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MULTIPRODUCT MODEL DECOMPOSITION OF COMPONENTS OF RUSSIAN GDP

BASIC RESEARCH PROGRAM

WORKING PAPERS

SERIES: ECONOMICS
WP BRP 111/EC/2015

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Abstract

This paper proposes a method for a multiproduct model decomposition of GDP components by expenditure which allows the use of several different price indices in the same model. The decomposition does not link the products to imports or exports, therefore, it imposes no restrictions on the behaviour of these series and their deflators. The theoretical reasoning, the estimation methodology and the estimation results for Russian GDP data are presented. A method of the decomposition of changes in inventories is also presented.

JEL classification: C65, C68

Keywords: GDP components by expenditure, decomposition, utility tree

1 Introduction

This paper presents a technique for a multiproduct model decomposition of components of GDP. Decomposition here is the disaggregation of observed statistics into separate components. The use of decomposition is motivated by the fact that one-product macroeconomic models (the most common today [4]), where GDP is treated as the only product, can not take into account the differences in price deflators of different components of GDP by expenditure (such as consumptio or investment). The one-product model can take into account only one deflator, other deflators will not be modelled correctly, causing errors either in the series in constant or current prices. The accuracy of the model will be limited by the differences in the deflators of GDP components, which can be quite significant. Decomposition allows us to separate the data into several products with different prices, so that deflators of GDP components will be

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[4]The shift towards one-product models can be caused by the erosion of boundaries between products.

represented as a linear combination of the deflators of products, and so all the different deflators needed can be modelled.

In the context of a macroeconomic time series data disaggregation can be understood in several ways. The first is to treat it as the disaggregation of low-frequency data to obtain data of higher frequency. It can be used, for example, to get monthly GDP series from quarterly series to estimate short-term forecasting models, as it is done in [9]; or to incorporate lower-frequency data into higher frequency models (see [7]).

Another understanding of disaggregation is into lower-level components, for example, of economy-level data into sector-level data (more common) or into region-level data (less common, see, e.g. [11]). Disaggregated models which operate with sectors, sector outputs and prices are not uncommon in the literature. [18] and [19] present disaggregated agent-based models. [18] demonstrates the general framework and provides examples of models with 2 and 4 products, in the second case with products of different types (investment and consumption goods). Different types of goods lead to different output behiviour of these goods. In [19] the behaviour of the financial system is studied using an example of a 3 sectoral model. Generally, disaggregation helps to make model descriptions more detailed and improves the forecasting power of the models.

Forecasting is one of the most important fields where disaggregation techniques are applied. For example, [17] discusses issues of density forecasting of the US real GDP growth rate. He shows that forecasts using disaggregated models are more precise than using their aggregated versions. Disaggregation here is conducted with the use of available data on different sectors of the economy. [24] investigates the forecasting performance of aggregated and disaggregated models of median growth rates of 18 industrialized countries. In [15] the authors compare the quality of the forecasts of different types of disaggregated models (modelling the aggregate directly, but using the disaggregated data or modelling disaggregated components to obtain the forecast of the aggregate as a sum of the disaggregated forecasts). In [21] disaggregated models are applied to forecast US aggregate inflation. Forecasting on disaggregated data can be performed differently in some cases the forecasting of a disaggregated series with their further aggregation, in others forecasting the aggregated indicator using a disaggregated series, in most of these studies it is shown that disaggregation can lead to a significant improvement of the forecasting performance of models.

Unfortunately, in many cases we do not have sufficient data on output and prices in tne sectors of the economy needed to estimate disaggregated models. For example, for Russian macroeconomic statistics, where input-output balances have not been calculated for years, the available statistics on the output of sectors of the economy are scarce and cannot be used to disaggregate even GDP itself, let alone its components. Moreover, the tendency over the last few decades is for the boundaries between products to slowly erode and become less stable as economies evolve and services begin to constitute a more important part of the economy. Taking these considerations into account, we devise a method for rational disaggregation which allows us to obtain a disaggregated series based on a set for a priori assumptions with no need of additional data. The idea of rational decomposition is presented in [1] and [4] and is largely based on aggregation techniques in [2] and [3]. Disaggregation in these papers was performed with the use of available import and export data so that the decomposed series can be treated as kinds of internal and external (import and export) products. This makes the results easily tractable, but imposes serious restrictions on the behaviour of observable series. The method, proposed here, is based on different assumptions (the rational behaviour of macroeconomic agents and a specific kind of utility functions – the utility tree of Strotz [22], [23] and Gorman [13]) and does not impose any restrictions on the behaviour of the indicators of interest.

