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Braided differential operators on quantum algebras

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ABSTRACT

We propose a general scheme of constructing braided differential algebras via algebras of "quantum exponentiated vector fields" and those of "quantum functions". We treat a reflection equation algebra as a quantum analog of the algebra of vector fields. The role of a quantum function algebra is played by a general quantum matrix algebra. As an example we mention the so-called RTT algebra of quantized functions on the linear matrix group $GL(m)$. In this case our construction essentially coincides with the quantum differential algebra introduced by S. Woronowicz. If the role of a quantum function algebra is played by another copy of the reflection equation algebra we get two different braided differential algebras. One of them is defined via a quantum analog of (co)adjoint vector fields, the other algebra is defined via a quantum analog of right-invariant vector fields. We show that the former algebra can be identified with a subalgebra of the latter one. Also, we show that "quantum adjoint vector fields" can be restricted to the so-called "braided orbits" which are counterparts of generic $GL(m)$-orbits in $gl^*(m)$. Such braided orbits endowed with these restricted vector fields constitute a new class of braided differential algebras.

1. Introduction

Since the creation of the quantum group theory plenty of different quantum algebras related to R-matrices (i.e., solutions of the quantum Yang–Baxter equation) have been introduced in the mathematical and physical literature. A remarkable family of such algebras was introduced in [1] under the name of Heisenberg doubles.

As an associative algebra the Heisenberg double is generated by elements of two dual Hopf algebras H and H^*. In order to define an associative product on the space $H \otimes H^*$ one needs a permutation operator

$$H \otimes H^* \to H^* \otimes H$$

transposing elements of two components. Such an operator can be defined via the pairing

$$H \otimes H^* \to \mathbb{K}$$

putting the algebras H and H^* in the duality ($\mathbb{K}$ is the ground field). Also, assuming one of these algebras to be the quantized function algebra, namely, the famous RTT algebra, one can extract from the dual object a space of "quantum exponentiated vector fields" via a construction similar to that of [2].

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A close approach to constructing a quantum version of differential calculus on a matrix pseudogroup was initiated by Woronowicz [3]. In that paper the central object — quantum differential algebra — consists of three ingredients: a quantized function algebra, an algebra of quantum exponentiated vector fields which is in fact the reflection equation (RE) algebra[1] and an algebra of "quantum differential forms".

We would also refer the reader to the papers [6,7] where a different approach to the quantum calculus was suggested. In particular, it was shown that the classical Leibnitz rule for the external differential must be modified.

In the present paper we disregard quantum differential forms and generalize the other components of this calculus as follows. We always keep the RE algebra[2] $\mathcal{L}(R)$ as an algebra of quantum exponentiated vector fields. However, we introduce different candidates on the role of a quantum function algebra $\mathcal{M}$, endowed with an appropriate action of the RE algebra. For any of such a candidate the key point consists in constructing a permutation operator

$$\mathsf{R}:\ \mathcal{L}(R)\otimes\mathcal{M}\to\mathcal{M}\otimes\mathcal{L}(R) \tag{1.1}$$

which enables us to endow the space $\mathcal{L}(R)\otimes\mathcal{M}$ with an associative product. We denote the resulting associative algebra by $\mathcal{B}(\mathcal{L}(R),\mathcal{M})$ and call it *a braided differential (BD) algebra*.

We emphasize that the operator (1.1) also enables us to write down a sort of the Leibnitz rule for elements of $\mathcal{L}(R)$. Having defined an action of these elements on generators of the algebra $\mathcal{M}$ and extending this action to the higher components of $\mathcal{M}$ via such a "Leibnitz rule", we get a representation of $\mathcal{L}(R)$ into the algebra $\mathcal{M}$. Note that in this construction we have no need of a bialgebra structure (usual or braided) of the algebra $\mathcal{M}$. Instead, we use a braided bialgebra structure of $\mathcal{L}(R)$ in order to apply its representation theory developed in [8].[3]

Now, describe in more detail different types of the quantum algebras $\mathcal{M}$ we are dealing with. First, we consider algebras $\mathcal{M}$ generated by the basic objects V and V^* of the $\mathcal{L}(R)$ representation category constructed in [8]. Note that the free tensor algebras $T(V)$ and $T(V^*)$, as well as the "R-symmetric" and "R-skew-symmetric" algebras of the space V (or V^*), are examples of such algebras $\mathcal{M}$.

Second, we consider quantum *matrix* algebras $\mathcal{M}$, each constructed via a pair of compatible R-matrices (see definition (2.11) in Section 2). As a particular case of such an algebra $\mathcal{M}$ we get the RTT algebra. In this case the resulting BD algebra coincides with the Heisenberg double studied in [9]. If we take another copy of the RE algebra as $\mathcal{M}$ we get one more example of a BD algebra. In this case two copies of the RE algebras are involved — one of them (denoted $\mathcal{L}(R)$) plays the role of quantum exponentiated vector fields, the other one (denoted $\mathcal{M}(R)$) plays the role of a quantum function algebra.

The characteristic property of these two and other similar examples of BD algebras is that the elements of $\mathcal{L}(R)$ act on the quantum matrix algebra $\mathcal{M}$ on the left side and are in a sense analogs of right-invariant exponentiated vector fields. In what follows such BD algebras are denoted $\mathcal{B}_r(\mathcal{L}(R),\mathcal{M})$ where the subscript r means "right-invariant".

However, if the quantum matrix algebra $\mathcal{M}$ is just the RE algebra $\mathcal{M}(R)$ we can define another action of $\mathcal{L}(R)$ on $\mathcal{M}(R)$, namely the "adjoint" action, which is an analog of the usual adjoint action of one copy of $gl(m)$ onto another one. The corresponding BD algebra is denoted $\mathcal{B}_{ad}(\mathcal{L}(R),\mathcal{M}(R))$. Thus, we have two versions of the BD algebra composed from the algebras $\mathcal{L}(R)$ and $\mathcal{M}(R)$. One of them $\mathcal{B}_r(\mathcal{L}(R),\mathcal{M}(R))$ is based on the right-invariant action, the other one — $\mathcal{B}_{ad}(\mathcal{L}(R),\mathcal{M}(R))$ — on the adjoint action. We show that the algebra $\mathcal{B}_{ad}(\mathcal{L}(R),\mathcal{M}(R))$ can be embedded into the properly extended algebra $\mathcal{B}_r(\mathcal{L}(R),\mathcal{M}(R))$ as a subalgebra.

Similarly to the classical case the adjoint action of the algebra $\mathcal{L}(R)$ onto $\mathcal{M}(R)$ preserves central elements of the latter algebra. Using this fact it is possible to reduce the quantum adjoint vector fields to "braided orbits", i.e. the quotients of $\mathcal{M}(R)$ which are quantum counterparts of generic $GL(m)$ orbits in $gl^*(m)$ (see [10,11] for detail). Thus, we get one more family of BD algebras, in which the role of quantum function algebras is played by "braided orbits".

The paper is organized as follows. In the next Section we recall some elements of "braided geometry" as presented in [12]. In particular, we exhibit a regular way of constructing a quantum matrix algebra via a pair of compatible R-matrices. In Section 3 we concentrate ourselves on properties of the RE algebra including its representation category. This enables us to construct some examples of the BD algebras considered in Section 4. In Section 5 we present a construction of the BD algebra over a general quantum matrix algebra and exhibit the mentioned relation between BD algebra based on two copies of the RE algebra but equipped with different types of action. We complete the paper with an example of BD algebra on a quantum hyperboloid.

2. $GL(m)$-type R-matrices and quantum matrix algebras

In this section we give a short list of definitions and notation to be used below. More details and proofs can be found in the cited literature.

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1 This algebra was studied by Majid under the name of braided matrix algebra (see [4] and the references therein). The term "reflection equation algebra" was introduced by Kulish and his coauthors (see [5]).

2 The RE algebra is parameterized by an R-matrix R. Below we are dealing with R-matrices of $GL(m)$-type defined in Section 2. However, a big part of our results can be generalized to algebras associated with R-matrices of a more general form.

3 In [8] we constructed a representation category for the so-called modified RE algebra which in fact coincides with the non-modified RE algebra but written in another basis. All construction and results of [8] can be directly adapted to the non-modified form of the RE algebra. Below, we refer to this paper without saying it each time.

Let $\mathbb{K}$ denote the field of complex or real numbers and V be a finite dimensional vector space over the field $\mathbb{K}$: $\dim_{\mathbb{K}} V = N$. Given a linear operator $X \in \mathrm{End}(V^{\otimes k})$, $\forall k \geqslant 1$, we extend it up to different operators belonging to $\mathrm{End}(V^{\otimes(k+p)})$, $p \geqslant 0$, in a natural way

$$X_{i\ldots i+k-1} = I_V^{\otimes(i-1)} \otimes X \otimes I_V^{\otimes(p-i+1)}, \quad 1 \leqslant i \leqslant p+1, \tag{2.1}$$

where I_V stands for the identity operator on V. In what follows we shall abbreviate I_V to I and simplify $X_{i\,i+1}$ to X_i for $X \in \mathrm{End}(V^{\otimes 2})$. Hereafter, all tensor products are taken over the ground field $\mathbb{K}$.

An invertible operator $R \in \mathrm{Aut}(V^{\otimes 2})$ is called an *R-matrix* if it satisfies the *Yang–Baxter equation* in $\mathrm{End}(V^{\otimes 3})$

$$R_1R_2R_1 - R_2R_1R_2 = 0. \tag{2.2}$$

In the present paper we are dealing with *Hecke type* R-matrices which obey the quadratic Hecke condition

$$(R - q\,I^{\otimes 2})(R + q^{-1}\,I^{\otimes 2}) = 0, \quad q \in \mathbb{K} \setminus 0. \tag{2.3}$$

A numerical parameter q is assumed to be *generic*, that is either $q = 1$ or $q^k \neq 1$, $\forall\, k \in \mathbb{N}$. In particular, for a generic value of the parameter the q-analogs of integers

$$k_q = q^{k-1} + q^{k-3} + \cdots + q^{1-k} = \frac{q^k - q^{-k}}{q - q^{-1}}$$

are non-zero for any integer $k \in \mathbb{Z}$.

An example of the Hecke type R-matrix for $q = 1$ is given by the flip (transposition operator):

$$P : V^{\otimes 2} \to V^{\otimes 2}, \quad P(v_1 \otimes v_2) = v_2 \otimes v_1. \tag{2.4}$$

A well-known example for $q \neq 1$ is the Drinfeld–Jimbo R-matrix

$$R(q) = q\sum_{i=1}^{N} E_{ii} \otimes E_{ii} + \sum_{i\neq j}^{N} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{1\leqslant i<j\leqslant N} E_{ii} \otimes E_{jj}, \tag{2.5}$$

where $E_{ij} \in \mathrm{Mat}_N(\mathbb{K})$ are the standard matrix units. Note that $R(q)$ is a continuous matrix function in q and $\lim_{q\to 1} R(q) = P$. There are known other Hecke type R-matrices which are continuous matrix functions in q and turn into the flip P at the limit $q \to 1$. All such Hecke type R-matrices will be referred to as deformations of the flip P.

