17), inequality $u\le v$, Lemma 2.16(a) and (2.7) from Lemma 2.13, we get

$$C^+_1(x)\oplus F_C(S^*\setminus\{x\}) = (w\overline{\ominus}v)\oplus u \le (w\overline{\ominus}u)\oplus u = w = f_C(\mathcal{S}^*_{-x}),$$

and it remains to take into account (4.21).

The inequality $C(x)\le C^+_1(x)$ is a consequence of (4.25), (2.4) and (4.17).

(c) First, we establish two auxiliary inequalities (under general conditions on $\oplus$). By virtue of Lemma 2.17(a) and (4.17), we find

$$[C(x)\oplus f_C(\mathcal{S}^*_{-x})]\underline{\ominus}f_C(S^*) \le f_C(\mathcal{S}^*_{-x})\overline{\ominus}[f_C(S^*)\overline{\ominus}C(x)] = C^+_1(x), \tag{4.26}$$

and (4.18) and Lemma 2.17(c) yield

$$C^+_2(x) = f_C(\mathcal{S}^*_{-x})\overline{\ominus}[f_C(S^*)\underline{\ominus}C(x)] \le [C(x)\oplus f_C(\mathcal{S}^*_{-x})]\overline{\ominus}f_C(S^*). \tag{4.27}$$

1. Suppose that the A-operation $\oplus$ is strict on $\mathbb{R}^+$. Then, by (2.10) from Lemma 2.13, $\overline{\ominus}=\underline{\ominus}$ on $D(\overline{\ominus})$, and so, the first three equalities in (4.19) follow from Theorem 4.7(b), (4.17), (4.18), (4.26) and (4.27).

Now, given $\gamma\in[C(x), C^+_{S^*}(x)]$, let us show that (4.2) (or (4.13)) holds. If $S\in\mathcal{S}$, then either $S\in\mathcal{S}_{-x}$ or $S\in\mathcal{S}_x$. Let $S\in\mathcal{S}_{-x}$. Since $C(x)\le\gamma\le C^+_{S^*}(x)$, we have $\gamma\in\Gamma_{x,S^*}(\mathcal{S}_{-x})$, which implies the inequality in (4.13). In more details, by (4.20), (4.21), (4.16) and (4.4), we have

$$\begin{aligned} f(C_{x,\gamma})(S^*) = \gamma\oplus F_C(S^*\setminus\{x\}) \le C^+_{S^*}(x)\oplus F_C(S^*\setminus\{x\}) \le \\ \le f_C(\mathcal{S}^*_{-x}) \le f_C(S) = f(C_{x,\gamma})(S). \end{aligned}$$

If $S\in\mathcal{S}_x$ (and $\gamma\in\mathbb{R}^+$), then it follows from (4.5) and (3.6) that

$$C(x)\oplus F_C(S^*\setminus\{x\}) = f_C(S^*) \le f_C(S) = C(x)\oplus F_C(S\setminus\{x\}),$$

and so, by the cancellation law (2.3), $F_C(S^*\setminus\{x\}) \le F_C(S\setminus\{x\})$. Taking into account the monotonicity of $\oplus$ and (4.4), we obtain

$$f(C_{x,\gamma})(S^*) = \gamma\oplus F_C(S^*\setminus\{x\}) \le \gamma\oplus F_C(S\setminus\{x\}) = f(C_{x,\gamma})(S).$$

This proves also that $\Gamma_{x,S^*}(\mathcal{S}_x)=\mathbb{R}^+$, whence

$$\Gamma_{x,S^*}(\mathcal{S})=\Gamma_{x,S^*}(\mathcal{S}_{-x})\cap\Gamma_{x,S^*}(\mathcal{S}_x)=\Gamma_{x,S^*}(\mathcal{S}_{-x}),$$

and so, $C^+_{S^*}(x)=\max\Gamma_{x,S^*}(\mathcal{S})$, which is the fourth equality in (4.19).

2. Now assume that $\oplus=\max$ on $\mathbb{R}^+=[0,\infty)$. Since $x\in S^*$, by (4.6), we have $C(x)\le f_C(S^*)=f_C(\mathcal{S}^*)$, and so, taking into account Example 2.12(b), we get $f_C(\mathcal{S}^*)\overline{\ominus}C(x)=f_C(\mathcal{S}^*)$, whereas $f_C(\mathcal{S}^*)\underline{\ominus}C(x)=f_C(\mathcal{S}^*)$ if $C(x)<f_C(\mathcal{S}^*)$, and $f_C(\mathcal{S}^*)\underline{\ominus}C(x)=0$ if $C(x)=f_C(\mathcal{S}^*)$. Since S^* is an optimal solution to (1.1), we get $f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*_{-x})$, and so, (4.17) implies

$$C^+_1(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)=f_C(\mathcal{S}^*_{-x}).$$

By virtue of (4.18), we find: if $C(x)<f_C(\mathcal{S}^*)$, then

$$C^+_2(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)=f_C(\mathcal{S}^*_{-x}),$$

and if $C(x)=f_C(\mathcal{S}^*)$, then

$$C^+_2(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,0=f_C(\mathcal{S}^*_{-x}).$$

Thus, $C^+_1(x)=C^+_2(x)=f_C(\mathcal{S}^*_{-x})$ and, by Theorem 4.7(b), $C^+_{S^*}(x)=f_C(\mathcal{S}^*_{-x})$, which establishes the first two equalities in (4.19).

Since $C(x)\le f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*_{-x})$, we get (with $\oplus=\max$)

$$[C(x)\oplus f_C(\mathcal{S}^*_{-x})]\,\overline{\ominus}\,f_C(\mathcal{S}^*)=\max\{C(x),f_C(\mathcal{S}^*_{-x})\}=f_C(\mathcal{S}^*_{-x}),$$

and the third equality in (4.19) follows.

Now we prove (4.13) for all $\gamma\in[C(x),C^+_{S^*}(x)]$. Since $\gamma\le C^+_{S^*}(x)$, (4.13) for $S\in\mathcal{S}_{-x}$ follows from the inclusion $\gamma\in\Gamma_{x,S^*}(\mathcal{S}_{-x})$. In more details, given $S\in\mathcal{S}_{-x}$, (4.12) implies

$$\begin{aligned}f(C_{x,\gamma})(S^*)=\max\{\gamma,f_C(S^*)\}\le\max\{C^+_{S^*}(x),f_C(S^*)\}=\\ =C^+_{S^*}(x)=f_C(\mathcal{S}^*_{-x})\le f_C(S)=f(C_{x,\gamma})(S).\end{aligned}$$

If $S\in\mathcal{S}_x$ (and $\gamma\in\mathbb{R}^+$), then the monotonicity of max and (4.12) yield

$$f(C_{x,\gamma})(S^*)=\max\{\gamma,f_C(S^*)\}\le\max\{\gamma,f_C(S)\}=f(C_{x,\gamma})(S).$$

As in step 1 of item (c), this proves also that $C^+_{S^*}(x)=\max\Gamma_{x,S^*}(\mathcal{S})$. □

4.8. Remark. By Theorem 4.7(a) and definition (4.14), the value $C^+_{S^*}(x)$ depends on the optimal solution S^* to problem (1.1). However, if $\oplus$ is strict or $\oplus=\max$, then, by Theorem 4.7(c), the value $C^+_{S^*}(x)=C^+_1(x)$ is independent of optimal solutions S^* to (1.1) such that $x\in S^*$ in the following sense: given $S^*_1,S^*_2\in\mathcal{S}^*$, if $x\in S^*_1\cap S^*_2$, then $C^+_{S^*_1}(x)=C^+_{S^*_2}(x)$ (in fact, these two quantities are given by the same formula (4.19), which does not involve neither S^*_1 nor S^*_2).

4.9. Having the upper stability interval $[C(x),C^+_{S^*}(x)]$ (with $C(x)\le C^+_{S^*}(x)$) for $S^*\in\mathcal{S}^*$ and $x\in S^*$, it may look quite natural to define the *upper tolerance* $u_{S^*}(x)$ of $x\in S^*$ as a "measure" (= some generalized length) of the upper stability interval. This can be done in many ways. For instance, if $\varphi:[0,\infty)\to[0,\infty)$ is a φ-function, then we may have set $\mu([C(x),C^+_{S^*}(x)])=\varphi(C^+_{S^*}(x))-\varphi(C(x))$. However, this is irrelevant for our purposes, because the upper stability interval has been generated via the A-operation $\oplus$ or, more precisely (cf. Theorem 4.7(a)), by the upper subtraction $\overline{\ominus}$ for $\oplus$. Having this in mind, as well as the translation invariance of $\overline{\ominus}$ and $\underline{\ominus}$ (cf. Lemma 2.18), we adopt the following definition.

4.10. Definition. Assume that the A-operation $\oplus$ is strict on $\mathbb{R}^+$. Given $S^*\in\mathcal{S}^*$ and $x\in S^*$, the *upper tolerance* of x is defined by

$$u_{S^*}(x)=C^+_{S^*}(x)\,\overline{\ominus}\,C(x)\in\mathbb{R}^+. \tag{4.28}$$

4.11. Theorem. *If $\oplus$ is a strict A-operation on $\mathbb{R}^+$, $S^*\in\mathcal{S}^*$ and $x\in S^*$, then the value $u_{S^*}(x)$ is well-defined,*

$$u_{S^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)\geq\boldsymbol{e}, \tag{4.29}$$

and $u_{S^*}(x)=\boldsymbol{e}$ iff $C^+_{S^*}(x)=C(x)$ iff $f_C(\mathcal{S}^*_{-x})=f_C(\mathcal{S}^*)$, where $\boldsymbol{e}\in\mathbb{R}^+$ is the neutral element with respect to $\oplus$.

