

PLANE TREES, SHABAT-ZAPPONI POLYNOMIALS AND JULIA SETS

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ABSTRACT. A tree, embedded into plane, is a dessin d'enfant and its Belyi function is a polynomial — Shabat polynomial. Zapponi form of this polynomial is unique, so we can correspond to an embedded tree the Julia set of its Shabat-Zapponi polynomial. In this purely experimental work we study relations between the form of a tree and properties (form, connectedness, Hausdorff dimension) of its Julia set.

1. INTRODUCTION

Shabat polynomial of a plane bipartite tree is not unique, but we can make it unique, if we demand that: a) critical values are $+1$ and -1 ; b) sum of coordinates of white vertices (i.e. inverse images of 1) is 1 ; c) sum of coordinates of black vertices (i.e. inverse images of -1) is -1 . Shabat polynomial with these properties will be called Shabat polynomial in Zapponi form, or Shabat-Zapponi polynomial, or SZ-polynomial [7]. Thus, we can correspond to a tree the Julia set, i.e. the Julia set of its SZ-polynomial. We want to understand is there a correspondence between geometry of a plane tree and such properties of its Julia set as form, connectedness and Hausdorff dimension?

Remark 1.1. At first it was expected that Julia set of a Shabat polynomial is something simple with Hausdorff dimension approximately 1 (because Shabat polynomial is a generalized Chebyshev polynomial). This assumption turned out to be wrong. So, we decided to study the Zapponi form of Shabat polynomial, because if there exists a SZ-polynomial for a given bipartite tree, then such polynomial is unique.

In the course of this experimental work we found that: a) there is some similarity between the form of a given tree and the form of its Julia set; b) the connectedness of Julia set is probably the main characteristic of an embedded tree.

2. DEFINITIONS AND NOTATIONS

2.1. Zapponi form of Shabat polynomials and its properties. We consider plane bipartite trees, i.e. trees embedded into plane, with vertices properly colored in black and white. A polynomial p with exactly two finite critical values — one and minus one will be called *Shabat polynomial* [5]. The inverse image $T(p) = p^{-1}[-1, 1]$ of segment $[-1, 1]$ is a plane bipartite tree, where white vertices are images of 1 and black — of -1 . For each plane bipartite tree T there exists a Shabat polynomial p such that trees T and $T(p)$ are isotopic. Such polynomial will be called a Shabat polynomial of the tree T . If polynomials p and q are Shabat polynomials of the same tree T , then $q(z) = p(\alpha z + \beta)$ for some constants $\alpha \neq 0$ and β .

A Shabat polynomial is in Zapponi form [7], if the sum of coordinates of white vertices is 1 and black vertices — -1 .

Proposition 2.1. *Let T be a bipartite tree and $p = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ — its SZ-polynomial. Then $a_{n-1} = 0$.*

Proof. Let x_1, \dots, x_s be roots of polynomial $p - 1$ with multiplicities k_1, \dots, k_s , respectively, and $y_1, \dots, y_t, l_1, \dots, l_t$ be roots of $p + 1$ and their multiplicities. Then

$$\sum_{i=1}^s k_i x_i = -\frac{a_{n-1}}{a_n} = \sum_{j=1}^t l_j y_j \Rightarrow \sum_{i=1}^s k_i x_i + \sum_{j=1}^t l_j y_j = -2 \frac{a_{n-1}}{a_n}. \quad (1)$$

Also we have, that

$$p'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + a_1 = n a_n \prod_{i=1}^s (z - x_i)^{k_i-1} \prod_{j=1}^t (z - y_j)^{l_j-1}.$$

Hence,

$$\sum_{i=1}^s k_i x_i + \sum_{j=1}^t l_j y_j = \sum_{i=1}^s k_i x_i - \sum_{i=1}^s x_i + \sum_{j=1}^t l_j y_j - \sum_{j=1}^t y_j = -2 \cdot \frac{(n-1) a_{n-1}}{n a_n}. \quad (2)$$

From (1) and (2) we have that $a_{n-1} = 0$. \square

Corollary 2.1. If p is a Shabat polynomial and $a_{n-1} = 0$, then

$$\sum_{i=1}^s x_i = - \sum_{j=1}^t y_j.$$

Corollary 2.2. Let T be a bipartite tree. If there exist its SZ-polynomial p , then p is unique and its field of definition coincides with the field of definition of the tree T [5].