The Hecke type R-matrices are closely connected with the representation theory of the A_{n-1} series Hecke algebras $\mathcal{H}_n(q)$, $n \geqslant 2$. Recall, that the Hecke algebra $\mathcal{H}_n(q)$ is the quotient of the group algebra $\mathbb{K}[\mathcal{B}_n]$ of the braid group $\mathcal{B}_n$, $n \geqslant 2$,

$$\mathcal{B}_n = \langle \{\sigma_i^{\pm}\}_{1\leqslant i\leqslant n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ i \neq j \pm 1,\ \sigma_i^{\pm 1}\sigma_i^{\mp 1} = 1_{\mathcal{B}} \rangle$$

over the two sided ideal generated by the elements

$$\sigma_i^{-1} - \sigma_i + (q - q^{-1})1_{\mathcal{B}}$$

where $1_{\mathcal{B}}$ stands for the unit element of the braid group.

At a generic value of q the algebra $\mathcal{H}_n(q)$ is known to be semisimple and isomorphic to the group algebra $\mathbb{K}[\mathfrak{S}_n]$ of the n-th order permutation group. As a consequence, the primitive idempotents $e_\lambda^a : e_\lambda^a e_\mu^b = \delta_{\lambda\mu}\delta^{ab}e_\lambda^a$ of the Hecke algebra $\mathcal{H}_n(q)$ are labeled by partitions $\lambda \vdash n$ and by an integer index $a : 1 \leqslant a \leqslant d_\lambda$ where d_λ equals to the number of standard Young tableaux corresponding to the partition λ.

Any Hecke type R-matrix R realizes a so-called *local R-matrix representation* ρ_R of a Hecke algebra $\mathcal{H}_n(q)$ by the following rule:

$$\rho_R : \mathcal{H}_n(q) \to \mathrm{End}(V^{\otimes n}), \qquad \rho_R(\sigma_i) = R_i, \quad 1 \leqslant i \leqslant n-1. \tag{2.6}$$

The detailed treatment of the Hecke algebra and its representations with a list of original papers can be found, e.g., in the review [13].

We constrain ourselves to considering a subfamily of the Hecke type R-matrices — so-called R-matrices of the $GL(m)$-type.

A Hecke type R-matrix R is said to be of the $GL(m)$-type if the operator

$$A_m(R) = \rho_R(e_{(1^m)}(\sigma))$$

is a rank one projector in $\mathrm{End}(V^{\otimes m})$, while $A_{m+1}(R) \in \mathrm{End}(V^{\otimes(m+1)})$ is the zero operator (explicit formulae for $A_k(R)$, $k \geqslant 1$, can be found, for example, in [13]). Note, that the $N^2 \times N^2$ Drinfeld–Jimbo R-matrix (2.5) is of the $GL(N)$-type, but for general $GL(m)$-type R-matrix we have $m \leqslant N$ (see [14] for examples).

Also, we need the following definition. An operator $R \in \mathrm{End}(V^{\otimes 2})$ is called *skew-invertible* provided that there exists an operator $\Psi_R \in \mathrm{End}(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{(2)}R_{12}\Psi_{R23} = P_{13} = \mathrm{Tr}_{(2)}\Psi_{R12}R_{23}, \tag{2.7}$$

where the subscript in the notation of the trace indicates the factor in the tensor product $V^{\otimes 3}$, where the trace operation is applied.

The operators

$$B_R = \mathrm{Tr}_{(1)}\Psi_{R\,12}, \qquad C_R = \mathrm{Tr}_{(2)}\Psi_{R\,12}, \tag{2.8}$$

possess the property

$$\mathrm{Tr}_{(1)}B_{R\,1}R_{12} = I = \mathrm{Tr}_{(2)}C_{R\,2}R_{12}. \tag{2.9}$$

A skew-invertible R-matrix is called *strictly* skew-invertible if the corresponding operator B_R (or C_R) is invertible.

As was shown in [14], any $GL(m)$-type R-matrix is skew-invertible. Moreover, the operators B_R and C_R are subject to the relation [8]

$$B_R \cdot C_R = q^{-2m} I.$$

Consequently, any $GL(m)$-type R-matrix is strictly skew-invertible.

The operators B_R and C_R play an important role in the theory of quantum matrix algebras considered below. In particular, the operator C_R appears in definition of the *R-trace* Tr_R:

$$\mathrm{Tr}_R : \mathrm{Mat}_N(A) \to A, \qquad \mathrm{Tr}_R(X) = \mathrm{Tr}(C_R X), \quad \forall X \in \mathrm{Mat}_N(A), \tag{2.10}$$

where A is a vector space over the field $\mathbb{K}$ and $\mathrm{Mat}_N(A) = \mathrm{Mat}_N(\mathbb{K}) \otimes_{\mathbb{K}} A$.

Turn now to the definition of a quantum matrix algebra [15]. An *ordered* pair $\{R, F\}$ of two R-matrices $R, F \in \mathrm{Aut}(V^{\otimes 2})$ is called *compatible* if they satisfy the following relations (compatibility conditions):

$$R_1F_2F_1 = F_2F_1R_2, \qquad F_1F_2R_1 = R_2F_1F_2. \tag{2.11}$$

Definition 1 (*[15]*). Given a compatible pair $\{R, F\}$ of strictly skew-invertible R-matrices $R, F \in \mathrm{Aut}(V^{\otimes 2})$, $\dim_{\mathbb{K}} V = N$, the *quantum matrix (QM) algebra* $\mathcal{M}(R, F)$ is a unital associative algebra generated by a unit element $1_{\mathcal{M}}$ and by N^2 entries of the matrix $M = \|M_i^j\|_{1\leqslant i,\, j\leqslant N}$ subject to the relations

$$R_1 M_{\overline{1}} M_{\overline{2}} - M_{\overline{1}} M_{\overline{2}} R_1 = 0, \tag{2.12}$$

where we use the notation

$$M_{\overline{1}} = M_1, \qquad M_{\overline{k+1}} = F_k M_{\overline{k}} F_k^{-1}, \quad k \geqslant 1, \tag{2.13}$$

for the "copies" of the matrix M. In what follows the matrix M will be called a generating matrix of the QM algebra $\mathcal{M}(R, F)$. (In the above formulae R_1 and F_k, $k \geqslant 1$, are treated in the sense of formula (2.1).)

Also, observe that by fixing a basis $\{x_i\}_{1\leqslant i\leqslant N}$ in the space V and the corresponding basis $\{x_i \otimes x_j\}$ in that $V^{\otimes 2}$ we can represent the operators $R, F \in \mathrm{Aut}(V^{\otimes 2})$ by *numerical* matrices $\|R_{ij}^{rs}\|$ and $\|F_{ij}^{rs}\|$ where

$$R(x_i \otimes x_j) = R_{ij}^{rs}\, x_r \otimes x_s, \qquad F(x_i \otimes x_j) = F_{ij}^{rs}\, x_r \otimes x_s. \tag{2.14}$$

The defining relations (2.12) and compatibility conditions (2.11) imply the same type relations for consecutive pairs of the copies of M [15]

$$R_k\, M_{\overline{k}} M_{\overline{k+1}} - M_{\overline{k}} M_{\overline{k+1}}\, R_k = 0. \tag{2.15}$$

It follows from Definition 1 that any QM algebra $\mathcal{M}(R, F)$ is a finitely generated quadratic (in the generators M_i^j) graded algebra and consequently, it can be presented as a sum of homogeneous components

$$\mathcal{M}(R, F) = \sum_{p\geqslant 0} \mathcal{M}^p(R, F), \qquad \mathcal{M}^0(R, F) \cong \mathbb{K}.$$

The widely known example of QM algebra is the quantized algebra of functions on the matrix algebra $M_N(\mathbb{K})$ [2]. This algebra is associated with a compatible pair $\{R, P\}$, P being a flip (2.4). Let $T = \|T_i^j\|_1^N$ be the generating matrix of $\mathcal{M}(R, P)$. Then the relations (2.12) take the form

$$R_1T_1T_2 - T_1T_2R_1 = 0, \tag{2.16}$$

since $T_{\overline{2}} = P_{12}T_1P_{12} = T_2$. We call the QM algebra $\mathcal{M}(R, P)$ the RTT algebra and denote it by $\mathcal{T}(R)$. At any choice of an R-matrix R the RTT algebra $\mathcal{T}(R)$ is a bialgebra with the coproduct Δ and counit ε

$$\Delta(T) = T \,\dot{\otimes}\, T, \qquad \varepsilon(T) = I. \tag{2.17}$$

The symbol $\dot{\otimes}$ stands for the following operation

$$(A \,\dot{\otimes}\, B)_i^j = \sum_{k=1}^{N} A_i^k \otimes B_k^j \tag{2.18}$$

where A and B are arbitrary $N_1 \times N$ and $N \times N_2$ matrices respectively.

Note that for any R-matrix of $GL(m)$-type a quantum determinant $\det_q T$ can be defined (see [14]). This enables us to introduce an antipode and a Hopf algebra structure in the algebra $\mathcal{T}(R)$ extended by $(\det_q T)^{-1}$. The resulting algebra is the most popular quantum analog of the function algebra on $GL(m)$ (see [2]).

Below we consider a family of the QM algebras associated with a compatible pair $\{R, F\}$ where R is a $GL(m)$-type R-matrix. The corresponding algebra $\mathcal{M}(R, F)$ is referred to as a $GL(m)$-type QM algebra.

A useful tool for studying the structure of the $GL(m)$-type QM algebra is the *characteristic subalgebra* $\mathrm{Char}(\mathcal{M}) \subset \mathcal{M}(R, F)$ [15]. By definition, this is a linear span of the unit element and the following elements

$$x(h_k) = \mathrm{Tr}_{R(1\dots k)}(M_{\bar{1}}\dots M_{\bar{k}}\,\rho_R(h_k)), \quad k \in \mathbb{N},$$

where h_k runs over all elements of the Hecke algebra $\mathcal{H}_k(q)$ and ρ_R is the R-matrix representation (2.6) of $\mathcal{H}_k(q)$ in $\mathrm{End}(V^{\otimes k})$.

Among all elements of the characteristic subalgebra we distinguish the following families:

- The *power sums* of the quantum matrix

$$p_0(M) = 1_{\mathcal{M}}\mathrm{Tr}_R(I), \qquad p_k(M) = \mathrm{Tr}_{R(1\dots k)}(M_1 M_{\bar{2}}\dots M_{\bar{k}}\rho_R(\sigma_{k-1}\dots\sigma_2\sigma_1)), \quad k \geqslant 1. \tag{2.19}$$

- The *elementary symmetric functions*

$$a_0(M) = 1_{\mathcal{M}}, \qquad a_k(M) = \mathrm{Tr}_{R(1\dots k)}(M_1 M_{\bar{2}}\dots M_{\bar{k}}\rho_R(e_{(1^k)})), \quad 1 \leqslant k \leqslant m. \tag{2.20}$$

The main properties of a $GL(m)$-type QM algebra to be used below are collected in the following proposition.