Proof. If $\oplus$ is a generalized addition (i.e., satisfies (A.1)–(A.3) and (A.5)), then inequality $C(x)\leq C^+_{S^*}(x)$ from Theorem 4.7(b), (c) and Lemma 2.11(b) imply $(C(x),C^+_{S^*}(x))\in D(\overline{\ominus})$, and if $\oplus$ is a generalized multiplication (i.e., (A.1)–(A.4) and (A.6) are satisfied), then the same inclusion is a consequence of Lemma 2.11(c). It follows from (2.4) that the quantity (4.28) is well-defined.

Since $\boldsymbol{e}\oplus C(x)=C(x)\leq C^+_{S^*}(x)$, definition (2.4) implies

$$\boldsymbol{e}\leq C^+_{S^*}(x)\,\overline{\ominus}\,C(x)=u_{S^*}(x).$$

If $u_{S^*}(x)=\boldsymbol{e}$, then, by virtue of (2.7), we get

$$C^+_{S^*}(x)=(C^+_{S^*}(x)\,\overline{\ominus}\,C(x))\oplus C(x)=u_{S^*}(x)\oplus C(x)=\boldsymbol{e}\oplus C(x)=C(x).$$

(Note that all assertions above do not rely on the strictness of $\oplus$.) Now, if equality $C^+_{S^*}(x)=C(x)$ holds, then we claim that $u_{S^*}(x)=\boldsymbol{e}$, for, otherwise, if $u_{S^*}(x)>\boldsymbol{e}$, then (2.7) and the strictness $(\mathrm{A.3_s})$ of $\oplus$ imply

$$C^+_{S^*}(x)=(C^+_{S^*}(x)\,\overline{\ominus}\,C(x))\oplus C(x)=u_{S^*}(x)\oplus C(x)>\boldsymbol{e}\oplus C(x)=C(x),$$

which contradicts the assumption.

Finally, let us establish the equality in (4.29). Setting

$$u=C(x),\quad v=f_C(\mathcal{S}^*)\quad\text{and}\quad w=f_C(\mathcal{S}^*_{-x}),$$

we find from (4.28) and (4.19) that

$$u_{S^*}(x)=C^+_{S^*}(x)\,\overline{\ominus}\,C(x)=\big([u\oplus w]\,\overline{\ominus}\,v\big)\,\overline{\ominus}\,u.$$

By (2.7), $w=(w\,\overline{\ominus}\,v)\oplus v$, and so, (2.10) and (2.9) yield

$$[u\oplus w]\,\overline{\ominus}\,v=[u\oplus(w\,\overline{\ominus}\,v)\oplus v]\,\overline{\ominus}\,v=u\oplus(w\,\overline{\ominus}\,v)$$

and

$$\big([u\oplus w]\,\overline{\ominus}\,v\big)\,\overline{\ominus}\,u=\big(u\oplus(w\,\overline{\ominus}\,v)\big)\,\overline{\ominus}\,u=w\,\overline{\ominus}\,v,$$

and it remains to take into account that $w\,\overline{\ominus}\,v=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)$. $\square$

4.12. Examples of upper stability intervals and upper tolerances. In accordance with Examples 2.12(a) and (c), (4.19) and (4.29), given $S^*\in\mathcal{S}^*$ and $x\in S^*$, we have:

(a) if $u\oplus v=(u^p+v^p)^{1/p}$ on $\mathbb{R}^+=[0,\infty)$ with $p>0$, then

$$C^+_{S^*}(x)=\big(C(x)^p+f_C(\mathcal{S}^*_{-x})^p-f_C(\mathcal{S}^*)^p\big)^{1/p},$$
$$u_{S^*}(x)=\big(f_C(\mathcal{S}^*_{-x})^p-f_C(\mathcal{S}^*)^p\big)^{1/p}.$$

In particular, if $p=1$, then for the OP $f_C(S)=\sum_{y\in S}C(y)\to\min$ with $S\in\mathcal{S}$ and an element $x\in S^*$ from its optimal solution $S^*\in\mathcal{S}^*$ we find

$$u_{S^*}(x)=f_C(\mathcal{S}^*_{-x})-f_C(\mathcal{S}^*)=\min_{S\in\mathcal{S}\text{ with }x\notin S}f_C(S)-\min_{S\in\mathcal{S}}f_C(S),$$

which gives a formula due to Libura [17] (cf. also [5], [7], [24]–[26]);

(b) if $u\oplus v=uv$ on $\mathbb{R}^+$ and $C(y)>0$ for all $y\in X$, then

$$C^+_{S^*}(x)=\frac{C(x)\cdot f_C(\mathcal{S}^*_{-x})}{f_C(\mathcal{S}^*)}\quad\text{and}\quad u_{S^*}(x)=\frac{f_C(\mathcal{S}^*_{-x})}{f_C(\mathcal{S}^*)}.$$

5. Lower stability intervals. Let the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$ of the form (1.1) be given and $\mathcal{S}^*$ be the set of its optimal solutions from (3.6).

In order to define the lower tolerance $\ell_{S^*}(x)$ of an $x\in X$ with respect to an $S^*\in\mathcal{S}^*$ (cf. Section 4.1), we have to decrease the cost $C(x)$ to the value $\gamma\le C(x)$ so that S^* is also an optimal solution to the perturbed $\mathrm{OP}(X,\mathcal{S},\oplus,C_{x,\gamma})$, i.e., the implication (4.2) holds.

5.1. Lower stability intervals. One may (intuitively) feel that for minimization problems (1.1) the cost of an element $x\in X\setminus S^*$ cannot be decreased "unboundedly" in such a way that the implication (4.2) is valid. It is our aim now to obtain (restrictions on the A-operation $\oplus$ and) the largest closed interval of costs $[C^-_{S^*}(x),C(x)]$ with $C^-_{S^*}(x)\le C(x)$, termed the *lower stability interval*, such that (4.2) holds for all $\gamma\in[C^-_{S^*}(x),C(x)]$.

Given $S^*\in\mathcal{S}^*$ and $x\in X\setminus S^*$, we set (cf. (4.3) and (4.15))

$$C^-_{S^*}(x)=\min\Gamma_{x,S^*}(\mathcal{S}_x). \tag{5.1}$$

In order to estimate and/or evaluate $C^-_{S^*}(x)$ (see Theorem 5.2), we apply notation (4.16) and note that there exists an $S_x\in\mathcal{S}_x$ (i.e., $S_x\in\mathcal{S}$ and $x\in S_x$) such that

$$F_C(S_x\setminus\{x\})=\min_{S\in\mathcal{S}_x}F_C(S\setminus\{x\}) \tag{5.2}$$

(actually, the value at the right in (5.2) and, hence, the quantity at the left in (5.2), are independent of the set S_x). Moreover, we have

$$f_C(S_x)=f_C(\mathcal{S}^*_x). \tag{5.3}$$

In fact, since $S_x\in\mathcal{S}_x$, (4.16) implies $f_C(S_x)\ge f_C(\mathcal{S}^*_x)$; on the other hand, given $S\in\mathcal{S}_x$, since $x\in S$ and $F_C(S_x\setminus\{x\})\le F_C(S\setminus\{x\})$, we find, by (4.5) and (A.3),

$$\begin{aligned}f_C(S_x)&=C(x)\oplus F_C(S_x\setminus\{x\})\le\\&\le C(x)\oplus F_C(S\setminus\{x\})=f_C(S),\end{aligned} \tag{5.4}$$

and so, again by (4.16), $f_C(S_x)\le f_C(\mathcal{S}^*_x)$, which yields equality (5.3).

In the next theorem $\overline{\ominus}$ and $\underline{\ominus}$ denote (as usual) the upper and lower subtractions for $\oplus$, respectively.

5.2. Theorem. *Given $S^*\in\mathcal{S}^*$ and $x\in X\setminus S^*$, we have:*

(a) $C^-_{S^*}(x)=f_C(\mathcal{S}^*)\,\underline{\ominus}\,F_C(S_x\setminus\{x\})=\min\Gamma_{x,S^*}(\mathcal{S})$ *(cf. (5.2)), and the implication* (4.2) *holds for all* $\gamma\in[C^-_{S^*}(x),C(x)]$;

(b) $C^-_1(x)\le C^-_{S^*}(x)\le C^-_2(x)\le C(x)$ *and* $C^-_2(x)\in\Gamma_{x,S^*}(\mathcal{S})$, *where*

$$C^-_1(x)=f_C(\mathcal{S}^*)\,\underline{\ominus}\,[f_C(\mathcal{S}^*_x)\,\overline{\ominus}\,C(x)],\tag{5.5}$$

$$C^-_2(x)=f_C(\mathcal{S}^*)\,\underline{\ominus}\,[f_C(\mathcal{S}^*_x)\,\underline{\ominus}\,C(x)];\tag{5.6}$$

(c) *if the* A*-operation* $\oplus$ *is strict on* $\mathbb{R}^+$, *then*

$$C^-_{S^*}(x)=C^-_1(x)=C^-_2(x)=[C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}^*_x);\tag{5.7}$$

(d) *if* $\oplus=\max$ *on* $\mathbb{R}^+=[0,\infty)$, *then* $C^-_1(x)=0$,

$$C^-_{S^*}(x)=\begin{cases} f_C(\mathcal{S}^*) & \text{if } F_C(S_x\setminus\{x\})<f_C(\mathcal{S}^*),\\ 0 & \text{if } F_C(S_x\setminus\{x\})\ge f_C(\mathcal{S}^*),\end{cases}\tag{5.8}$$

and

$$C^-_2(x)=\begin{cases} f_C(\mathcal{S}^*) & \text{if } C(x)=f_C(\mathcal{S}^*_x),\\ 0 & \text{if } C(x)<f_C(\mathcal{S}^*_x).\end{cases}\tag{5.9}$$

Proof. (a) Since $S^*\in\mathcal{S}_{-x}$, given $S\in\mathcal{S}_x$ and $\gamma\in\mathbb{R}^+$, by (4.4), we find

$$f(C_{x,\gamma})(S^*)=f_C(S^*)=f_C(\mathcal{S}^*)\quad\text{and}\quad f(C_{x,\gamma})(S)=\gamma\oplus F_C(S\setminus\{x\}).$$

It follows from (4.15) with $\mathcal{S}_1=\mathcal{S}_x$ and (5.2) that

$$\begin{aligned}\Gamma_{x,S^*}(\mathcal{S}_x)&=\big\{\gamma\in\mathbb{R}^+: f_C(S^*)\le\gamma\oplus F_C(S\setminus\{x\})\text{ for all }S\in\mathcal{S}_x\big\}=\\ &=\big\{\gamma\in\mathbb{R}^+: f_C(S^*)\le\gamma\oplus F_C(S_x\setminus\{x\})\big\}.\end{aligned}\tag{5.10}$$

Since $S^*\in\mathcal{S}^*$, $f_C(S^*)\le f_C(S_x)$, and so, (5.4) implies $C(x)\in\Gamma_{x,S^*}(\mathcal{S}_x)$. Now, (5.1), (5.10) and definition (2.5) of the lower subtraction $\underline{\ominus}$ for $\oplus$ yield inequality $C^-_{S^*}(x)\le C(x)$ and the first equality in (a).