Proof. If $p = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ is a Shabat polynomial of a tree T and $a_{n-1} = 0$, then p is unique up to variable change $z := \alpha z$ and the unique choice of α in this variable change gives us SZ-polynomial.

Let now K be the field of definition of a tree T and $q = b_n z^n + \dots + b_0 \in K[z]$ be its Shabat polynomial. The variable change $z := z - b_{n-1}/b_n$ preserves the field of definition, but turns coefficient at z^{n-1} to zero. If $X = \sum x_i$, then $X \in K$. If $X \neq 0$, then the variable change $z := X \cdot z$ also preserves the field of definition, but turns (in new coordinates) X to one. Then $Y = \sum y_j = -1$. If $X = 0$, then Shabat polynomial in Zapponi form does not exist for the tree T . \square

Remark 2.1. In [7] it was proved that SZ-polynomial always exists for trees with prime number of edges. SZ-polynomial obviously does not exist, if the tree is symmetric, i.e. if it has a nontrivial rotation automorphism with the center in one of vertices.

Conjecture. *SZ-polynomial exists for non-symmetric trees.*

In what follows SZ-polynomial for a tree T will be denoted p_T .

2.2. Julia sets and Hausdorff dimension. Definitions of Fatou and Julia sets see, for example, in the book [6]. For us the following properties of Julia sets will be important.

- Julia set of a polynomial p is the boundary of the basin of infinity, i.e. the boundary of open set of those points, whose iterations converge to infinity.
- Let A_0 be the set of stationary repelling points of p and let $A_i = p^{-1}(A_{i-1})$, $i > 0$. Then Julia set of p is the closure of $\cup_i A_i$.
- Julia set of p is connected if and only if iterations of critical points of p constitute a bounded set. In the case of a SZ-polynomial it means that connectedness of Julia set is equivalent to the boundedness of iterations of 1 and -1 . If iterations of 1 and -1 are both unbounded, then Julia set is totally disconnected (two dimensional Cantor compact). If iterations of 1, for example, are bounded, and iterations of -1 — not, then Julia set is a union of infinite number of connected components [2].

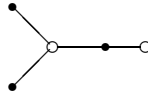
Definition of Hausdorff dimension see in [3]. There are several methods of its computation. Description of "box counting" and "packing dimension" methods see in [3]. Description of Jenkinson-Pollicott algorithm (JP-algorithm) see in [4].

Remark 2.2. It must be noted that performance of these algorithms differs from case to case. Box counting method does not work, if Julia set is totally disconnected. It also demonstrate bad performance, if Hausdorff dimension of Julia set is > 1.5 . If there is a stationary point and derivative in this point is close to 1, then JP-algorithm demonstrates a bad convergence.

2.3. Julia sets of SZ-polynomials. Let T be a bipartite tree and let \mathfrak{T} be the same tree, but with inverse colors (i.e. white vertices in T are black in \mathfrak{T} and black vertices in T are white in \mathfrak{T}). Then $p_{\mathfrak{T}}(z) = -p_T(-z)$. Let a_0 be an arbitrary point and $p_T(a_0) = a_1$, $p_T(a_1) = a_2$, $p_T(a_2) = a_3$ and so on. Then $p_{\mathfrak{T}}(-a_0) = -a_1$, $p_{\mathfrak{T}}(-a_1) = -a_2$, $p_{\mathfrak{T}}(-a_2) = -a_3$, and so on. It means that Julia sets of polynomials $p_{\mathfrak{T}}$ and p_T are the same up to rotation on π around the origin, i.e. characteristics of Julia set depends only on tree and not on its coloring. In what follows we will study one tree from the pair (T, \mathfrak{T}) .

Remark 2.3. Let T be a bipartite tree. By fixing some white vertex of degree > 1 at 1 and some black vertex of degree > 1 at -1 we uniquely define Shabat polynomial p of T . In this case p will be a postcritically finite polynomial (a pcf-polynomial), i.e. a polynomial with finite orbit of set of critical points (see [1]). It must be noted that Shabat pcf-polynomial of a tree T is not unique.