Proposition 2 (*[15]*). *Let $\mathcal{M}(R, F)$ be a $GL(m)$-type quantum matrix algebra. Then the following statements hold true.*

1. *The characteristic subalgebra* $\mathrm{Char}(\mathcal{M})$ *is abelian.*
2. *The characteristic subalgebra is generated by the set of* power sums $\{p_k(M)\}_{0\leqslant k\leqslant m}$ *or, equivalently, by the set of elementary symmetric functions* $\{a_k(M)\}_{0\leqslant k\leqslant m}$.
3. *The power sums are related with elementary symmetric functions by the quantum Newton identities*

$$(-1)^{k-1}k_q a_k(M) = \sum_{i=0}^{k-1}(-q)^i p_{k-i}(M)a_i(M), \quad 1 \leqslant k \leqslant m. \tag{2.21}$$

4. *The generating matrix M satisfies the Cayley–Hamilton identity*

$$\sum_{k=0}^{m}(-q)^k M^{\overline{m-k}} a_k(M) = 0, \tag{2.22}$$

where

$$M^{\bar{0}} = 1_{\mathcal{M}}I, \qquad M^{\bar{k}} = \mathrm{Tr}_{R(2\dots k)}(M_{\bar{1}}M_{\bar{2}}\dots M_{\bar{k}}\rho_R(\sigma_{k-1}\sigma_{k-2}\dots\sigma_1)).$$

3. RE algebra and its representation theory

In this section we give a short review of a particular case of the QM algebra — the RE algebra [5,4]. We discuss its structure and the representation theory. Our exposition will mainly follow the paper [8] where the reader can find detailed proofs and further references to the literature on the RE algebra.

The $GL(m)$-type RE algebra $\mathcal{L}(R)$ is associated with the compatible pair $\{R, R\}$, where $R \in \mathrm{End}(V^{\otimes 2})$ is a $GL(m)$-type R-matrix (recall, that $\dim_{\mathbb{K}} V = N \geqslant m$). Denoting the matrix of the generators of the RE algebra by $L = \|L_i^j\|_1^N$ we rewrite the general commutation relations (2.12) in the equivalent form:

$$R_1L_1R_1L_1 - L_1R_1L_1R_1 = 0. \tag{3.1}$$

Note, that the algebra $\mathcal{L}(R)$ has the structure of the left coadjoint comodule over the RTT algebra $\mathcal{T}(R)$ defined by (2.16). On the first order homogeneous component $\mathcal{L}^1(R)$ (the linear span of the generators of the RE algebra) the coaction $\delta_\ell : \mathcal{L}(R) \to \mathcal{T}(R) \otimes \mathcal{L}(R)$ reads

$$\delta_\ell(L_i^j) = \sum_{k,p=1}^{N} T_i^k S(T_p^j) \otimes L_k^p, \tag{3.2}$$

where $S(T)$ stands for the antipodal mapping of the RTT Hopf algebra.

The main peculiarities of the RE algebra (comparing with a general QM algebra) are listed below.

1. The quantum matrix powers $L^{\bar{k}}$, $k \geqslant 1$, (2.22) are simplified to the usual matrix products $L^{\bar{k}} = L^k = L \cdot L^{k-1}$, where $L^0 = 1_{\mathcal{L}}I$. Consequently, the power sums $p_k(L)$ (2.19) take the form $p_k(L) = \mathrm{Tr}_R(L^k)$, $k \geqslant 0$.

2. The power sums $p_k(L)$, $k \geqslant 0$ are *central* elements in $\mathcal{L}(R)$ [5]. As a consequence, the abelian characteristic subalgebra $\mathrm{Char}(\mathcal{L})$ is *central* in the RE algebra $\mathcal{L}(R)$.
3. Due to the property 1 the Cayley–Hamilton identity (2.22) for the matrix L takes the usual form:

$$\sum_{i=0}^{m}(-q)^i a_i(L)L^{m-i} = 0. \tag{3.3}$$

For the $GL(m)$-type RE algebra we introduce m spectral values μ_i, $1 \leqslant i \leqslant m$, of the quantum matrix L considered as elements of a central extension of the $\mathrm{Char}(\mathcal{L})$ (see [16] for more detail and a generalization to the case of $GL(m|n)$-type R-matrix). These spectral values are defined by the following system of polynomial relations

$$a_k(L) = q^{-k}\sum_{1\leqslant i_1<i_2<\cdots i_k\leqslant m}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}, \quad 1\leqslant k\leqslant m. \tag{3.4}$$

Then, any element of the characteristic subalgebra can be parameterized by a symmetric polynomial in spectral values. In particular, the parameterization of power sums $p_k(L)$ reads [10]:

$$p_k(L) = \sum_{i=1}^{m} d_i\mu_i^k, \quad d_i = q^{-m}\prod_{j\neq i}^{m}\frac{q\mu_i - q^{-1}\mu_j}{\mu_i-\mu_j}.$$

Besides, (3.3) and (3.4) allow the Cayley–Hamilton identity (3.3) to be written in a factorized form

$$\prod_{i=1}^{m}(L-\mu_i I) = 0.$$

The next property of the RE algebra $\mathcal{L}(R)$ allows us to construct a category of its finite dimensional representations. Namely, the RE algebra has a structure of a *braided bialgebra* [4,8]. To define this structure we need some more notation.

Introduce a finite dimensional vector space $W(L)$:

$$W(L) = \mathrm{span}_{\mathbb{K}}\{L_i^j, 1\leqslant i,j\leqslant N\}, \qquad \dim_{\mathbb{K}} W(L) = N^2, \tag{3.5}$$

and consider the free tensor algebra $TW(L)$ generated by the space $W(L)$. In each homogeneous component $W(L)^{\otimes k} \subset TW(L)$, $k \geqslant 1$, we take the basis, formed by entries of the $N^k \times N^k$ matrix $L_{1\to\overline{k}}$:

$$L_{1\to\overline{k}} \stackrel{\mathrm{def}}{=} L_1 \dot{\otimes} L_{\overline{2}} \dot{\otimes} \cdots \dot{\otimes} L_{\overline{k-1}} \dot{\otimes} L_{\overline{k}}, \qquad L_{\overline{i+1}} = R_i L_{\overline{i}} R_i^{-1}. \tag{3.6}$$

Another possible choice of the basis set in $W(L)^{\otimes k}$ is given by elements of the matrix

$$L_{\underline{k}\to 1} \stackrel{\mathrm{def}}{=} L_{\underline{k}} \dot{\otimes} L_{\underline{k-1}} \dot{\otimes} \cdots \dot{\otimes} L_{\underline{2}} \dot{\otimes} L_1, \qquad L_{\underline{i+1}} = R_i^{-1} L_{\underline{i}} R_i. \tag{3.7}$$

The RE algebra $\mathcal{L}(R)$ is isomorphic to the quotient

$$\mathcal{L}(R) \cong TW(L)/\langle R_1 L_{1\to\overline{2}} - L_{1\to\overline{2}} R_1\rangle,$$

where $\langle J\rangle$ denotes the two-sided ideal in the tensor algebra $TW(L)$ generated by a subset $J \subset TW(L)$. Being projected from $TW(L)$ to the algebra $\mathcal{L}(R)$, the sets (3.6) and (3.7) have coinciding images

$$L_1 L_{\overline{2}}\ldots L_{\overline{k-1}} L_{\overline{k}} = L_{\underline{k}} L_{\underline{k-1}}\ldots L_{\underline{2}} L_1.$$

This can be proved with the use of permutation relations (3.1) and Yang–Baxter equation for R. The homogeneous component $\mathcal{L}^k(R)$ is a linear span of matrix elements of the matrix

$$L_{1\to\overline{k}} = L_1 L_{\overline{2}}\ldots L_{\overline{k}}$$

where we keep the same notation as in (3.6).

The braided bialgebra structure of the RE algebra $\mathcal{L}(R)$ is defined by two homomorphic maps: the *coproduct* $\Delta : \mathcal{L}(R) \to \mathbf{L}(R)$ and the *counit* $\varepsilon : \mathcal{L}(R) \to \mathbb{K}$. Here an associative unital algebra $\mathbf{L}(R)$ has the following structure [8]:

1. As a vector space over the field $\mathbb{K}$ the algebra $\mathbf{L}(R)$ is isomorphic to the tensor product of two copies of the RE algebra

$$\mathbf{L}(R) \cong \mathcal{L}(R) \otimes \mathcal{L}(R).$$

2. The algebra $\mathbf{L}(R)$ is endowed with a vector space automorphism

$$\mathsf{R} : \mathcal{L}^1(R) \otimes \mathcal{L}^1(R) \to \mathcal{L}^1(R) \otimes \mathcal{L}^1(R)$$

where $\mathcal{L}^k(R)$ stands for the homogeneous component of degree k. On the basis elements of the space $\mathcal{L}^1(R) \otimes \mathcal{L}^1(R)$ the automorphism is defined by the rule

$$\mathsf{R}(L_1 \dot{\otimes} L_{\overline{2}}) = L_{\overline{2}} \dot{\otimes} L_1. \tag{3.8}$$

It can be extended to the set of automorphisms

$$\mathsf{R}_{(k,p)}:\ \mathcal{L}^k(R)\otimes\mathcal{L}^p(R)\to\mathcal{L}^p(R)\otimes\mathcal{L}^k(R),\quad k\geqslant 0, p\geqslant 0$$

by the following relations

$$\mathsf{R}_{(k,p)}(L_{1\to\bar{k}}\dot{\otimes}L_{\overline{(k+1)}\to\overline{(k+p)}})=L_{\overline{(k+1)}\to\overline{(k+p)}}\dot{\otimes}L_{1\to\bar{k}}. \tag{3.9}$$

The subspace $\mathcal{L}^0(R)$ generated by the unit element $1_{\mathcal{L}}$ commutes with any $\mathcal{L}^k(R)$.

3. Let $a, d\in\mathcal{L}(R)$ be arbitrary elements of the RE algebra and $b\in\mathcal{L}^k(R)$, $c\in\mathcal{L}^p(R)$ be arbitrary elements of homogeneous components of $\mathcal{L}(R)$. Then by definition the product $(a\otimes b)*(c\otimes d)$ is given by the rule

$$(a\otimes b)*(c\otimes d)=ac_{(1)}\otimes b_{(2)}d, \tag{3.10}$$

where $ac_{(1)}$ and $b_{(2)}d$ are products of elements of the RE algebra, while $c_{(1)}$ and $b_{(2)}$ are the Sweedler's notation for the image of the automorphism $\mathsf{R}_{(k,p)}$:

$$\mathsf{R}_{(k,p)}(b\otimes c)=c_{(1)}\otimes b_{(2)}=\sum_i c_i\otimes b_i,\quad c_i\in\mathcal{L}^p(R),\ b_i\in\mathcal{L}^k(R).$$

The product of arbitrary elements of the algebra $\mathbf{L}(R)$ is obtained from the above definition by linearity.

The following proposition holds true [8] (in an equivalent form the proposition was proved in [4]).

Proposition 3. *Consider two linear maps $\Delta:\mathcal{L}(R)\to\mathbf{L}(R)$ and $\varepsilon:\mathcal{L}(R)\to\mathbb{K}$ defined by the relations*

$$\Delta(1_{\mathcal{L}})=1_{\mathcal{L}}\otimes 1_{\mathcal{L}},\qquad \Delta(L_{1\to\bar{k}})=L_{1\to\bar{k}}\dot{\otimes}L_{1\to\bar{k}},\quad k\geqslant 1 \tag{3.11}$$

and

$$\varepsilon(1_{\mathcal{L}})=1,\qquad \varepsilon(L_{1\to\bar{k}})=I_{12\ldots k}. \tag{3.12}$$

Then the maps Δ and ε are homomorphisms of associative unital algebras and they define a braided bialgebra structure on the RE algebra $\mathcal{L}(R)$ with the coproduct Δ and the counit ε.