Given $\gamma\in[C^-_{S^*}(x),C(x)]$, let us show that (4.2) (or (4.13)) holds. For this, suppose $S\in\mathcal{S}$. If $S\in\mathcal{S}_{-x}$, then, taking into account that $S^*\in\mathcal{S}_{-x}\cap\mathcal{S}^*$ and (4.4), we have (even for all $\gamma\in\mathbb{R}^+$)

$$f(C_{x,\gamma})(S^*)=f_C(S^*)\le f_C(S)=f(C_{x,\gamma})(S).$$

This proves also that $\Gamma_{x,S^*}(\mathcal{S}_{-x})=\mathbb{R}^+$, whence

$$\Gamma_{x,S^*}(\mathcal{S})=\Gamma_{x,S^*}(\mathcal{S}_{-x})\cap\Gamma_{x,S^*}(\mathcal{S}_x)=\Gamma_{x,S^*}(\mathcal{S}_x),$$

and so, the second equality in (a) is a consequence of definition (5.1).

If $S\in\mathcal{S}_x$, then inequalities $C^-_{S^*}(x)\le\gamma\le C(x)$ imply $\gamma\in\Gamma_{x,S^*}(\mathcal{S}_x)$, which gives the inequality in (4.13). More directly, by virtue of (5.10), (5.2) and (4.4),

$$\begin{aligned}f(C_{x,\gamma})(S^*)&=f_C(S^*)\le C^-_{S^*}(x)\oplus F_C(S_x\setminus\{x\})\le\gamma\oplus F_C(S_x\setminus\{x\})=\\ &=\gamma\oplus\Big(\min_{S'\in\mathcal{S}_x}F_C(S'\setminus\{x\})\Big)\le\gamma\oplus F_C(S\setminus\{x\})=f(C_{x,\gamma})(S).\end{aligned}$$

(b) Taking into account (5.3) and (5.4) and applying inequalities (2.9) from Lemma 2.13 (with $w=F_C(S_x\setminus\{x\})$ and $v=C(x)$), we find

$$f_C(\mathcal{S}^*_x)\,\underline{\ominus}\,C(x)\le F_C(S_x\setminus\{x\})\le f_C(\mathcal{S}^*_x)\,\overline{\ominus}\,C(x).\tag{5.11}$$

Now, let us put $w=f_C(\mathcal{S}^*)$.

First, we set $u=f_C(\mathcal{S}_x^*)\,\underline{\ominus}\,C(x)$ and $v=F_C(S_x\setminus\{x\})$. By virtue of (5.11), $u\le v$, and so, Lemma 2.16(b) implies $w\,\underline{\ominus}\,v\le w\,\underline{\ominus}\,u$. This inequality, Theorem 5.2(a) and (5.6) give $C_{S^*}^-(x)\le C_2^-(x)$. Since $C_{S^*}^-(x)$ is the minimal element of the set $\Gamma_{x,S^*}(\mathcal{S})$, the last inequality yields $C_2^-(x)\in\Gamma_{x,S^*}(\mathcal{S})$. It is also worth-while to verify this inclusion directly: in fact, by virtue of (2.8), (A.3), (5.6) and (5.10), we have

$$f_C(\mathcal{S}^*)=w\le(w\,\underline{\ominus}\,v)\oplus v\le(w\,\underline{\ominus}\,u)\oplus v=C_2^-(x)\oplus F_C(S_x\setminus\{x\}).$$

To see that $C_2^-(x)\le C(x)$, we note that, by (2.8),

$$f_C(\mathcal{S}_x^*)\le C(x)\oplus[f_C(\mathcal{S}_x^*)\,\underline{\ominus}\,C(x)],$$

and so, the desired inequality follows from (2.5) and (5.6).

Second, we set $u=F_C(S_x\setminus\{x\})$ and $v=f_C(\mathcal{S}_x^*)\,\overline{\ominus}\,C(x)$. Then (5.11) implies $u\le v$, and so, by Lemma 2.16(b), $w\,\underline{\ominus}\,v\le w\,\underline{\ominus}\,u$, and it follows from (5.5) and Theorem 5.2(a) that $C_1^-(x)\le C_{S^*}^-(x)$.

(c) By the strictness of $\oplus$ and (2.10), $\overline{\ominus}=\underline{\ominus}$ on $D(\overline{\ominus})$, and so, (5.5), (5.6) and inequalities in item (b) imply $C_1^-(x)=C_{S^*}^-(x)=C_2^-(x)$, which proves the first two equalities in (5.7).

In order to prove the third equality in (5.7), we set

$$C_0^-(x)=[C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}_x^*) \tag{5.12}$$

and note that, by virtue of Lemma 2.17(b) and (5.5), we have (the following inequality, which is independent of the strictness of $\oplus$)

$$C_0^-(x)\le f_C(\mathcal{S}^*)\,\underline{\ominus}\,[f_C(\mathcal{S}_x^*)\,\overline{\ominus}\,C(x)]=C_1^-(x). \tag{5.13}$$

Had we shown that $C_0^-(x)\in\Gamma_{x,S^*}(\mathcal{S}_x)$, then, by (5.1), we would have $C_1^-(x)=C_{S^*}^-(x)\le C_0^-(x)$, which implies the third equality in (5.7). Setting $u=C(x)$, $v=F_C(S_x\setminus\{x\})$ and $w=f_C(\mathcal{S}^*)$ and noting that, by virtue of (5.3) and (5.4), $f_C(\mathcal{S}_x^*)=u\oplus v$, we find

$$C_0^-(x)\oplus F_C(S_x\setminus\{x\})=\big[(u\oplus w)\,\underline{\ominus}\,(u\oplus v)\big]\oplus v. \tag{5.14}$$

If $u_1=(u\oplus w)\,\underline{\ominus}\,(u\oplus v)$, then inequality (2.8) implies

$$u\oplus w\le[(u\oplus w)\,\underline{\ominus}\,(u\oplus v)]\oplus(u\oplus v)=u_1\oplus u\oplus v,$$

and so, cancelling by u (by the strictness of $\oplus$ and (2.3)), we get $w\le u_1\oplus v$. It follows from definition (2.5) of $\underline{\ominus}$ that $w\,\underline{\ominus}\,v\le u_1$. Hence, (2.8), (A.3) and (5.14) yield

$$f_C(\mathcal{S}^*)=w\le(w\,\underline{\ominus}\,v)\oplus v\le u_1\oplus v=C_0^-(x)\oplus F_C(S_x\setminus\{x\}),$$

which, by virtue of (5.10), gives $C_0^-(x)\in\Gamma_{x,S^*}(\mathcal{S}_x)$.

(d) Suppose $\oplus=\max$ on $\mathbb{R}^+=[0,\infty)$. Let us evaluate the values of $C_{S^*}^-(x)$, $C_1^-(x)$ and $C_2^-(x)$ from Theorem 5.2(b) and the value of $C_0^-(x)$ from (5.12). Note that, by virtue of item (b) and (5.13), we have

$$C_0^-(x)\le C_1^-(x)\le C_{S^*}^-(x)\le C_2^-(x)\le C(x). \tag{5.15}$$

First, equality (5.8) is a straightforward consequence of the first equality in Theorem 5.2(a) and Example 2.12(b).

Second, in order to evaluate $C_1^-(x)$, we note that $C(x)\le f_C(\mathcal{S}_x^*)$; in fact, if $S\in\mathcal{S}_x$, then $x\in S$, and so, $C(x)\le\max_{y\in S}C(y)=f_C(S)$, which, by (4.16), implies the desired inequality. It follows from Example 2.12(b) that

$$f_C(\mathcal{S}_x^*)\,\overline{\ominus}\,C(x)=f_C(\mathcal{S}_x^*),$$

and, since $f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*_x)$, (5.5) and Example 2.12(b) give

$$C_1^-(x)=f_C(\mathcal{S}^*)\,\underline{\ominus}\,[f_C(\mathcal{S}^*_x)\,\overline{\ominus}\,C(x)]=f_C(\mathcal{S}^*)\,\underline{\ominus}\,f_C(\mathcal{S}^*_x)=0.$$

Third, by the previous step and (5.15), we get $C_0^-(x)=0$. Also, this can be seen directly as follows: since $C(x)\le f_C(\mathcal{S}^*_x)$ and $f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*_x)$, we find

$$C(x)\oplus f_C(\mathcal{S}^*)=\max\{C(x),f_C(\mathcal{S}^*)\}\le f_C(\mathcal{S}^*_x),$$

and so, (5.12) and Example 2.12(b) imply $C_0^-(x)=0$.