Example 2.1. Let T be a tree with four edges:



Then

$$p = -\frac{(z+1)^3(3z-8)}{8} - 1$$

is its pcf-polynomial and

$$p_T = \frac{2(2z+1)^3(2z-3)}{27} + 1$$

is its SZ-polynomial. Julia sets of p and p_T are quite different:

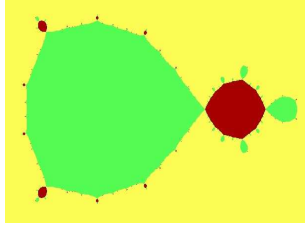


Figure 1.

Julia set of pcf-polynomial p .

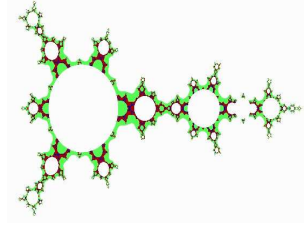


Figure 2.

Julia set of SZ-polynomial p_T .

In the left figure iterations of yellow points converge to infinity, of green points — to -1 , of red points — to 1 . Julia set is connected. Its Hausdorff dimension approximately equals 1.17 (box counting method) or 1.13 (packing dimension method).

SZ-polynomial p_T has a weakly attracting 10-cycle. Let O be a union of the domain $\{z \mid \text{abs}(z) > 3\}$ and 0.01 -neighborhood of the attracting cycle. In the right figure points that get into O in 5 steps or less are white, in 6 or 7 steps — green, in 8, 9 or 10 steps — red. All other points (including points of Julia set) are blue. Julia set is connected. For its Hausdorff dimension box counting method gives estimation ≈ 1.62 , packing dimension method — ≈ 1.35 , JP-algorithm — ≈ 1.22 .

3. GENERAL REMARKS

Let T be a tree and p_T — its SZ-polynomial. Characteristics of Julia set $J(p_T)$ depend on behavior of iterations of ± 1 . There are several types of this behavior.

3.1. "Generic" types.

- g1:** Iterations on ± 1 converge to an attracting point p . Here Julia set is a common border of two basins: the basin B_∞ of infinity and the basin B_p of attracting point p . As all vertices of T belong to B_p , then the form of Julia set resembles the form of the tree (in some general manner). Hausdorff dimension here is close to 1.
- g2:** Iterations on ± 1 converge to an attracting 2-cycle. Julia set approximates the tree better, than in the previous case. Also the "fractality" of set is greater, hence the Hausdorff dimension is greater (in average). Julia sets in this case are more "interesting", than in previous. Good example see in Figure 4.
- g3:** Iterations on ± 1 converge to an attracting k -cycle, where $k > 2$. Julia sets can be very "interesting". The form of Julia set even more closely resembles the form of the tree. "Fractality" is great and Hausdorff dimension is greater, than 1.5. Good examples see in Figures 2, 3 and 6.
- g4:** Iterations on ± 1 converge to infinity. Julia set here is totally disconnected, but some small similarity to the form of the tree remains. Hausdorff dimension can be smaller, than 1. Julia sets are rather "uninteresting". See example of s3 case in Figure 5.

3.2. "Special" types.

- s1:** Iterations of 1 (for example) converge to an attracting point and iterations of -1 converge to an attracting k -cycle, $k > 1$.
- s2:** Iterations of 1 converge to one attracting k -cycle, $k > 1$, and iterations of -1 converge to another attracting l -cycle, $l > 1$.
- s3:** Iterations of 1 (for example) converge to an attracting point and iterations of -1 converge to infinity. Julia set here is a union of infinite number of connected components. Good example see in Figure 5.

Remark 3.1. In what follows we will give several most interesting examples of Julia sets of SZ-polynomials.

4. TREES WITH FIVE EDGES

In this section for each 5-edge tree T we will compute its SZ-polynomial p_T and find characteristics of $J(p_T)$. The passport of a tree T is the list of degrees of white vertices (in non increasing order) and the list of degrees of black vertices (also in non increasing order). We will always assume that "white" list is lexicographically not less, than "black" list.

Estimations of Hausdorff dimension we will write in order: the box counting estimation, the packing estimation and the JP-algorithm estimation. If some method is inapplicable, then we will put "?" in the corresponding position.

- (1) Passport $\langle 4, 1 \mid 2, 1, 1, 1 \rangle$.

$$T: \begin{array}{c} \bullet \\ | \\ \bullet - \circ - \bullet - \circ \\ | \\ \bullet \end{array} \Rightarrow p_T = \frac{(3z+1)^4(3z-4)}{128} + 1.$$

Polynomial p_T has an attracting 24-cycle. Iterations of ± 1 converge to this cycle, i.e. p_T is of g3-type. The set $J(p_T)$ is very similar to the set in Figure 2. Hausdorff dimension: $\approx 1.65, \approx 1.32, ?$.