Another indicator of interest is the change in inventories. Because of the different nature of this indicator (not the flow, but the change in stock) it can not be described and modelled in the same terms. In this paper we propose a method for its decomposition.

The paper is organized as follows. We begin with a brief review of main techniques and concepts used in the aggregation and disaggregation of economic indicators. Then we present the theoretical background of the proposed decomposition procedure. In the third part we discuss the estimation issues and present the results of our two-product decomposition.

2 Description of the model

2.1 Utility tree

One of the main concepts to aggregate and decompose data is the utility tree, presented in papers of Strotz [22], [23] and Gorman [13]. All the goods here are separated into nonoverlapping groups that sum up to the whole set of goods, so that the consumer first estimates the group utility and then estimates the general utility using the group utilities.

A utility tree is a less general, but more convenient type of utility function. It is based on the observation that people often distribute their income not between the separate commodities, but between groups of commodities (food, clothing, etc.) which are represented by functions $U_1, U_2, \ldots, U_k$ and only then distribute group expenditures between certain goods in the group.

This allows us to represent the consumer problem as a two-step problem: the distribution of income between groups and the distribution of group expenditure between different goods in the group. Moreover, some types of utility tree functions allow consumers to make decisions at the first step using only the price indices of the groups (not the goods' prices) and this fact, in turn, facilitates the solution of the aggregation and decomposition problems.

Denoting agent's utility as U:

$$U = U(q_1, q_2, \ldots, q_n)$$

where $q_1, \ldots, q_n$ are the consumption of goods.

The utility tree function, proposed in [22], can be written as:

$$U = W(U_1(q_{1\,1}, q_{1\,2}, \ldots, q_{1\,n_1}), U_2(q_{2\,1}, q_{2\,2}, \ldots, q_{2\,n_2}), \ldots, U_k(q_{k\,1}, q_{k\,2}, \ldots, q_{k\,n_k}))$$

where $U_1, \ldots, U_k$ are the utilities of groups of goods.

The key feature of a utility tree here is the dependence (or independence) of the consumption of groups from each other. Under the weak definition of a utility tree the marginal rate of substitution of goods in the same group does not depend on the consumption of goods from other groups; under the strong definition the marginal rate of substitution of any two goods (not necessarily from the same group) does not depend on the consumption of goods from other groups. This leads to the dependence of the marginal rate of substitution of goods in different groups on the consumption of other groups for the weak definition and to independence for the strong definition.

A utility tree allows us to use a relatively small number of groups instead of a large number of separate goods. It is very convenient both from a theoretical point of view as it simplifies models and calculations, and from a practical point of view because it allows us to use aggregated information when we estimate models. An explicit specification of the utility function can give us a direct description of the aggregation procedure.

A survey of different types of utility functions that satisfy our requirements is presented, for example, in [6]. Among all the functional forms an S-branch utility tree based on a CES utility function is one of most popular. In [8] it is defined as follows:

$$U = \left\{ \sum_{s=1}^{S} \alpha_s \left[\sum_{i \in s}^{n_s} \beta_{si} (q_{si} - \gamma_{si})^{\rho_s} \right]^{\rho/\rho_s} \right\}^{1/\rho}$$

where γ_{si} is the minimally required consumption of goods, q_{si}, β and α are the parameters of the utility functions. Here both the total utility and the utilities of groups are specified with CES functions. [16] presents a nested utility function based on CES function.

This approach is often used in the estimation of consumer demand (see, e.g., [5], [2] or [10]).

2.2 Disaggregation

Many techniques, used in aggregation (see, e.g. [2] and [3]) can be used for decomposition. A decomposition technique similar to the one presented here was introduced in [1] and [4]. It is based on the assumption that observed indicators (such as GDP components) are comprised of several products. These papers propose the decomposition on domestic product (produced and consumed inside the country), export product (produced inside the country for export) and import product. So, every GDP component can be written as:

$$C(t) = X_C(t) + I_C(t)$$

where $C(t)$ is the total consumption, $X_C(t)$ is the production of domestic product, associated with consumption, $I_C(t)$ is import, associated with consumption. We can present other components of the GDP the same way. So, the total domestic production (ignoring government expenditure) $X(t)$:

$$X(t) = J_X(t) + C_X(t)$$

and total import $I(t)$:

$$I(t) = J_I(t) + C_I(t)$$

Domestic production and the components of consumption and investment are not observed in the statistics.