Fourth, since $C(x)\le f_C(\mathcal{S}^*_x)$, it follows from Example 2.12(b) that

$$f_C(\mathcal{S}^*_x)\,\underline{\ominus}\,C(x)=\begin{cases}0 & \text{if } C(x)=f_C(\mathcal{S}^*_x),\\ f_C(\mathcal{S}^*_x) & \text{if } C(x)<f_C(\mathcal{S}^*_x),\end{cases}$$

and so, (5.6) yields

$$C_2^-(x)=\begin{cases}f_C(\mathcal{S}^*)\,\underline{\ominus}\,0 & \text{if } C(x)=f_C(\mathcal{S}^*_x),\\ f_C(\mathcal{S}^*)\,\underline{\ominus}\,f_C(\mathcal{S}^*_x) & \text{if } C(x)<f_C(\mathcal{S}^*_x),\end{cases}$$

whence (5.9) follows if we take into account that $f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*_x)$. □

5.3. Remark. By (the first equality in) Theorem 5.2(a), the value $C^-_{S^*}(x)$ with $x\in X\setminus S^*$ does not depend on the optimal solution S^* to problem (1.1) in the following sense: if $S_1^*,S_2^*\in\mathcal{S}^*$ and $x\in(X\setminus S_1^*)\cap(X\setminus S_2^*)$, then $C^-_{S_1^*}(x)=C^-_{S_2^*}(x)$.

In our next result we treat the case of "unrestricted" lower tolerances: for certain elements x from X (e.g., $x\in S^*$) the implication (4.2) always holds for all costs $\gamma\le C(x)$. In particular, this clarifies definition (5.1). However, in contrast with Lemma 4.5, we will have to assume that the A-operation $\oplus$ is strict or $\oplus=\max$.

5.4. Theorem. *Given $S^*\in\mathcal{S}^*$ and $x\in S^*$, if one of the following two conditions* (a) *or* (b) *holds*:

(a) *the A-operation $\oplus$ is strict on $\mathbb{R}^+$, or*
(b) *$\oplus=\max$ on $[0,\infty)$ and either*
 (i) *$C(x)<f_C(\mathcal{S}^*)$, or*
 (ii) *$C(x)=f_C(\mathcal{S}^*)$ and $\mathcal{S}^*=\{S^*\}$, or*
 (iii) *$C(x)=f_C(\mathcal{S}^*)\le F_C(S_x\setminus\{x\})$, or*
 (iv) *$f_C(\mathcal{S}^*)\le F_C(S_x\setminus\{x\})$,*

then $f(C_{x,\gamma})(S^*)\le f(C_{x,\gamma})(S)$ for all $S\in\mathcal{S}$ and $\gamma\in\mathbb{R}^+$ with $\gamma\le C(x)$.

Proof. (a) Let $\oplus$ be strict and $\gamma\in\mathbb{R}^+$, $\gamma\le C(x)$, be arbitrarily fixed. By (3.6), $S^*\in\mathcal{S}$ and $f_C(S^*)\le f_C(S)$ for all $S\in\mathcal{S}$. Since $x\in S^*$ iff $S^*\in\mathcal{S}_x$, (4.4) implies $f(C_{x,\gamma})(S^*)=\gamma\oplus F_C(S^*\setminus\{x\})$. Given $S\in\mathcal{S}$, we have either $S\in\mathcal{S}_{-x}$ or $S\in\mathcal{S}_x$. If $S\in\mathcal{S}_{-x}$, then the monotonicity (A.3) of $\oplus$, (4.5) and (4.4) yield

$$\begin{aligned}f(C_{x,\gamma})(S^*)&=\gamma\oplus F_C(S^*\setminus\{x\})\le C(x)\oplus F_C(S^*\setminus\{x\})=\\&=f_C(S^*)\le f_C(S)=f(C_{x,\gamma})(S).\end{aligned}$$

Now, let $S\in\mathcal{S}_x$, i.e., $x\in S$. Since, by (4.5) and (3.6),

$$C(x)\oplus F_C(S^*\setminus\{x\})=f_C(S^*)\le f_C(S)=C(x)\oplus F_C(S\setminus\{x\}),$$

the strictness of $\oplus$ and the cancellation law (2.3) imply $F_C(S^*\setminus\{x\})\le F_C(S\setminus\{x\})$, and so, by (A.3) and (4.4), we get (even for all $\gamma\in\mathbb{R}^+$)

$$f(C_{x,\gamma})(S^*)=\gamma\oplus F_C(S^*\setminus\{x\})\le\gamma\oplus F_C(S\setminus\{x\})=f(C_{x,\gamma})(S).$$

(b) Suppose $\oplus=\max$ on $[0,\infty)$, and let us fix $0\le\gamma\le C(x)$. Since $S^*\in\mathcal{S}^*$, $f_C(S^*)\le f_C(S)$ for all $S\in\mathcal{S}$. Because $x\in S^*$ iff $S^*\in\mathcal{S}_x$, (4.12) gives

$$f(C_{x,\gamma})(S^*)=\max\{f_C(S^*)+\gamma-C(x),F_C(S^*\setminus\{x\})\}. \tag{5.16}$$

Let $S\in\mathcal{S}$. If $S\in\mathcal{S}_{-x}$, then, by (4.12), $f(C_{x,\gamma})(S)=f_C(S)$. The inequality $f_C(S^*)\le f_C(S)$ can be rewritten as

$$f_C(S^*)=\max\{C(x),F_C(S^*\setminus\{x\})\}\le f_C(S),$$

which implies $F_C(S^*\setminus\{x\})\le f_C(S)$. Inequality $\gamma\le C(x)$ implies (cf. (5.16))

$$f_C(S^*)+\gamma-C(x)\le f_C(S)+\gamma-C(x)\le f_C(S), \tag{5.17}$$

and so, by (5.16),

$$f(C_{x,\gamma})(S^*)\le f_C(S)=f(C_{x,\gamma})(S).$$

Now, suppose that $S\in\mathcal{S}_x$.

(i) Let $C(x)<f_C(\mathcal{S}^*)=f_C(S^*)$. Taking into account (4.12), we have to show that

$$f(C_{x,\gamma})(S^*)\le f(C_{x,\gamma})(S)=\max\{f_C(S)+\gamma-C(x),F_C(S\setminus\{x\})\}. \tag{5.18}$$

Since $f_C(S^*)\le f_C(S)$ implies the left-hand side inequality in (5.17), we have

$$f_C(S^*)+\gamma-C(x)\le f(C_{x,\gamma})(S). \tag{5.19}$$

It follows from inequalities $C(x)<f_C(S^*)$ and

$$f_C(S^*)=\max\{C(x),F_C(S^*\setminus\{x\})\}\le f_C(S)=\max\{C(x),F_C(S\setminus\{x\})\}$$

that $F_C(S^*\setminus\{x\})\le F_C(S\setminus\{x\})\le f(C_{x,\gamma})(S)$, which together with (5.16) and (5.19) implies (5.18).

(ii) Let $C(x)=f_C(\mathcal{S}^*)$ and $\mathcal{S}^*=\{S^*\}$. By virtue of (5.16), we have

$$f(C_{x,\gamma})(S^*)=\max\{\gamma,F_C(S^*\setminus\{x\})\}. \tag{5.20}$$

Two cases are possible for $S\in\mathcal{S}_x$ (cf. (4.8)): 1) $C(x)=f_C(S)$, or 2) $f_C(S)=F_C(S\setminus\{x\})$. If case 1) holds, then $S\in\mathcal{S}^*=\{S^*\}$, and so, $S=S^*$ and inequality (5.18) is clear. In case 2), by virtue of (4.11), we have $\gamma\le F_C(S\setminus\{x\})$, and so,

$$f(C_{x,\gamma})(S)=\max\{\gamma,F_C(S\setminus\{x\})\}=F_C(S\setminus\{x\})=f_C(S). \tag{5.21}$$

Thus, it follows from (5.20), inequality $\gamma\le C(x)$ and (5.21) that

$$f(C_{x,\gamma})(S^*)\le\max\{C(x),F_C(S^*\setminus\{x\})\}=f_C(S^*)\le f_C(S)=f(C_{x,\gamma})(S).$$

(iii) Assume that $C(x)=f_C(\mathcal{S}^*)\le F_C(\mathcal{S}_x\setminus\{x\})$. Then, by (5.2), we find $C(x)\le F_C(S\setminus\{x\})$ for $S\in\mathcal{S}_x$, and so,

$$f_C(S)=\max\{C(x),F_C(S\setminus\{x\})\}=F_C(S\setminus\{x\}). \tag{5.22}$$

The rest of this proof is as in case 2) of step (ii).

(iv) Let $f_C(\mathcal{S}^*)\le F_C(\mathcal{S}_x\setminus\{x\})$. Since $x\in S^*$, $C(x)\le f_C(S^*)\le F_C(S\setminus\{x\})$, and so, (5.22) holds and, by (4.7) and (4.11), $f(C_{x,\gamma})(S)=f_C(S)$. Now, it follows from (5.16) and inequality $\gamma\le C(x)$ that

$$f(C_{x,\gamma})(S^*)\le\max\{f_C(S^*),F_C(S^*\setminus\{x\})\}=f_C(S^*)\le f_C(S)=f(C_{x,\gamma})(S),$$

which was to be proved. $\square$

Now we are in a position to define the notion of the lower tolerance.