The representation theory of the $GL(m)$-type RE algebra $\mathcal{L}(R)$ generated by N^2 generators L_i^j can be developed in a monoidal rigid quasitensor (provided $q\neq 1$) category (see [17] for terminology) generated by an N-dimensional vector space V, $R\in\mathrm{Aut}(V^{\otimes 2})$ ([18], see also [8] for generalization to $GL(m|n)$ case). Following [8] we shall call this category the Schur–Weyl category.

The term "quasitensor" means, that for any couple of objects U_1 and U_2 of the category the functorial commutativity (iso)morphism $\mathsf{R}_{(U_1,U_2)}:U_1\otimes U_2\to U_2\otimes U_1$ is *not* involutive (unless one of the objects is the field $\mathbb{K}$). The above mappings $\mathsf{R}_{(k,p)}$ give examples of such morphisms.

The rigidity means that for any object U its dual U^* is also an object of the category and moreover, there exist a left $\langle U^*\otimes U\rangle_l\to\mathbb{K}$ and a right $\langle U\otimes U^*\rangle_r\to\mathbb{K}$ pairings which are morphisms of the category (evaluation morphisms). Besides, there exist embeddings of the field $\mathbb{K}\to U\otimes U^*$ and $\mathbb{K}\to U^*\otimes U$ which are also morphisms (co-evaluation morphisms).

Given a basis $\{x_i\}_{1\leqslant i\leqslant N}$ (2.14) in the generating space V, then a basis $\{y^i\}_{1\leqslant i\leqslant N}$ in the dual space V^* can be chosen in such a way that

$$\langle x_i, y^j\rangle_r=\delta_i^j,\qquad \langle y^i, x_j\rangle_l=(B_R)_j^i. \tag{3.13}$$

The aforementioned co-evaluation morphisms in these basis sets read

$$\mathbb{K}\to V\otimes V^*:\ 1\to(C_R)_i^j x_j\otimes y^i,\qquad \mathbb{K}\to V^*\otimes V:\ 1\to y^i\otimes x_i.$$

In [8] it was argued that the space $W(L)$ defined in (3.5) can be treated as an object of the Schur–Weyl category isomorphic to $V\otimes V^*$. This fact allows us to construct the categorical commutativity morphisms $\mathsf{R}_{(W(L),V^{\otimes p})}$ which play a crucial role in extending the $\mathcal{L}(R)$-module structure from the space V to any its tensor power. In particular, as minimal "building blocks" we have the following relations

$$\mathsf{R}_{(W(L),V)}(L_{\underline{2}}\dot{\otimes}x_1)=x_1\otimes L_2, \tag{3.14}$$

$$\mathsf{R}_{(W(L),V^*)}(L_2\otimes y_1)=y_1\dot{\otimes}L_{\underline{2}}. \tag{3.15}$$

For two copies of the space $W(L)$ we get [8] (compare with (3.9))

$$\mathsf{R}_{(W(L),W(L))}(L_1\dot{\otimes}L_{\bar{2}})=L_{\bar{2}}\dot{\otimes}L_1. \tag{3.16}$$

Also, the isomorphism $W(L)\cong V\otimes V^*$ enables us to define the *adjoint* representation of the RE algebra $\mathcal{L}(R)$ on the space $W(L)$. Besides, the adjoint action sends to zero the ideal $\mathcal{J}$ generated by the left-hand side of relations (3.1). So, the adjoint action, being extended to the whole algebra $\mathcal{L}(R)\cong TW(L)/\mathcal{J}$ via the braided coproduct (3.11) and the commutativity morphism (3.16), respects the algebraic structure of the RE algebra.

Now, we present some explicit formulae of the $\mathcal{L}(R)$ representations in various spaces. In the basis $\{x_i\}_{1\leqslant i\leqslant N}$ (2.14) the action $\triangleright$ of the linear operator L_i^j is defined as follows

$$L_i^j \triangleright x_p = \delta_i^j x_p - (q - q^{-1})(B_R)_p^j x_i$$

where $(B_R)_p^j = \sum_a (\Psi_R)_{ap}^{aj}$ according to (2.8). Using the property (2.9) one can easily show that the above action provides the space V with the left $\mathcal{L}(R)$-module structure. We rewrite the above action in an equivalent covariant matrix form

$$L_1 R_1 \triangleright x_1 = R_1^{-1} x_1. \tag{3.17}$$

The compact formula (3.17) is a concise notation for the following expression

$$\sum_{a,b=1}^{N} (L_{i_1}^a R_{ai_2}^{bj_2}) \triangleright x_b = \sum_{a=1}^{N} (R^{-1})_{i_1 i_2}^{aj_2} x_a.$$

The representation (3.17) is irreducible provided that the matrix B_R is nonsingular.

Remark 4. The RE algebra $\mathcal{L}(R)$ is defined by the quadratic relations (3.1), so it admits an evident rescaling automorphism $L \mapsto \eta L$, with arbitrary non-zero $\eta \in \mathbb{K}$. As a consequence, the action

$$L_1 R_1 \triangleright x_1 = \eta\, R_1^{-1} x_1, \quad \eta \in \mathbb{K} \setminus 0 \tag{3.18}$$

is also a representation of the algebra $\mathcal{L}(R)$.

To extend the $\mathcal{L}(R)$-module structure to $V^{\otimes p}$, $p \geqslant 2$, we use the coproduct operation (3.11) and an inductive procedure. Let spaces U and W be left $\mathcal{L}(R)$-modules with the corresponding representations $\rho_U : \mathcal{L}(R) \to \mathrm{End}(U)$ and $\rho_W : \mathcal{L}(R) \to \mathrm{End}(W)$. To define the action of the RE algebra

$$\mathcal{L}(R) \otimes U \otimes W \to U \otimes W : \qquad a \otimes u \otimes w \to a \triangleright (u \otimes w),$$

where $a \in \mathcal{L}(R)$, $u \otimes w \in U \otimes W$, we apply the coproduct $\Delta(a) = a_{(1)} \otimes a_{(2)}$ (in the Sweedler's notation), then permute $a_{(2)}$ with the vector u by means of the categorical commutativity morphism $\mathsf{R}_{(\mathcal{L}(R),U)}$:

$$\mathsf{R}_{(\mathcal{L}(R),U)}(a_{(2)} \otimes u) = u_{(3)} \otimes a_{(23)},$$

and, finally, apply the representations $\rho_U(a_{(1)})$ and $\rho_W(a_{(23)})$ to the corresponding modules:

$$a \otimes u \otimes w \xrightarrow{\Delta} a_{(1)} \otimes a_{(2)} \otimes u \otimes w \xrightarrow{\mathsf{R}_{\mathcal{L}(R),U}} a_{(1)} \otimes u_{(3)} \otimes a_{(23)} \otimes w \xrightarrow{\triangleright} (a_{(1)} \triangleright u_{(3)}) \otimes (a_{(23)} \triangleright w). \tag{3.19}$$

Below we shall use the following notation for a product of R-matrices:

$$R_{i\to j}^{\pm 1} = \begin{cases} R_i^{\pm 1} R_{i+1}^{\pm 1} \dots R_j^{\pm 1} & i < j \\ R_i^{\pm 1} R_{i-1}^{\pm 1} \dots R_j^{\pm 1} & i > j. \end{cases}$$

Taking the linear combinations $R_{1\to p} x_1 \otimes \cdots x_p$ as the basis vectors of $V^{\otimes p}$ and using (3.14), (3.18) and (3.19) we get

$$\begin{aligned} L_1 \triangleright R_{1\to p} x_1 \otimes x_2 \otimes \dots x_p &= (L_1 \triangleright R_1 x_1) \dot{\otimes} L_2 \triangleright (R_{2\to p} x_2 \otimes \dots x_p) = \cdots \\ &= \eta^p R_{1\to p}^{-1} x_1 \otimes x_2 \otimes \dots \otimes x_p. \end{aligned} \tag{3.20}$$

The chain (3.19) specialized to $a = L_i^j$, $u = x_k$ leads to an important consequence. Taking into account (3.14), we find for any $w \in W$

$$L_1 R_1 \triangleright (x_1 \otimes w) = \eta\, R_1^{-1} x_1 \dot{\otimes} (L_2 \triangleright w),$$

or, omitting an arbitrary w, we come to the "permutation rule" of the operators $L_i^j \triangleright$ and basis vectors x_p of the space V:

$$R_1 (L_1 \triangleright) R_1 x_1 = \eta\, x_1 (L_2 \triangleright). \tag{3.21}$$

This formula includes the action of L on V and the categorical commutativity morphism (3.14) and gives a simple way of extending the module structure over the RE algebra to the tensor power $V^{\otimes p}$. It serves us as the key relation for definition of the BD algebra (4.1) in the next section.

For the dual vector space V^* and its tensor powers the representation structure is as follows. The representation of the RE algebra in V^* is given by the operators

$$L_i^j \triangleright y^k = \tilde{\eta} \sum_{s=1}^{N} y^s (R^2)_{si}^{kj}$$

or, in a compact matrix form,

$$L_2 \triangleright y_1 = \tilde{\eta}\, y_1 R_1^2, \tag{3.22}$$

where $\tilde{\eta}$ is another (nonzero) numerical parameter.

The categorical commutativity morphism (3.15) and the action (3.22) leads to the corresponding operator-vector "permutation rule" (analogous to (3.21))

$$(L_{\overline{2}} \triangleright) y_1 = \tilde{\eta}\, y_1 R_1 (L_{\overline{1}} \triangleright) R_1. \tag{3.23}$$

There exists a remarkable connection between the set of $\mathcal{L}(R)$-submodules in $V^{\otimes p}$ and the R-matrix representation of the Hecke algebra $\mathcal{H}_p(q)$ in $\mathrm{End}(V^{\otimes p})$.

Proposition 5. *For any given $p \geqslant 2$ the $\mathcal{L}(R)$-module $V^{\otimes p}$ is reducible. The invariant subspaces $V_\lambda \subset V^{\otimes p}$, $\lambda \vdash p$, are extracted by the action of projection operators $P^a_\lambda = \rho_R(e^a_\lambda)$, $1 \leqslant a \leqslant d_\lambda$, where $e^a_\lambda(\sigma)$ is the* primitive *idempotent of the Hecke algebra $\mathcal{H}_p(q)$ corresponding to a standard Young tableau associated with a partition λ (there are d_λ of such tableaux in all). Thus, we have the following expansion:*

$$V^{\otimes p} \cong \bigoplus_{\lambda \vdash p} d_\lambda V_\lambda, \quad V_\lambda \cong \mathrm{Im} P^a_\lambda,\ 1 \leqslant \forall a \leqslant d_\lambda.$$

Here the coefficient d_λ in the direct sum of vector spaces stands for the multiplicity of the module V_λ in the tensor power $V^{\otimes p}$. A similar decomposition is true for an $\mathcal{L}(R)$-module $(V^*)^{\otimes p}$.

