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SEEDING, COMPETITIVE INTENSITY AND QUALITY IN KNOCK-OUT TOURNAMENTS

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What is the optimal way to seed a knock-out tournament in order to maximize the overall spectator interest in it? Seeding affects the set of matches being played in the tournament, while neutral spectators tend to prefer to watch (i) close and intense matches; (ii) matches that involve strong teams. We formulate a discrete optimization problem that takes into account both these effects for every match of the tournament. With deterministic outcomes and linear objective function, we solve this problem analytically for any number of participants. It turns out that, depending on parameters, only two special classes of seedings can be optimal. While one of the classes includes a seeding that is often used in practice, the seedings in the other class are very different. When we relax the assumptions, we find that these classes of seedings are in fact optimal in a sizable number of cases.

JEL Classification: C61

Keywords: knock-out tournaments, seeding, competitive intensity, quality

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1 Introduction

Knock-out tournament (also known as elimination tournament) is one of the most frequently used sports tournament formats. After each game the winner advances for the next round and the loser is out. Contrary to round-robin competitions (each team plays each other), in a knock-out tournament teams play against only a very limited number of competitors. Usually, before the start of the competition a draw is held to fill in the tournament bracket. The role of the draw is crucial because even for a comparatively strong team an unlucky draw may lead to early elimination. In order to protect the best teams from meeting each other at the early stages, favorites — they are called seeded teams — are drawn at different parts of the tournament bracket. Such design has its reasons because a loss of a strong and well-known team in the first rounds may reduce the spectator interest in the whole tournament. As Wright (2014) put it, the reason for seeding "is clearly to ensure high competitive intensity, especially at the latter stages of a tournament".

The term "competitive intensity" (hereafter CI) refers to the degree of balancedness of a match. A match between two equally strong teams is said to have high CI, while a match between a strong team and a weak team is said to have low CI.

While the traditional method of seeding does ensure high CI in late stages of a tournament, it must do so at a price of having relatively low CI in the early stages of the tournament. Indeed, the best teams have to meet relatively weak teams in the early rounds if they are to meet with each other only at the late rounds. On the other hand, one can imagine methods of seeding (or simply, *seedings*) that let strong teams play with strong teams and weak teams play with weak teams already in the first round (an extreme example of this would be the situation in which in the first round the strongest team is paired with the second strongest, the third strongest with the fourth strongest, etc). In contrast to the traditional seeding, such seedings generate high CI in the beginning of the tournament while low CI in the end.

Which seeding, then, should the organizers choose if they wish to maximize the tournament's overall competitive intensity? In this paper, we formulate the corresponding optimization problem and solve it analytically under a number of assumptions.

Although, as asserted again by Wright (2014), "maintaining high CI is a key goal of tournament organizers ... to keep spectator interest at a high level", CI itself is not everything that even neutral spectators derive excitement from. Indeed, a close match between two weak teams is not nearly as exciting to watch as a close match between two strong teams even if the values of a measure of CI for the two matches are identical. Therefore, we augment our objective function with a term that accounts for the strength of teams playing in a match. We call this term *quality*. Thus, we consider the problem of finding a seeding that maximizes an increasing function of both CI and quality, aggregated over all the matches of the tournament, and keep the maximization of pure competitive intensity as an important special case. This may be viewed as a problem of maximizing the overall spectator interest in watching the matches of a tournament.

Our results are as follows. If spectator interest function is linear in both CI and quality and in any match, a stronger team wins with probability one, then, depending on the parameters, only two classes of seedings can be optimal. We call these classes *close* seedings and *distant* seedings. We postpone the formal definitions of these classes to Section 2. Informally, close seedings are such that, if a stronger team always beats a weaker one, in every round of the tournament the strongest team out of the remaining participants faces the second strongest, the third strongest faces the fourth strongest, and so on. By contrast, distant seedings are such that at any stage of the tournament each of the top half participants meets one of the bottom half ones. The examples of close and distant seedings in a tournament with 8 participants are provided on Figure 1. (Number i means the i-th strongest team for all $i = 1, ..., 8$.)

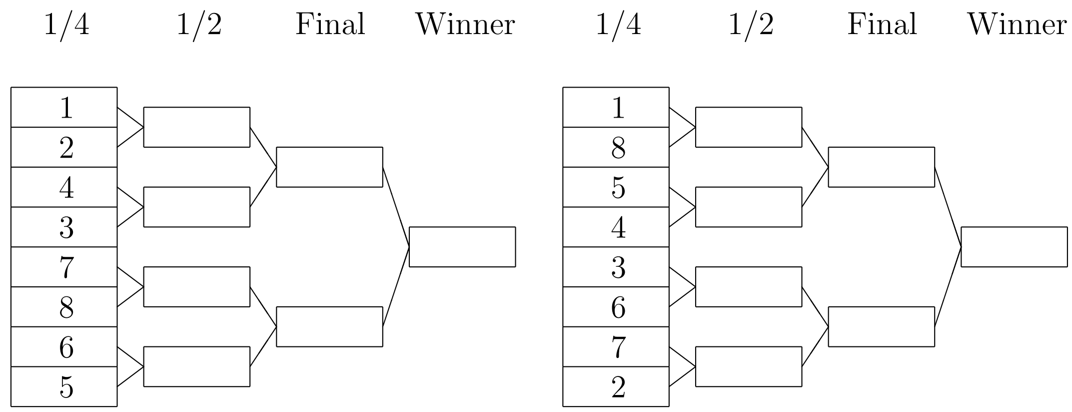

Figure 1: The examples of close (left) and distant (right) seedings.

We derive a simple condition which governs which seedings — close or distant — will be optimal in the linear case. It turns out that distant seedings are optimal whenever there is a sufficiently strong preference for quality (as reflected by its relative weight in the objective function) and if, other things being equal, the spectator interest increases at a sufficiently high rate as the tournament proceeds. Otherwise, close seedings are optimal. Our results hold for all cardinal values of teams' strength.

We then drop the linearity assumption and show that close and distant seedings remain optimal in a large number of cases. We provide sufficient conditions on the functional form of the objective under which distant seedings are optimal as well as the sufficient conditions under which close seedings are optimal. In the case of close seedings, the sufficient conditions turn out to be "almost" necessary. Finally, we explore the robustness of our findings with respect to the assumption of deterministic match outcomes. While we show by means of a counterexample that our results do no longer hold generally in the probabilistic case, we prove that they continue to hold when a stronger team beats a weaker one with sufficiently high probability.

Note that one of the distant seedings (the one depicted in Figure 1) is the very seeding that we have referred to as "traditional" above; this seeding is widely used in practice and has been subject to much analysis in the literature (see, for example, Hwang (1982) or Schwenk (2000)). By contrast, close seedings are, to the best of our knowledge, never employed by tournament organizers. Our results, then, suggest that for a certain set of parameters the existing practices may be far from optimal. However, even though close seedings sometimes turn out to be optimal in our model, there are certain reasons for avoiding such seedings that are beyond our framework. One of those reasons stems from the fact that seeding is usually determined on a basis of teams' historical rankings. If, under a certain seeding, a highly ranked team is assured to face another strong competitor already in the first round, the strong teams may have a perverse incentive to manipulate their rankings downwards by exerting less effort or even deliberately losing matches in previous competitions. Therefore, our analysis suggests that the provision of the right incentives by means of the traditional seeding may have a cost in terms of the tournament's overall competitive balance and quality.

The articles on the topic differ in four main dimensions. First, there exist several different metrics which the organizers may find reasonable to maximize. These metrics include the probability of the best team winning the tournament, overall quality of the games and overall competitive intensity of the tournament. Second, the number of teams considered in a model can differ much. The more teams compete each other, the harder it is to find an analytical solution to the organizers optimization problem. Third, there are different tournament formats: round-robin tournaments, knockout tournaments, mixed tournaments with preliminary group stages and the following knockout phase, and so on. Fourth, the authors either search for the exact solution analytically or run simulations in order to obtain an approximate solution.