5.5. Definition. Suppose $\oplus$ is a strict A-operation on $\mathbb{R}^+$. Given $S^* \in \mathcal{S}^*$ and $x \in X \setminus S^*$, the *lower tolerance* of x is defined by

$$\ell_{S^*}(x) = C(x) \,\overline{\ominus}\, C^-_{S^*}(x) \in \mathbb{R}^+, \tag{5.23}$$

where $\overline{\ominus}$ is the upper subtraction for $\oplus$.

5.6. Theorem. *Let the* A*-operation $\oplus$ be strict on $\mathbb{R}^+$, $S^* \in \mathcal{S}^*$ and $x \in X \setminus S^*$. Then the value $\ell_{S^*}(x)$ is well-defined,*

$$\boldsymbol{e} \le \ell_{S^*}(x) \le f_C(\mathcal{S}^*_x) \,\overline{\ominus}\, f_C(\mathcal{S}^*) \equiv \overline{\ell}_{S^*}(x), \tag{5.24}$$

and $\ell_{S^*}(x) = \boldsymbol{e}$ iff $C^-_{S^*}(x) = C(x)$, where $\boldsymbol{e}$ is the neutral element with respect to $\oplus$.

Moreover, if $\oplus$ is a strict A*-operation of addition on $[0, \infty)$, then*

$$\ell_{S^*}(x) = \begin{cases} f_C(\mathcal{S}^*_x) \,\overline{\ominus}\, f_C(\mathcal{S}^*) & \text{if } F_C(S_x \setminus \{x\}) \le f_C(\mathcal{S}^*), \\ C(x) & \text{if } F_C(S_x \setminus \{x\}) \ge f_C(\mathcal{S}^*), \end{cases} \tag{5.25}$$

and if $\oplus$ is a strict A*-operation of multiplication on $(0, \infty)$, then*

$$\ell_{S^*}(x) = f_C(\mathcal{S}^*_x) \,\overline{\ominus}\, f_C(\mathcal{S}^*) = \overline{\ell}_{S^*}(x). \tag{5.26}$$

Proof. That $\ell_{S^*}(x)$ is well-defined can be established along the same lines as in the proof of Theorem 4.11.

Theorem 5.2(b), (in)equalities $\boldsymbol{e} \oplus C^-_{S^*}(x) = C^-_{S^*}(x) \le C(x)$ and definition (2.4) imply

$$\boldsymbol{e} \le C(x) \,\overline{\ominus}\, C^-_{S^*}(x) = \ell_{S^*}(x).$$

It follows from the definition of $\ell_{S^*}(x)$ and (2.7) that

$$\ell_{S^*}(x) \oplus C^-_{S^*}(x) = (C(x) \,\overline{\ominus}\, C^-_{S^*}(x)) \oplus C^-_{S^*}(x) = C(x). \tag{5.27}$$

If $\ell_{S^*}(x) = \boldsymbol{e}$, then (5.27) implies $C^-_{S^*}(x) = C(x)$; now, if $C^-_{S^*}(x) = C(x)$, then (5.27) yields equality $\ell_{S^*}(x) \oplus C(x) = \boldsymbol{e} \oplus C(x)$, and so, taking into account the strictness of $\oplus$ and cancelling by $C(x)$, we get $\ell_{S^*}(x) = \boldsymbol{e}$.

Let us prove the right-hand side inequality in (5.24). We set

$$u = C(x), \quad v = f_C(\mathcal{S}^*) \quad \text{and} \quad w = f_C(\mathcal{S}^*_x). \tag{5.28}$$

Then it follows from (5.23) and the third equality in (5.7) that

$$\ell_{S^*}(x) = C(x) \,\overline{\ominus}\, C^-_{S^*}(x) = u \,\overline{\ominus}\, [(u \oplus v) \,\underline{\ominus}\, w].$$

By virtue of Lemmas (2.17)(c) and 2.18(a), we find

$$u \,\overline{\ominus}\, [(u \oplus v) \,\underline{\ominus}\, w] \le (u \oplus w) \,\overline{\ominus}\, (u \oplus v) = w \,\overline{\ominus}\, v,$$

and it remains to note that $w \,\overline{\ominus}\, v = f_C(\mathcal{S}^*_x) \,\overline{\ominus}\, f_C(\mathcal{S}^*) = \overline{\ell}_{S^*}(x)$.

Let us establish (5.25). By virtue of (5.7), Lemma 2.11(b) and (5.28), we have

$$C^-_{S^*}(x) = (u \oplus v) \,\underline{\ominus}\, w = (u \oplus v) \,\overline{\ominus}\, w \quad \text{if} \quad u \oplus v \ge w, \tag{5.29}$$

and $C^-_{S^*}(x)=0$ otherwise. Taking into account (5.3), (5.4) and the strictness of $\oplus$, we find that $u\oplus v\ge w$ iff

$$C(x)\oplus f_C(\mathcal{S}^*)\ge f_C(\mathcal{S}^*_x)=C(x)\oplus F_C(S_x\setminus\{x\})\quad\text{iff}\quad f_C(\mathcal{S}^*)\ge F_C(S_x\setminus\{x\}),$$

and so, applying Lemmas 2.17(a) and 2.18(a) as well as definition (5.23), in this case we get

$$\ell_{S^*}(x)=u\,\overline{\ominus}\,[(u\oplus v)\,\overline{\ominus}\,w]=(u\oplus w)\,\overline{\ominus}(u\oplus v)=w\,\overline{\ominus}\,v=f_C(\mathcal{S}^*_x)\,\overline{\ominus}\,f_C(\mathcal{S}^*). \tag{5.30}$$

If $u\oplus v\le w$, i.e., $f_C(\mathcal{S}^*)\le F_C(S_x\setminus\{x\})$, then $\ell_{S^*}(x)=C(x)\,\overline{\ominus}\,0=C(x)$, which establishes equality (5.25).

Finally, let us prove (5.26). If $\oplus$ is an A-operation of multiplication, then equalities in (5.29) follow from (5.7) and Lemma 2.11(a), (c) and hold with no restrictions on u, v and w. This and Lemmas 2.17(a) and 2.18(a) imply equalities (5.30). □

5.7. Examples of lower tolerances. Taking into account Examples 2.12(a), (c) and 3.4(a), (c), (5.7), (5.23), (5.25) and (5.29), given $S^*\in\mathcal{S}^*$ and $x\in X\setminus S^*$, we have:

(a) if $u\oplus v=(u^p+v^p)^{1/p}$ on $\mathbb{R}^+=[0,\infty)$ with $p>0$, then

$$C^-_{S^*}(x)=\big(C(x)^p+f_C(\mathcal{S}^*)^p-f_C(\mathcal{S}^*_x)^p\big)^{1/p}\quad\text{if}\quad F_C(S_x\setminus\{x\})\le f_C(\mathcal{S}^*),$$

and $C^-_{S^*}(x)=0$ otherwise,

$$\ell_{S^*}(x)=\big(f_C(\mathcal{S}^*_x)^p-f_C(\mathcal{S}^*)^p\big)^{1/p}\quad\text{if}\quad F_C(S_x\setminus\{x\})\le f_C(\mathcal{S}^*),$$

and $\ell_{S^*}(x)=C(x)$ otherwise, where (cf. (5.2))

$$F_C(S_x\setminus\{x\})=\min_{S\in\mathcal{S}_x}\Big(\sum_{y\in S\setminus\{x\}}C(y)^p\Big)^{1/p}\quad\text{and}\quad f_C(\mathcal{S}^*)=\Big(\sum_{y\in S^*}C(y)^p\Big)^{1/p};$$

(b) if $u\oplus v=u\cdot v$ on $\mathbb{R}^+=(0,\infty)$ and $C(y)>0$ for all $y\in X$, then

$$C^-_{S^*}(x)=\frac{C(x)\cdot f_C(\mathcal{S}^*)}{f_C(\mathcal{S}^*_x)}\quad\text{and}\quad \ell_{S^*}(x)=\frac{f_C(\mathcal{S}^*_x)}{f_C(\mathcal{S}^*)}=\overline{\ell}_{S^*}(x).$$

5.8. Example. For a strict A-operation of addition $\oplus$ the right-hand side inequality in (5.24) may be strict as can be seen from the following simple example (as well as from formula (5.25)).

We set $X=\{x_1,x_2,x_3,x_4\}$ with $C(x_1)=C(x_2)=C(x_3)=1$ and $C(x_4)=3$, $\mathcal{S}=\{S_1,S_2\}$ with $S_1=\{x_1,x_2\}$ and $S_2=\{x_3,x_4\}$, and $\oplus=+$ (ordinary addition), and so, $f_C(S)=F_C(S)=\sum_{y\in S}C(y)$ for $S\in\mathcal{S}$ (see Table 1).

x	x_1	x_2	x_3	x_4
$C(x)$	1	1	1	3
$u_{S^*}(x)$	2	2		
$\ell_{S^*}(x)$			1	2
$\overline{\ell}_{S^*}(x)$			2	2

Table 1

Since $f_C(S_1)=C(x_1)+C(x_2)=2$ and $f_C(S_2)=C(x_3)+C(x_4)=4$, we find $\mathcal{S}^*=\{S^*\}$ with $S^*=S_1$ and $f_C(\mathcal{S}^*)=f_C(S^*)=2$. Noting that $\mathcal{S}_{-x}=\{S_2\}$ and $f_C(\mathcal{S}^*_{-x})=f_C(S_2)=4$ for $x=x_1,x_2$ and taking into account Examples 2.12(a) and 4.12(a) with $p=1$, we have:

$$u_{S^*}(x)=f_C(\mathcal{S}^*_{-x})-f_C(\mathcal{S}^*)=4-2=2 \quad \text{if} \quad x=x_1,\,x_2.$$