We assume that C_X, C_I, J_X and J_I are determined by agent preferences, described by the homogeneous utility functions:

$$C = g(C_I, C_X)$$

$$J = h(J_I, J_X)$$

Production is described by the homogeneous function:

$$Y = f(X, E)$$

Financial balances are held for the consumer, investor and producer:

$$p_c(t)g(C_I(t), C_X(t)) = p_I(t)C_I(t) + p_X(t)C_X(t)$$

where $p_C(t), p_I(t), p_X(t)$ are the deflators of consumption, import and domestic products respectively.

From agent rationality we get:

$$\frac{\partial_1 g(C_I(t), C_X(t))}{\partial_2 g(C_I(t), C_X(t))} = \frac{p_I(t)}{p_X(t)}$$

and analogues for the investor and producer.

At the stage of calibration CES functions are used for preferences, errors are introduced into agent optimality conditions and the parameters of functions and unobserved variable values which minimize the sum of squared errors are found.

This decomposition has one important drawback. If the simplest balance in current prices for an indicator is held:

$$p_t^X X_t = p_t^A X_t^A + p_t^B X_t^B,\ X \in \{Y, C, G, J, Ex, Im\}$$

and the similar balance in constant prices:

$$X_t = X_t^A + X_t^B,\ X \in \{Y, C, G, J, Ex, Im\}$$

where X_t^A and X_t^B are quantities of products A and B in indicator X_t, respectively, then every observed deflator is the weighted mean of the deflators of these two products:

$$p_t^X = \frac{X_t^A}{X_t} p_t^A + \frac{X_t^B}{X_t} p_t^B,\ X \in \{Y, C, G, J, Ex, Im\}$$

Hence, the deflators of all indicators should lie between the deflators of these two products at all time. As observed import is one of the products, all the other deflators must be between the unobserved deflator of domestic product and the observed deflator of import. This means that the deflator of import should be either the minimal or the maximal of all the observed deflators at all time. As we can see from statistics (see figure 1) this requirement does not always hold.

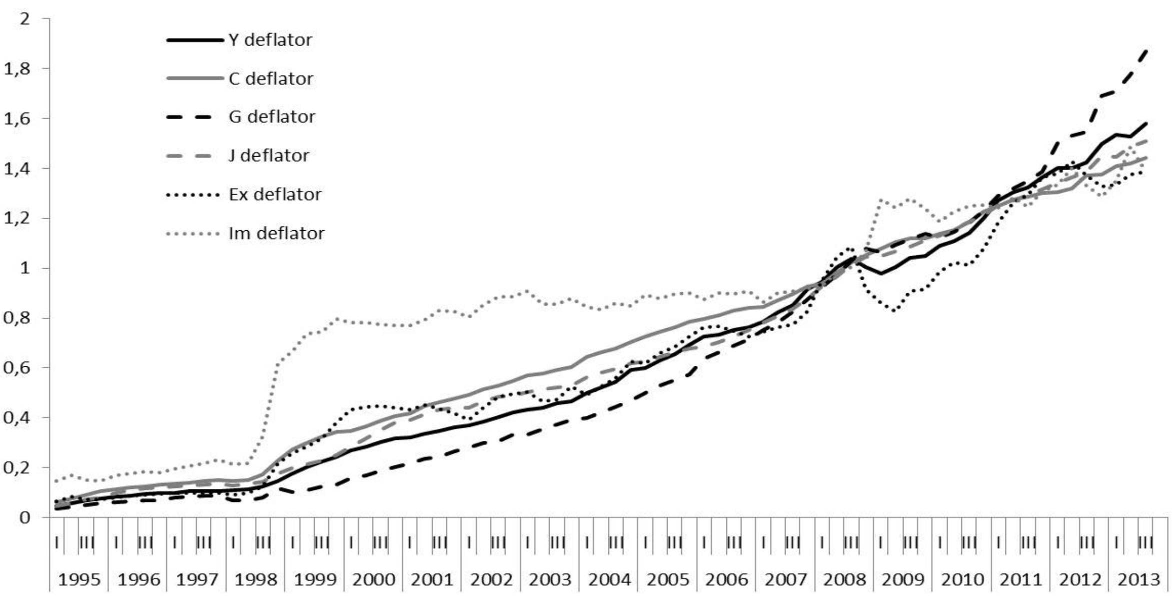

Figure 1: Deflators of GDP components

Hence, the simple decomposition can not be performed on part of the series and we need a method, which does not link one of the products to the observed values.

The method of decomposition proposed here differs from that described above in several ways. Firstly, it does not require the linking of one of the products to import or any other observed series. Secondly, the decomposition is performed on seasonally corrected series. Seasonality in data, among other problems, can lead to the estimation procedure adjusting seasonal peaks in different series to each other, which can lead to a bias in estimated coefficients. Seasonal correction can remove this problem and improve the accuracy of estimates.