A complete and up-to-date general survey of the sporting rules problems may be found in (Wright , 2014). A more detailed discussion of some of these problems is provided in (Szymanski , 2003).

The importance of seeding procedures is nicely illustrated in (Baumann, Matheson and Howe , 2012). The authors show that in the NCAA March Madness basketball tournament teams seeded 10-th and 11-th statistically go further than the teams seeded 8-th and 9-th. It is suggested that this effect is due to a suboptimal seeding procedure. Note that such a situation generates the above-mentioned perverse incentives for competing teams. In order to prevent the unwanted issues, sometimes organizers engage the services of researchers to choose the optimal format for the competition. The Royal Belgian Football Association asked a group of researchers to compare several formats using the sum of match importances as the main criterion (Goossens, Beliën and Spieksma , 2012). The authors ran Monte-Carlo simulations and answered the question. A similar approach is adopted in (Scarf, Yusof and Bilbao , 2009) and (Scarf, Yusof and Bilbao , 2011).

Fundamental analytical study of optimal seedings in knock-out tournaments starts with 4-teams knock-out tournament. Horen and Riezman (1985) provide necessary and sufficient conditions on winning probabilities for the optimality of seeding (1, 4, 2, 3), wherein the criteria they applied were 1) probability of the best team winning the tournament; 2) monotonicity of tournament win probabilities in team rank; 3) probability of the final between two best teams; 4) the expected value of the tournament winner. Also, the authors prove some extensions for 8-teams knock-out tournament. Groh et al. (2012) endogenize the winning probabilities by including teams' valuations and effort in the model of knock-out tournaments with 4 teams. The role of effort was highlighted by Rosen (1985) who studied the optimal structure of prizes in knock-out tournaments and showed that an unproportional increase of the first places prizes may lead to an increase in the overall effort exerted by the participants of the tournament. Knock-out tournaments with an arbitrary number of participants but specific Bradley-Terry type winning probabilities were studied in (Chung and Hwang , 1973). It was shown that in this case the stronger team has the greater probability to win the tournament. A counterexample for another set of winning probabilities was provided.

The different nature of tournament quality and competitive intensity metrics was underscored in (Palomino and Rigotti , 2000), where they were included simultaneously in a demand function for watching the tournament. The authors studied revenue sharing mechanisms in the league through consideration of the effort game. They described equilibria in the game depending on the parameters of the model — weights of the terms in the demand function. A similar objective function was used by Vu (2010) in connection with the problem of optimal seeding in a knock-out tournament. For a particular number of participants ($n = 8$) Vu searches all different seedings, runs 1 million simulations for each seeding and averages the achieved values of the objective function for different values of teams strengths which are i.i.d. random variables uniformly distributed on the interval [0,1] and for random winning probabilities distributed in a specific manner. The weight of each round is a linear function of its number. It is found experimentally that one of the distant seedings (namely, (1, 8, 4, 5, 2, 7, 3, 6), the traditional seeding) is optimal. The author also considers several other metrics. For $n > 8$ the author provides upper and lower estimates for winning probabilities and the corresponding estimates for the values of objective function.

However, simulations do not provide the exact solution for the optimal seedings problem. Also, one can compare several given formats, but will hardly find the best one from the set of all possible formats when the number of participants is large enough. In contrast, we provide the exact set of optimal seedings for an arbitrary number of participants.

The rest of the paper is organized as follows. Section 2 contains the formal setup. In Section 3, we solve the optimization problem when the objective function is linear in competitive intensity and quality and outcomes are deterministic (i.e., a stronger team always beats a weaker one). In Section 4, we discuss the case of arbitrary objective functions. We derive necessary and sufficient conditions on the form of objective function for the optimality of close seedings and a sufficient condition for the optimality of distant seedings. The consequences of allowing for probabilistic outcomes are investigated in Section 5. Finally, Section 6 concludes by outlining some possible directions of further research.

2 Framework

2.1 Preliminaries

We consider a standard knock-out tournament with exactly 2^n teams, where $n \geq 1$ is the total number of rounds. Let Q be the set of all teams. For any two teams x_i and x_j playing with each other, let $W(x_i, x_j)$ be the winner of the match between x_i and x_j and $L(x_i, x_j)$ be the loser.

Fix an arbitrary 2^n-tuple $x = (x_1, ..., x_{2^n})$, where $\forall i \; x_i \in Q$ and $\forall i, j \; x_i \neq x_j$. We call such a tuple *a seeding*. We give a formal definition of a knock-out tournament T_x with 2^n participants by induction over n.

Definition 1. *(Knock-out tournament)*

1. *For any* 2*-tuple* $x = (x_1, x_2)$ *a knock-out tournament* T_x *with two participants* x_1, x_2 *is a single match between teams* x_1 *and* x_2*.*

2. *Suppose that knock-out tournaments* T_x *are defined for all* 2^n*-tuples* x*, and for all* $n \leq l - 1$*, where* $l \geq 2$*. For any* 2^l*-tuple* $x = (x_1, ..., x_{2^l})$ *a knock-out tournament* T_x *is a composition of* 2^{l-1} *knock-out tournaments* $T_{(x_i, x_{i+1})}$ *with two participants* x_i *and* x_{i+1}*,* $i = 2k + 1$*,* $0 \leq k \leq 2^{l-1} - 1$*, and one following knock-out tournament with* 2^{l-1} *participants* T_y*, with the seeding* $y = (W(x_1, x_2), ..., W(x_{2^l - 1}, x_{2^l}))$*.*

We assume that the teams are strictly ranked by their strength, so it is convenient to use their ranks as names. That is, let the set of teams in the grand tournament be $Q = \{1, 2, \ldots, 2^n\}$, where i is the i^{th} strongest team. A seeding, then, can be represented by a vector of integers which is a permutation of the vector $(1, 2, \ldots, 2^n)$[1]. For a fixed n, denote the set of permutations of $(1, 2, \ldots, 2^n)$ by X_n. For the formalization of the organizers' objective function, we also need to introduce numerical levels of strength. Let $s = (s_1, s_2, \ldots, s_{2^n})$ be a vector where $s_i \in \mathbb{R}$ is the strength of the team i (for example, it is team's rating), with $s_1 > s_2 > \ldots > s_{2^n}$.

2.2 The probability model

It is usually assumed that for any two teams i and j, there exists a fixed probability p_{ij} that i beats j which is nonincreasing in the rank of i and nondecreasing in the rank of j. For most results in this paper, we use the following stronger assumption.

Assumption 1. *For any two teams* i *and* j*,* $W(i, j) = \min\{i, j\}$ *with probability one, i.e. a stronger team always beats a weaker one.*

While this assumption might seem restrictive, we consider it a reasonable first approximation. Having paid the price of this restriction, we are able to provide analytical results for

[1]Many permutations of $(1, 2, 3, \ldots, 2^n)$ correspond to the same set of matches in the tournament regardless of the outcomes of matches, so such a formalization of the notion of seeding is superfluous. We use it for notational convenience only.

an arbitrary number of teams. In Section 5 we show that while our results do not generally hold without this assumption, they hold when the probabilities p_{ij} are sufficiently close to $\mathbf{1}\{i < j\}$.

2.3 The optimization problem

The organizers seek to maximize the overall spectator interest in watching the matches of the tournament. We assume that spectator interest in a single match depends positively on (i) the strength of the teams involved and (ii) the degree of balancedness of the match. Such a formulation conforms to intuition and is in accord with several previous treatments (see expressions for "demand" in (Palomino and Rigotti , 2000) and "revenue" in (Vu , 2010)). It is also plausible that, other things being equal, a match later in the tournament attracts more attention than the same match in the beginning of the tournament.