Now, since $S_{x_3}=S_{x_4}=S_2$, $S_{x_3}\setminus\{x_3\}=\{x_4\}$ and $S_{x_4}\setminus\{x_4\}=\{x_3\}$, and so,

$$F_C(S_{x_3}\setminus\{x_3\})=C(x_4)=3>f_C(\mathcal{S}^*) \quad \text{and} \quad F_C(S_{x_4}\setminus\{x_4\})=C(x_3)=1<f_C(\mathcal{S}^*),$$

it follows from (5.25) and Example 5.8(a) that

$$\ell_{S^*}(x_3)=C(x_3)=1 \quad \text{and} \quad \ell_{S^*}(x_4)=f_C(\mathcal{S}^*_{x_4})-f_C(\mathcal{S}^*)=4-2=2.$$

On the other hand, since $\mathcal{S}_x=\{S_2\}$ for $x=x_3,x_4$, we find (cf. (5.24))

$$\overline{\ell}_{S^*}(x)=f_C(\mathcal{S}^*_x)-f_C(\mathcal{S}^*)=4-2=2 \quad \text{if} \quad x=x_3,x_4.$$

5.9. Remark. The quantity $\overline{\ell}_{S^*}(x)$ from (5.24), which may be called the *extended lower tolerance* of x, differs from $\ell_{S^*}(x)$ in the following way. Because $C^-_{S^*}(x)\in\mathbb{R}^+$ and $C^-_{S^*}(x)\le C(x)$, elements of the interval $[C^-_{S^*}(x),C(x)]$ are still *costs* (i.e., nonnegative). Moreover, by virtue of (5.23) and (2.7), $\ell_{S^*}(x)\oplus C^-_{S^*}(x)=C(x)$, and so, $C^-_{S^*}(x)=C(x)\overline{\ominus}\,\ell_{S^*}(x)$ is restored from $\ell_{S^*}(x)$ as the lowest possible stability cost of x. From this point of view the quantity $\widetilde{C}^-_{S^*}(x)=C(x)\overline{\ominus}\,\overline{\ell}_{S^*}(x)$ may not be defined as a cost. For instance, in Example 5.8 we have $C^-_{S^*}(x_3)=1-1=0$, whereas $\widetilde{C}^-_{S^*}(x_3)=1-2=-1\notin\mathbb{R}^+$.

6. Tolerance functions. Throughout this section we assume that the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$ of the form (1.1) is given, $\oplus$ is *strict* and $\mathcal{S}^*=\mathcal{S}^*_C$ is from (3.6).

We are going to define a function T_C on X, which is an *invariant* of the OP under consideration in the sense that it is *independent* of optimal solutions $S^*\in\mathcal{S}^*$.

If $\oplus$ is an A-operation of addition on $\mathbb{R}^+=[0,\infty)$ and $u\ge 0$, then we set $u^{-1}=-u$, and if $\oplus$ is an A-operation of multiplication on $\mathbb{R}^+=(0,\infty)$ with the neutral element $\boldsymbol{e}\in\mathbb{R}^+$ and $u>0$, then, taking into account Lemma 2.11(c), we set $u^{-1}=\boldsymbol{e}\,\overline{\ominus}\,u$. It is to be noted that in the latter case we have

$$u^{-1}\oplus u=\boldsymbol{e}, \quad \boldsymbol{e}^{-1}=\boldsymbol{e}, \quad (u^{-1})^{-1}=u, \quad \text{and} \quad u>\boldsymbol{e} \;\text{ iff }\; u^{-1}<\boldsymbol{e}. \tag{6.1}$$

The last three properties in (6.1) also hold in the former case with $\boldsymbol{e}=0$.

6.1. Definition. Given $S^*\in\mathcal{S}^*$ and $x\in X$, we set

$$T_C(x)=\begin{cases} u_{S^*}(x) & \text{if } x\in S^*,\\ (\ell_{S^*}(x))^{-1} & \text{if } x\in X\setminus S^*.\end{cases} \tag{6.2}$$

The function T_C on X is said to be the *tolerance function* of the $\mathrm{OP}(X,\mathcal{S},\oplus,C)$. Replacing $\ell_{S^*}(x)$ by $\overline{\ell}_{S^*}(x)$ in the second line of (6.2) we get the notion of the *extended tolerance function*, denoted by $\overline{T}_C(x)$. Both tolerance functions assume their values in $\mathbb{R}$ if $\oplus$ is an A-operation of addition and in $\mathbb{R}^+=(0,\infty)$ if $\oplus$ is an A-operation of multiplication. Note also that $T_C(x)$ can be represented as

$$T_C(x)=u_{S^*}(x)\chi_{S^*}(x)+(\ell_{S^*}(x))^{-1}\chi_{X\setminus S^*}(x), \qquad x\in X,$$

where, given $Y\subset X$, χ_Y is the characteristic function of the set Y (i.e., $\chi_Y(x)=1$ if $x\in Y$ and $\chi_Y(x)=0$ if $x\in X\setminus Y$).

The correctness of this definition is justified by the following

6.2. Theorem. *The tolerance function T_C on X is well-defined, independent of optimal solutions $S^*\in\mathcal{S}^*$ and has the following properties*:

$$T_C|_{S^*}(\cdot)=u_{S^*}(\cdot)\geq\boldsymbol{e}\quad and\quad T_C|_{X\setminus S^*}(\cdot)=(\ell_{S^*}(\cdot))^{-1}\leq\boldsymbol{e}\quad for\ all\ S^*\in\mathcal{S}^*,\tag{6.3}$$

where $T_C|_Y(\cdot)$ denotes the restriction of T_C to the set $Y\subset X$.

In order to prove Theorem 6.2, we need a lemma, which is of interest in its own.

6.3. Lemma. *Given $S_1^*,S_2^*\in\mathcal{S}^*$ and $x\in X$, we have*:
(a) *if $x\in S_1^*\cap S_2^*$, then $u_{S_1^*}(x)=u_{S_2^*}(x)$ (and $\geq\boldsymbol{e}$)*;
(b) *if $x\in(X\setminus S_1^*)\cap(X\setminus S_2^*)$, then $\ell_{S_1^*}(x)=\ell_{S_2^*}(x)$ (and $\geq\boldsymbol{e}$)*;
(c) *if $x\in S_1^*\setminus S_2^*$, then $u_{S_1^*}(x)=\boldsymbol{e}=\ell_{S_2^*}(x)$.*

Proof. (a) By virtue of (4.29), we have

$$u_{S_1^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)=u_{S_2^*}(x).$$

(b) It follows from (5.23) and (5.7) that

$$\begin{aligned}\ell_{S_1^*}(x)&=C(x)\,\overline{\ominus}\,C^-_{S_1^*}(x)=C(x)\,\overline{\ominus}\big([C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}^*_x)\big)=\\&=C(x)\,\overline{\ominus}\,C^-_{S_2^*}(x)=\ell_{S_2^*}(x).\end{aligned}$$

(c) Since $x\in S_1^*$ iff $S_1^*\in\mathcal{S}_x$, and $x\in X\setminus S_2^*$ iff $S_2^*\in\mathcal{S}_{-x}$, (4.16) and (3.7) imply $f_C(\mathcal{S}^*_{-x})\leq f_C(S_2^*)=f_C(\mathcal{S}^*)$. Now, it follows from Theorem 4.11 and Lemma 2.15(a) that

$$\boldsymbol{e}\leq u_{S_1^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*)\leq f_C(\mathcal{S}^*)\,\overline{\ominus}\,f_C(\mathcal{S}^*)=\boldsymbol{e},$$

and so, $u_{S_1^*}(x)=\boldsymbol{e}$. At the same time, by virtue of (4.16), we find $f_C(\mathcal{S}^*_x)\leq f_C(S_1^*)=f_C(\mathcal{S}^*)$, and so, Lemma 2.16(b), the strictness of $\oplus$ and (2.9) yield

$$[C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}^*_x)\geq[C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}^*)=C(x).$$

Now, it follows from Theorem 5.6, (5.23), (5.7) and Lemma 2.16(a) that

$$\boldsymbol{e}\leq\ell_{S_2^*}(x)=C(x)\,\overline{\ominus}\big([C(x)\oplus f_C(\mathcal{S}^*)]\,\underline{\ominus}\,f_C(\mathcal{S}^*_x)\big)\leq C(x)\,\overline{\ominus}\,C(x)=\boldsymbol{e},$$

which implies the equality $\ell_{S_2^*}(x)=\boldsymbol{e}$. □

Now we are in a position to prove Theorem 6.2.

Proof of Theorem 6.2. Formally, the tolerance function, as it is defined in (6.2), depends on $S^*\in\mathcal{S}^*$, and so, we temporarily write T_{C,S^*} in place of T_C. Only the case $|\mathcal{S}^*|\geq 2$ is to be considered, and so, assuming that $S_1^*,S_2^*\in\mathcal{S}^*$ are arbitrarily chosen, let us show that $T_{C,S_1^*}(x)=T_{C,S_2^*}(x)$ for all $x\in X$.