- (2) Passport $\langle 3, 2 \mid 2, 1, 1, 1 \rangle$.

$$T: \begin{array}{c} \bullet \\ \diagdown \\ \bullet - \circ - \bullet - \circ \\ \diagup \\ \bullet \end{array} \Rightarrow p_T = -\frac{(z+2)^3(z-3)^2}{54} + 1.$$

Polynomial p_T has an attracting 2-cycle:

$$0.607872363 \rightarrow 0.879463661 \rightarrow 0.607872363$$

Iterations $+1$ and -1 converge to this cycle, i.e. p_T is of g2-type. Hausdorff dimension: $\approx 1.24, \approx 1.19, ?$.

- (3) Passport $\langle 3, 1, 1 \mid 3, 1, 1 \rangle$.

$$T: \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet - \circ - \bullet - \circ \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \Rightarrow p_T = -12z^5 + 10z^3 - \frac{15z}{4}.$$

Iterations of $+1$ and -1 converge to infinity, i.e. p_T is of g4-type. Julia set is totally disconnected. Hausdorff dimension: $?, \approx 0.85, \approx 0.83$.

(4) Passport $\langle 3, 1, 1 \mid 2, 2, 1 \rangle$.

$$T: \begin{array}{c} \bullet \\ | \\ \bullet - \circ - \bullet \\ | \quad | \\ \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \Rightarrow p_T = \frac{(2z+1)^3(2z^2-3z+18)}{432} + 1.$$

Iterations of $+1$ and -1 converge to infinity, i.e. p_T is of g4. Hausdorff dimension: $?$, ≈ 1.11 , ≈ 1.15 .

(5) Passport $\langle 2, 2, 1 \mid 2, 2, 1 \rangle$.

$$T: \bullet - \circ - \bullet - \circ - \bullet - \circ \Rightarrow p_T = \frac{z^5 - 5z^3 + 5z}{2}.$$

The polynomial p_T has two attracting 4-cycles:

$$0.500469 \rightarrow 0.953491 \rightarrow 0.610612 \rightarrow 0.999810 \rightarrow 0.500469$$

and

$$-0.500469 \rightarrow -0.953491 \rightarrow -0.610612 \rightarrow -0.999810 \rightarrow -0.500469$$

Iterations of 1 converge to the first cycle and iterations of -1 — to the second, i.e. p_T is of s2-type. Hausdorff dimension: ≈ 1.60 , ≈ 1.50 , $?$.

5. TREES WITH SIX EDGES

Only one non-symmetric 6 edge tree generates a connected Julia set:

$$T: \begin{array}{c} \bullet \\ | \\ \bullet - \circ - \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \Rightarrow p_T = \frac{-z^6 + 6z^4 + 4z^3 - 9z^2 - 12z + 4}{8}.$$

The polynomial p_T has a superattracting 2-cycle: $1 \leftrightarrow -1$, $p'_T(1) = p'_T(-1) = 0$, i.e. p_T is of g2-type. Hausdorff dimension: ≈ 1.21 , ≈ 1.15 , $?$. Julia set is similar to Julia set in Figure 4.

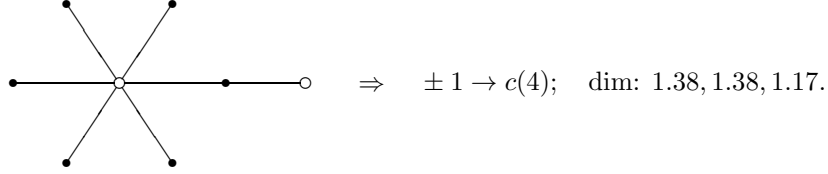
6. TREES WITH SEVEN EDGES

Here we have many trees that generate connected Julia sets. For such tree T we will present behavior of iterations of ± 1 , characteristics of Julia set $J(p_T)$ and the picture of this set in interesting cases. In the picture of Julia set points that quite fast come into attracting domain of infinity (or into attracting domain of attracting point or a cycle) are white, points that come there more slowly are yellow, even more slowly are green, then light red, then deep red.