Therefore, we model the spectator interest in a single match between teams i and j happening in round r (denoted by D_{ij}^r) as follows:

$$D_{ij}^r = \alpha^r f\left[\gamma \cdot (s_i + s_j) - |s_i - s_j|\right], \tag{1}$$

where $\alpha \geq 1$ is a coefficient that reflects how rapidly the attention to the tournament increases as it unfolds. The term $(s_i + s_j)$ is *the quality* of the match while the term $-|s_i - s_j|$ is *the competitive intensity*. The relative importance of quality as compared to competitive intensity is given by the coefficient $\gamma \geq 0$. The function f captures possible nonlinearities in the effect of quality and competitive intensity on the spectator interest.

Assumption 2. *The function $f(x)$: $\mathbb{R} \to \mathbb{R}$ is strictly increasing and continuously differentiable.*[2] *As a normalization, put $f(0) = 0$.*

Under Assumption 1, the seeding x determines unambiguously the set of matches played in every round r. Denoting this set by $M_r(x)$, we can write the organizers' optimization problem as follows:

$$\max_x \sum_{r=1}^{n} \sum_{(ij) \in M_r(x)} D_{ij}^r, \tag{2}$$

where the notation $(ij) \in M_r(x)$ means that the match between teams i and j belongs to the set $M_r(x)$.

[2]The assumption that f is continuously differentiable is for ease of exposition only. Our results hold as long as (1) f is continuous; (2) f is differentiable except at most a countable number of points; (3) $f'_-(0)$ exists and $\lim\limits_{x \to 0-} f'(x) = f'_-(0)$.

Problem (2) is a discrete optimization problem. As the feasible set X_n is finite, a solution must exist; however, as the effective number of alternatives is $\frac{(2^n)!}{2^{2^n-1}}$, the exhaustive search may be difficult to carry out for large n. In following sections, we solve Problem (2) exactly when f is linear and explore under which nonlinear functions f the solution is the same.

2.4 Close and distant seedings

Two groups of seedings play a special role in our analysis, as they turn out be the only possible solutions to the problem (2) when f is linear. We first discuss them informally.

As noted in the introduction, *close seedings* are seedings such that under Assumption 1, in every round any team faces an opponent closest to it in rank out of the participants remaining in the tournament. Hence, in the first round the team i where i is odd is paired with the team $i+1$ and these pairs are placed within the bracket in such a way that in the second round the team 1 faces the team 3, the team 5 faces the team 7, and so on.

By contrast, *distant seedings* are such seedings that under Assumption 1 a team from the strongest half of remaining teams meets a team from the weakest half of remaining teams in each match of the tournament. For example, for $n = 3$ a popular seeding $(1, 8, 4, 5, 3, 6, 2, 7)$ is a distant seeding. Figure 2 shows again the examples of a close and a distant seeding, this time we fill in the tournament bracket with the winners of each match under Assumption 1.

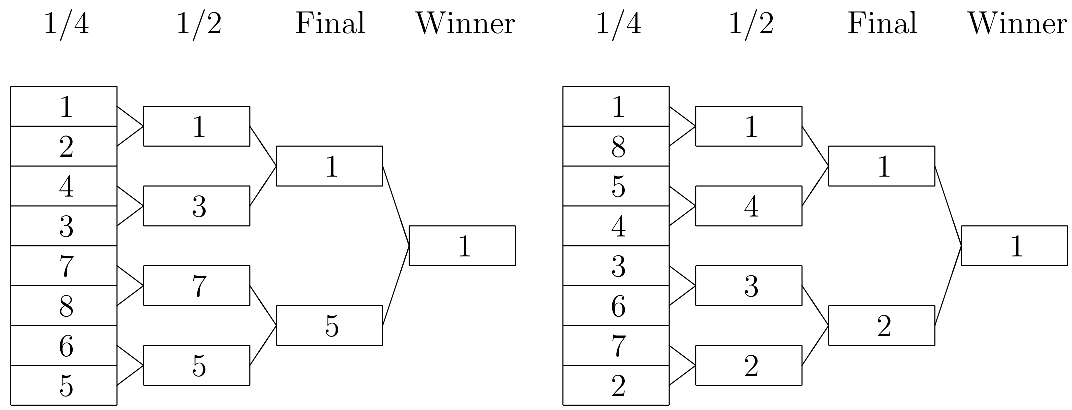

Figure 2: The examples of close (left) and distant (right) seedings; the tournament bracket is filled in as anticipated under Assumption 1.

We now give formal definitions of close and distant seedings that do not make direct use of Assumption 1. We define “close vectors” and “distant vectors”. A close seeding is a seeding that can be represented by a close vector, and a distant seeding is a seeding that can be represented by a distant vector.

Let A and B be two finite subsets of $\mathbb{R}$. We say that A and B *do not overlap* if either the smallest element of A is greater than the largest element of B or the smallest element of B is greater than the largest element of A. Otherwise we say that A and B overlap.

We give formal definitions of close and distant vectors by induction over the number of rounds, n.

Definition 2. *(Close vectors in $\mathbb{R}^{2^k}$)*

1. *If $k = 1$, any vector $x \in \mathbb{R}^{2^k}$ is close;*
2. *Suppose we defined close vectors for some $k = l \geq 1$. A vector $x \in \mathbb{R}^{2^{l+1}}$ is close if and only if two conditions hold:*
   (a) *The vectors $(x_1, x_2, \ldots, x_{2^l})$ and $(x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}})$ are close;*
   (b) *The sets $\{x_1, x_2, \ldots, x_{2^l}\}$ and $\{x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}}\}$ do not overlap.*

For a vector $x \in \mathbb{R}^{2^n}$, let $W(x)$ be the set $\{t \in \mathbb{R} : \exists k : t = \min\{x_{2k-1}, x_{2k}\}\}$. Analogously, let $L(x)$ be the set $\{t \in \mathbb{R} : \exists k : t = \max\{x_{2k-1}, x_{2k}\}\}$. That is, under Assumption 1 $W(x)$ would be the set of teams that win of the first round and $L(x)$ would be the set of teams that lose in first round if the seeding is x.

Definition 3. *(Distant vectors in $\mathbb{R}^{2^k}$)*

1. *If $k = 1$, any vector $x \in \mathbb{R}^{2^k}$ is distant;*
2. *Suppose we defined distant vectors for $k = l$. A vector $x \in \mathbb{R}^{2^{l+1}}$ is distant if and only if two conditions hold:*
   (a) *The vector $(\min\{x_1, x_2\}, \min\{x_3, x_4\}, \ldots, \min\{x_{2^{l+1}-1}, x_{2^{l+1}}\})$ is distant;*
   (b) *The sets $W(x)$ and $L(x)$ do not overlap.*

Note that in case of four participants, any seeding is either close or distant; note also that under Assumption 1 any close seeding leads to the same set of matches being played (and thus the same value of the objective function), whereas different distant seedings can involve different pairings of teams from the stronger half and the weaker half, and thus yield different values of the objective function. Under linearity of f, however, all distant seedings generate the same level of spectator interest.

3 Results: linear f

Denote the set of close seedings by C, and the set of distant seedings by D. Denote by $X^*(s,\alpha,\gamma)$ the set of optimal seedings as a function of the vector of strengths s, the later-round preference parameter α and the quality preference parameter γ.

