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The threshold decision making

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Abstract

The problem of axiomatic and algorithmic constructions of the threshold decision making is studied in the case when individual opinions are given as m -graded strict preferences (with $m \geq 3$). It is shown that the only rule satisfying the introduced axioms is the threshold rule. Two explicit algorithms are presented: the ordering algorithm, under which the vector-grades of alternatives are successively written out, and an enumerating social decision function corresponding to the natural order of the weak order equivalence classes.

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1. Introduction

We investigate the following problem of construction of a social decision rule. Given a set of n agents, each agent evaluates alternatives from a finite set X using complete and transitive preferences, and we look for a complete and transitive social preference over the alternatives. This kind of aggregation has been considered in many publications, beginning with the seminal work by Arrow [6]. In order to solve the problem, two ways have been in general proposed. Arrow's kind of axiomatics can be described as the local aggregation [1]; in other words, the aggregation is done on the basis of pairwise comparisons. Another way is to use certain non-local procedures, e.g., positional rules, for which axiomatic approaches have been developed rather poorly [8, 9]. On the other hand, an application of the Borda rule is often not adequate, since any summation of ranks has a compensatory nature: a low evaluation of some alternative by one agent can be compensated by a high evaluation of another agent. Thus, if we would like to take carefully into account low evaluations of alternatives, the Borda rule or its counterparts cannot be used.

We present an axiomatic construction of the new aggregation procedure called the 'threshold rule'. The axioms used are Pairwise compensation, Pareto domination, Non-compensatory threshold and Contraction ([2]–[5], [7]).

The Pairwise compensation axiom means that if all agents but two evaluate two alternatives equally, and the two agents put 'mutually inverse' grades, then the two alternatives have the same rank in the social decision.

The Pareto domination axiom states that if the grades of all agents for one alternative are not less than for the second alternative and the grade of at least one agent for the first alternative is strictly greater than that of the second one, then in the social ranking the first alternative has a higher rank than the second one.

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The Non-compensatory threshold and Contraction axiom reveals the very idea of the threshold aggregation (we illustrate it for the case of three-graded preferences, that is, when the ranks 1, 2, 3 mean ‘bad’, ‘average’ and ‘good’, respectively, cf. [4, 5]): if at least one agent evaluates an alternative as ‘bad’, then, no matter how many ‘good’ grades it admits, in the social ranking this alternative has lower rank than any alternative evaluated as ‘average’ by all agents. In this context the Contraction means that if for two alternatives the evaluations of some agent are equal, then the agent may be excluded from the consideration when the social ranking is constructed, and the social decision is achieved by the remaining agents’ evaluations.

We show that the threshold rule is the only rule satisfying the three axioms. The threshold rule aggregates individual preferences as follows (for $m = 3$): first, if the number of ‘bad’ evaluations in the first alternative is greater than that in the second one, then the first alternative has lower rank in the social ranking; if the numbers of ‘bads’ for both alternatives are equal, then the comparison is made with respect to ‘average’ evaluations.

Since individual opinions are orderings as well as a social decision, the next question is whether it is possible to define the rank of an alternative using its ranks in the individual orderings. We present an affirmative answer to this question showing that it can indeed be done by using an explicit aggregation function defined in a combinatorial way. This function has the following nice features: the value of this function on an alternative is the ordinal number of this alternative in the axiomatic ranking, which in turn is equal to the ordinal number of the weak order equivalence class of the alternative and at the same time the ordinal number of the ordered vector grades of the alternative with respect to the algorithmic order. This gives us the ability to work with precise numerical expressions in evaluation of equivalence classes as well as the indifference classes and restore the vector grades of the alternative from the knowledge of its ordinal number as the result of the social decision. This is a new aspect of our theory as compared to the mostly axiomatic approach generally adopted in the literature.

The paper is organized as follows. Section 2 addresses axiomatics of social decision functions. Section 3 treats the algorithms. Section 4 concludes.

2. Axiomatics

Let X be a finite set of alternatives, $N = \{1, 2, \dots, n\}$ a set of $n \geq 2$ agents, $M = \{1, 2, \dots, m\}$ the set of ordered grades, $1 < 2 < \dots < m$, with $m \geq 3$, M^n the set of all n -dimensional vectors with components from M and $\widehat{X} = \{\widehat{x} : x \in X\} \subset M^n$ an individual (evaluation) profile of X , where $\widehat{x} = (x_1, \dots, x_n) \in M^n$ with each $x_i \in M$.

The problem is to rank the elements of X making use of the individual profile \widehat{X} of X . By a *ranking* of X we mean a complete and transitive binary relation on X .

Given $x \in X$ and $j \in M$, we denote by $v_j(x) = |\{i \in N : x_i = j\}|$ the number of grades j corresponding to x (equivalently, in the vector \widehat{x}). Note that $\sum_{j=1}^m v_j(x) = n$ for all $x \in X$.

We look for a social decision function $\varphi : X \rightarrow \mathbb{R}$ satisfying the following m axioms ([2, 3, 7]).

