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The Groups of Basic Automorphisms of Complete Cartan Foliations

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Abstract—For a complete Cartan foliation (M, F) we introduce two algebraic invariants $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_1(M, F)$ which we call structure Lie algebras. If the transverse Cartan geometry of (M, F) is effective then $\mathfrak{g}_0(M, F) = \mathfrak{g}_1(M, F)$. We prove that if $\mathfrak{g}_0(M, F)$ is zero then in the category of Cartan foliations the group of all basic automorphisms of the foliation (M, F) admits a unique structure of a finite-dimensional Lie group. In particular, we obtain sufficient conditions for this group to be discrete. We give some exact (i.e. best possible) estimates of the dimension of this group depending on the transverse geometry and topology of leaves. We construct several examples of groups of all basic automorphisms of complete Cartan foliations.

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1. INTRODUCTION

The automorphism group is associated with every object of a category. One of the central problems is the question whether the group of all automorphisms of the object may be endowed with a finite-dimensional Lie group structure [1] (see also [2]).

According to the results of Cartan, Myers and Steenrod, Nomizy, Kobayashi, Ehresmann and others the groups of all automorphisms of different geometries are often Lie groups of transformations (an overview can be found, for example, in [3] and [1]).

In the theory of foliations with transverse geometries an isomorphism is a diffeomorphism which maps leaves onto leaves and preserves transverse geometries. We study the category of foliations with transverse Cartan geometries which are referred to as Cartan foliations.

When investigating foliations (M, F) admit a transverse geometry ξ it is natural ask if there exists a finite-dimensional Lie group structure for the group of all basic automorphisms of (M, F).

Leslie [4] was the first who solved a similar problem for smooth foliations on compact manifolds. For foliations with complete transversal projectable affine connection this problem was formulated by Belko [5]. Groups of basic automorphisms of complete foliations (M, F) with effective transverse rigid geometries were investigated in [6].

Ineffective Cartan geometries have nontrivial gauge groups which are very important in theoretical physics [7]. For example, spin geometries are based on ineffective models. Some parabolic geometry of rang one has finite gauge groups [8].

The goal of this paper is to give sufficient conditions for the group of basic automorphisms of complete Cartan foliations to admit a finite-dimensional Lie group structure without assumption of effectiveness of the transverse Cartan geometry.

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We wish to stress that in our work the gauge group of a transverse Cartan geometry may be indiscrete in general case which is in contrast to [9] where assumptions imply the discreteness of the gauge group.

Assumptions. Throughout this paper we assume the manifolds and the maps to be C^∞ smooth and all neighborhoods to be open.

Notations. Let $\mathfrak{X}(N)$ denote the Lie algebra of smooth vector fields on a manifold N. If $\mathfrak{M}$ is a smooth distribution on M and $f : K \to M$ is a submersion then let $f^*\mathfrak{M}$ be the distribution on the manifold K such that $(f^*\mathfrak{M})_z := \{X \in T_zK | f_{*z}(X) \in \mathfrak{M}_{f(z)}\}$ where $z \in K$. Let $\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \forall u \in M\}$.

Let $\mathfrak{Fol}$ be the category of foliations whose morphisms are smooth maps sending leaves into leaves.

If $\alpha : G_1 \to G_2$ is a group homomorphism, then $Im(\alpha) := \alpha(G_1)$. Let the symbol $\cong$ denotes an isomorphism of objects in any category (according to the context).

We denote $P(N, H)$ a principal H-bundle over the manifold N with the projection $P \to N$.

2. CATEGORIES OF CARTAN FOLIATIONS AND FOLIATED BUNDLES

Spaces, now called Cartan geometries ([1, 9] and [10]), were introduced by Elie Cartan in the 1920s under the name of *espaces généralisés*.

Let G be a Lie group and H be a closed subgroup of G. Denote $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups G and H respectively. Let N be a smooth manifold. *A Cartan geometry* on N of type (G, H) (or of type $\mathfrak{g}/\mathfrak{h}$) is a principal H-bundle $P(N, H)$ with the projection $p : P \to N$ together with a $\mathfrak{g}$-valued 1-form ω on P such that ω is nondegenerate and H-equivariant, and $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field defined by the element A.

The Cartan geometry is denoted $\xi = (P(N, H), \omega)$. The pair (N, ξ) is called a *Cartan manifold*.

Cartan geometry $\xi = (P(N, H), \omega)$ is referred to as complete if any vector field $X \in \mathfrak{X}(P)$ such that $\omega(X) = const$ is complete [1].

Let $\xi = (P(N, H), \omega)$ and $\xi' = (P'(N', H), \omega')$ be two Cartan geometries of same type (G, H). A smooth map $\Gamma : P \to P'$ is called a morphism from ξ to ξ' if $\Gamma^*\omega' = \omega$ and $R_a \circ \Gamma = \Gamma \circ R_a, a \in H$. If $\Gamma \in Mor(\xi, \xi')$ then the projection $\gamma : N \to N'$ is defined such that $p' \circ \Gamma = \gamma \circ p$ where $p : P \to N$ and $p' : P' \to N'$ are the projections of the corresponding H-bundles. When $\xi = \xi'$ the projection γ is called an *automorphism* of the Cartan manifold (N, ξ). Denote $Aut(N, \xi)$ the group of all automorphisms group of (N, ξ). The category of Cartan geometries is denoted $\mathfrak{Car}$. Let $A(P, \omega) := \{\Gamma \in Diff(P) | \Gamma^*\omega = \omega\}$ be the automorphism group of the parallelizable manifold (P, ω). Let $A^H(P, \omega) := \{\Gamma \in A(P, \omega) | \Gamma \circ R_a = R_a \circ \Gamma \,\forall a \in H\}$, then $A^H(P, \omega)$ is a closed Lie subgroup of the Lie group $A(P, \omega)$ and $Aut(\xi) = A^H(P, \omega)$ is the automorphism group of the Cartan geometry ξ. The map $\sigma : A^H(P, \omega) \to Aut(N, \xi) : \Gamma \mapsto \gamma$ sending Γ to its projection γ is a Lie group epimorphism. The group $\mathrm{Gau}(\xi) := ker(\sigma) = \{\Gamma \in Aut(\xi) \mid p \circ \Gamma = p\}$ is called *of the gauge transformation group of the Cartan geometry* ξ. Remark that a Cartan geometry ξ is effective if and only if the gauge transformation group $\mathrm{Gau}(\xi)$ is trivial.

We use the notion of a Cartan foliation introduced in [11]. As parabolic, conformal, Weyl, projective, transversally homogeneous, pseudo-Riemannian, Lorentzian, Riemannian foliations and foliations with transverse linear connection are Cartan foliations, our results are valid for all these classes of foliations.