For more detailed treatment and technical results the reader is referred to [19].

To complete the section, we consider another particular module over the RE algebra, namely, the module $V \otimes V^*$. As was mentioned above, the corresponding representation can be treated as the action of the RE algebra $\mathcal{L}(R)$ on the generating space $W(L)$ and can be extended to the whole algebra $\mathcal{L}(R)$ while preserving the algebraic structure of the RE algebra. Due to this reason we call this representation *adjoint*. Such a terminology is also justified by the classical limit $q \to 1$ considered below.

So, we consider the "second copy" of the space $V \otimes V^*$ and denote its basis elements as $M_i^j = x_i \otimes y^j$. Thus, the space $W(M) = \mathrm{span}_{\mathbb{K}}(M_i^j)$ is isomorphic to the space $W(L)$ generating the RE algebra and plays the role of the adjoint representation space for the RE algebra.

The commutativity morphism $\mathsf{R}_{(W(L),W(M))}$ is given by (3.16) with the corresponding change of notation for the second factor:

$$\mathsf{R}_{(W(L),W(M))}(L_1 \dot{\otimes} M_{\overline{2}}) = M_{\overline{2}} \dot{\otimes} L_1.$$

Then formulae (3.21) and (3.22) allow us to get the action of the RE algebra $\mathcal{L}(R)$ on the space $W(M) = \mathrm{span}_{\mathbb{K}}\{M_i^j\}$

$$L_1 \triangleright M_{\overline{2}} = \eta\tilde{\eta} M_{\underline{2}}. \tag{3.24}$$

Finally, the adjoint action (3.24) together with the above commutativity morphism leads to the "permutation rule" for the operators $L \triangleright$ and the basis vectors M of the representation space $W(M)$:

$$(L_1 \triangleright) M_{\overline{2}} = \eta\tilde{\eta} M_{\underline{2}}\,(L_1 \triangleright). \tag{3.25}$$

This formula is consistent with the braided bialgebra structure of the RE algebra and the adjoint action on the space $W(M)$. It gives a way of extending the left module structure to the whole tensor algebra $TW(M)$.

4. Braided differential algebras arising from the representation theory of RE algebra

In this section we consider the construction of unital associative algebras $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$, containing two subalgebras — a $GL(m)$-type RE algebra $\mathcal{L}(R)$ and an $\mathcal{L}(R)$-algebra $\mathcal{M}$ which (as a vector space) is the direct sum of some $\mathcal{L}(R)$-modules. The subalgebra $\mathcal{M}$ will be interpreted as a noncommutative function algebra endowed with an action of "exponentiated" differential operators which form the subalgebra $\mathcal{L}(R)$. Due to this reason, we call the algebras $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$ the *braided differential algebras* (or BD algebras for short) in what follows. To clarify the reasons for using such a terminology we consider a classical limit ($q \to 1$) of some algebras $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$ and suggest the differential-geometric interpretation of constructions obtained in this way.

In defining the associative algebra structure in $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$ a decisive role belongs to the permutation rule of elements of $\mathcal{L}(R)$ and $\mathcal{M}$. This should be an analog of the classical Leibnitz rule, since it embraces the action of a differential operator on a function and their mutual permutation (see (3.21), (3.23) and (3.25) as examples). We shall refer to this rule as the *operator-function permutation* (OFP) rule.

We impose two natural requirements on the OFP rule. First, it should respect the algebraic structures of $\mathcal{L}(R)$ and $\mathcal{M}$ as subalgebras of the BD algebra. This means, that the subalgebra $\mathcal{M}$ is an $\mathcal{L}(R)$-module and the action of the RE algebra $\mathcal{L}(R)$ is compatible with the multiplication in $\mathcal{M}$ (that is $\mathcal{M}$ is an $\mathcal{L}(R)$-algebra).

Second, the OFP relation must be compatible with possible additional symmetries of $\mathcal{L}(R)$ and $\mathcal{M}$. As an example of such a symmetry we can point out the coadjoint comodule structure of $\mathcal{L}(R)$ over the RTT algebra (see (3.2)). The "quantum function"algebra $\mathcal{M}$ can also bear coadjoint or (co)vector comodule structure over the RTT algebra.[4]

4 The last case was considered in the paper [9] devoted to the Heisenberg double over the quantum group.

It turns out that the first requirement restricts considerably possible forms of the OFP relation. Besides, the RE algebra representation theory and the structure of the Schur–Weyl category allows one to find all possible OFP rules up to a renormalization isomorphism.

Below we give several important examples of BD algebras. In the next section we use them in order to construct a BD algebra involving general quantum *matrix* algebra $\mathcal{M}$ and the RE algebra acting on $\mathcal{M}$ by quantum right-invariant differential operators.

Example 1. Let an N-dimensional vector space V be a left $\mathcal{L}(R)$-module with the action (3.18) of the $\mathcal{L}(R)$ generators on a given basis set $\{x_i\}_{1\leqslant i\leqslant N}$ of the space V. Consider a unital associative $\mathbb{K}$-algebra $\mathcal{X}(V)$ freely generated by elements x_i:

$$\mathcal{X}(V) = \mathbb{K}\langle x_1, x_2, \ldots, x_N\rangle$$

and its p-th order homogeneous component $\mathcal{X}^p(V) \simeq V^{\otimes p}$. The $\mathcal{L}(R)$-module structure is introduced by an analog of the relation (3.21). This formula is the key point for constructing the BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}(V))$ — it leads to the OFP relation we need.

Definition 6. Let $\mathcal{X}(V) = \mathbb{K}\langle x_i\rangle_{1\leqslant i\leqslant N}$ be an algebra of noncommutative polynomials freely generated by elements x_i, $\mathcal{L}(R)$ be the RE algebra generated by N^2 elements L_i^j subject to multiplication rules (3.1) with a $GL(m)$-type R-matrix. Then the *free braided differential algebra* is the unital associative algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}(V))$ generated by $\{x_i\}$ and $\{L_i^j\}$ subject to the additional permutation rule

$$R_1L_1R_1x_1 = \eta\, x_1L_2, \quad \eta \in \mathbb{K}\setminus 0. \tag{4.1}$$

In order to provide the subalgebra $\mathcal{X}(V)$ with the structure of a module over the RE algebra, we should only define an action of L generators on the unit element $1_{\mathcal{B}}$. Since this action should realize a one-dimensional representation of the RE algebra, we naturally set

$$L \triangleright 1_{\mathcal{B}} = \varepsilon(L)\, 1_{\mathcal{B}}. \tag{4.2}$$

Then OFP relation (4.1) together with (4.2) allows us to get the action of L on any homogeneous monomial in x_i: we should move the element L to the most right position and then apply (4.2). For example, for a p-th order homogeneous monomial we find

$$\begin{aligned}(R_{p\to 1}L_1R_{1\to p}) \triangleright (x_1x_2\ldots x_p) &\equiv (R_{p\to 1}L_1R_{1\to p}\, x_1x_2\ldots x_p) \triangleright 1_{\mathcal{B}}\\ &= (R_{p\to 2}(R_1L_1R_1x_1)R_{2\to p}\, x_2\ldots x_p) \triangleright 1_{\mathcal{B}}\\ &\overset{(4.1)}{=} \eta x_1(R_{p\to 3}(R_2L_2R_2x_2)R_{3\to p}\, x_3\ldots x_p) \triangleright 1_{\mathcal{B}}\\ &= \cdots = \eta^p x_1x_2\ldots x_p(L_{p+1} \triangleright 1_{\mathcal{B}}) \overset{(4.2)}{=} \eta^p x_1x_2\ldots x_p\, I_{p+1}.\end{aligned}$$

Clearly, this is the same action as (3.20) in full agreement with the isomorphism $\mathcal{X}^p(V) \simeq V^{\otimes p}$.

It is evident, that the free BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}(V))$ contains all the $\mathcal{L}(R)$-modules V_λ, $\lambda \vdash p \geqslant 1$. Any such a module is a subspace of the corresponding homogeneous component $\mathcal{X}^p(V)$:

$$V_\lambda \cong \mathrm{Im}(\rho_R(e_\lambda^a)) \subset \mathcal{X}^p(V), \quad \lambda \vdash p,\ 1 \leqslant a \leqslant d_\lambda$$

with the multiplicity d_λ (see Proposition 5).

We can decrease the size of the free BD algebra by passing to a quotient

$$\mathcal{X}_J(V) = \mathcal{X}(V)/\langle J\rangle, \quad J \subset \mathcal{X}(V).$$

Recall, that $\langle J\rangle$ stands for the two-sided ideal, generated by a subset J. Assuming the ideal $\langle J\rangle$ to be invariant w.r.t. the action of $\mathcal{L}(R)$ we can define its action on the quotient $\mathcal{X}_J(V)$.

A systematic way to get a set of relations on x_i with the desired properties consists in choosing J to be equal to the image of a *central* idempotent $e_\lambda(\sigma) \in \mathcal{H}_p(q)$ for some $p \geqslant 2$:

$$J_\lambda = \mathrm{Im}(\rho_R(e_\lambda)) \subset \mathcal{X}^p(V), \quad \lambda \vdash p.$$

Basing on the properties of idempotents e_λ one can show that at the canonical projection $\pi_\lambda : \mathcal{X}(V) \to \mathcal{X}_{J_\lambda}(V)$ all the $\mathcal{L}(R)$-submodules $V_\mu \in \mathcal{X}(V)$ corresponding to partitions $\mu \supset \lambda$ are mapped to zero:

$$\pi_\lambda(V_\mu) = 0, \quad \forall \mu \supset \lambda.$$

For example, if we want to impose *quadratic* relations on the generators x_i we have only two possibilities: to annihilate the q-antisymmetric component

$$J_{(1^2)} \subset \mathcal{X}^2(V): \qquad J_{(1^2)} = \mathrm{Im}((q - R)) \tag{4.3}$$

or q-symmetric component

$$J_{(2)} \subset \mathcal{X}^2(V): \qquad J_{(2)} = \mathrm{Im}((q^{-1} + R)). \tag{4.4}$$

The choice (4.3) gives rise to a BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_s(V))$ of the RE algebra $\mathcal{L}(R)$ over the "quantum plane" $\mathcal{X}_s(V)$ [2]

$$\begin{aligned} &R_1x_1x_2 - qx_1x_2 = 0\\ &R_1L_1R_1L_1 - L_1R_1L_1R_1 = 0\\ &R_1L_1R_1x_1 = \eta\, x_1L_2. \end{aligned} \tag{4.5}$$

The action $\mathcal{L}(R) \triangleright \mathcal{X}_s(V)$ is induced by (4.2) together with the third relation in system (4.5). The BD algebra (4.5) contains only the $\mathcal{L}(R)$-submodules isomorphic to $V_{(p)}$, where (p) is a single row partition of an integer $p \geqslant 1$.

The BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_s(V))$ is covariant with respect to the left coaction of the RTT bialgebra

$$\delta_\ell(L_i^j) = \sum_{k,p=1}^{N} T_i^k S(T_p^j) \otimes L_k^p, \qquad \delta_\ell(x_i) = \sum_{k=1}^{N} T_i^k \otimes x_k.$$

S being the antipodal map in the RTT algebra. In practical calculations it is convenient to use a loose notation $\delta_\ell(L_1) = T_1L_1S(T_1)$ and $\delta_\ell(x_1) = T_1x_1$ and treat T_a to be commutative with L_b and x_c if $a \neq b$ and $a \neq c$.