We will use the following notations:

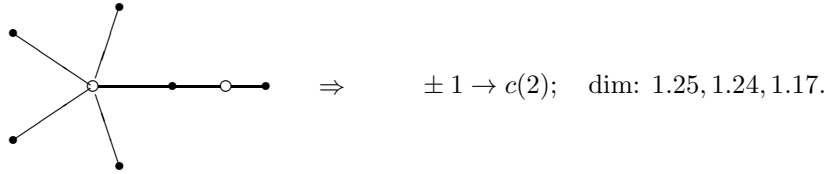
- $\pm 1 \rightarrow \infty$ means that iterations of 1 and -1 converge to infinity, so Julia set is totally disconnected;
- "p" means that SZ-polynomial has an attracting point;
- " $c(k)$ " means that SZ-polynomial has an attracting k -cycle;
- " $1 \rightarrow c_1(2), -1 \rightarrow c_2(3)$ " means that iterations of 1 converge to attracting 2-cycle and iterations of -1 to attracting 3-cycle.

1) $\langle 6, 1 \mid 2, 1, 1, 1, 1, 1 \rangle$.



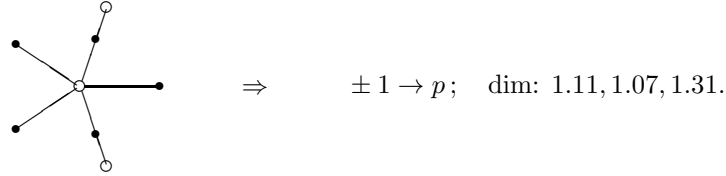
The polynomial p_T is of g3-type. Julia set here is similar to Julia set in Figure 2.

2) $\langle 5, 2 \mid 2, 1, 1, 1, 1, 1 \rangle$.



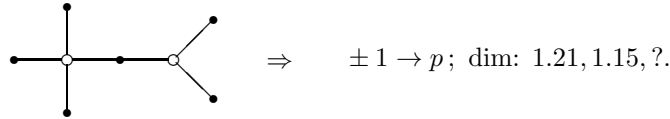
The polynomial p_T is of g2-type. Julia set here is similar to Julia set in Figure 4.

3) $\langle 5, 1, 1 \mid 2, 2, 1, 1, 1 \rangle$.



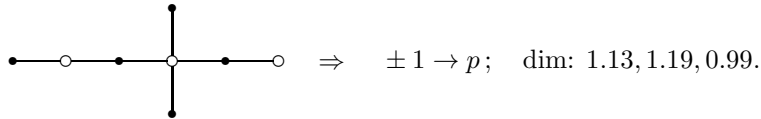
Polynomial p_T is of g1-type. Convergence rate is quite good: $|p'_T(p)| \approx 0.35$.

4) $\langle 4, 3 \mid 2, 1, 1, 1, 1, 1 \rangle$.



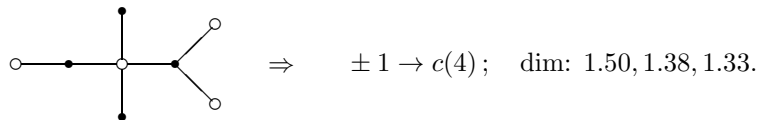
Polynomial p_T is of g1-type. Convergence rate is weak: $|p'_T(p)| \approx 0.95$.

5) $\langle 4, 2, 1 \mid 2, 2, 1, 1, 1 \rangle$.



Polynomial p_T is of g1-type.

6) $\langle 4, 1, 1, 1 \mid 3, 2, 1, 1 \rangle$.



Polynomial p_T is of g3-type. Here we have a high rate of convergence to the attracting cycle: the product of derivatives in cycle points is around 10^{-4} .

$J(p_T) :$

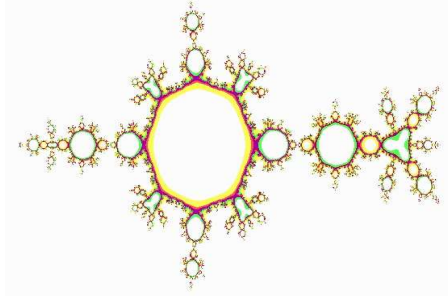
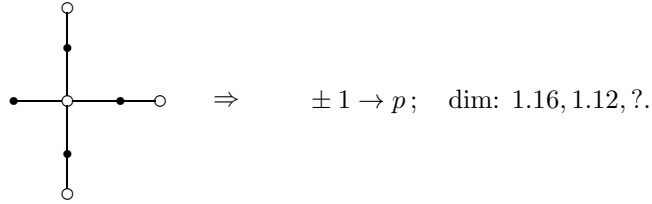