In fact, we have the following decomposition of the set X:

$$X=[S_1^*\cap S_2^*]\cup[(X\setminus S_1^*)\cap(X\setminus S_2^*)]\cup[(S_1^*\cup S_2^*)\setminus(S_1^*\cap S_2^*)],$$

where, by de Morgan's laws, $(X\setminus S_1^*)\cap(X\setminus S_2^*)=X\setminus(S_1^*\cup S_2^*)$, and the sets on the right in square brackets are pairwise disjoint. Suppose $x\in X$. If $x\in S_1^*\cap S_2^*$, then (6.2) and Lemma 6.3(a) imply

$$T_{C,S_1^*}(x)=u_{S_1^*}(x)=u_{S_2^*}(x)=T_{C,S_2^*}(x).$$

If $x\in(X\setminus S_1^*)\cap(X\setminus S_2^*)$, then it follows from (6.2) and Lemma 6.3(b) that

$$T_{C,S_1^*}(x)=(\ell_{S_1^*}(x))^{-1}=(\ell_{S_2^*}(x))^{-1}=T_{C,S_2^*}(x).$$

Now, assume that $x\in(S_1^*\cup S_2^*)\setminus(S_1^*\cap S_2^*)=(S_1^*\setminus S_2^*)\cup(S_2^*\setminus S_1^*)$ and, to be more specific, $x\in S_1^*\setminus S_2^*$. Then, by virtue of (6.2), Lemma 6.3(c) and (6.1), we find

$$T_{C,S_1^*}(x)=u_{S_1^*}(x)=\boldsymbol{e}=\boldsymbol{e}^{-1}=(\ell_{S_2^*}(x))^{-1}=T_{C,S_2^*}(x).$$

Now it is correct to set back $T_C(x)=T_{C,S^*}(x)$ for all $x\in X$ and $S^*\in\mathcal{S}^*$, which together with (6.2) implies $T_C|_{S^*}(x)=T_C(x)=u_{S^*}(x)$ if $x\in S^*$ and $T_C|_{X\setminus S^*}(x)=T_C(x)=(\ell_{S^*}(x))^{-1}$ if $x\in X\setminus S^*$. This completes the proof. □

6.4. Remark. An assertion similar to Theorem 6.2 holds for the extended tolerance function $\overline{T}_C$ with obvious modifications. In fact, it is to be noted only that in Lemma 6.3(b) we have, by virtue of (5.24), $\overline{\ell}_{S_1^*}(x)=f_C(\mathcal{S}_x^*)\,\overline{\ominus}\,f_C(\mathcal{S}^*)=\overline{\ell}_{S_2^*}(x)$, and under conditions of Lemma 6.3(c), we get

$$\boldsymbol{e}\le\overline{\ell}_{S_2^*}(x)=f_C(\mathcal{S}_x^*)\,\overline{\ominus}\,f_C(\mathcal{S}^*)\le f_C(\mathcal{S}^*)\,\overline{\ominus}\,f_C(\mathcal{S}^*)=\boldsymbol{e}.$$

6.5. Example. Let us illustrate Theorem 6.2 by the following example of small cardinality $|X|$ and the simplest possible A-operation $\oplus=+$ (so that calculations are not cumbersome and all the details can be clearly seen).

Let $X=\{x_1,x_2,x_3,x_4,x_5,x_6\}$ with $C(x_1)=C(x_2)=C(x_3)=2$, $C(x_4)=1$, $C(x_5)=3$ and $C(x_6)=5$, and $\mathcal{S}=\{S_1,S_2,S_3\}$ with $S_1=\{x_1,x_2,x_3\}$, $S_2=\{x_2,x_4,x_5\}$ and $S_3=\{x_1,x_4,x_6\}$. Since the objective function is of the form $f_C(S)=\sum_{y\in S}C(y)$, $S\in\mathcal{S}$, we find $f_C(S_1)=f_C(S_2)=6$ and $f_C(S_3)=8$, and so, $\mathcal{S}^*=\{S_1^*,S_2^*\}$ with $S_1^*=S_1$ and $S_2^*=S_2$. The corresponding values of upper and lower tolerances and the tolerance function are presented in the following Table 2:

x	x_1	x_2	x_3	x_4	x_5	x_6
$C(x)$	2	2	2	1	3	5
S_1	*	*	*			
S_2		*		*	*	
S_3	*			*		*
$u_{S_1^*}(x)$	0	2	0			
$\ell_{S_1^*}(x)$				0	0	2
$T_C(x)$	0	2	0	0	0	−2
$u_{S_2^*}(x)$		2		0	0	
$\ell_{S_2^*}(x)$	0		0			2

Table 2

Given $i\in\{1,2,3\}$ and $j\in\{1,2,3,4,5,6\}$, we put $*$ in row S_i and column x_j provided $x_j\in S_i$. Then setting $S^*=S_1^*$ we calculate the values $u_{S_1^*}(x)$ for $x\in S_1^*$ and $\ell_{S_1^*}(x)$ for $x\in X\setminus S_1^*$ in accordance with Theorems 4.11 and 5.6 and Lemma 6.3(c). By virtue of (6.2), we form the tolerance function

$$T_C(x)=\big(T_C(x_1),T_C(x_2),T_C(x_3),T_C(x_4),T_C(x_5),T_C(x_6)\big)=(0,2,0,0,0,-2).$$

Now, making use of Theorem 6.2 and taking into account row S_2, in which elements of the optimal solution S_2^* and outside of it are marked, we extract from the vector $T_C(x)$ the corresponding values of $u_{S_2^*}(x)$ for $x\in S_2^*$ and $\ell_{S_2^*}(x)$ for $x\in X\setminus S_2^*$.

Thus, Theorem 6.2 says that only upper and lower tolerances with respect to any fixed optimal solution to the OP under consideration are to be calculated, the other tolerances being determined uniquely via the tolerance function.

6.6. Convention. In what follows we assume that $C(x)>0$ for all $x\in X$.

Definition 6.1 is also motivated by the fact that the set of optimal solutions $\mathcal{S}^*$ to the discrete optimization problem (1.1) can be characterized by means of the tolerance function(s) in the following way.

6.7. Theorem. (a) $\{x\in X:T_C(x)=\boldsymbol{e}\}=(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)$.
(b) $\{x\in X:T_C(x)>\boldsymbol{e}\}=\cap\mathcal{S}^*$.
(c) $\{x\in X:T_C(x)\ge\boldsymbol{e}\}=\cup\mathcal{S}^*$.
(d) $\{x\in X:T_C(x)<\boldsymbol{e}\}=X\setminus(\cup\mathcal{S}^*)$.
(e) $\{x\in X:T_C(x)\le\boldsymbol{e}\}=X\setminus(\cap\mathcal{S}^*)$.

Proof. (a) ($\supset$) If $x\in(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)$, then there exist two optimal trajectories $S_1^*,S_2^*\in\mathcal{S}^*$ such that $x\in S_1^*\setminus S_2^*$, and so, by (6.2) and Lemma 6.3(c), we get

$$T_C(x)=u_{S_1^*}(x)=(\ell_{S_2^*}(x))^{-1}=\boldsymbol{e}.$$

(a) ($\subset$) Suppose $x\in X$ and $T_C(x)=\boldsymbol{e}$. Let us fix an $S^*\in\mathcal{S}^*$. If $x\in S^*$, then, by virtue of (6.2) and (4.29), we have

$$\boldsymbol{e}=T_C(x)=u_{S^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*),$$

which implies $f_C(\mathcal{S}^*_{-x})=f_C(\mathcal{S}^*)$ (cf. (2.7)). Taking into account (4.16), we may choose $S_1\in\mathcal{S}_{-x}$ such that $f_C(S_1)=f_C(\mathcal{S}^*)$, whence $S_1\in\mathcal{S}^*$. Since $x\notin S_1$ and $x\in S^*$, we get $x\in S^*\setminus S_1\subset(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)$.

Now, assume that $x\notin S^*$. By virtue of (6.2), (5.25) and (5.26), we claim that

$$\boldsymbol{e}=\boldsymbol{e}^{-1}=(T_C(x))^{-1}=\ell_{S^*}(x)=f_C(\mathcal{S}^*_x)\,\overline{\ominus}\,f_C(\mathcal{S}^*). \tag{6.4}$$

Only the last equality in (6.4) is to be justified in the case when $\oplus$ is an A-operation of addition on $[0,\infty)$: on the contrary, if this is not so, then (5.25) and (6.2) imply

$$C(x)=\ell_{S^*}(x)=(T_C(x))^{-1}=\boldsymbol{e}^{-1}=\boldsymbol{e}=0,$$

which contradicts our convention 6.6. Making use of (6.4), we get $f_C(\mathcal{S}^*_x)=f_C(\mathcal{S}^*)$, and so, by virtue of (4.16), there exists $S_2\in\mathcal{S}_x$ such that $f_C(S_2)=f_C(\mathcal{S}^*)$ and, hence, $S_2\in\mathcal{S}^*$. Since $x\in S_2$ and $x\notin S^*$, we find $x\in S_2\setminus S^*\subset(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)$.

(b) ($\subset$) Let $x\in X$ and $T_C(x)>\boldsymbol{e}$. We claim that $x\in\cap\mathcal{S}^*$. On the contrary, assume that $x\notin\cap\mathcal{S}^*$, and so, $x\notin S^*$ for some $S^*\in\mathcal{S}^*$. By Theorem 5.6, we have $\ell_{S^*}(x)\ge\boldsymbol{e}$, and so, (6.2) gives $T_C(x)=(\ell_{S^*}(x))^{-1}\le\boldsymbol{e}$, which is a contradiction.

(b) ($\supset$) Let $x\in\cap\mathcal{S}^*$. Choose an $S^*\in\mathcal{S}^*$. Since $x\in S^*$, (6.2) and (4.29) imply $T_C(x)=u_{S^*}(x)\ge\boldsymbol{e}$. Taking into account item (a), we infer that $T_C(x)>\boldsymbol{e}$.

(c) is a consequence of (a) and (b): $T_C(x)\ge\boldsymbol{e}$ iff $T_C(x)=\boldsymbol{e}$ or $T_C(x)>\boldsymbol{e}$, i.e., iff $x\in(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)$ or $x\in\cap\mathcal{S}^*$.

(d) follows immediately from (c):

$$\{x\in X:T_C(x)<\boldsymbol{e}\}=X\setminus\{x\in X:T_C(x)\ge\boldsymbol{e}\}=X\setminus(\cup\mathcal{S}^*).$$

(e) is a straightforward consequence of (b). $\square$

Tolerance functions can be effectively applied for the characterization of uniqueness and nonuniqueness of optimal trajectories:

6.8. Corollary. (a) *The* OP (1.1) *admits a unique optimal solution* (*i.e.*, $|\mathcal{S}^*|=1$) *if and only if* $T_C(\cdot)\neq \boldsymbol{e}$ *on* X.