For a Cartan foliation (M, F) modelled on a Cartan geometry $\xi = (P(N, H), \omega)$ of type $\mathfrak{g}/\mathfrak{h}$ according to ([11], Proposition 2) there exists the principal H-bundle with the projection $\pi : \mathcal{R} \to M$, the H-invariant foliation $(\mathcal{R}, \mathcal{F})$ and the $\mathfrak{g}$-valued H-equivariant 1-form β on $\mathcal{R}$ such that:

(i) the projection $\pi : \mathcal{R} \to M$ is a morphism $(\mathcal{R}, \mathcal{F}) \to (M, F)$ in the category of foliations $\mathfrak{Fol}$, and the restriction $\pi|_{\mathcal{L}} : \mathcal{L} \to L$ on a leaf $\mathcal{L}$ of $(\mathcal{R}, \mathcal{F})$ is a regular covering map onto some leaf L of (M, F);

(ii) $\beta(A^*) = A$ for any $A \in \mathfrak{h}$;

(iii) the map $\beta_u : T_u\mathcal{R} \to \mathfrak{g} \;\forall u \in \mathcal{R}$ is surjective, and $ker(\beta_u) = T_u\mathcal{F}$;

(iv) the foliation $(\mathcal{R}, \mathcal{F})$ is e-foliation;

(v) the Lie derivative $L_X\beta$ is equal to zero for every vector field X tangent to the foliation $(\mathcal{R}, \mathcal{F})$.

Definition 1. *The principal H-bundle $\mathcal{R}(M, H)$ is said to be the foliated bundle. The foliation $(\mathcal{R}, \mathcal{F})$ is called the lifted foliation for (M, F).*

If $\mathcal{R}$ is disconnected, then we consider a connected component of $\mathcal{R}$.

Let (M, F) and (M', F') are Cartan foliations modeled on transverse Cartan geometries $\xi = (P(N, H), \omega)$ and $\xi' = (P'(N', H), \omega')$ of same type $\mathfrak{g}/\mathfrak{h}$. Let $\pi : \mathcal{R} \to M$ and $\pi' : \mathcal{R}' \to M'$ be their foliated bundles with the lifted foliations $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$. Denote β and β' the $\mathfrak{g}$-valued 1-forms defined on $\mathcal{R}$ and $\mathcal{R}'$ indicated above.

Definition 2. *A smooth map* $\Gamma : \mathcal{R} \to \mathcal{R}'$ *is referred to as a morphism of the Cartan foliations* (M, F) *and* (M', F') *of the same type* (G, H) *if* Γ *is a morphism of the foliated* H*-bundles* $\mathcal{R}(M, H)$ *and* $\mathcal{R}'(M', H)$ *such that* $\Gamma^*\beta' = \beta$.

Note that Γ is a morphism of $(\mathcal{R}, \mathcal{F})$ to $(\mathcal{R}', \mathcal{F}')$ in the category of foliations. Let us denote $\mathfrak{CF}$ *the category of Cartan foliations*. In the case when every Cartan foliation has the dimension zero the category of Cartan foliations $\mathfrak{CF}$ coincides with the category $\mathfrak{Car}$.

Denote $A^{\xi}(M, F)$ the group of all automorphisms of a Cartan foliation (M, F) with a transverse Cartan geometry ξ in the category $\mathfrak{CF}$. Since every $\Gamma \in A^{\xi}(M, F)$ is an automorphism of the H-bundle $\mathcal{R}(M, H)$, there exists the projection $\gamma \in Diff(M)$ satisfying the equality $\pi \circ \Gamma = \gamma \circ \pi$. The natural homomorphism of the groups $\nu : A^{\xi}(M, F) \to Diff(M) : \Gamma \mapsto \gamma$ is defined. Let $A(M, F)_{\xi} := \nu(A^{\xi}(M, F))$. The group $A^{\xi}_L(M, F) := \{f \in A^{\xi}(M, F) \mid f(\mathcal{L}) = \mathcal{L} \forall \mathcal{L} \in \mathcal{F}\}$ which we call the *group leaf automorphisms* of (M, F), is a normal subgroup of the group $A^{\xi}(M, F)$. We say the quotient group $A^{\xi}(M, F)/A^{\xi}_L(M, F)$ to be *the group of basic automorphisms* of the Cartan foliation (M, F) denoted $A^{\xi}_B(M, F)$. Example 3 shows that the group $A^{\xi}_B(M, F)$ depends on the transverse Cartan geometry ξ.

Denote $A_L(M, F)_{\xi} := \{f \in A(M, F)_{\xi} \mid f(L) = L \forall L \in F\}$ the leaf automorphism subgroup of the group $A(M, F)_{\xi}$. We say the quotient group $A_B(M, F)_{\xi} := A(M, F)_{\xi}/A_L(M, F)_{\xi}$ to be the *projection of the group* $A^{\xi}_B(M, F)$ of the basic automorphisms to the leaf space M/F. Point out that if the transverse Cartan geometry ξ is effective then the groups $A^{\xi}_B(M, F)$ and $A_B(M, F)_{\xi}$ are canonically isomorphic. The group $A^{\xi}_B(M, F)$ of basic automorphisms of a Cartan foliation (M, F) as well as its projection $A_B(M, F)_{\xi}$ are invariants in the category $\mathfrak{CF}$.

Denote TF the distribution tangent to the foliation (M, F). Fix a distribution $\mathfrak{M}$ transversal to (M, F), i. e., $T_xM = T_xF \oplus \mathfrak{M}_x$, $x \in M$. Then the distribution $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$ is transversal to the lifted foliation $(\mathcal{R}, \mathcal{F})$.

The Cartan foliation (M, F) is said to be $\mathfrak{M}$-*complete* if any vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R}, \mathcal{F})$, $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$, such that $\beta(X) = \mathrm{const}$ is complete. In other words, (M, F) is an $\mathfrak{M}$-complete foliation if and only if the lifted e-foliation $(\mathcal{R}, \mathcal{F})$ is complete with respect to the distribution $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ in the sense of Conlon [12].