Let us verify the covariance of the OFP rule in the system (4.5). We get

$$\begin{aligned} \delta_\ell(R_1L_1R_1x_1) &= R_1T_1L_1\underline{S(T_1)R_1T_1}x_1 = R_1T_1L_1T_2R_1S(T_2)x_1 = \underline{R_1T_1T_2}L_1R_1x_1S(T_2)\\ &= T_1T_2\underline{R_1L_1R_1x_1}S(T_2) = \eta\, T_1x_1T_2L_2S(T_2) = \delta_\ell(\eta\, x_1L_2). \end{aligned}$$

Assuming a given R-matrix of $GL(m)$-type to be a deformation of the usual flip P (then $m = N = \dim_{\mathbb{K}} V$), consider the classical limit $q \to 1$ of the BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_s(V))$ (4.5). For this purpose we pass to a different set $\{K_i^j\}_{1\leqslant i,\, j,\leqslant N}$ of the RE algebra generators:

$$L = I - (q - q^{-1})K, \quad K = \|K_i^j\|. \tag{4.6}$$

Taking into account the Hecke condition (2.3), we rewrite the defining relations (3.1) in terms of the new generators

$$R_1K_1R_1K_1 - K_1R_1K_1R_1 = R_1K_1 - K_1R_1. \tag{4.7}$$

The bialgebra structure now reads

$$\Delta(K) = 1 \otimes K + K \otimes 1 - (q - q^{-1})K \otimes K, \qquad \varepsilon(K) = 0. \tag{4.8}$$

Then, according to the first line of (4.5), the generators x_i of the subalgebra $\mathcal{X}_s(V)$ turn into commutative elements

$$x_2x_1 - x_1x_2 = 0. \tag{4.9}$$

So, at $q \to 1$ we have $\mathcal{X}_s(V) = \mathbb{K}[V^*]$.

The multiplication rules (4.7) turns into defining relations of the universal enveloping algebra $U(gl(m))$

$$\kappa_1\kappa_2 - \kappa_2\kappa_1 = \kappa_1P_{12} - P_{12}\kappa_1, \tag{4.10}$$

where the matrix $\kappa = \|\kappa_i^j\|$ is the limit of generating matrix K at $q \to 1$.

In order to get the limit of the RE algebra action (the third relation in (4.5)) we additionally suppose the following behavior of the parameter $\eta = 1 - (q - q^{-1})\eta_0 + o(q^2 - 1)$. Under this assumption the OFP relation in (4.5) gives rise to

$$\kappa_2x_1 - x_1\kappa_2 = \eta_0x_1 + P_{12}x_1. \tag{4.11}$$

Together with the commutation relations (4.10) this formula allows us to interpret the generators κ_i^j as the following vector fields on the $\mathbb{K}[V^*]$:

$$\kappa_i^j = x_i\partial_x^j + \eta_0\delta_i^j\,(x \cdot \partial_x), \tag{4.12}$$

where we denote

$$\partial_x^k = \frac{\partial}{\partial x_k}, \qquad (x \cdot \partial_x) = \sum_{k=1}^{m} x_k\partial_x^k.$$

If V is the left fundamental vector $GL(m)$-module

$$x_i \mapsto x_jM_i^j, \quad M = \|M_i^j\| \in GL(m)$$

then the fields κ_i^j in (4.12) are invariant with respect to the $GL(m)$ action on the right side.

Remark 7. In considering the classical limit $q \to 1$ it is convenient to parameterize $q = e^{\frac{\tau}{2}}$ and treat the classical limit as $\tau \to 0$. In this limit the shift formula (4.6) turns into $L = I - \tau\kappa + o(\tau^2)$. Together with the group-like coproduct (3.11) for L generators and their "Weyl-type" commutation with the generators x_i (the third relation in system (4.5)) it allows us to interpret the generators L_i^j as exponentiated quantized differential operators κ (4.12).

Note, that $x\cdot\partial = \mathrm{Tr}\,\kappa$ is a *central* element of the Lie algebra $gl(m)$. Therefore, on adding to κ_i^j (4.12) a term proportional to this central element, we can specialize the parameter η_0 in (4.11) to any given value (for example, we can get $\eta_0 = 0$). Such an operation changes the multiplicative parameter η in the OFP relation of (4.5). This is another evidence of exponential-like dependence of quantum differential operators L on classical differential operators κ: a linear shift of κ leads to a multiplicative renormalization of L.

Consider now the choice (4.4) for the permutation rules on x_i. We come to the BD algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_a(V))$ with the following relations on the generators

$$\begin{aligned} &R_1x_1x_2 + q^{-1}x_1x_2 = 0\\ &R_1L_1R_1L_1 - L_1R_1L_1R_1 = 0\\ &R_1L_1R_1x_1 = \eta\, x_1L_2. \end{aligned} \tag{4.13}$$

This algebra has the same RTT-comodule property as the algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_s(V))$ considered above. But contrary to the BD algebra (4.5), now we have only a finite number of the $\mathcal{L}(R)$-submodules in $\mathcal{B}(\mathcal{L}(R), \mathcal{X}_a(V))$. Namely, the vector space $\mathcal{X}_a(V)$ is isomorphic to the direct sum of the modules $V_{(1^p)}$, $1 \leqslant p \leqslant m$.

Again, assuming a given R-matrix R to be a deformation of the flip P and applying the shift (4.6) we get at the classical limit $q \to 1$ the following system of relations

$$\begin{aligned} &x_1x_2 + x_2x_1 = 0\\ &\kappa_1\kappa_2 - \kappa_2\kappa_1 = \kappa_1P_{12} - P_{12}\kappa_1\\ &\kappa_2x_1 - x_1\kappa_2 = \eta_0x_1 + P_{12}x_1. \end{aligned}$$

The classical algebra defined by the above relations on the generators has a transparent geometrical interpretation. The elements x_i generate an external subalgebra and are treated as one-forms — the differentials of coordinate functions of $\mathbb{K}[V^*]$

$$x_i = \mathrm{d}y_i, \quad 1 \leqslant i \leqslant m,$$

while κ_i^j is of the form

$$\kappa_i^j = \mathfrak{L}_i^j + \eta_0\, \delta_i^j \sum_{k=1}^m \mathfrak{L}_k^k,$$

where $\mathfrak{L}_i^j$ is the *Lie derivative* along the vector field $y_i\partial_y^j$.

Example 2. We can start from a more interesting module $V \otimes V^*$ (called *adjoint*) with the linear basis $M_i^j = x_i \otimes y^j$. The action of the RE algebra is given by (3.24). Formula (3.25) provides a recipe for extending the module's structure on tensor powers of the adjoint module $V \otimes V^*$. In analogy with constructions of Example 1, we consider a unital associative algebra $\mathcal{M}$, generated by N^2 free elements M_i^j and define the algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$ by imposing the following multiplication rules of the free generators M_i^j and RE algebra generators L_i^j

$$L_1M_{\overline{2}} = M_{\underline{2}}L_1. \tag{4.14}$$

This formula stems from the relation (3.25) which is defined by the RE algebra representation theory. Then the subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{L}(R), \mathcal{M})$ can be given the structure of a module over the RE algebra by relation (4.2). Note, that the requirement (4.2) fixes the constant $\tilde\eta$ as $\eta\,\tilde\eta = 1$.

We can restrict the algebra $\mathcal{B}(\mathcal{L}(R), \mathcal{M})$ by setting some relations on the generators M_i^j which are consistent with the OFP relation (4.14). We consider the case of *quadratic* relations. In [8] a pair of orthogonal projectors $\mathcal{A}_q, \mathcal{S}_q : \mathcal{M}^{(2)} \to \mathcal{M}^{(2)}$ was constructed. Here $\mathcal{M}^{(2)}$ is the subspace of $\mathcal{M}$ spanned by the quadratic monomials in generators M_i^j. The projectors $\mathcal{A}_q$ and $\mathcal{S}_q$ have the natural interpretation as a q-antisymmetrizer and a q-symmetrizer on the space $\mathcal{M}^{(2)}$. The images of these operators are invariant subspaces with respect to the RE algebra action. So, a consistent quadratic relation on the free generators M_i^j can be chosen as $\mathrm{Im}\,\mathcal{A}_q = 0$ or $\mathrm{Im}\,\mathcal{S}_q = 0$.

Consider the first case. It can be shown that the requirement $\mathrm{Im}\,\mathcal{A}_q = 0$ is equivalent to the RE algebra type relations on the generators M_i^j. So, we come to the BD algebra defined by the following relations on generators

$$\begin{aligned} &R_1M_1R_1M_1 - M_1R_1M_1R_1 = 0\\ &R_1L_1R_1L_1 - L_1R_1L_1R_1 = 0\\ &R_1L_1R_1M_1 = M_1R_1L_1R_1. \end{aligned} \tag{4.15}$$

We denote this BD algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{L}(R), \mathcal{M}(R))$. Both the RE algebras $\mathcal{L}(R)$ and $\mathcal{M}(R)$ are subalgebras of $\mathcal{B}_{\mathrm{ad}}(\mathcal{L}(R), \mathcal{M}(R))$, the subalgebra $\mathcal{M}(R)$ is endowed with a $\mathcal{L}(R)$-module structure by means of (4.2) and by the third relation of the above system. Besides, the algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{L}(R), \mathcal{M}(R))$ has the left coadjoint comodule structure over the RTT algebra (2.16).

The algebraic properties of this BD algebra will be considered in more detail in the next section. Here we only point out that the R-traces $\mathrm{Tr}_R M^k$, $k \geqslant 0$, are central in the whole BD algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{L}(R), \mathcal{M}(R))$ (not only in the RE subalgebra $\mathcal{M}(R)$) and therefore are invariant under the action of the subalgebra $\mathcal{L}(R)$. This means that this action can be restricted to quotients of $\mathcal{M}(R)$ over ideals generated by relations $\mathrm{Tr}_R M^k = c_k$, $1 \leqslant k \leqslant m$, where c_k are fixed constants. Recall, that in [10,20] such like quotients were interpreted as quantum (braided) analogs of $GL(m)$ orbits in $gl^*(m)$. Therefore, the subalgebra $\mathcal{L}(R)$ in the BD algebra $\mathcal{B}_{ad}(\mathcal{L}(R), \mathcal{M}(R))$ can be treated as the quantized algebra of differential operators generated by the vector fields tangential to the mentioned orbits.

To justify this interpretation we consider the classical limit $q \to 1$ of the BD algebra (4.15) by assuming R to be a deformation of the flip P.