2.3 Decomposition procedure – the theoretical background

2.3.1 Goods (microproducts) and macro-indicators

Let there be a full list of goods $\mathcal{G}$. Flows of these goods at every moment t participate in following processes:

$$\mathcal{P} = \{\underset{\text{production}}{X}, \underset{\text{intermediate consumption}}{V}, \underset{\text{investment}}{J}, \underset{\text{consumption}}{C}, \underset{\text{government spending}}{G}, \underset{\text{export}}{E}, \underset{\text{import}}{I}\}$$

Microproducts will be called goods and groups of goods will be called products.

Corresponding flows at moment t are written as:

$$\mathbf{x}_t^A = \left\{\mathbf{x}_t^A(i)\right\}_{i\in\mathcal{G}}, \quad A \in \mathcal{P} \tag{1}$$

Flows that correspond to part $\mathcal{M}$ of list $\mathcal{G}$ will be written as:

$$\mathbf{x}_t^A(\mathcal{M}) = \left\{\mathbf{x}_t^A(i)\right\}_{i\in\mathcal{M}\subset\mathcal{G}} \tag{2}$$

The time index will be often omitted. Index $t = 0$ corresponds to the base period.

At this stage we ignore the statistical discrepancy and change in inventories (though we will return to inventories later), so the flows satisfy the balance relations:

$$\mathbf{x}_t^X + \mathbf{x}_t^I = \mathbf{x}_t^V + \mathbf{x}_t^J + \mathbf{x}_t^C + \mathbf{x}_t^G + \mathbf{x}_t^E \tag{3}$$

2.3.2 Prices, macroindicators and deflators

All goods are bought and sold at the same price $\mathbf{p}_t(i)$ for all processes.

Thus, we can define the macroindicators in the following way:

- Real – by the scalar multiplication of flow vectors by base prices:

$$A_t = \left\langle \mathbf{p}_0, \mathbf{x}_t^A \right\rangle, A \in \mathcal{P}. \tag{4}$$

- Nominal – by the scalar multiplication of flows vectors on current prices:

$$\tilde{A}_t = \left\langle \mathbf{p}_t, \mathbf{x}_t^A \right\rangle, A \in \mathcal{P}. \tag{5}$$

- The deflator p_t^A of a process is the quotient of the corresponding nominal and real macroindicators:

$$\tilde{A}_t = p_t^A A_t, A \in \mathcal{P}. \tag{6}$$

So we can define

- Total production X_t,
- Import I_t,
- Intermediate consumption V_t,
- Gross capital formation J_t,
- Private consumption C_t,
- Government (= public) consumption G_t,
- Export E_t.

As the quantities of production and intermediate consumption are not invariant in the sense of the separation of production and trade on individual firms, the indicators of **real and nominal GDP** are introduced:

$$Y_t = X_t - V_t, \quad \tilde{Y}_t = \tilde{X}_t - \tilde{V}_t,$$

From (3) - (6) we can get macroeconomic balance in real and nominal expressions:

$$Y_t + I_t = C_t + G_t + J_t + E_t, \tag{7}$$

$$p_t^Y Y_t + p_t^I I_t = p_t^C C_t + p_t^G G_t + p_t^J J_t + p_t^E E_t. \tag{8}$$

where p_t^Y is the **GDP deflator**.

As we ignore the statistical discrepancy, (8) is the definition of the GDP deflator.

All the indicators in (7) and (8) are statistically observed values.

Data on the production quantity X_t and intermediate consumption V_t are presented separately in the statistics.

2.3.3 Rationalization of consumption flows

The problem of maximization of homogeneous utility Function $U : R_+^n \to R_+^1$ belongs to class U, if it is:

Smooth;[5]

Concave;

Homogeneous: $U(\alpha\,\mathbf{x}) = \alpha\,U(\mathbf{x}) \quad \forall \mathbf{x} \in \mathrm{R}_+^{|G|}, \alpha \geq 0$ and;

Not negative and strictly positive in at least one point.

Hence (with all these requirements fulfilled), it is monotone.

Theorem 1. *Maximum of homogeneous transferable utility*

We introduce the standard problem of utility maximization under budget constraint.