Definition 3. *A Cartan foliation* (M, F) *of a codimension* q *is said to be complete if there exists a* q*-dimensional transverse distribution* $\mathfrak{M}$ *on* M *such that* (M, F) *is* $\mathfrak{M}$*-complete [11].*

Denote (G, H) the pair of Lie groups, where H is a closed subgroup of G. The maximal normal subgroup K of G belonging to H is called the kernel of the pair (G, H), and K is denoted $Ker(G, H)$. A Cartan foliation modeled on an ineffective Cartan geometry $\xi = (P(N, H), \omega)$ of type (G, H) admits the effective transverse Cartan geometry $\widehat{\xi} = (\widehat{P}(N, \widehat{H}), \widehat{\omega})$ of type $(\widehat{G}, \widehat{H})$, where $\widehat{G} = G/K$, $\widehat{H} = H/K$ and K is the kernel of the pair (G, H) ([11], Proposition 1). We say $\widehat{\xi}$ to be the effective Cartan geometry associated with ξ.

Let $\widehat{\mathcal{R}}(M, \widehat{H})$ be the principal $\widehat{H}$-bundle of the foliated bundle constructed for the effective Cartan geometry $\widehat{\xi}$. The foliation $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ is called the *associated lifted foliation* for the Cartan foliation (M, F).

Let (M, F) be a complete Cartan foliation, $(\mathcal{R}, \mathcal{F})$ be the lifted e-foliation and $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ be the associated lifted e-foliation. Repeating and summing up the relevant results of Molino [13] we obtain:

(i) the closure of the leaves of the foliation $(\mathcal{R}, \mathcal{F})$ are fibers of a certain locally trivial fibration $\pi_b \colon \mathcal{R} \to W$;

(ii) the foliations $(Cl(\mathcal{L}), \mathcal{F}|_{Cl(\mathcal{L})})$ and $(Cl(\widehat{\mathcal{L}}), \widehat{\mathcal{F}}|_{Cl(\widehat{\mathcal{L}})})$ induced on the closures $Cl(\mathcal{L})$ and $Cl(\widehat{\mathcal{L}})$ correspondingly are Lie foliations with dense leaves with the structural Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively, which do not depend of a choice of $\mathcal{L} \in \mathcal{F}$ and $\widehat{\mathcal{L}} \in \widehat{\mathcal{F}}$.

Definition 4. *The structural Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ of the Lie foliations $(Cl(\mathcal{L}), \mathcal{F}|_{Cl(\mathcal{L})})$ and $(Cl(\widehat{\mathcal{L}}), \widehat{\mathcal{F}}|_{Cl(\widehat{\mathcal{L}})})$ respectively are called the structural Lie algebras of the complete Cartan foliation (M, F) and are denoted $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_1(M, F)$.*

For a Cartan foliation (M, F) modeled on an effective Cartan geometry we have $\mathfrak{g}_0(M, F) = \mathfrak{g}_1(M, F)$. For Riemannian foliations on compact manifolds this notion was introduced by Molino [13]. Emphasize that the structural Lie algebras $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_1(M, F)$ are invariants in the category $\mathfrak{CF}$.

3. MAIN RESULTS

First, we prove the following theorem which gives us a sufficient condition for the existence of a unique structure of a Lie group in the group of basic automorphisms of a complete Cartan foliation.

Theorem 1. *Let (M, F) be a complete Cartan foliation modeled on a Cartan geometry ξ of type $\mathfrak{g}/\mathfrak{h}$. If the structural Lie algebra $\mathfrak{g}_0(M, F)$ is zero, then the group $A_B^{\xi}(M, F)$ of basic automorphisms of this foliation is a Lie group with dimension*

$$\dim(A_B^{\xi}(M, F)) \leq \dim(\mathfrak{g}), \tag{1}$$

and the Lie group structure in $A_B^{\xi}(M, F)$ is unique. The estimate (1) is exact (i.e. best possible).

Recall that a leaf L of a foliation (M, F) is proper if L is an embedded submanifold in M. A foliation is called proper if all its leaves are proper. A leaf L is said to be closed if L is a closed subset of M. Any closed leaf is known to be proper.

Theorem 2. *Let (M, F) be a complete Cartan foliation modeled on a Cartan geometry ξ of type $\mathfrak{g}/\mathfrak{h}$ and let $\widehat{\xi}$ be the associated effective transverse Cartan geometry. Suppose that both structural Lie algebras $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_1(M, F)$ are zero. Then*

(i) both the group $A_B^{\xi}(M, F)$ of basic automorphisms and its projection $A_B(M, F)_{\xi}$ admit unique Lie group structures, and

$$\dim(A_B(M, F)_{\xi}) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{k}), \tag{2}$$

where $\mathfrak{k}$ is the kernel of the pair of the Lie algebras $(\mathfrak{g}, \mathfrak{h})$;

(ii) if there exists an isolated proper leaf (or there exists an isolated closed leaf) or if the set of proper leaves (or the set of closed leaves) is countable, then

$$\dim(A_B(M, F)_{\xi}) \leq \dim(A_B(M, F)_{\widehat{\xi}}) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{k}); \tag{3}$$

(iii) if the set of proper leaves (or the set of closed leaves) is countable and dense, then

$$\dim(A_B(M, F)_{\xi}) = \dim(A_B(M, F)_{\widehat{\xi}}) = 0. \tag{4}$$

The estimates (2), (3) are exact (i.e. the best possible) and there exist foliations (M, F) for which (4) holds.

Examples 1–3 show the exactness of estimates (2) and (3).

Theorem 3. *Let (M, F) be a complete Cartan foliation with a transverse Cartan geometry ξ of type $\mathfrak{g}/\mathfrak{h}$ and let $\widehat{\xi}$ be the associated Cartan geometry. Assume that the kernel of the pair of the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ is zero and $\mathfrak{g}_1(M, F) = 0$. Then $\mathfrak{g}_0(M, F) = 0$ and the groups $A_B^{\xi}(M, F)$, $A_B(M, F)_{\xi}$, $A_B^{\widehat{\xi}}(M, F)$ and $A_B(M, F)_{\widehat{\xi}}$ admit unique Lie group structures of the same dimension.*

In the proves of theorems we use of notation of an Ehresmann connection $\mathfrak{M}$ for a smooth foliation (M, F) which belongs to Blumenthal and Hebda, where $\mathfrak{M}$ is a transversal distribution to (M, F) [14].

4. PROOFS OF THEOREMS

4.1. Proof of Theorem 1

Let (M, F) be an $\mathfrak{M}$-complete Cartan foliation and let $\mathcal{R}(M, H)$ be the foliated bundle with the lifted foliation $(\mathcal{R}, \mathcal{F})$. As $\mathfrak{g}_0(M, F) = 0$, so the leaves of $(\mathcal{R}, \mathcal{F})$ are fibers the locally trivial bundle $\pi_b : \mathcal{R} \to W$. The map

$$W \times H \to W : (w, a) \mapsto \pi_b(R_a(u)) \; \forall (w, a) \in W \times H, u \in \pi_b^{-1}(w),$$

defines a locally free action of the Lie group H on the basic manifold W, and the orbits space W/H is homeomorphic to the leaf space M/F. Identify W/H with M/F. The equality $\pi_b^*\sigma = \beta$ defines an $\mathfrak{g}$-valued 1-form σ on W such that $\sigma(A_W^*) = A$, where A_W^* is the fundamental vector field on W defined by $A \in \mathfrak{h} \subset \mathfrak{g}$.