Making the shift (4.6) for L and passing to the limit $q \to 1$ in the BD algebra (4.15) we come to the following permutation rules for the generators $m_i^j = \lim_{q\to 1} M_i^j$ and $\kappa_i^j = \lim_{q\to 1} K_i^j$:

$$m_1 m_2 - m_2 m_1 = 0$$
$$\kappa_1\kappa_2 - \kappa_2\kappa_1 = \kappa_1 P_{12} - P_{12}\kappa_1$$
$$\kappa_2 m_1 - m_1\kappa_2 = P_{12} m_1 - m_1 P_{12}.$$

The two last lines in this system of permutation rules show that κ_i^j are coadjoint vector fields on the space $gl^*(m)$:

$$\kappa_i^j = m_i^s \frac{\partial}{\partial m_j^s} - m_s^j \frac{\partial}{\partial m_s^i}, \tag{4.16}$$

where the summation over the index s is understood. As is well known, the vector fields (4.16) are tangent to the $GL(m)$ orbits in the linear space $gl^*(m)$.

5. The braided differential algebra over QM algebra

In the preceding section we gave an example (4.15) of a BD algebra over a quantum matrix algebra. In the classical limit the RE algebra generators turned into the adjoint vector fields (4.16). In the example (4.15) the algebra of "quantum functions" was taken to be the second copy of RE algebra.

Now we are going to define the braided differential algebra $\mathcal{B}_r(\mathcal{L}(R), \mathcal{M})$ of the $GL(m)$-type RE algebra $\mathcal{L}(R)$ over an arbitrary quantum matrix algebra $\mathcal{M}(R, F)$. Therefore, we should supply the BD algebra with the OFP relation similar to the third relation in (4.5). The construction is presented in the following definition.

Definition 8. Let $\mathcal{L}(R)$ be the RE algebra associated with a $GL(m)$-type R-matrix R and $\mathcal{M}(R, F)$ be the QM algebra, associated with a compatible pair of R-matrices $\{R, F\}$ (see Section 2). Define a unital associative algebra $\mathcal{B}_r(\mathcal{L}(R), \mathcal{M})$ over the field $\mathbb{K}$ generated by the elements L_i^j of the RE algebra and by elements M_i^j of the QM algebra subject to the following system of relations

$$R_1 M_1 M_{\bar 2} - M_1 M_{\bar 2} R_1 = 0$$
$$R_1 L_1 R_1 L_1 - L_1 R_1 L_1 R_1 = 0$$
$$R_1 L_1 R_1 M_1 = \eta M_1 L_{\bar 2}, \tag{5.1}$$

where the "matrix copies" $M_{\bar 2}$ and $L_{\bar 2}$ are produced by the R-matrix F in accordance with (2.13):

$$L_{\bar 2} = F_1 L_1 F_1^{-1}, \qquad M_{\bar 2} = F_1 M_1 F_1^{-1}.$$

The nonzero number η is a parameter of the algebra.

We introduce an action of the RE algebra generators on the unit element $1_{\mathcal{B}}$ by the rule

$$a \triangleright 1_{\mathcal{B}} = \varepsilon(a) 1_{\mathcal{B}}, \quad \forall a \in \mathcal{L}(R), \tag{5.2}$$

where ε is the counit map of the braided bialgebra $\mathcal{L}(R)$: $\varepsilon(L) = I$ (see (3.12)).

Note, that the Heisenberg double, considered in [9] corresponds to the pair $\{R, P\}$ of the compatible R-matrices, where R is a $GL(m)$-type R-matrix. In this case the QM algebra $\mathcal{M}(R, P)$ turns into the Hopf algebra of quantum functions on $GL(m)$.

Remark 9. Let us shortly explain how one can get the OFP relation (5.1) in the above definition. As we mentioned in Section 3, the RE algebra generators can be treated as basis elements of the space $V \otimes V^*$ of the Schur–Weyl category SW(V) (the category of finite dimensional modules over the RE algebra): $L_i^j = x_i \otimes y^j$, $\{x_i\}_{1\le i\le N}$ and $\{y^j\}_{1\le j\le N}$ being the respective basis (3.13) of V and V^*. In order to get the RE algebra action on generators M_i^j which would be not the adjoint one (as in (4.15)) but rather similar to (4.5) we proceed as follows.

Let us enlarge the class of objects of the category SW(V) by another pair of N dimensional vector spaces

$$U = \mathrm{span}_{\mathbb{K}}(t_i)_{1\le i\le N}, \qquad U^* = \mathrm{span}_{\mathbb{K}}(z^i)_{1\le i\le N}$$

dual to each other. Then the matrix elements M_i^j are taken to be the basis elements of $V \otimes U^*$: $M_i^j = x_i \otimes z^j$. In order to get the OFP relation among L_i^j and M_r^s we have to take into account the (known) OFP relation among L and x and the categorical permutation morphism of L (treated as $V \otimes V^*$) and U^* in accordance with our general recipe described in (3.19).

The commutativity morphism $\mathsf{F}: V\otimes U^* \to U^*\otimes V$ is defined via the operator F:

$$\mathsf{F}(x_i\otimes z^j) = z^k\otimes x_s F^{sj}_{ki}.$$

This gives rise to F appearing in (5.1) in the formula for $L_{\overline{2}}$.

Note also, that there exists another choice of the "function algebra". Namely, we could take as the basis of function algebra the elements $M_i^j = t_i\otimes y^j \in U\otimes V^*$. Such a choice would give rise to another form of the OFP relation (compare with (5.1))

$$L_{\overline{2}}M_1 = \tilde{\eta}\, M_1R_1L_1R_1. \qquad (5.3)$$

Actually, we can get the corresponding BD algebra starting from (5.1). If we introduce the matrix $\hat{L} = M^{-1}LM$ and the new R-matrix $\hat{R} = F^{-1}R^{-1}F$, then the system of relations (5.1) leads to

$$\hat{R}_1M_{\underline{2}}M_{\underline{1}} - M_{\underline{2}}M_{\underline{1}}\hat{R}_1 = 0$$
$$\hat{R}_1\hat{L}_1\hat{R}_1\hat{L}_1 - \hat{L}_1\hat{R}_1\hat{L}_1\hat{R}_1 = 0$$
$$\hat{L}_{\underline{2}}M_1 = \eta M_1\hat{R}_1\hat{L}_1\hat{R}_1,$$

where $M_{\underline{2}} = F^{-1}M_1F$. The OFP relation standing in the third line of the above system coincides with (5.3) (up to the nonessential change $F\to F^{-1}$).

Strictly speaking, the BD algebra generated by $\hat{L}$ and M is not a subalgebra of the algebra (5.1) since we have to use the inverse matrix M^{-1} in passing from L to $\hat{L}$. As follows from the Cayley–Hamilton identity (2.22) this requires some extension of the initial algebra, namely, we have to demand the invertibility of the element $a_m(M)$ (see [9] for more detail).

From the viewpoint of the representation theory of the RE algebra, the BD algebra introduced in Definition 8 consists of the direct sum of modules over the RE algebra isomorphic to those of the BD algebra (4.5). To be more precise, the following proposition holds true.

Proposition 10. *Relation (5.2) allows us to define the $\mathcal{L}(R)$-module structure on the subalgebra $\mathcal{M}(R,F)$ of the BD algebra $\mathcal{B}_r(\mathcal{L}(R),\mathcal{M})$ introduced in Definition 8. The action of the RE algebra generators L_i^j on the basis vectors of the p-th order homogeneous component $\mathcal{M}^p(R,F)$ reads*

$$L_1 \triangleright R_{(1\to p)}M_{\overline{1}}M_{\overline{2}}\dots M_{\overline{p}} = \eta^p R^{-1}_{(1\to p)}M_{\overline{1}}M_{\overline{2}}\dots M_{\overline{p}}, \qquad (5.4)$$

where $M_{\overline{k}} = F_{k-1}M_{\overline{k-1}}F^{-1}_{k-1}$.

Proof. The proof consists in direct calculations. First of all, using the compatibility conditions (2.11) we can transform relation (5.1) to

$$R_kL_{\overline{k}}R_kM_{\overline{k}} = \eta\, M_{\overline{k}}L_{\overline{k+1}}, \quad \forall k\geqslant 1,$$

where the copies $L_{\overline{k}}$ and $M_{\overline{k}}$ are defined with the help of the R-matrix F in accordance with (2.13). Then we get

$$L_1\triangleright (R_{(1\to p)}M_{\overline{1}}\dots M_{\overline{p}}) = \eta^p\, R^{-1}_{(1\to p)}M_{\overline{1}}\dots M_{\overline{p}}(L_{\overline{p+1}}\triangleright 1_{\mathcal{B}}).$$

Since $L_{\overline{p+1}}\triangleright 1_{\mathcal{B}} = \varepsilon(L_{\overline{p+1}})1_{\mathcal{B}} = I_{12\dots p+1}1_{\mathcal{B}}$, the result (5.4) follows. This should be compared with (3.20).

In a similar manner one can prove that the RE algebra action respects the algebraic structure of the QM algebra $\mathcal{M}(R,F)$, that is

$$a\triangleright (R_kM_{\overline{k}}M_{\overline{k+1}} - M_{\overline{k}}M_{\overline{k+1}}R_k) = 0, \quad \forall a\in\mathcal{L}(R),\ \forall k\geqslant 1. \quad \square$$

Now, we consider the case of the BD algebra over the RE algebra in more detail. This means that we put $F=R$. Note, that we do not come to the algebra (4.15) since the OFP relation (5.1) takes the form

$$R_1L_1R_1M_1 = \eta M_1R_1L_1R_1^{-1} \qquad (5.5)$$

which differs from the third relation of the BD algebra (4.15) by the inverse R in the last place. As a consequence, the traces Tr_RM^k are not central in the BD algebra (5.5), and the action of the braided differential operators from the subalgebra $\mathcal{L}(R)$ does not preserve the quantum orbits which are quotients of the RE algebra $\mathcal{M}(R)$ over the ideals generated by conditions on these traces [10]. It is not a surprise, since as can be easily seen from the classical limit $q\to 1$, the relation (5.5) defines the right-invariant vector fields on the $gl^*(m)$:

$$L = I-(q-q^{-1})K, \qquad K_i^j \xrightarrow{q\to 1} m_i^a\frac{\partial}{\partial m_j^a}.$$

Here we neglect the possible central term proportional to η_0 (see (4.12)).

The BD algebra (4.15) consisting of the quantum differential operators generated by coadjoint vector fields can be subtracted as a subalgebra of the algebra $\mathcal{B}_r(\mathcal{L}(R),\mathcal{M}(R))$ with the OFP relation (5.5). (More precisely, we extend this algebra by L^{-1} and M^{-1}.)

Let us introduce the matrices

$$Q = LM^{-1}L^{-1}M, \qquad N = M^{-1}Q. \tag{5.6}$$

Here we have to impose the invertibility condition on the elements $a_m(L)$ and $a_m(M)$ (the polynomial a_m is defined in (2.20)) and extend our algebra by the elements $a_m^{-1}(L)$ and $a_m^{-1}(M)$. Then, the Cayley–Hamilton identities (3.3) for L and M guarantee the invertibility of the matrices involved. The following proposition is a direct consequence of the multiplication rule (5.5).