Axiom 1 (Pairwise compensation): if $x, y \in X$ and $v_j(x) = v_j(y)$ for all $1 \leq j \leq m - 1$, then $\varphi(x) = \varphi(y)$.

Axiom 2 (Pareto domination): if $x, y \in X$ and $\widehat{x} > \widehat{y}$ in M^n , then $\varphi(x) > \varphi(y)$, where $\widehat{x} > \widehat{y}$ in M^n means that $x_i \geq y_i$ for all $1 \leq i \leq n$ and $x_{i_0} > y_{i_0}$ for some $1 \leq i_0 \leq n$.

Axiom k for $k = 3, \dots, m$ (Non-compensatory threshold and Contraction): if $x, y \in X$, $v_j(x) = v_j(y)$ for all $1 \leq j \leq m - k$ (no condition if $k = m$), $v_{m-k+1}(x) + 1 = v_{m-k+1}(y)$, $v_j(x) = 0$ for all $m - k + 3 \leq j \leq m$, $v_j(y) = 0$ for all $m - k + 2 \leq j \leq m - 1$ and $v_m(y) \neq 0$, then $\varphi(x) > \varphi(y)$.

Setting $v(x) = (v_1(x), \dots, v_{m-1}(x))$ for $x \in X$, we denote by $v(x) \angle v(y)$ the property of $x, y \in X$ meaning that there exists a $1 \leq k \leq m - 1$ such that $v_k(x) < v_k(y)$ and $v_j(x) = v_j(y)$ for all $1 \leq j \leq k - 1$ (no second condition if $k = 1$), which will be called the *threshold rule* for comparison of x and y (note that \angle is the lexicographic order on \mathbb{R}^{m-1}). We say that a binary relation $P \subset X \times X$ is *generated* by the threshold rule provided that $P = \{(x, y) \in X \times X : v(x) \angle v(y)\}$. The relation P is a weak order on X , i.e., it is transitive ($(x, y) \in P$ and $(y, z) \in P$ imply $(x, z) \in P$), irreflexive ($(x, x) \notin P$) and negatively transitive ($(x, y) \notin P$ and $(y, z) \notin P$ imply $(x, z) \notin P$).

It is well known (e.g., [1]) that a weak order P on X is completely characterized by the family of its equivalence classes, whose construction is recalled now. Set $X'_1 = \pi(X)$ where, given $A \subset X$, $\pi(A) = \{x \in A : \neg \exists y \in A \text{ such that } (y, x) \in P\}$. Inductively, if $\ell \geq 2$ and nonempty subsets $X'_1, \dots, X'_{\ell-1}$ of X are already defined, we put $X'_\ell = \pi(X \setminus (\bigcup_{k=1}^{\ell-1} X'_k))$. Since X is finite, we have $X = \bigcup_{\ell=1}^s X'_\ell$ for some positive integer s . Setting

$X_\ell = X'_{s-\ell+1}$ for $\ell = 1, \dots, s$, the disjoint collection $\{X_\ell\}_{\ell=1}^s$ is said to be the *family of equivalence classes* of the weak order P , and has the following characterization property: $(x, y) \in P$ if and only if there exist $1 \leq \ell_1 < \ell_2 \leq s$ such that $x \in X_{\ell_2}$ and $y \in X_{\ell_1}$.

We say that a function $\varphi : X \rightarrow \mathbb{R}$ is *coherent* with the family $\{X_\ell\}_{\ell=1}^s$ of equivalence classes of weak order P if, given $x, y \in X$, $\varphi(x) > \varphi(y)$ if and only if there exist $1 \leq \ell_1 < \ell_2 \leq s$ such that $x \in X_{\ell_2}$ and $y \in X_{\ell_1}$.

The first main result is the following

Theorem 1. *A social decision function $\varphi : X \rightarrow \mathbb{R}$ satisfies Axioms 1 through m if and only if it is coherent with the family of equivalence classes of the weak order P generated by the threshold rule $v(x) \angle v(y)$.*

3. Algorithms

We define the *indifference* relation I on X by $(x, y) \in I$ whenever $v(x) = v(y)$, $x, y \in X$, i.e., when the vector grades \widehat{x} and \widehat{y} can be transformed to each other by permutations of their coordinates. Clearly, I is an equivalence relation on X . We denote by $I[x] = \{y \in X : (y, x) \in I\}$ the indifference class of $x \in X$ and by X/I the set of all indifference classes. Given $x \in X$, we denote by $x^* = (x_1^*, \dots, x_n^*) \in M^n$ the unique vector with $x_1^* \leq x_2^* \leq \dots \leq x_n^*$, which is obtained from \widehat{x} by a permutation of its coordinates. Note that $v_j(x) = |\{i \in N : x_i^* = j\}|$ for all $j \in M$. We call x^* the *monotone representative* of x and denote by X^* the set of all monotone representatives and by M^{n*} the subset of M^n of all vectors, whose coordinates are ordered in ascending order, so that, $X^* \subset M^{n*}$. It can be shown that ([2])

$$|X/I| = |X^*| \leq |M^{n*}| = \binom{n+m-1}{m-1}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

$$\text{and} \quad \{X_\ell\}_{\ell=1}^s = X/I, \quad \text{whence} \quad s = |X^*|.$$

In what follows we assume that $X^* = M^{n*}$, and so, $s = \binom{n+m-1}{m-1}$.