Denote by $A(W, \sigma)$ the Lie group of automorphisms of the parallelizable manifold (W, σ), i.e., $A(W, \sigma) = \{f \in Diff(W) | f^*\sigma = \sigma\}$. Let $A^H(W, \sigma) = \{f \in A(W, \sigma) | f \circ R_a = R_a \circ f \; \forall a \in H\}$. Then $A^H(W, \sigma)$ and its unity component $A_e^H(W, \sigma)$ are Lie groups as closed subgroups of the Lie group $A(W, \sigma)$.

Since $\Gamma, \Gamma \in A^{\xi}(M, F)$ preserves $(\mathcal{R}, \mathcal{F})$, there exists $\phi \in Diff(W)$ satisfying the equality $\pi_b \circ \Gamma = \phi \circ \pi_b$. It is not difficult to show that we have the map

$$\kappa : A_B^{\xi}(M, F) = A_B^{\xi}(M, F)/A_L^{\xi}(M, F) \to A^H(W, \sigma) : \Gamma \cdot A_L^{\xi}(M, F) \mapsto \phi$$

which is a group monomorphism. Let us show that $Im(\kappa)$ is the open-closed Lie subgroup of the Lie group $A^H(W, \sigma)$.

Suppose that $A^H(W, \sigma)$ is a discrete Lie group, then $A_B^{\xi}(M, F)$ is also discrete Lie group and the required statement is true. Further we assume that $\dim(A^H(W, \sigma)) \geq 1$. Let $\mathfrak{a}$ be the Lie algebra of the Lie group $A^H(W, \sigma)$ and let B^* be the fundamental vector field defined by an element $B \in \mathfrak{a}$. Hence $X := B^*$ is a complete vector field on W which defines the 1-parameter group $\varphi_t^X, t \in (-\infty, +\infty)$, of transformations from $A^H(W, \sigma)$. Consequently 1)$L_X\sigma = 0$ and 2)$L_X A_W^* = 0$ where A_W^* is the fundamental vector field on W defined by an element $A \in \mathfrak{h}$.

Let f be any element from the identity component $A_e^H(W, \sigma)$ of the Lie group $A^H(W, \sigma)$. Then there exist $B \in \mathfrak{a}$ and $t_0 \in (-\infty, +\infty)$ such that $f = \varphi_{t_0}^X$ where $X = B^*$. Note that $\pi_b : \mathcal{R} \to W$ is the submersion with the Ehresmann connection $\widetilde{\mathfrak{M}}$, where $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$. Therefore there exists the unique vector field $Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $\pi_{b^*}Y = X$, and Y is a foliated vector field, i.e. $[Y, Z] \in \mathfrak{X}_{T\mathcal{F}}(\mathcal{R})$ for all $Z \in \mathfrak{X}_{T\mathcal{F}}(\mathcal{R})$. Completeness of the vector field X implies completeness of the vector field Y. Hence Y defines the 1-parameter group $\psi_t^Y, t \in (-\infty, +\infty)$, of diffeomorphisms of the manifold $\mathcal{R}$. Let us show that $\psi_t^Y \in A^{\xi}(M, F)_e$ for all $t \in (-\infty, +\infty)$, i.e., we have to check the validity of the following two facts: 1) $L_Y\beta = 0$; 2) $L_Y A^* = 0$ for all $A \in \mathfrak{h}$.

1. Let us denote $Z_W := \pi_{b*}Z$ for any foliated vector field $Z \in \mathfrak{X}(\mathcal{R})$.

Case I: The vector field $Z \in \mathfrak{X}(\mathcal{R})$ satisfies the condition $\beta(Z) = const$. The relation $\beta = \sigma \circ \pi_{b^*}$ implies that $\sigma(Z_W) = \beta(Z) = const$, so $X(\sigma(Z_W)) = 0$. In accordance with the choice of X we have $\varphi_t^X \in A^H(W, \sigma)$, hence $L_X\sigma = 0$. Therefore using formula for the Lie derivative $L_X\sigma$ we have

$$0 = (L_X\sigma)(Z_W) = X(\sigma(Z_W)) - \sigma([X, Z_W]) = -\sigma([X, Z_W]). \tag{5}$$

In the equality $(L_Y\beta)(Z) = Y(\beta(Z)) - \beta([Y, Z])$ the first term $Y(\beta(Z))$ is zero because $\beta(Z) = const$. The relations $\beta = \sigma \circ \pi_{b^*}$ and (5) imply the following chain of equalities:

$$\beta([Y, Z]) = \sigma(\pi_{b^*}[Y, Z]) = \sigma([\pi_{b^*}Y, \pi_{b^*}Z]) = \sigma([X, Z_W]) = 0.$$

Therefore $(L_Y\beta)(Z) = 0$.

Case II: Consider any vector field Z in $\mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$. Let $E_i, i = 1, ..., \dim \mathfrak{g}$, be a fixed basis of the vector space $\mathfrak{g}$ and let Z_i be the vector fields in $\mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $\beta(Z_i) = E_i$. In this case $Z = h^i Z_i$ where

h^i are smooth functions on $\mathcal{R}$. Due to linearity of β and the property of the Lie brackets we have the following chain of equalities:

$$(L_Y\beta)(Z) = Y(\beta(h^iZ_i)) - \beta([Y, h^iZ_i]) = h^i(Y(\beta(Z_i)) - \beta([Y, Z_i]))$$
$$+ Y(h^i)\beta(Z_i) - Y(h^i)\beta(Z_i) = h^i(Y(\beta(Z_i) - \beta([Y, Z_i]))) = h^i(L_Y\beta)(Z_i).$$

Since $\beta(Z_i) = const$ we apply Case I and get $(L_Y\beta)(Z) = 0\ \forall Z \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$.