Proposition 11. *The matrix elements of Q and M satisfy the following multiplication rules*

$$\begin{aligned} &R_1M_1R_1M_1 - M_1R_1M_1R_1 = 0\\ &R_1Q_1R_1Q_1 - Q_1R_1Q_1R_1 = 0\\ &R_1Q_1R_1M_1 - M_1R_1Q_1R_1 = 0. \end{aligned} \tag{5.7}$$

For the pair M and N we have

$$\begin{aligned} &R_1M_1R_1M_1 - M_1R_1M_1R_1 = 0\\ &R_1N_1R_1N_1 - N_1R_1N_1R_1 = 0\\ &R_1^{-1}N_1R_1M_1 - M_1R_1^{-1}N_1R_1 = 0. \end{aligned} \tag{5.8}$$

It is clear from the above proposition that the Q and M generate a subalgebra in the BD algebra $\mathcal{B}_r(\mathcal{L}, \mathcal{M})$ (5.1). Moreover, we can calculate the action of the generators Q_i^j on the unit element $1_{\mathcal{B}}$ and turn the above subalgebra into a new BD algebra.

Proposition 12. *Given the action (4.2) of the generators L, for the elements Q defined in (5.6) one gets:*

$$Q_1 \triangleright 1_{\mathcal{B}} = \xi\, I_1 1_{\mathcal{B}}, \quad \xi = \eta^{-1}q^{2m}. \tag{5.9}$$

We call the BD algebra, generated by Q and M subject to the system of relations (5.7) and the action (5.9) the adjoint BD algebra and denote it as $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$. Note, that OFP relation in the adjoint BD algebra does not depend on the parameter η entering the OFP relation in the algebra (5.1). This parameter appears only in the action of the adjoint generators Q_i^j on the unit element in (5.9).

It is evident that the BD algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$ defined by relations (5.7) differs from the algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{L}(R), \mathcal{M}(R))$ of example (4.15) only by notation and general normalization (the parameter ξ) of the $\mathcal{L}(R)$-action.

Based on this result, we can reveal the structure of the $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$ as a module over the RE algebra.

Proposition 13. *Given the adjoint BD algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$ defined by relations (5.7) and (5.9), the subalgebra $\mathcal{M}(R)$ generated by M_i^j is endowed with the $\mathcal{Q}(R)$-module structure with the following action of the basis elements of $\mathcal{Q}(R)$ on basis elements of p-th order homogeneous component $\mathcal{M}^p(R)$*

$$(Q_1Q_{\overline{2}}\dots Q_{\overline{k}}) \triangleright (M_{\overline{k+1}}M_{\overline{k+2}}\dots M_{\overline{k+p}}) = \xi^k(M_{\underline{k+1}}M_{\underline{k+2}}\dots M_{\underline{k+p}}), \quad \forall k, p \geqslant 1. \tag{5.10}$$

Recall, that in the above formula the copies of matrices are defined via the R-matrix R:

$$M_{\overline{k}} = R_{k-1}M_{\overline{k-1}}R_{k-1}^{-1}, \qquad M_{\underline{k}} = R_{k-1}^{-1}M_{\underline{k-1}}R_{k-1}.$$

Now, assuming R to be a deformation of the flip P, we discuss the restriction of the adjoint BD algebra to some quotients of the RE algebra $\mathcal{M}(R)$ which can be interpreted as a quantization of the coordinate algebra of $GL(m)$ orbits in $gl^*(m)$ [10,11]. Such a quantum (braided) orbit is defined by an ideal $J_{\{c\}}$, generated by elements

$$\mathrm{Tr}_R(M^k) - c_k, \quad 1 \leqslant k \leqslant m.$$

(In order to get analogs of *regular* orbits we have to impose some restrictions on the parameters c_i, see [10].)

The systems of relations (5.8) allow us to conclude, that the elements $\mathrm{Tr}_R(M^k)$ and $\mathrm{Tr}_R(N^k)$ are central in the adjoint BD algebra. This is a consequence of the following property of the trace

$$\mathrm{Tr}_{R(2)}(R_1^{\pm 1}X_1R_1^{\mp 1}) = \mathrm{Tr}_R(X)\, I_1,$$

valid for an arbitrary $N \times N$ matrix X. Therefore, the quantum orbits are preserved by the action of the RE algebra $\mathcal{Q}(R)$. Having restricted the adjoint BD algebra $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$ on the orbit $\mathcal{M}(R)/J_{\{c\}}$ we get the nontrivial relations on the differential operators. They appear as the corresponding fixation of another set of central elements — $\mathrm{Tr}_R(N^k) = \mathrm{Tr}_R((M^{-1}Q)^k)$.

Definition 14. A restriction of the adjoint BD algebra (5.7) on a quantum orbit $\mathcal{M}(R)/J_{\{c\}}$ is the quotient of $\mathcal{B}_{\mathrm{ad}}(\mathcal{Q}, \mathcal{M})$ over the ideal generated by the relations

$$\mathrm{Tr}_R(M^k) = c_k, \qquad \mathrm{Tr}_R((M^{-1}Q)^k) = \mathrm{Tr}_R((M^{-1}Q\ \triangleright)^k)1_{\mathcal{B}}|_{J_{\{c\}}}, \quad 1 \leqslant k \leqslant m, \tag{5.11}$$

where in the last relation we assume that traces of M should be specified to corresponding constants c_i after calculating the action of Q.

Remark 15. The restriction $1 \leqslant k \leqslant m$ in the above definition is due to the fact that for a $GL(m)$-type R-matrix the quantum matrices M and $M^{-1}Q$ satisfy the Cayley–Hamilton identity. The order of the Cayley–Hamilton polynomial is m, so all traces $\mathrm{Tr}_R(M^p)$ and $\mathrm{Tr}_R((M^{-1}Q)^p)$ with $p > m$ can be expressed in terms of the first m traces.

Note also, that the restriction on central elements $\mathrm{Tr}_R((M^{-1}Q)^k)$ given in (5.11) is compatible with the operator action of Q, presented in (5.10). Namely, one can show, that $\mathrm{Tr}_R((M^{-1}Q\ \triangleright)^k)$ is a scalar operator on any homogeneous component $\mathcal{M}^p(R)$. For example,

$$\mathrm{Tr}_R(M^{-1}Q\ \triangleright)M_1M_{\overline{2}}\dots M_{\overline{p}} = \xi\, \mathrm{Tr}_R(M^{-1})M_1M_{\overline{2}}\dots M_{\overline{p}}.$$

At the classical level the corresponding restrictions have rather simple form $\mathrm{Tr}(M^kK) = 0$. These relations mean that the $gl^*(m)$-module generated by infinitesimal vector fields arising from the $GL(m)$ action on $gl^*(m)$ is a quotient of a free $gl^*(m)$-module. (However, its restriction to a generic orbit is projective.) The more complex formulae (5.11) are due to the "exponentiated character" of quantum differential operators Q.

Consider a simple example, corresponding to a $GL(2)$-type R-matrix. Besides, we take the generating matrices M and L to be of 2×2 size.

The Cayley–Hamilton identity (3.3) for the matrix M reads

$$M^2 - qa_1(M)M + q^2a_2(M)I = 0,$$

where

$$a_1(M) = \mathrm{Tr}_R(M), \qquad 2_q a_2(M) = q(\mathrm{Tr}_R(M))^2 - \mathrm{Tr}_R(M^2)$$

in accordance with (2.21).

Let us consider the "braided orbit" $\mathcal{O}(r)$ defined by the following values of the parameters c_k, $k = 1,\ 2$:

$$\mathrm{Tr}_R(M) = 0, \quad \mathrm{Tr}_R(M^2) = -\frac{2_q}{q^2}\, r^2, \tag{5.12}$$

where r is a nonzero *real* number. Then the Cayley–Hamilton identity gives us the inverse matrix M^{-1} in the form

$$M^{-1} = c\, M, \quad c = -r^{-2}.$$

According to the definition (5.11), we can calculate the restriction conditions for differential operators. The first condition is as follows

$$\mathrm{Tr}_R(M^{-1}Q) = \mathrm{Tr}_R(M^{-1}Q\ \triangleright)1_{\mathcal{B}|_{\mathcal{O}(r)}} = \xi \mathrm{Tr}_R(M^{-1})_{|_{\mathcal{O}(r)}} = \xi c\, \mathrm{Tr}_R(M)_{|_{\mathcal{O}(r)}} = 0, \tag{5.13}$$

where we used (5.9) for the action of Q operator and the above explicit form of M^{-1} on the orbit $\mathcal{O}(r)$. Passing to the shifted set of generators $Q_i^j = 1_{\mathcal{B}}\delta_i^j - (q - q^{-1})K_i^j$ we rewrite restriction (5.13) in the form

$$\mathrm{Tr}_R(MK) = 0. \tag{5.14}$$

At the classical limit $q \to 1$ the entries of the matrix K become generators of the Lie algebra $sl(2)$. We pass to the compact form of this algebra by introducing new generators in the matrices M and K. Namely, we put

$$M = \begin{pmatrix} \mathrm{i}x_3 & \mathrm{i}x_1 - x_2 \\ \mathrm{i}x_1 + x_2 & -\mathrm{i}x_3 \end{pmatrix}.$$

Also we have

$$x_1^2 + x_2^2 + x_3^2 = r^2 \tag{5.15}$$

as a consequence of the above equation $\mathrm{Tr}_R(M^2) = -2_q q^{-2} r^2$.

Since at the limit $q \to 1$ $\mathrm{Tr}(K)$ is a central element, we can also take the matrix K to be traceless and parameterize its matrix elements as follows

$$K = \begin{pmatrix} \mathrm{i}X_3 & \mathrm{i}X_1 - X_2 \\ \mathrm{i}X_1 + X_2 & -\mathrm{i}X_3 \end{pmatrix}.$$

As follows from (5.7) the operators X_i satisfy the $su(2)$ commutation relations:

$$[X_i, X_j] = \varepsilon_{ijk}X_k.$$

ε_{ijk} being the completely antisymmetric tensor.

Taking into account the relations among x_i and X_i

$$[x_i, X_j] = -\varepsilon_{ijk}x_k$$

(which can also be extracted as a classical limit of the third equation in (5.7)) we interpret generators X_i as adjoint vector fields $X_i = \varepsilon_{ijk} x_j \partial_k$, tangent to the sphere of radius r described by the condition (5.15). At the classical limit the condition (5.14) leads to the equality

$$x_1X_1 + x_2X_2 + x_3X_3 = 0.$$

This is the well-known identity which is satisfied by the tangent vector fields X_i.

We emphasize that all higher order relations following from (5.11) are satisfied automatically at the classical limit. Indeed, calculating the second restriction for $\mathrm{Tr}_R((M^{-1}Q)^2)$ at the condition $\xi = 1$ we get:

$$\mathrm{Tr}_R(M^{-1}KM^{-1} + M^{-2}K - (q - q^{-1})M^{-1}KM^{-1}K) = \mathrm{Tr}_R(M^{-2})\mathrm{Tr}_R(I) - (\mathrm{Tr}_R(M^{-1}))^2.$$

On the orbit $\mathcal{O}(r)$ we have $\mathrm{Tr}_R(M^{-1}) = c\,\mathrm{Tr}_R(M) = 0$, $M^{-2} = c\,I$. So, at the classical limit we find:

$$\mathrm{Tr}(MKM) = 2\,\mathrm{Tr}(M^2),$$

or, using our parameterization for the matrices M and K:

$$\sum_{i,j,k} \varepsilon_{ijk} x_i X_j x_k = -2 \sum_i x_i^2.$$

The above relation indeed turns into an identity with the choice $X_i = \varepsilon_{ijk} x_j \partial_k$.

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