Theorem 1. *Suppose f is linear, and Assumptions 1,2 hold. Then:*

1. *If $\alpha(\gamma+1) < 2$, then $\forall s$ $X^\star(s,\alpha,\gamma) = C$;*
2. *If $\alpha(\gamma+1) > 2$, then $\forall s$ $X^\star(s,\alpha,\gamma) = D$;*
3. *If $\alpha(\gamma+1) = 2$, then any seeding is optimal.*

Denote by P_r the set of participants of round r. Denote by W_r the set of winners of round r and by $L_r = P_r \setminus W_r$ the set of losers of round r. Consider special sets

$$W_r^* = \{1, 1+2^r, 1+2\cdot 2^r, 1+3\cdot 2^r, \dots, 2^n+1-2^r\} \text{ and } W_r^{**} = \{1,2,3,\dots,2^{n-r}\}.$$

Lemma 1. *For any round r, $\sum_{i\in W_r^*} s_i \le \sum_{i\in W_r} s_i \le \sum_{i\in W_r^{**}} s_i$.*

Proof. Proof of Lemma 1. The inequality $\sum_{i\in W_r} s_i \le \sum_{i\in W_r^{**}} s_i$ is obvious. As for the inequality $\sum_{i\in W_r^*} s_i \le \sum_{i\in W_r} s_i$, consider the weakest team that wins in round r. It is a winner of a sub-tournament with 2^r participants, so there are at least 2^r-1 teams weaker than it. Hence, its strength is at least $s_{2^n+1-2^r}$. Consider the second weakest team that wins in round r. It is a winner of another sub-tournament with 2^r participants, so there exist *another* 2^r-1 teams that are weaker than it. Overall, it must be stronger than 2^r-1+2^r teams, so its strength is at least $s_{2^n+1-2\cdot 2^r}$. Proceeding in this fashion, we see that the strength of the i^{th} weakest winner of round r is at least $s_{2^n+1-i\cdot 2^r}$ which implies the result. □

Proof. Proof of Theorem 1. Without loss of generality, consider $f(t)=t$. Consider the quantities $u_r = \sum_{(ij)\in M_r(x)} (\gamma(s_i+s_j) - |s_i-s_j|)$. Note that for any r,

$$u_r = \gamma\sum_{i\in P_r} s_i - \left(\sum_{i\in W_r} s_i - \sum_{i\in L_r} s_i\right) = \gamma\sum_{i\in P_r} s_i - \left(2\sum_{i\in W_r} s_i - \sum_{i\in P_r} s_i\right) = (\gamma+1)\sum_{i\in P_r} s_i - 2\sum_{i\in W_r} s_i. \tag{3}$$

The objective function is equal to $\sum_{r=1}^{n} \alpha^r u_r$. Substituting u_r from (3) and using the fact that by the nature of a knock-out tournament $W_{r-1} = P_r$, one gets that

$$\sum_{r=1}^{n} \alpha^r u_r = \alpha(\gamma+1)\sum_{i\in P_1} s_i + \left(\alpha^2(\gamma+1) - 2\alpha\right)\sum_{i\in W_1} s_i + \left(\alpha^3(\gamma+1) - 2\alpha^2\right)\sum_{i\in W_2} s_i + \dots$$
$$+ \left(\alpha^n(\gamma+1) - 2\alpha^{n-1}\right)\sum_{i\in W_{n-1}} s_i - 2\sum_{i\in W_n} s_i. \tag{4}$$

Under Assumption 1, $2\sum_{i\in W_n} s_i = 2s_1$ and $\sum_{i\in P_1} s_i$ are just constants so eventually we should maximize the expression

$$(\alpha(\gamma+1)-2)\sum_{r=1}^{n-1}\alpha^r\sum_{i\in W_r} s_i. \tag{5}$$

Then there are three cases.

Case 1. $\alpha(\gamma+1) < 2$, so we should minimize the expression $\sum_{r=1}^{n-1}\alpha^r\sum_{i\in W_r} s_i$. By Lemma 1, $\sum_{i\in W_r} s_i$ is minimized when $W_r = W_r^*$. The key point is that it is feasible to set $W_r = W_r^*$ simultaneously for all r. It is evident that this happens if and only if the seeding is close.

Case 2. $\alpha(\gamma+1) > 2$, so we should maximize the expression $\sum_{r=1}^{n-1}\alpha^r\sum_{i\in W_r} s_i$. By Lemma 1, $\sum_{i\in W_r} s_i$ is maximized when $W_r = W_r^{**}$. Again, it is feasible to set $W_r = W_r^{**}$ simultaneously for all r. This happens if and only if the seeding is distant.

Case 3. $\alpha(\gamma+1) = 2$. The objective function is constant, so any seeding is optimal. □

Theorem 1 shows that if the effect of a match's quality and competitive intensity on spectator interest is linear, only two types of seedings — close seedings or distant seedings — can possibly maximize the objective function. It also elucidates the way the solution to Problem (2) depends on the parameters α and γ.

This relationship is intuitive. First, note that there is a trade-off between competitive intensity at early stages and late stages of the tournament; close seedings generate great intensity in first rounds, but a very unbalanced final, whereas distant seedings create unbalanced matches early in the tournament, but guarantee a final between the top two teams. As a result, close seedings are optimal when α is relatively low while distant seedings are optimal when α is relatively high. (Close seedings can be optimal even if $\alpha > 1$ because the sheer number of matches in the beginning of the tournament is greater than the number of matches at later stages.)

Second, notice that the longer strong teams are not eliminated from the tournament the higher its overall quality is (as high levels of strength are counted more times in (2)). Close seedings eliminate top teams quickly (except the strongest one) while distant seedings favor strong teams by pairing them with weak ones. As a result, close seedings are optimal when γ is relatively low while distant seedings are optimal when γ is relatively high. When the spectators care only about quality ($\gamma = \infty$), distant seedings are always optimal; by contrast, when the spectators care only about competitive intensity ($\gamma = 0$) and $\alpha < 2$, close seedings are optimal. Thus, when $\alpha < 2$ so that spectator preference for later-stage matches is not too strong, there also exists a trade-off between the tournament's overall quality and competitive intensity.

Finally, note that the set of optimal seedings does not depend on cardinal levels of the teams' strength even though they enter the objective function explicitly. Thus, in order to implement the solution stated in Theorem 1, the organizers would have to know only the relative ranking of the teams, and the value of the parameters α and γ.

4 Results: general f

To which extent do the results of the previous section generalize when f is not longer linear? In this section, we show that close and distant seedings can arise as a solution to Problem (2) for a nontrivial set of functions.

4.1 Optimality of close seedings

First, suppose that the spectators care only about competitive intensity, i.e. $\gamma = 0$. In this case we are able to give, for a fixed n and α, both necessary conditions and sufficient conditions on f for close seedings to be the only optimal seedings. These necessary and sufficient conditions differ only insignificantly; in a sense, what we provide is almost a charaterization of the set of functions f such that $\forall s \; X^*(s, \alpha, \gamma) = C$.

Note that for $\gamma = 0$ only negative arguments enter f in (1). Therefore, in this subsection f is understood as a function from the set of nonpositive real numbers to $\mathbb{R}$.

Recall that function f is called *subadditive* if $\forall \; u, v$ from the domain $u + v$ also belongs to the domain and $f(u + v) \leq f(u) + f(v)$.

Theorem 2. *(A sufficient condition for the optimality of close seedings)*

Suppose Assumptions 1, 2 hold and $\gamma = 0$.

1. *Fix* $n = 2$ *and suppose that:*

   (a) f *is subadditive;*

   (b) $\inf f'(t) > (\alpha - 1) \sup f'(t)$.

   Then $\forall s \; X^*(s, \alpha, \gamma) = C$.

2. *Fix* $n \geq 3$ *and suppose that:*

   (a) f *is subadditive;*

   (b) $\inf f'(t) > \frac{\alpha^2}{2+\alpha} \sup f'(t)$.