2. Let us pick any $A \in \mathfrak{h}$. We denote A^* and A^*_W the fundamental vector fields on $\mathcal{R}$ and W defined by $A \in \mathfrak{h}$ respectively. As $\beta(A^*) = \sigma(A^*_W) = A$, the vector field A^* is foliated with respect to $(\mathcal{R}, \mathcal{F})$. So the application of the equality $L_XA^*_W = 0$ allows us to get the following chain of equalities $\pi_{b^*}[A^*, Y] = [\pi_{b^*}A^*, \pi_{b^*}Y] = [A^*_W, X] = L_XA^*_W = 0$, hence $[A^*, Y] \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$. Let a_t, $t \in (-\infty, +\infty)$, be the 1-parametric transformation group of $\mathcal{R}$ generated by the fundamental vector field A^*. Then $R_{a_t}(u) = ua_t$ for every t. We get

$$[A^*, Y] = \lim_{t\to 0}\frac{1}{t}[(R_{a_t})_*Y - Y]. \tag{6}$$

By the definition $Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$. Since $\pi_*\widetilde{\mathfrak{M}} = \mathfrak{M}$, the distribution $\widetilde{\mathfrak{M}}$ is H-invariant. Therefore, $(R_{a_t})_*Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ and

$$(R_{a_t})_*Y - Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})\ \forall t \in (-\infty, +\infty). \tag{7}$$

Let us pick any point $u \in \mathcal{R}$, put $x = \pi(u) \in M$. Due to (6), (7) and continuity of π, using the fact that the subspace $\mathfrak{M}_x$ is closed in the topology of the tangent space $T_xM \cong \mathbb{R}^n$ we get the following chain of relations

$$\pi_{*u}([A^*, Y]) = \pi_{*u}(\lim_{t\to 0}\frac{1}{t}[(R_{a_t})_*Y - Y]) = \lim_{t\to 0}\frac{1}{t}\pi_{*u}[(R_{a_t})_*Y - Y] \in \mathfrak{M}_x, \quad \text{hence}$$

$$\pi_{*u}([A^*, Y]) \in \mathfrak{M}_x. \tag{8}$$

It is followed from (8) that $\pi_*[A^*, Y] \in \mathfrak{X}_{\mathfrak{M}}(X)$. Consequently $[A^*, Y] \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$.

The relations $[A^*, Y] \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$ and $[A^*, Y] \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ imply the equality $[A^*, Y] = 0$ for all $A \in \mathfrak{h}$. This completes the check that $\psi^Y_t \in A^\xi_e(M, F)$.

Thus $A^H_e(W, \sigma) \subset Im(\kappa)$ and $\kappa : A^\xi_B(M, F) \to A^H(W, \sigma)$ is the group isomorphism onto the open-closed Lie subgroup $Im(\kappa)$ of the Lie group $A^H(W, \sigma)$.

Since the group $A^\xi_B(M, F)$ is realized as a group of diffeomorphisms of the manifold W, the Lie group structure in $A^\xi_B(M, F)$ is unique ([15], Theorem VI).

4.2. Proof of Theorem 2

(i). Let $A^\xi_B(M, F)$ and $A^{\widehat{\xi}}_B(M, F)$ be the groups of basic automorphisms corresponding to the original Cartan geometry ξ and to the associated Cartan geometry $\widehat{\xi}$. In accordance with the conditions $\mathfrak{g}_0(M, F) = 0$ and $\mathfrak{g}_1(M, F) = 0$ due to Theorem 1 both groups $A^\xi_B(M, F)$ and $A^{\widehat{\xi}}_B(M, F)$ are Lie groups, and their Lie group structures are unique. Besides all leaves of the associated lifted foliation $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ are closed and the manifold of the leaf space $\widehat{W} := \widehat{\mathcal{R}}/\widehat{\mathcal{F}}$ with the projection $\widehat{\pi}_b : \widehat{\mathcal{R}} \to \widehat{W}$ are defined. Denote by $\widehat{\mathfrak{g}}$ the quotient algebra $\mathfrak{g}/\mathfrak{k}$, where $\mathfrak{k}$ is the kernel of the pair of the Lie algebras $(\mathfrak{g}, \mathfrak{k})$. Denote by $\widehat{\beta}$ the respective $\widehat{g}$-valued 1-form on $\widehat{\mathcal{R}}$. Let $\widehat{\sigma}$ be the $\widehat{\mathfrak{g}}$-valued 1-form on $\widehat{W}$ satisfying the equality $\widehat{\pi}^*_b\widehat{\sigma} := \widehat{\beta}$. Let $K = Ker(G, H)$ and let $\tau : \mathcal{R} \to \widehat{\mathcal{R}} = \mathcal{R}/K$ be the quotient map. Pick any leaf $\mathcal{L}$ of the foliation $(\mathcal{R}, \mathcal{F})$, then $\tau(Cl(\mathcal{L})) = Cl(\tau(\mathcal{L}))$, hence $\tau(Cl(\mathcal{L})) = \widehat{\mathcal{L}}$ is a leaf of $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ and there is a map $\widehat{\tau} : W \to \widehat{W}$ satisfying the following equation $\widehat{\tau} \circ \pi_b = \widehat{\pi}_b \circ \tau$. Since $\widehat{W} = W/K$, every orbit $w \cdot K$, $w \in W$, is closed. Therefore $\eta = (W(\widehat{W}, K), \sigma)$ is the Cartan geometry with the projection $\widehat{\tau} : W \to \widehat{W}$. Observe that $\widehat{\mathfrak{g}}$-valued 1-form $\widehat{\sigma}$ satisfies the equality $\widehat{\tau}^*\widehat{\sigma} = \sigma$, because $\tau^*\widehat{\beta} = \beta$.

For any $\varphi \in A^{\xi}_B(M,F)$ which is considered as an element of $A^H(W,\sigma)$ we have $\varphi \circ R_b = R_b \circ \varphi$ $\forall b \in K$. Then there exists $\widehat{\varphi} \in Diff(\widehat{W})$ such that $\widehat{\tau} \circ \varphi = \widehat{\varphi} \circ \widehat{\tau}$. The direct check shows that $\widehat{\varphi} \in A^{\widehat{\xi}}_B(M,F)$. Thus the map $\mu : A^{\xi}_B(M,F) \to A^{\widehat{\xi}}_B(M,F) : \varphi \mapsto \widehat{\varphi}$ is defined, and μ is a group homomorphism.