(b) $|\mathcal{S}^*|\geq 2$ *iff* $T_C(x)=\boldsymbol{e}$ *for some* $x\in X$.

Proof. (a) By virtue of Theorem 6.7(a), $|\mathcal{S}^*|=1$ iff $\cup\mathcal{S}^*=\cap\mathcal{S}^*$ iff $(\cup\mathcal{S}^*)\setminus(\cap\mathcal{S}^*)=\varnothing$ iff $T_C(x)\neq\boldsymbol{e}$ for all $x\in X$.

(b) is simply the negation of item (a). □

At the end of this section we are going to establish certain relationships between the values of T_C on $S^*\in\mathcal{S}^*$ and on $X\setminus S^*$.

6.9. Covering trajectories. Given $Y\subset X$, it is convenient to introduce the collection $\mathcal{S}_c(Y)$ (possibly, empty) of those trajectories $S\in\mathcal{S}$, which cover the set Y:

$$\mathcal{S}_c(Y)=\{S\in\mathcal{S}: Y\subset S \text{ and } S\neq Y\}.$$

It is to be noted that $\mathcal{S}_c(Y)=\varnothing$ iff $Y\setminus S\neq\varnothing$ for all $S\in\mathcal{S}$ with $S\neq Y$, and if $\mathcal{S}_c(Y)\neq\varnothing$, then $S\notin\mathcal{S}_c(Y)$ iff $Y\setminus S\neq\varnothing$.

We say that the set of trajectories $\mathcal{S}$ *consists of nonembedded sets* provided that $\mathcal{S}_c(S)=\varnothing$ for all $S\in\mathcal{S}$. In other words (cf. [5, Theorem 1]), the last condition is equivalent to saying that $S_1\setminus S_2\neq\varnothing$ for all $S_1,S_2\in\mathcal{S}$, $S_1\neq S_2$. For instance, in Examples 5.8 and 6.5 collections of trajectories $\mathcal{S}$ consist of nonembedded sets.

6.10. Theorem. *Assume that* $C(x)>0$ *for all* $x\in X$ *if* $\oplus$ *is an* A*-operation of addition on* $\mathbb{R}^+=[0,\infty)$ *and* $C(x)\geq\boldsymbol{e}$ *for all* $x\in X$ *if* $\oplus$ *is an* A*-operation of multiplication on* $\mathbb{R}^+=(0,\infty)$. *If* $S^*\in\mathcal{S}^*$ *is the unique optimal solution to the* OP (1.1), *then we have the inequalities*

$$\min_{y\in X\setminus S^*}(T_C(y))^{-1}\leq \min_{y\in X\setminus S^*}(\overline{T}_C(y))^{-1}\leq\min_{x\in S^*}T_C(x) \tag{6.5}$$

and

$$\min_{x\in S^*}T_C(x)\leq \min_{y\in X\setminus[S^*\cup(\cup\mathcal{S}_c(S^*))]}(\overline{T}_C(y))^{-1}. \tag{6.6}$$

Proof. 1. We begin by proving the right-hand side inequality in (6.5) (the left-hand side inequality in (6.5) is always valid by virtue of (6.2) and (5.24)). Given $x\in S^*$, it follows from (6.2) and (4.29) that

$$T_C(x)=u_{S^*}(x)=f_C(\mathcal{S}^*_{-x})\,\overline{\ominus}\,f_C(\mathcal{S}^*),$$

and so, by virtue of (4.16), there exists $S_1\in\mathcal{S}_{-x}$ such that

$$T_C(x)=f_C(S_1)\,\overline{\ominus}\,f_C(\mathcal{S}^*).$$

Also, it follows from Theorem 6.7(c) and Corollary 6.8(a) that $T_C(x)>\boldsymbol{e}$, and so, by (2.7) and (A.3$_\mathrm{s}$),

$$f_C(S_1)=T_C(x)\oplus f_C(\mathcal{S}^*)>\boldsymbol{e}\oplus f_C(\mathcal{S}^*)=f_C(\mathcal{S}^*), \tag{6.7}$$

i.e., $S_1\notin\mathcal{S}^*$. We claim that $S_1\setminus S^*\neq\varnothing$. On the contrary, assume that $S_1\setminus S^*=\varnothing$, and so, $S_1\subset S^*$, say, $S_1=\{x_1,\dots,x_n\}$ and $S^*=\{x_1,\dots,x_n,y_1,\dots,y_m\}$. Then

$$f_C(S_1)=\bigoplus_{i=1}^n C(x_i)=\left(\bigoplus_{i=1}^n C(x_i)\right)\oplus\left(\bigoplus_{j=1}^m \boldsymbol{e}\right)\leq$$
$$\leq\left(\bigoplus_{i=1}^n C(x_i)\right)\oplus\left(\bigoplus_{j=1}^m C(y_j)\right)=f_C(S^*)=f_C(\mathcal{S}^*),$$

which contradicts to inequality (6.7). Now, pick $y_0 \in S_1 \setminus S^*$. Then $y_0 \in S_1$ and $y_0 \notin S^*$ or, in other words, $S_1 \in \mathcal{S}_{y_0}$ and $y_0 \in X \setminus S^*$. By virtue of (6.2), (5.24), (4.16) and Lemma 2.15(a), we get

$$\min_{y \in X \setminus S^*} (\overline{T}_C(y))^{-1} \leq (\overline{T}_C(y_0))^{-1} = \overline{\ell}_{S^*}(y_0) = f_C(\mathcal{S}^*_{y_0}) \overline{\ominus} f_C(\mathcal{S}^*) \leq \\ \leq f_C(S_1) \overline{\ominus} f_C(\mathcal{S}^*) = T_C(x),$$

from which the right-hand side inequality in (6.5) follows if we take into account the arbitrariness of $x \in S^*$.

2. Now we establish inequality (6.6). Let $y \in X \setminus [S^* \cup (\cup \mathcal{S}_c(S^*))]$. Since $y \in X \setminus S^*$, it follows from (6.2) and (5.24) that

$$(\overline{T}_C(y))^{-1} = \overline{\ell}_{S^*}(y) = f_C(\mathcal{S}^*_y) \overline{\ominus} f_C(\mathcal{S}^*),$$

and so, by (4.16), there exists $S_2 \in \mathcal{S}_y$ (i.e., $S_2 \in \mathcal{S}$ and $y \in S_2$) such that

$$(\overline{T}_C(y))^{-1} = f_C(S_2) \overline{\ominus} f_C(\mathcal{S}^*).$$

By Theorem 6.7(e) and Corollary 6.8(a), $T_C(y) < \boldsymbol{e}$, and so, (6.1), (2.7) and (A.3$_\mathrm{s}$) yield that $(\overline{T}_C(y))^{-1} > \boldsymbol{e}$ and

$$f_C(S_2) = (\overline{T}_C(y))^{-1} \oplus f_C(\mathcal{S}^*) > \boldsymbol{e} \oplus f_C(\mathcal{S}^*) = f_C(\mathcal{S}^*)$$

i.e., $S_2 \notin \mathcal{S}^*$ implying $S_2 \neq S^*$. We claim that $S^* \setminus S_2 \neq \varnothing$. There are two possibilities: either $\mathcal{S}_c(S^*) = \varnothing$ or $\mathcal{S}_c(S^*) \neq \varnothing$. If $\mathcal{S}_c(S^*) = \varnothing$, then no $S \in \mathcal{S}$, $S \neq S^*$, covers S^*, and so, $S^* \setminus S_2 \neq \varnothing$. Assume that $\mathcal{S}_c(S^*) \neq \varnothing$. Because $y \notin \cup \mathcal{S}_c(S^*)$, we have $y \notin S$ for all $S \in \mathcal{S}_c(S^*)$ (i.e., for all $S \in \mathcal{S}$ such that $S^* \subset S$ and $S \neq S^*$). Taking into account that $y \in S_2$ and $S_2 \neq S^*$, we find $S_2 \notin \mathcal{S}_c(S^*)$, and so, S_2 does not cover S^* and $S^* \setminus S_2 \neq \varnothing$. Now, choose an $x_0 \in S^* \setminus S_2$. This gives $x_0 \in S^*$ and $S_2 \in \mathcal{S}_{-x_0}$, and so, applying (6.2), (4.29), (4.16) and Lemma 2.15(a), we find

$$\min_{x \in S^*} T_C(x) \leq T_C(x_0) = u_{S^*}(x_0) = f_C(\mathcal{S}^*_{-x_0}) \overline{\ominus} f_C(\mathcal{S}^*) \leq \\ \leq f_C(S_2) \overline{\ominus} f_C(\mathcal{S}^*) = (\overline{T}_C(y))^{-1},$$

and it remains to take into account the arbitrariness of y as above. □

6.11. Corollary. *Under the assumptions of Theorem* 6.10*, if the set of trajectories* $\mathcal{S}$ *consists of nonembedded sets, then*

$$\min_{y \in X \setminus S^*} (T_C(y))^{-1} \leq \min_{y \in X \setminus S^*} (\overline{T}_C(y))^{-1} = \min_{x \in S^*} T_C(x),$$

and in the case of the A*-operation of multiplication* $\oplus$ *on* $\mathbb{R}^+ = (0, \infty)$ *the inequality* $\leq$ *above turns out to be the equality.*

Proof. Since $\mathcal{S}_c(S^*) = \varnothing$, the (in)equalities follow from (6.5) and (6.6). □

A discussion of issues as in Theorem 6.10 is presented in [5, Sections 5–7] in the case $\oplus = +$ (cf. also [8]). Also, Examples 1 and 2 from [5] show that inequalities (6.5) and (6.6) may be strict.

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