Then $\forall s\ X^*(s,\alpha,\gamma) = C$.

The sufficient conditions stated in the Theorem show that close seedings are indeed optimal for any numerical levels of strength in a considerable number of cases. The conditions (b) ensure that the variation in the derivative of f is not too high. In a sense, they give a precise statement of the idea that f should not differ too much from a linear function. Note that when f is linear, $\inf f'(t) = \sup f'(t)$ and so both conditions (b) become just $\alpha \leq 2$, which is in accord with Theorem 1. Subadditivity ensures that in every two-round sub-tournament, the close structure $(1,2,3,4)$ is weakly better than the structure $(1,4,2,3)$.

To prove Theorem 2 for the case $n \geq 3$, we need a lemma that resembles the dynamic programming principle.

Lemma 2. *Suppose Assumption 1 holds, and a seeding* x_0 *is optimal. Then all the seedings induced by* x_0 *in all sub-tournaments are optimal in the corresponding sub-tournaments, i.e. they maximize spectator interest in the corresponding sub-tournaments given the sets of participants in the sub-tournaments.*

The proof of the lemma is evident so we omit it.

Proof. Proof of Theorem 2. **Part 1** ($n = 2$).

It is sufficient to prove that the seeding (1,2,3,4) is strictly better than both seedings (1,4,2,3) and (1,3,2,4). As for the seeding (1,3,2,4), we should prove the inequality (15).

By Mean Value Theorem, it is true that for an increasing differentiable function f and any two points a and b, $a < b$, $\inf f'(t)(b-a) \leq f(b) - f(a) \leq \sup f'(t)(b-a)$.

So we have

$$f(s_4 - s_3) - f(s_4 - s_2) \geq \inf f'(t)(s_2 - s_3) > (\alpha - 1)\sup f'(t)(s_2 - s_3) \geq$$
$$\geq (\alpha - 1)[f(s_2 - s_1) - f(s_3 - s_1)], \quad (6)$$

where the second inequality is by assumption.

As for the seeding (1,4,2,3), we should prove the inequality (13). The inequality

$$f(s_3 - s_1) + f(s_4 - s_3) \geq f(s_4 - s_1). \quad (7)$$

is true by subadditivity. However, it is also true that

$$f(0) - f(s_3 - s_2) \geq \inf f'(t)(s_2 - s_3) > (\alpha - 1)\sup f'(t)(s_2 - s_3) \geq$$
$$\geq (\alpha - 1)[f(s_2 - s_1) - f(s_3 - s_1)], \quad (8)$$

where the second inequality is again by assumption. Given that $f(0) = 0$, adding inequalities (7) and (8) proves the desired inequality (13).

Part 2 ($n \geq 3$).

The proof is by induction. We prove the statement not only for $n \geq 3$, but for all $n \geq 2$.

Induction base ($n = 2$). The inequality $\inf f'(t) > \frac{\alpha^2}{2+\alpha} \sup f'(t)$ implies that $1 \leq \alpha < 2$ as an infimum cannot be strictly greater than a supremum. But for $1 \leq \alpha < 2$ we have $\frac{\alpha^2}{2+\alpha} > (\alpha - 1)$ so the condition in the Part 1 is satisfied. Hence, the result of Part 1 applies.

Induction step. Suppose close seedings are strictly optimal for any tournament (and sub-tournament) with $n = l \geq 2$ rounds. Consider a tournament with $n = l + 1$ rounds.

Take any optimal seeding $x^\star = (x_1, x_2, \ldots, x_{2^{l+1}})$. Suppose $x^\star$ is not close. By definition, there are three possibilities: (i) the vector $(x_1, x_2, \ldots, x_{2^l})$ is not close; (ii) the vector $(x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}})$ is not close; (iii) the sets $\{x_1, x_2, \ldots, x_{2^l}\}$ and $\{x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}}\}$ overlap.

Note that the vector $(x_1, x_2, \ldots, x_{2^l})$ is a seeding in the upper-bracket sub-tournament of the grand tournament. By Lemma 2, this seeding should be optimal in the sub-tournament. But there are 2^l participants in this sub-tournament, so by induction hypothesis, the vector $(x_1, x_2, \ldots, x_{2^l})$ should be close. Analogously, the vector $(x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}})$ should be close. This rules out the first two possibilities.

We are left with the third one: suppose the sets

$$A = \{x_1, x_2, \ldots, x_{2^l}\} \quad \text{and} \quad B = \{x_{2^l+1}, x_{2^l+2}, \ldots, x_{2^{l+1}}\}$$

overlap. Without loss of generality, assume that the strongest team belongs to A. There are again two cases.

Case 1. There are two or more teams in A which are weaker than the strongest team in B. Let u be the weakest team in A and v be the second weakest. As the vector $(x_1, x_2, \ldots, x_{2^l})$ is close, these teams play against each other in the first round, and v wins. As v is weaker than the strongest team in B, the vector $w = (W(x_1, x_2), W(x_3, x_4), \ldots, W(x_{2^{l+1}-1}, x_{2^{l+1}}))$ is not close. However, w is a seeding in a sub-tournament of the grand tournament (this sub-tournament includes all the matches of the grand tournament except first-round matches). There are 2^l participants in this subtournament, so by Lemma 2 and the induction hypothesis, w should be close. Contradiction.

Case 2. There exists exactly one team in A which is weaker than the strongest team in B. Let this team be $u > 2^l$. The strongest team in B should be the team 2^l.

Now generate a new seeding $\hat{x}$ by switching the positions of u and the team 2^l in $x^\star$. We claim that $\hat{x}$ generates greater spectator interest than $x^\star$.

Subcase 1. $u = 2^l + 1$. The only differences in matches between x^* and $\hat{x}$ are as follows: in round 1, $\hat{x}$ assigns matches $(2^l - 1, 2^l)$ and $(2^l + 1, 2^l + 2)$ whereas x^* assigns matches $(2^l - 1, 2^l + 1)$ and $(2^l, 2^l + 2)$. In rounds 2 through $n - 1$, $\hat{x}$ assigns the team $2^l + 1$ to play against various weaker teams t_i, whereas x^* assigns the team 2^l to play against the very same teams. Finally, in round $n = l + 1$, $\hat{x}$ assigns team 1 to play with $2^l + 1$ whereas x^* assigns team 1 to play with 2^l.

So the objective function at $\hat{x}$ is strictly greater than at x^* if and only if

$$f(s_{2^l} - s_{2^l-1}) + f(s_{2^l+2} - s_{2^l+1}) + \sum_{i=2}^{l} \alpha^{i-1} f(s_{t_i} - s_{2^l+1}) + \alpha^l f(s_{2^l+1} - s_1) >$$

$$> f(s_{2^l+1} - s_{2^l-1}) + f(s_{2^l+2} - s_{2^l}) + \sum_{i=2}^{l} \alpha^{i-1} f(s_{t_i} - s_{2^l}) + \alpha^l f(s_{2^l} - s_1). \quad (9)$$

Rearranging terms, we get

$$[f(s_{2^l} - s_{2^l-1}) - f(s_{2^l+1} - s_{2^l-1})] + [f(s_{2^l+2} - s_{2^l+1}) - f(s_{2^l+2} - s_{2^l})]+$$

$$+ \sum_{i=2}^{l} \alpha^{i-1} [f(s_{t_i} - s_{2^l+1}) - f(s_{t_i} - s_{2^l})] > \alpha^l [f(s_{2^l} - s_1) - f(s_{2^l+1} - s_1)]. \quad (10)$$

Applying bounds given by Mean Value Theorem (similar to that in Part 1) and dividing both parts by $(s_{2^l} - s_{2^l+1})$ we see that the inequality (10) is implied by the inequality

$$\left(2 + \sum_{i=1}^{l-1} \alpha^i\right) \inf f'(t) > \alpha^l \sup f'(t).$$

But this inequality follows directly from the assumption of the Theorem and the fact that for all $l \geq 2$ and $1 \leq \alpha < 2$

$$\frac{\alpha^l}{2 + \alpha + \alpha^2 + \ldots + \alpha^{l-1}} \leq \frac{\alpha^2}{2 + \alpha}.$$

Hence, $\hat{x}$ generates greater spectator interest, and x^* is not optimal. Contradiction.