Consider any 1-parametric group $\varphi^X_t \subset A^H_e(W,\sigma), t \in \mathbb{R}^1$. According to the proof of Theorem 1 we may identify $A^{\xi}_B(M,F)_e$ with $A^H_e(W,\sigma)$ through the group isomorphism κ. Using this identification it is easy to get the inclusion $\mu(A^{\xi}_B(M,F)_e) \subset A^{\widehat{\xi}}_B(M,F)_e$ and to see that $\mu : A^{\xi}_B(M,F) \to A^{\widehat{\xi}}_B(M,F)$ is the Lie groups homomorphism. Therefore

$$\dim(Im(\mu)) \leq \dim(A^{\widehat{\xi}}_B(M,F)) \leq \dim(\widehat{\mathfrak{g}}) = \dim(\mathfrak{g}) - \dim(\mathfrak{k}).$$

Efficiency of the associated Cartan geometry $\widehat{\xi}$ implies the existence of a group isomorphism ([6], Proposition 9) $\theta : A^{\widehat{\xi}}_B(M,F) \to A_B(M,F)_{\widehat{\xi}}$. Note that $A_B(M,F)_{\xi} \cong \theta(Im(\mu))$. Thus $A_B(M,F)_{\xi}$ is a Lie subgroup of the Lie group $A_B(M,F)_{\widehat{\xi}}$, and

$$\dim(A_B(M,F)_{\xi}) = \dim(Im(\mu)) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{k}).$$

Since the group $A_B(M,F)_{\widehat{\xi}}$ is isomorphic to an open-closed subgroup of $A^{\widehat{H}}(\widehat{W},\widehat{\sigma})$, the groups $A_B(M,F)_{\widehat{\xi}}$ and $A_B(M,F)_{\xi}$ are realized as groups of diffeomorphisms of the manifold $\widehat{W}$, hence their Lie group structures are unique.

(ii) Let $s : \widehat{W} \to \widehat{W}/\widehat{H}$ be the canonical projection onto the orbit space and $q : M \to M/F$ be the canonical projection onto the leaf space. Assume now that there exists an isolated proper leaf L of the foliation (M,F). Let $x \in L$, $v = \widehat{\pi}^{-1}(x)$ and $z = \widehat{\pi}_b(v) \in \widehat{W}$. Observe, that any automorphism of a foliation transforms a proper leaf to the corresponding proper leaf. Since the orbit $A^{\widehat{H}}_e(\widehat{W},\widehat{\sigma}) \cdot z$ of the point z is connected subnanifold of $\widehat{W}$ and $q(L) = s(z)$, it is necessary that the orbit $A^{\widehat{H}}_e(\widehat{W},\widehat{\sigma}) \cdot z$ belongs to $s^{-1}(s(z))$. Consequently we have

$$\begin{aligned}\dim(A_B(M,F)_{\xi}) \leq \dim(A_B(M,F)_{\widehat{\xi}}) &= \dim(A^{\widehat{H}}_e(\widehat{W},\widehat{\sigma}) \cdot z) \leq \dim(\widehat{H}) \\ &= \dim(\widehat{\mathfrak{h}}) = \dim(\mathfrak{h}) - \dim(\mathfrak{k}).\end{aligned}$$

Thus $\dim(A_B(M,F)_{\xi}) \leq \dim(\mathfrak{h}) - \dim(\mathfrak{k})$.

Suppose now that the set of proper leaves of (M,F) is countable (nonempty). Consider any 1-parametric group φ_t, $t \in (-\infty,+\infty)$, from the Lie group $A^{\widehat{H}}_e(\widehat{W},\widehat{\sigma}) \cong A_B(M,F)_{\widehat{\xi},e}$. Let $L = L(x)$ be any proper leaf, $v = \widehat{\pi}^{-1}(x)$ and $z = \widehat{\pi}_b(v) \in \widehat{W}$. Let $z \cdot \widehat{H}$ be the orbit of z respectively $\widehat{H}$. Since for any fixed t the automorphism φ_t transforms the proper leaf L to the proper leaf $\varphi_t(L)$, the countability of the set of proper leaves implies that $\varphi_t(z \cdot \widehat{H}) = z \cdot \widehat{H}$. Hence, by analogy with the previous case we have the estimate (3).

(iii) Now we suppose that the set of proper leaves $\{L_n | n \in \mathbb{N}\}$ of (M,F) is countable and dense. Let $x_n \in L_n$, $v_n = \widehat{\pi}^{-1}(x_n)$ and $z_n = \widehat{\pi}_b(v_n) \in \widehat{W}$. Assume that $\dim(A_B(M,F)_{\xi}) \geq 1$. Let φ_t, $t \in (-\infty,+\infty)$, be any 1-parametric subgroup of the automorphism group $A^{\widehat{H}}_e(\widehat{W},\widehat{\sigma}) \cong A_B(M,F)_{\widehat{\xi},e}$. As it was proved above, it is necessary $\varphi_t(z_n \cdot \widehat{H}) = z_n \cdot \widehat{H}$ for all $t \in (-\infty,+\infty)$ and $n \in \mathbb{N}$. Remark that the leaf space M/F is homeomorphic to the orbit space $\widehat{W}/\widehat{H}$. Denote $\widetilde{\varphi}_t$ the induced 1-parametric group of homeomorphisms of the leaf space M/F. Therefore, for every $t \in (-\infty,+\infty)$, $t \neq 0$, the homeomorphism $\widetilde{\varphi}_t$ has dense subset $\{[L_n] | [L_n] = s(z_n \cdot \widehat{H}), n \in \mathbb{N}\}$ of fixed points in $M/F \cong \widehat{W}/\widehat{H}$.

Continuity and openness of the canonical projection $q : M \to M/F$ imply that the leaf space M/F is a first-countable topological space, that is every its point has a countable basis of neighborhoods. Then for every fixed $t \in (-\infty,+\infty)$ the transformation $\widetilde{\varphi}_t$ of M/F is sequentially continuous. Therefore the existence of the dense subset of fixed points of the homeomorphism $\widetilde{\varphi}_t$ implies $\widetilde{\varphi}_t = Id_{M/F}$. Hence $\varphi_t = Id_{\widehat{W}}$ for every $t \in (-\infty, +\infty)$ that contradicts to the assumption.

Thus, $\dim(A_B(M,F)_\xi) = \dim(A_B(M,F)_{\widehat{\xi}}) = 0$ and (4) is proved.

4.3. Proof of Theorem 3

Use the notations from the proof of Theorem 2. By definition, K is the kernel of the pair (G, H). The condition $\mathfrak{k} = 0$ implies that K is a discrete normal subgroup in both the Lie groups G and H. Therefore the submersion $\tau : \mathcal{R} \to \widehat{\mathcal{R}} := \mathcal{R}/K$ is a regular covering map. Observe that due to the condition $\mathfrak{g}_1(M,F) = 0$ this implies $\mathfrak{g}_0(M,F) = 0$, hence the leaves of the lifted foliation $(\mathcal{R}, \mathcal{F})$ are fibres of the basic fibration $\pi_b : \mathcal{R} \to W$. Moreover, the quotient map $\widehat{\tau} : W \to W/K$ satisfies the equality $\widehat{\tau} \circ \pi_b = \widehat{\pi}_b \circ \tau$. Therefore the map $\widehat{\tau} : W \to \widehat{W} := W/K$ is the regular covering map with the deck transformation group $K^W := \{R_b | b \in K\}$.