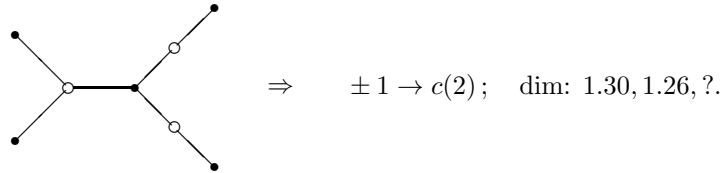
Figure 3

7) $\langle 4, 1, 1, 1 \mid 2, 2, 2, 1 \rangle$.



The polynomial p_T is of g1-type.

8) $\langle 3, 2, 2 \mid 3, 1, 1, 1, 1 \rangle$.



The polynomial p_T is of g2-type. Here we have a medium rate of convergence to the attracting cycle: the product of derivatives in cycle points is around 0.38.

$J(p_T) :$

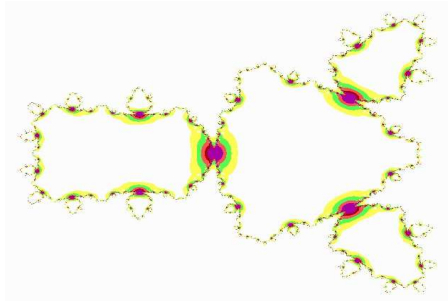
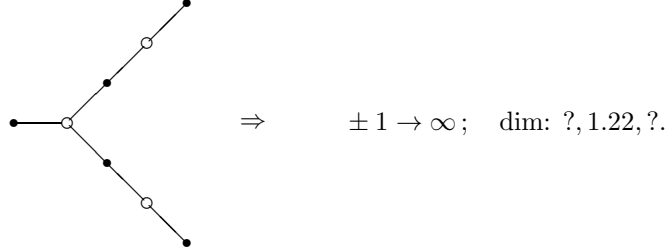


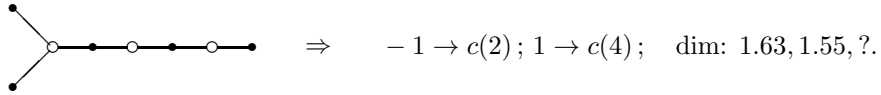
Figure 4

9) $\langle 3, 2, 2 \mid 2, 2, 1, 1, 1 \rangle$.



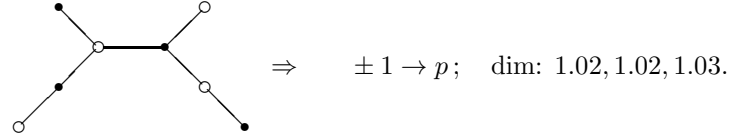
The polynomial p_T is of g4-type.

10) $\langle 3, 2, 2 \mid 2, 2, 1, 1, 1 \rangle$.



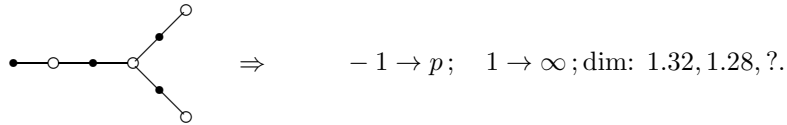
The polynomial p_T is of s2-type.

11) $\langle 3, 2, 1, 1 \mid 3, 2, 1, 1 \rangle$.



The polynomial p_T is of g1-type.

12) $\langle 3, 2, 1, 1, \mid 2, 2, 2, 1 \rangle$.



The polynomial p_T is of s3-type.