We also note that the binary relation R on X given by $R = P \cup I$ is complete and transitive (i.e., it is a ranking of X).

We define the *algorithmic order* on M^{n*} as follows: write out one by one a string of vectors of the form:

$$\underbrace{(1, \dots, 1)}_{n-n_1}, \underbrace{(2, \dots, 2)}_{n_1-n_2}, \underbrace{(3, \dots, 3)}_{n_2-n_3}, \dots, \underbrace{(m-1, \dots, m-1)}_{n_{m-2}-n_{m-1}}, \underbrace{(m, \dots, m)}_{n_{m-1}} \quad (1)$$

(the numbers under the braces being the lengths of the underbraced subvectors) in such a way that n_1 assumes successively the values $0, 1, \dots, n$, and if n_1 is fixed, n_2 assumes successively the values $0, 1, \dots, n_1$, and if n_2 is fixed, n_3 assumes successively the values $0, 1, \dots, n_2$, and so on, and finally, if n_1, n_2, \dots, n_{m-2} are fixed, n_{m-1} assumes successively the values $0, 1, \dots, n_{m-2}$. Since the algorithmic order is linear, to each vector $x^* \in X^* = M^{n*}$ corresponds a unique natural number $\text{alg}(x^*)$, which is defined as the ordinal number of x^* with respect to the algorithmic order on M^{n*} .

Our second main result is as follows.

Theorem 2. *$(x, y) \in P$ or, equivalently, $x \in X_{\ell_2}$ and $y \in X_{\ell_1}$ for some $1 \leq \ell_1 < \ell_2 \leq s$ if and only if $\text{alg}(x^*) > \text{alg}(y^*)$.*

Note that, since each vector $x^* \in M^{n*}$ can be uniquely written in the form (1) with $n_j = n_j(x^*)$ depending on x^* , $1 \leq j \leq m-1$, and satisfying $0 \leq n_j(x^*) \leq n_{j-1}(x^*)$ for all $1 \leq j \leq m-1$ and $n_0(x^*) = n$, then setting $\bar{n}(x^*) = (n_1(x^*), \dots, n_{m-1}(x^*))$, we find that $\text{alg}(x^*) > \text{alg}(y^*)$ if and only if $\bar{n}(y^*) \angle \bar{n}(x^*)$ in \mathbb{R}^{m-1} .

Finally, we define explicitly the social decision function Φ , which characterizes the equivalence classes of P , the indifference classes of I and the algorithmic order on M^{n*} , according to the rule:

$$\Phi(x) = 1 + v_m(x) + \sum_{i=1}^{m-2} \left(i + \sum_{j=m-i}^m v_j(x) \right), \quad x \in X,$$

where $\binom{i}{i+1} = 0$ for all $1 \leq i \leq m-2$.

Our third main result is the following

Theorem 3. *The function Φ has the properties:*

- (1) Φ maps X onto $\{1, 2, \dots, s\}$ with $s = \binom{n+m-1}{m-1}$ and satisfies Axioms 1 through m ;
- (2) $\Phi(x) = \text{alg}(x^*)$ for all $x \in X$;
- (3) $X_\ell = \{x \in X : \Phi(x) = \ell\}$ for all $1 \leq \ell \leq s$;
- (4) $I[x] = X_{\Phi(x)}$ for all $x \in X$;
- (5) given $1 \leq \ell \leq s$, we have: $x \in X_\ell$, or $\Phi(x) = \ell$, if and only if there exists a uniquely determined collection of numbers n_1, n_2, \dots, n_{m-2} satisfying $0 \leq n_j \leq n_{j-1}$ for all $1 \leq j \leq m-2$ with $n_0 = n$ and $L+1 \leq \ell \leq n_{m-2} + L+1$, where $L = \sum_{i=1}^{m-2} \binom{i+n_{m-i-1}}{i+1}$, such that $v_j(x) = n_{j-1} - n_j$ for all $1 \leq j \leq m-2$, $v_{m-1}(x) = n_{m-2} + L+1 - \ell$ and $v_m(x) = \ell - L - 1$.

4. Conclusion

We have considered a model of aggregation, in which ‘low’ evaluations of agents were taken into account extremely carefully (in a different interpretation, ‘punished’ rather severely), since low grades cannot in any way be compensated by high grades of the other agents. This is exactly the situation when the quality or perfectness of alternatives are of great value and interest. On the other hand, it is clear that an aggregation procedure can be made taking carefully into account high grades of agents: this is the case when we are interested in at least one good feature of alternatives. It is exactly the dual model to that considered above, and it has all advantages of the dual model including the construction of a social decision function similar to Φ from Section 3.

Yet, one more remark ought to be made concerning an interpretation of the Non-compensatory property. Under this property, any agent giving a low grade to an alternative puts it down in the social decision as compared to an alternative with average grades. Thus, marginal opinions can strongly influence the social decision. Correspondingly, it opens the door to a rather high manipulability of the threshold rule.

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