In the case when $\dim(A_B(M,F)_{\widehat{\xi}}) = 0$ the groups $A_B^{\xi}(M,F)$, $A_B(M,F)_\xi$, $A_B^{\widehat{\xi}}(M,F)$ and $A_B(M,F)_{\widehat{\xi}}$ are discrete and Theorem 3 is valid.

Let $\dim(A_B(M,F)_{\widehat{\xi}}) \geq 1$. According to the proof of Theorem 2 we have $A_B^{\xi}(M,F)_e \cong A_e^H(W,\sigma)$ and $A_B^{\widehat{\xi}}(M,F)_e \cong A_e^{\widehat{H}}(\widehat{W},\widehat{\sigma})$. Let us show that in this case μ induces the group epimorphism $\varepsilon : A_e^H(W,\sigma) \to A_e^{\widehat{H}}(\widehat{W},\widehat{\sigma})$. Let $\varphi_t^{\widehat{Z}}$ be 1-parametric group in $A_e^{\widehat{H}}(\widehat{W},\widehat{\sigma})$. This is equivalent to the fulfillment of the following two conditions: 1)$[A^*_{\widehat{W}}, \widehat{Z}] = 0 \ \forall A \in \widehat{\mathfrak{h}}$ and 2)$L_{\widehat{Z}}\widehat{\sigma} = 0$. Since $\widehat{\tau} : W \to \widehat{W}$ is the covering map, there exists the unique vector field Z on W such that $\widehat{\tau}_* Z = \widehat{Z}$. Observe, that for any fixed point $\widehat{w} \in \widehat{W}$ and $w \in \widehat{\tau}^{-1}(\widehat{w})$ the curve $\varphi_t^Z(w)$ is the lift to point w of the curve $\varphi_t^{\widehat{Z}}(\widehat{w})$ with origin in $\widehat{w}$ respectively the covering map $\widehat{\tau} : W \to \widehat{W}$. Due to $\widehat{\tau} : W \to \widehat{W}$ is the covering map, for any point $\widehat{w} \in \widehat{W}$ there exists neighborhood U of $\widehat{w}$ such that $\widehat{\tau}|_U : U \to \widehat{\tau}(U)$ is a diffeomorphism. The Lie brackets of vector fields have an infinitesimal character, hence the conditions $[\widehat{X}, \widehat{Y}] = 0$ and $[X, Y] = 0$ are equivalent for every vector fields $\widehat{X}, \widehat{Y} \in \mathfrak{X}(\widehat{W})$ and their lifts $X, Y \in \mathfrak{X}(W)$. Let A^*_W be the lift of $A^*_{\widehat{W}}$, then the equality $[A^*_{\widehat{W}}, \widehat{Z}] = 0$ implies $[A^*_W, Z] = 0$. Note that the Lie derivative has also infinitesimal character. Since $\widehat{\tau}$ is a local diffeomorphism and $\sigma = \widehat{\tau}^*\widehat{\sigma}$, the equality $L_Z\sigma = 0$ is equivalent to the equality $L_{\widehat{Z}}\widehat{\sigma} = 0$. This means that φ_t^Z is 1-parametric subgroup of the Lie group $A^H(W,\sigma)$.

Thus, $\varepsilon : A_B(M,F)_{\xi,e} \to A_B(M,F)_{\widehat{\xi},e}$ is a Lie group epimorphism. Discreteness of fibers of $\widehat{\tau}$ implies discreteness of the kernel of ε. Therefore the kernel of $\delta : A_B^{\xi}(M,F) \to A_B^{\widehat{\xi}}(M,F)$ is also discrete and the following Lie groups $A_B^{\xi}(M,F)$, $A_B(M,F)_\xi$, $A_B^{\widehat{\xi}}(M,F)$ and $A_B(M,F)_{\widehat{\xi}}$ have the same dimension.

5. CONSTRUCTIONS OF EXAMPLES

Suspended foliations Let Q and T be smooth connected manifolds. Denote $\rho : G := \pi_1(Q, x) \to Diff(T)$ a group homomorphism. Let $\Psi := \rho(G)$. Consider the universal covering map $\widetilde{p} : \widetilde{Q} \to Q$ as a right G-space. A right action of the group G on the product of manifolds $\widetilde{Q} \times T$ is defined as follows: $\Theta : \widetilde{Q} \times T \times G \to \widetilde{Q} \times T : (x, t, g) \to (x \cdot g, \rho(g^{-1})(t))$, where the covering transformation $\widetilde{Q} \to \widetilde{Q} : x \to x \cdot g$ is induced by an element $g \in G$. The quotient manifold $M := (\widetilde{Q} \times T)/G$ with the canonical projection $f_0 : \widetilde{Q} \times T \to M = (\widetilde{Q} \times T)/G$ are determined. Then the projection $f_0 : \widetilde{Q} \times T \to M$ induced the smooth foliation $F = \{f_0(\widehat{Q} \times \{t\}) | t \in T\}$ on M. The pair (M, F) is called *a suspended foliation* and is denoted $Sus(T, Q, \rho)$.

The following easy proved statement will be useful further.

Proposition 8. *Let $\xi = (P(T,H),\omega)$ be a complete effective Cartan geometry on a simply connected manifold T and $\rho : \pi_1(N,x) \to Aut(\xi)$ be a group homomorphism. Suppose that $\Psi := Im(\rho)$ is a discrete subgroup of $Aut(\xi)$ and the normalizer $N(\Psi)$ of Ψ belongs to the centralizer $Z(\Psi)$ of Ψ in the group $Aut(\xi)$. Then $(M,F) := Sus(T,B,\rho)$ is a complete Cartan foliation modelled on ξ, and there exists a Lie group isomorphism between $A_B^{\xi}(M,F) \cong A_B(M,F)_{\xi}$ and the Lie quotient group $N(\Psi)/\Psi$.*

Example 1. Let G be a Lie group and H be a closed subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the Lie groups G and H respectively. Suppose that the kernel of the pair of Lie groups (G,H) is equal to the intersection $K = Z(G) \cap Z(H)$ of the centres of the groups G and H. Denote ω_G the Maurer–Cartan $\mathfrak{g}$-valued 1-form on the Lie group G, then $\xi^0 = (G(G/H,H),\omega_G)$ is a Cartan geometry. Consider any smooth manifold L. Denote M the product of manifolds $M = L \times (G/H)$ and $F = \{L \times \{x\} | x \in G/H\}$. Hence $A_B^{\xi^0}(M,F) \cong Aut(\xi^0)$ and $\dim(A_B^{\xi^0}(M,F)) = \dim \mathfrak{g}$. By assumption, $K = Z(G) \cap Z(H)$ is the kernel of the pair (G,H), hence $Gau(\xi^0) = \{L_b | b \in K\}$. Thus, $A_B(M,F)_{\xi^0} \cong Aut(\xi^0)/Gau(\xi^0) \cong G/K$, hence $\dim(A_B(M,F)_{\xi^0}) = \dim(\mathfrak{g}) - \dim(\mathfrak{k})$.