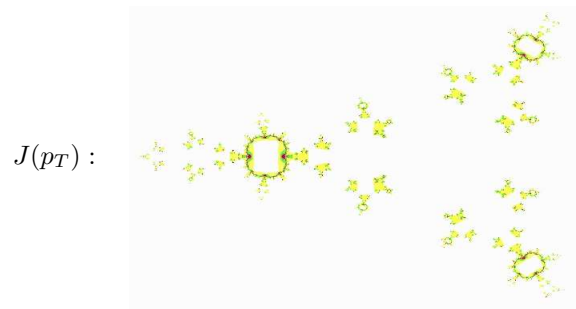
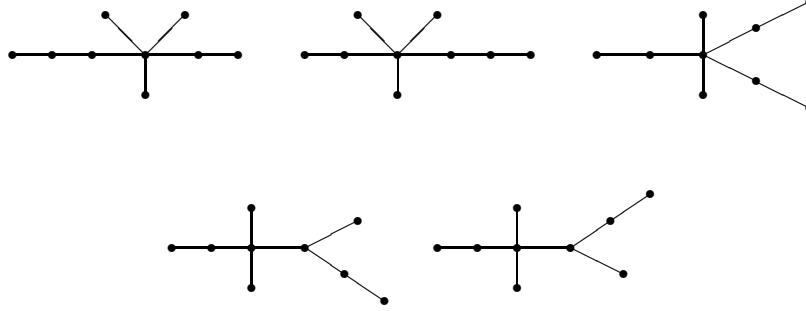


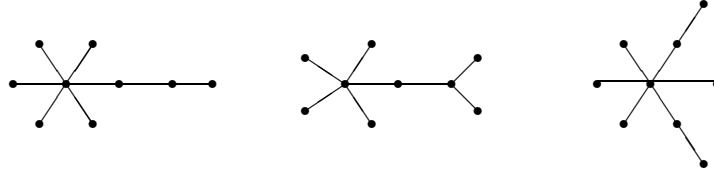
Figure 5

7. TREES WITH EIGHT EDGES

There are five trees whose SZ-polynomials have an attracting point, i.e. are of g1-type.



There are three trees whose SZ-polynomials has an attracting 2-cycle, i.e. are of g2-type.



Next three cases are more interesting.

$$1. \quad \text{Tree diagram} \Rightarrow -1 \rightarrow c(?), 1 \rightarrow p.$$

Polynomial p_T is of s1-type.

$$2. \quad \text{Tree diagram} \Rightarrow \pm 1 \rightarrow c(7)$$

Polynomial p_T is of g3-type and its Julia set is visually interesting.

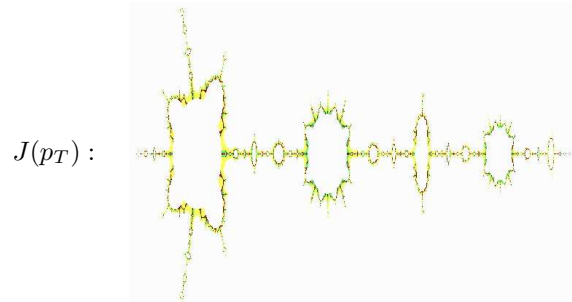
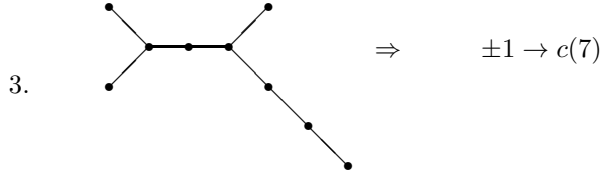


Figure 6



It is an interesting example of g3-type polynomial p_T that is not defined over \mathbb{R} , i.e. where T is not mirror symmetric.

8. SOME RESULTS ABOUT TREES WITH BIG NUMBER OF EDGES

If the passport is relatively simple, then SZ-polynomials can be computed for trees with big number of edges. Here are some examples.

Example 8.1. Passport $\langle n, 1 \mid 2, 1, \dots, 1 \rangle$. If $n \geq 7$, then Julia set is totally disconnected. Otherwise:

$$n = 3 : \pm 1 \rightarrow c(10); \quad n = 4 : \pm 1 \rightarrow c(24); \quad n = 5 : \pm 1 \rightarrow \infty; \quad n = 6 : \pm 1 \rightarrow c(4).$$