Subcase 2. $u > 2^l + 1$. The only differences in matches between $\hat{x}$ and x^* are as follows: in round 1, $\hat{x}$ assigns matches $(2^l - 1, 2^l)$ and $(2^l + 1, u)$ whereas x^* assigns matches $(2^l - 1, u)$ and $(2^l, 2^l + 1)$. In rounds 2 through $n - 1$, $\hat{x}$ assigns the team $2^l + 1$ to play against various weaker teams t_i, whereas x^* assigns the team 2^l to play against the very same teams. Finally, in round $n = l + 1$, $\hat{x}$ assigns team 1 to play with $2^l + 1$ whereas x^* assigns team 1 to play with 2^l.

So the objective function at $\hat{x}$ is strictly greater than at x^* if and only if

$$f(s_{2^l} - s_{2^l-1}) + f(s_u - s_{2^l+1}) + \sum_{i=2}^{l} \alpha^{i-1} f(s_{t_i} - s_{2^l+1}) + \alpha^l f(s_{2^l+1} - s_1) >$$
$$> f(s_u - s_{2^l-1}) + f(s_{2^l+1} - s_{2^l}) + \sum_{i=2}^{l} \alpha^{i-1} f(s_{t_i} - s_{2^l}) + \alpha^l f(s_{2^l} - s_1). \quad (11)$$

By subadditivity, $f(s_u - s_{2^l-1}) \leq f(s_{2^l+1} - s_{2^l-1}) + f(s_u - s_{2^l+1})$. Using this inequality and then applying the technique similar to those in Part 1 and Subcase 1 above, proves inequality (11).

Hence, again, x^* is not optimal. Contradiction.

This proves that any optimal seeding is close. To prove that any close seeding is optimal, note that an optimal seeding exists and all close seedings yield the same value of the objective function.

□

It is remarkable that the conditions stated in Theorem 2 are not only sufficient, but also almost necessary for the optimality of close seedings. The necessary conditions differ from the sufficient conditions only in a knife-edge case when $\inf f'(t) = (\alpha - 1) \sup f'(t)$ for $n = 2$ and $\inf f'(t) = \frac{\alpha^2}{2+\alpha} \sup f'(t)$ for $n \geq 3$.

Theorem 3. *(A necessary condition for the optimality of close seedings.)*

Suppose Assumptions 1, 2 hold and $\gamma = 0$.

1. *Fix* $n = 2$ *and suppose that* $\forall s\ X^*(s, \alpha, \gamma) = C$. *Then:*
   (a) *f is subadditive;*
   (b) $\inf f'(t) \geq (\alpha - 1) \sup f'(t)$.
2. *Fix* $n \geq 3$ *and suppose that* $\forall s\ X^*(s, \alpha, \gamma) = C$. *Then:*
   (a) *f is subadditive;*
   (b) $\inf f'(t) \geq \frac{\alpha^2}{2+\alpha} \sup f'(t)$.

Lemma 3. *Let f be a subadditive function $\mathbb{R}_- \to \mathbb{R}$ satisfying Assumption 2. Then* $\inf f'(x) = f'_-(0)$, *where $f'_-(0)$ is the left derivative at zero.*

Proof. Proof of Lemma 3. By definition of a derivative we have

$$f'(t) = \lim_{\Delta t \to 0-} \frac{f(t+\Delta t) - f(t)}{\Delta t} \geq \lim_{\Delta t \to 0-} \frac{f(t) + f(\Delta t) - f(t)}{\Delta t} = f'_{-}(0),$$

where the inequality follows from subadditivity of function f and the fact that $\Delta t < 0$. □

Proof. Proof of Theorem 3. **Part 1** ($n = 2$).

A close seeding (1,2,3,4) should be strictly better than the seeding (1,4,2,3):

$$f(-|s_1 - s_2|) + f(-|s_3 - s_4|) + \alpha f(-|s_1 - s_3|) > f(-|s_1 - s_4|) + f(-|s_2 - s_3|) + \alpha f(-|s_1 - s_2|). \tag{12}$$

This can be rewritten as

$$f(s_3 - s_1) + f(s_4 - s_3) > f(s_4 - s_1) + f(s_3 - s_2) + (\alpha - 1)[f(s_2 - s_1) - f(s_3 - s_1)]. \tag{13}$$

Fix s_1, s_3, s_4 and let $s_2 \to s_3$. As by assumption f is continuous and $f(0) = 0$, the term $f(s_3 - s_2) + (\alpha - 1)[f(s_2 - s_1) - f(s_3 - s_1)]$ can be made arbitrarily small so we should have

$$f(s_3 - s_1) + f(s_4 - s_3) \geq f(s_4 - s_1). \tag{14}$$

This is nothing but the subadditivity.

Also, (1,2,3,4) should be strictly better than the seeding (1,3,2,4). This boils down to:

$$f(s_4 - s_3) - f(s_4 - s_2) > (\alpha - 1)[f(s_2 - s_1) - f(s_3 - s_1)]. \tag{15}$$

Fix all strengths except s_2 and let $s_2 \to s_3$. Then use first-order Taylor expansions around the limit points. One gets:

$$f'(s_4 - s_3)(s_2 - s_3) > (\alpha - 1) f'(s_3 - s_1)(s_2 - s_3) + o(s_2 - s_3). \tag{16}$$

Now divide both parts by $(s_2 - s_3)$ and note that the term $o(s_2 - s_3)/(s_2 - s_3)$ can be ignored if the strict inequality is replaced with a weak one. Let $s_4 \to s_3$. As by assumption the derivative of f is continuous, one gets

$$f'_{-}(0) \geq (\alpha - 1) f'(s_3 - s_1). \tag{17}$$

By subadditiviy and Lemma 3, $f'_{-}(0) = \inf f'(t)$. So, by varying s_1 we immediately get the required result:

$$\inf f'(t) \geq (\alpha - 1) \sup f'(t).$$

Part 2 ($n \geq 3$).

Fix α and suppose $\forall s\ X^\star(s, \alpha, 0) = C$, i.e. close seedings are strictly optimal. In particular, this means that the close seeding $(1, 2, 3, 4, 5, 6, \dots, 2^n)$ is strictly better than the seeding $(1, 4, 2, 3, 5, 6, \dots, 2^n)$. Note that the sets of matches generated by these two seedings differ only in a two-round sub-tournament won by the team 1. Hence, we should have that the seeding $(1, 2, 3, 4)$ in this two-round sub-tournament should be strictly better than $(1, 4, 2, 3)$. Applying the argument from Part 1 to this subtournament, we get that f is subadditive.