Example 2. Let $\mathbb{E}^2 = (\mathbb{R}^2, g)$ be the Euclidean plane with the Euclidean metric g and ζ be the respective Cartan geometry. Let ψ_k be the rotation of the Euclidean plane $\mathbb{E}^2$ around the point $0 \in \mathbb{E}^2$ by the angle $\delta = 2\pi k$, $k \in \mathbb{R}$. Denote $\mathfrak{I}(\mathbb{E}^2)$ the full isometry group of $\mathbb{E}^2$. It is well known that $\mathfrak{I}(\mathbb{E}^2) \cong O(2) \ltimes \mathbb{R}^2$.

Let $\rho_k \colon \pi_1(S^1,b) \cong \mathbb{Z} \to \mathfrak{I}(\mathbb{E}^2)$ be defined by the equality $\rho_k(1) := \psi_k$, $1 \in \mathbb{Z}$. Then we have the suspended Riemannian foliation $(M,F_k) := Sus(\mathbb{E}^2, S^1, \rho_k)$ with the global holonomy group $\Psi_k := Im(\rho_k)$. This foliation has a unique closed leaf which is compact.

Since $N(\Psi_k) = Z(\Psi_k) = O(2)$, Proposition 8 is applicable. Consequently $A_B(M,F)_{\zeta} \cong N(\Psi_k)/\Psi_k = O(2)/\Psi_k$. Hence $A_B(M,F)_{\zeta}$ admits a Lie group structure if and only if Ψ_k is a closed subgroup of $O(2)$, that is equivalent to $\delta = 2\pi k$ for some rational number k.

If $\delta = 2\pi k$, where $k \neq 0$ is rational number, then $A_B(M,F)_{\zeta} \cong O(2)$.

Remark 3. *I. V. Belko ([5], Theorem 2) stated that the existence of a closed leaf of a foliation (M,F) with complete transversally projectable affine connection is sufficient for the group $A_B(M,F)_{\zeta}$ of basic automorphisms to admit a Lie group structure. Example 2 shows that this statement in general is not correct.*

Example 3. Let us consider the foliation (M,F_k), constructed in Example 2 for a rational number k as a transversally similar foliation modeled on the Cartan geometry $\xi := (G(G,H),\omega_G)$ where $G = CO(2) \ltimes \mathbb{R}^2$ and $H = CO(2)$. In this case $N(\Psi_k) = Z(\Psi_k) = CO(2)$ and according to Proposition 8 the Lie group $A_B(M,F_k)_{\xi}$ is isomorphic to the Lie quotient group $N(\Psi_k)/\Psi_k \cong CO(2)$. Therefore $A_B^{\xi}(M,F_k) \cong CO(2)$.

Thus, the group $A_B^{\xi}(M,F_k) \cong A_B(M,F_k)_{\xi} \cong CO(2)$ is not isomorphic to the group $A_B^{\zeta}(M,F_k) \cong A_B(M,F_k)_{\zeta} \cong O(2)$ where ζ is the Euclidean geometry on the plane considered in Example 2.

Example 4. Consider the standard 2-dimension torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map which is the universal covering of the torus. Let g be the flat Lorentzian metric on the torus $\mathbb{T}^2$ given in the standard basis by the matrix $t\begin{pmatrix} 2 & m \\ m & 2 \end{pmatrix}$, where $m \in \mathbb{Z}$, $|m| > 2$ and t is any non zero real number.

Denote ξ the effective Cartan geometry which is defined by g. Introduce notations $\mathfrak{I}(\mathbb{T}^2, g)$ for the full isometry group of this Lorentzian torus $(\mathbb{T}^2, g)$ and $\mathfrak{I}_0(\mathbb{T}^2, g)$ for the stationary subgroup of the group $\mathfrak{I}(\mathbb{T}^2, g)$ at point $0 = \Omega(0)$, $0 = (0,0) \in \mathbb{R}^2$. As it is known ([16], Example 3), $\mathfrak{I}(\mathbb{T}^2, g) = \mathfrak{I}_0(\mathbb{T}^2, g) \ltimes \mathbb{T}^2$, and the group $\mathfrak{I}_0(\mathbb{T}^2, g)$ is generated by f_A, $f_{\widetilde{A}}$ and $(-E)$, where $A = \begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$ and $\widetilde{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, hence $\mathfrak{I}(\mathbb{T}^2, g) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}) \ltimes \mathbb{T}^2$. Denote f_A the Anosov automorphism of the torus $\mathbb{T}^2$ determined by the matrix $A \in SL(2,\mathbb{Z})$, while by E the identity 2×2 matrix.

Let $Q = S^1$ and $T = \mathbb{T}^2$ in the definition of suspended foliation given above. Define the group homomorphism $\rho : \pi_1(S^1) \cong \mathbb{Z} \to \mathfrak{I}(\mathbb{T}^2, g)$ by the equality $\rho(k) := (f_A)^k$, $k \in \mathbb{Z}$. Then the foliation $(M, F) := Sus(\mathbb{T}^2, S^1, \rho)$ is Lorentzian, and its global holonomy group Ψ is the group of all lifts of transformations from the group $\Phi := \langle f_A \rangle$ respectively the universal covering map $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$. It is not difficult to show the existence of a group isomorphism $\theta : A_B(M, F)_\xi \to N(\Psi)/\Psi$.

The direct check shows that $N(\Psi)/\Psi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus

$$A_B^\xi(M, F) \cong A_B(M, F)_\xi \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Remark 4. *It is well known that the set of periodic orbits of an Anosov automorphism of the torus $\mathbb{T}^2$ is countable and dense. Therefore the foliation (M, F) constructed in Example 4 has a countable dense set of closed leaves and according to item (iii) of Theorem 2, its group of basic automorphisms $A_B(M, F)_\xi$ is a discrete Lie group. Our result $A_B(M, F)_\xi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ illustrates this statement.*

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