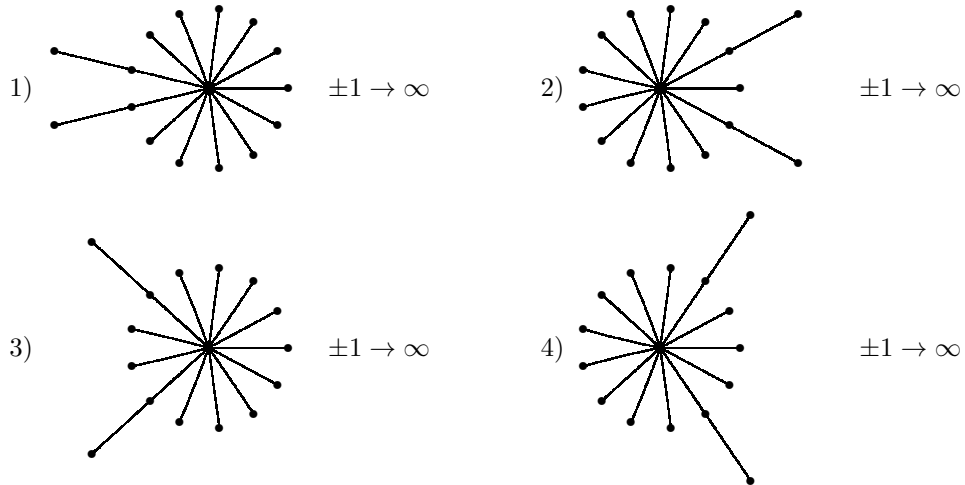
Passport $\langle n, 2 \mid 2, 1, \dots, 1 \rangle$. If $n \geq 13$, then Julia set is totally disconnected. Otherwise:

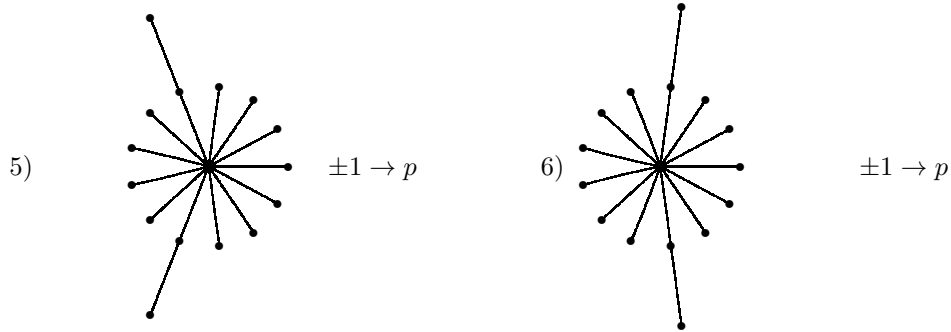
$$\begin{aligned} n = 3 : \pm 1 \rightarrow c(2); \quad n = 4 : \pm 1 \rightarrow c(2); \\ n = 5 : \pm 1 \rightarrow c(2); \quad n = 6 : \pm 1 \rightarrow c(2); \\ n = 7 : \pm 1 \rightarrow c(4); \quad n = 8 : \pm 1 \rightarrow c(16); \\ n = 9 : \pm 1 \rightarrow \infty; \quad n = 10 : \pm 1 \rightarrow c(3); \\ n = 11 : \pm 1 \rightarrow \infty; \quad n = 12 : \pm 1 \rightarrow c(5). \end{aligned}$$

Passport $\langle n, 3 \mid 2, 1, \dots, 1 \rangle$. If $4 \leq n \leq 10$ then $\pm 1 \rightarrow c(2)$. If $n \geq 19$, then Julia set is totally disconnected. Otherwise:

$$\begin{aligned} n = 11 : \pm 1 \rightarrow c(4); \quad n = 12 : \pm 1 \rightarrow c(4); \quad n = 13 : \pm 1 \rightarrow c(3); \\ n = 14 : \pm 1 \rightarrow c(5); \quad n = 15 : \pm 1 \rightarrow \infty; \quad n = 16 : \pm 1 \rightarrow c(3); \\ n = 17 : \pm 1 \rightarrow \infty; \quad n = 18 : \pm 1 \rightarrow c(6). \end{aligned}$$

Passport $\langle 13, 1, 1 \mid 2, 2, 1, \dots, 1 \rangle$.





This example demonstrates that when we consider a set of trees with the same passport, then almost all trees in this set have totally disconnected Julia sets, but for "nearly symmetric" trees this set is connected.

9. CONCLUSION

Further work in this field is related to the following problems.

Problem 1. Prove that SZ-polynomials exist for all non-symmetric trees, or find an example of non-symmetric tree, for which SZ-polynomial does not exist.

Problem 2. When SZ-polynomial p_T of a tree T has an attracting cycle $c(k)$, $k > 1$?

Problem 3. Construct an analogue of SZ-polynomial for genus zero maps and study Julia sets for them.

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