To prove the property

$$\inf |f'(x)| \geq \frac{\alpha^2}{2+\alpha} \sup |f'(x)|,$$

note that the close seeding $(1, 2, 3, 4, 5, 6, 7, 8, \dots, 2^n)$ should be strictly better than seeding $(1, 2, 3, 5, 4, 6, 7, 8, \dots, 2^n)$. The sets of matches generated by these two seedings differ only in a three-round sub-tournament won by team 1. The corresponding inequality, then, boils down to

$$[f(s_4 - s_3) - f(s_5 - s_3)] + [f(s_6 - s_5) - f(s_6 - s_4)] + \alpha[f(s_7 - s_5) - f(s_7 - s_4)] > \alpha^2[f(s_4 - s_1) - f(s_5 - s_1)]. \quad (18)$$

Fix all strengths except s_4 and let $s_4 \to s_5$. Then use first-order Taylor expansions around the limit points. One gets:

$$f'(s_5 - s_3)(s_4 - s_5) + f'(s_6 - s_5)(s_4 - s_5) + \alpha f'(s_7 - s_5)|(s_4 - s_5) > \alpha^2 f'(s_5 - s_1)(s_4 - s_5) + o(s_4 - s_5). \quad (19)$$

Now divide both parts by $(s_4 - s_5)$, ignore the term $o(s_4 - s_5)/(s_4 - s_5)$ and let all strengths except s_1 go to s_3. As by assumption the derivative of f is continuous, one gets

$$(2 + \alpha) f'_-(0) \geq \alpha^2 f'(s_3 - s_1). \quad (20)$$

By subadditivity and Lemma 3, $f'_-(0) = \inf f'(t)$. So, by varying s_1 we get the required result:

$$\inf f'(t) \geq \frac{\alpha^2}{2+\alpha} \sup f'(t).$$

□

An infimum cannot be strictly greater than a supremum; this implies that $(\alpha - 1) \leq 1$ and $\frac{\alpha^2}{\alpha+2} \leq 1$. Hence, we obtain the following corollary.

Corollary 1. *Suppose the spectators care only about competitive intensity. If $\alpha > 2$ there does not exist a function f satisfying Assumption 2 such that close seedings maximize spectator interest for any vector of strengths.*

We know already from Theorem 1 that if f is linear, $\gamma = 0$, and $\alpha > 2$, close seedings are not optimal. Corollary 1 shows that if the preference for later-stage matches is sufficiently strong, no nonlinear effect of competitive intensity on spectator interest can restore the optimality of close seedings.

Ignoring the knife-edge cases, Theorems 2 and 3, taken together, provide a characterization of the set of functions $F(\alpha, n)$ such that close seedings are the only optimal seedings in the n-round tournament given the parameter α. Note that $F(\alpha, 3)$ is substantially smaller than $F(\alpha, 2)$ since the condition $\inf f'(t) \geq \frac{\alpha^2}{2+\alpha} \sup f'(t)$ is strictly more restrictive than the condition $\inf f'(t) \geq (\alpha - 1) \sup f'(t)$. This is understandable given the fact that a three-round tournament is a more complex structure that a two-round tournament and thus more conditions must hold for close seedings to be optimal. However, it is not the case that $F(\alpha, 4)$ is substantially smaller than $F(\alpha, 3)$. Indeed, for $n > 3$, sets $F(\alpha, n)$ differ from $F(\alpha, 3)$ in at most the knife-edge case when the condition (b) in part 2 of Theorem 3 holds as equality. This counterintuitive result suggests that, in a certain sense, there is a qualitative increase in the complexity of tournament structure when the number of rounds rises from 2 to 3 only; further increases in the number of rounds have a less significant effect.

4.2 Optimality of distant seedings

In this subsection, we provide the result analogous to Theorem 2 that deals with the optimality of distant seedings.

Unlike close seedings, different distant seedings in general result in different sets of matches being played and thus generate different levels of spectator interest. This complicates the analysis, and the results for distant seedings are true only in a weaker form than the results for close seedings. Namely, we have to replace the statement $X^\star(s, \alpha, \gamma) = C$ with $X^\star(s, \alpha, \gamma) \subset D$. Moreover, we state only a sufficient condition, but not a necessary condition, for the optimality of distant seedings. However, in the case of distant seedings we do not have to assume that $\gamma = 0$; we provide results for an arbitrary nonnegative value of γ.

Theorem 4. *(A sufficient condition for the optimality of distant seedings)*

Suppose Assumptions 1, 2 hold. Fix $n \geq 1$ and suppose $\alpha(\gamma+1)\inf f'(x) > 2\sup f'(x)$. Then $\forall s\ X^\star(s,\alpha,\gamma) \subset D$.

Again, note that the above result is in total accord with Theorem 1 when f is linear.

Proof. Proof of Theorem 4.The proof is by induction.

Induction base $(n = 1)$ is obvious.

Induction step. Suppose that the statement has been proved for $n = l \geq 1$. Consider $n = l + 1$.

Take any optimal seeding $x^* = (x_1, x_2, \ldots, x_{2^{l+1}})$. Suppose x^* is not distant. By definition, there are two possibilities:

(i) the vector $w = (\min\{x_1, x_2\}, \min\{x_3, x_4\}, \ldots, \min\{x_{2^{l+1}-1}, x_{2^{l+1}}\})$ is not distant;

(ii) the sets $W(x^*)$ and $L(x^*)$ do overlap.

Analogously to the proof of Theorem 2, note that w is the seeding in a sub-tournament with 2^l participants. So, by induction hypothesis and Lemma 2, w should be close. Therefore, we are left with possibility (ii).

For any team u and seeding y, define the sets

$$\overline{W}(u|y) = \{w \in W(y) : w \text{ is stronger than } u\}$$

and

$$\underline{W}(u|y) = \{w \in W(y) : w \text{ is weaker than } u\}.$$

As $W(x^*)$ and $L(x^*)$ overlap, there exists $u \in L(x^*)$ such that $\underline{W}(u|x^*)$ is not empty. Call the set of all such u's U^*. (As $U^* \subset L(x^*)$, for any $u \in U^*$, $\overline{W}(u|x^*)$ is not empty either).

Lemma 4. *For any $u \in U^*$ there exists $w \in \underline{W}(u|x^*)$ such that w loses in round 2 to a team from $\overline{W}(u|x^*)$.*

Proof. Proof of Lemma 4. Suppose there is no such w. Hence, all teams in $\underline{W}(u|x^*)$ that lose in round 2, lose to a team from $\underline{W}(u|x^*)$. Thus, there is an even number of teams in $\underline{W}(u|x^*)$ and half of them win in round 2 and half of them lose. However, as we have seen above, the seeding in round 2 (vector w) is distant and any distant seeding has the following obvious property: if a team wins in a round, all stronger teams also win in that round. So it must be that all teams in $\overline{W}(u|x^*)$ also win in round 2. But this implies that there are more winning teams in round 2 than losing teams. Contradiction. □

Take any $u \in U^*$ and any $w \in \underline{W}(u|x^*)$ guaranteed by Lemma 4. Generate a new seeding $\hat{x}$ by switching the positions of u and w in x^*. We claim that $\hat{x}$ is strictly better than x^*.

As w loses to a team from $\overline{W}(u|x^*)$ in round two in the original seeding, u will also lose to it. Hence the switch will produce changes in matches played only in the first two rounds. Call the team that u loses to under y, a; the team that w wins in round 1 under y, b, and the team that w loses to in round 2 under y, c.

By simple bookkeeping, $\hat{x}$ is strictly better than x^* whenever

$$\alpha[f(\gamma(s_c+s_u)-(s_c-s_u))-f(\gamma(s_c+s_w)-(s_c-s_w))]>[f(\gamma(s_a+s_u)-(s_a-s_u))-$$
$$-f(\gamma(s_a+s_w)-(s_a-s_w))]+[f(\gamma(s_w+s_b)-(s_w-s_b))-f(\gamma(s_u+s_b)-(s_u-s_b))]. \tag{21}$$

However, (21) is true because the following three inequalities are true:

$$\alpha[f(\gamma(s_c+s_u)-(s_c-s_u))-f(\gamma(s_c+s_w)-(s_c-s_w))]\geq\alpha(\gamma+1)\inf f'(t)(s_w-s_u), \tag{22}$$

$$\alpha(\gamma+1)\inf f'(t)(s_w-s_u)>2\sup f'(t)(s_w-s_u), \tag{23}$$

$$2\sup f'(t)(s_w-s_u)\geq[f(\gamma(s_a+s_u)-(s_a-s_u))-f(\gamma(s_a+s_w)-(s_a-s_w))]+$$
$$+[f(\gamma(s_w+s_b)-(s_w-s_b))-f(\gamma(s_u+s_b)-(s_u-s_b))], \tag{24}$$

where the first and the third inequalities are our usual bounds guaranteed by Mean Value Theorem and the second inequality is by assumption. Hence, x^* is not optimal. Contradiction.

□

5 Extensions

What happens if we relax Assumption 1? Instead of assuming that a stronger team always beats a weaker one, assume that there exists a general probability matrix P where p_{ij} is the probability that the team i beats the team j in a match, so that $p_{ij}+p_{ji}=1$ for all $i\neq j$. Assume that a plausible monotonicity condition holds: p_{ij} is nonincreasing in i and nondecreasing in j. Such probability matrices are sometimes called *doubly monotonic.*

In this case, the optimization problem of the organizers should be modified as follows:

$$\max_x \mathbb{E}\sum_{r=1}^{n}\sum_{(ij)\in M_r(x)} D^r_{ij}, \tag{25}$$

where, for every seeding x, the random element is $M_r(x)$, the set of matches played in the round r. (The probability distribution of $M_r(x)$ can be computed given P, x and the rules of the knock-out tournament.)

Do the results stated in Sections 3 and 4 generally apply to the solutions of problem (25)? The following simple example shows that the answer is negative.

Example 1. *Consider four teams with strengths 100, 6, 5 and 1. (In this example, we abuse the notation slightly by using strengths as the team names.) That is, the 100 is the obvious leader, and the others are too weak to compete with it. Let the probabilities p_{ij} reflect this. Suppose that team 100 beats any other team with certainty, whereas in any match involving two of the three other teams, any team can win with probability 0.5. Note that not only the described probability matrix is doubly monotonic, but a stronger monotonicity condition holds: for any two pairs of teams (i, j) and (k, t) such that $i < j$ and $k < t$, $s_i - s_j \geq s_k - s_t$ implies $p_{ij} \geq p_{kt}$. Suppose that $\gamma = 0$ so that the spectators care only about competitive intensity, and $\alpha = 1$ so that there is no preference for the final as compared to the semi-finals. Finally, suppose f is linear.*

By Theorem 1, if the outcomes of the matches were deterministic, close seedings such as $(100, 6, 5, 1)$ would be optimal. Under the given matrix of win probabilities, the objective function evaluated at $(100, 6, 5, 1)$ is equal to $-94 - 4 + 0.5 \cdot (-95) + 0.5 \cdot (-99) = -195$. However, the objective function evaluated at the seeding $(100, 1, 6, 5)$ is equal to $-99 - 1 - 0.5 \cdot (94) - 0.5 \cdot (95) = -194.5 > -195$. Hence, the close seeding $(100, 6, 5, 1)$ is not optimal and therefore Theorem 1 is no longer true.

However, many of the results of Sections 3 and 4 remain true if Assumption 1 holds "approximately", i.e. if the probabilities p_{ij} are sufficiently close to $\mathbf{1}\{i < j\}$. To state this result formally, note that the probability matrix P can alternatively be written as a vector $p \in \mathbb{R}^{\frac{n(n-1)}{2}}$. Assumption 1 corresponds to $p = (1, 1, \ldots, 1)$. Fix α, γ, f and the vector of strengths, s. Let $X^*(p)$ the set of optimal seedings given the probability vector p.

Proposition 1. *There exists an open neighborhood $\mathcal{U}$ of the point $(1, 1, \ldots, 1) \in \mathbb{R}^{\frac{n(n-1)}{2}}$, such that $\forall p \in \mathcal{U}$ $X^*(p) = X^*(1, 1, \ldots, 1)$.*

Proof. Proof of Proposition 1. Denote the vector $(1, 1, \ldots, 1)$ by p_0. Denote by $D(p|z)$ the value of the objective function if the seeding is z and the probability vector is p. Note that for any seeding z, $D(p|z)$ is continuous in p. Indeed, $D(p|z)$ is the expectation of a discrete random variable that takes a finite number of values $d_1(z), d_2(z), \ldots, d_H(z)$ with probabilities $q_1(z), q_2(z), \ldots, q_H(z)$. Each probability $q_h(z)$, $h = 1, 2, \ldots, H$, is a sum of several products of the components of p and hence is continuous in p. Therefore, $D(p|z)$ is continuous in p.

Take any $x \in X^*(p_0)$ and any $y \in Y(p_0) = X_n \setminus X^*(p_0)$. We must have $D(p_0|x) - D(p_0|y) > 0$. Due to the continuity of the left-hand side with respect to p, there exists an open neighborhood $\mathcal{U}_{xy}$ of p_0 such that $\forall p \in \mathcal{U}_{xy}$ $D(p|x) - D(p|y) > 0$. Now let

$$\mathcal{U} = \bigcap_{x \in X^*(p_0)} \bigcap_{y \in Y(p_0)} \mathcal{U}_{xy}.$$

We have that $\forall x \in X^*(p_0)$, $\forall y \in Y(p_0)$ and $\forall p \in \mathcal{U}$ $D(p|x) - D(p|y) > 0$, so $X^*(p) = X^*(p_0)$.

□

Proposition 1 implies that whenever, for a particular vector of strengths s, the set of close or distant seedings is optimal under Assumption 1, it will continue to be optimal if Assumption 1 holds only approximately. In this sense, Proposition 1 extends the results of Theorems 1, 2 and 4 to the case in which it is not necessarily true that a stronger team always beats a weaker one.

6 Concluding remarks

To conclude, we would like to outline some possible directions of further research. First, it is important to explore the problem for general doubly monotonic probability matrices P. Similar to Horen and Riezman (1985), it may turn out that, even with no further restrictions on P, only a limited set of seedings can be optimal. Here, one may consider to impose certain requirements on consistency between the probability matrix P and the vector of strengths s. By putting more structure on the problem, this may help make it more tractable. An extreme version of a consistency requirement is the existence of a functional relation like $p_{ij} = \frac{s_i}{s_i + s_j}$.

In reality, not all participants of a knock-out tournament are seeded. (For example, in ATP (Association of Tennis Professionals) Tour 2014 only 16 players are seeded in a tournament with 64 participants; the other 48 are placed randomly.) Effectively, this means that the organizers randomize on a less-than-full set of seedings. Therefore, an interesting question which generalizes the question of optimal seeding is what is the optimal *set* of seedings to randomize on. This problem arises naturally when spectators enjoy the uncertainty of the outcomes *per se* (see Ely, Frankel and Kamenica (2015)). However, even if one does not model this "love for uncertainty" explicitly, it may turn out that excluding certain seedings from randomization will improve competitive intensity (or other metrics) regardless of the parameters such as the matrix P. Identification of such "dominated" seedings can easily translate into concrete practical suggestions to tournament organizers.

Third, one may consider a full-fledged problem of profit maximization for tournament organizers, rather than just "spectator interest maximization". In this case, one will have to introduce prices into the model and be more explicit about spectator (or broadcaster) buying behavior. In such a rich model, one will (hopefully) be able to see the interactions between the optimal seeding and pricing decisions. Here, an exciting aspect to consider is the possibility of both reseeding and re-pricing of unbought tickets as the tournament unfolds and uncertainty realizes.

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