Under $U(\cdot) \in \mathrm{U} \;\; \mathbf{p} \in \mathrm{R}_+^n$ the problem is

$$\mathbf{x}^U \in \underset{x \in R_+^n}{Argmax}\{qU(x) - < \mathbf{p}, \mathbf{x} >\} \tag{9}$$

where $1/q$ is the dual variable. This problem has a non-negative solution iff

$$q = \bar{U}(\mathbf{p}) = \underset{x \in K^U}{min} \frac{< \mathbf{p}, \mathbf{x} >}{U(\mathbf{x})}. \tag{10}$$

The properties of the $\bar{U}$ function will be discussed later.

This solution is defined up to a factor and can be found from any of equations:

$$\nabla U(\mathbf{x}^U) = \frac{\mathbf{p}}{\bar{U}(\mathbf{p})}, \tag{11}$$

$$\frac{\mathbf{x}^U}{U(\mathbf{x}^U)} = \nabla \bar{U}(\mathbf{p})\,. \tag{12}$$

and

$$\left\langle \mathbf{p}, \mathbf{x}^U \right\rangle = \bar{U}(\mathbf{p}) \cdot U(\mathbf{x}^U) = q \cdot U(\mathbf{x}^U). \tag{13}$$

This theorem motivates the following definition:

[5]This requirement is not obligatory. The analogous proof without smoothness is considered below

Conjugate functions If $U(\cdot) \in \mathrm{U}$, then the **Young conjugate** function is defined as:

$$\bar{U}(\mathbf{p}) = \min_{x \in K^U} \frac{<\mathbf{p}, \mathbf{x}>}{U(\mathbf{x})}. \tag{14}$$

Theorem 2. *The existence and properties of a conjugate function*

The following properties hold for Young conjugate functions:

$$\bar{U}(\mathbf{p}) \in \mathrm{U}, \tag{15}$$

$$\bar{\bar{U}}(\mathbf{p}) = U(\mathbf{p}). \tag{16}$$

The last equality means that the function, conjugated to the conjugated function is the original function (in our case: conjugated to conjugated utility is the direct utility).

with $\mathbf{p} > 0$

$$U\left(\nabla \bar{U}(\mathbf{p})\right) = 1, \bar{U}\left(\nabla U(\mathbf{p})\right) = 1. \tag{17}$$

Using the Young conjugate function concept, we now understand the meaning of $\bar{U}$ function from (11): it is a Young conjugate to utility function. Thus, we can present the utility maximization problem in a slightly different form.

Theorem 3. *The maximum of a homogeneous utility under a budget constraint*

With $U(\cdot) \in \mathrm{U}$ $\mathbf{p} \in \mathrm{R}^n_+$ the problem

$$\mathbf{x}^U \in \underset{x}{Argmax}\{U(x)| <\mathbf{p}, \mathbf{x}> \leq \tilde{\mathbf{U}}\}, \mathbf{p} > \mathbf{0}, \tilde{\mathbf{U}} > \mathbf{0} \tag{18}$$

is reduced to the problem

$$\mathbf{x}^U \in \underset{x \in R^n_+}{Argmax}\{\bar{U}(\mathbf{p})U(x) - <\mathbf{p}, \mathbf{x}>\} \tag{19}$$

with a ray of solutions and normalization condition

$$\tilde{U} = \bar{U}(\mathbf{p}) \cdot U(\mathbf{x}^U) \tag{20}$$

that determines a single point on the ray.

Conjugate indices of price and quantity (20) allows us to treat $\bar{U}(\mathbf{p})$ as a price index and $U(\mathbf{x}^U)$ as a quantity index.

Index $\bar{U}(\mathbf{p})$ is a price of a utility unit $U(\mathbf{x}^U)$, under the optimal purchase the value saves, while under a nonoptimal purchase ($\mathbf{x} \neq \mathbf{x}^U$) the consumer pays more $\mathbf{p}\,\mathbf{x} \geq \bar{U}(\mathbf{p}) \cdot U(\mathbf{x})$ (see (13) and (10))

Unlike the Paasche and Laspeyres indices this index depends only on prices and does not depend on flows.

Quasi-linear rationalization The idea of the rationalization of demand is to assume that for all processes $A \in \mathcal{C} = \{J, C, G, E\}$

$$\mathbf{x}_t^A \in \underset{x}{Argmax}\{U^A(x) | < \mathbf{p_t}, \mathbf{x} > \leq \tilde{\mathbf{A}}_\mathbf{t}\} \tag{21}$$

for some $U^A(\cdot) \in$ U, $A \in \mathcal{C}$.

We also want the quantity index to correspond to real quantity that we either observe or want to reconstruct, because in contrast to utilities for real quantities balance relations hold.

We require that for optimal consumption (21)

$$U^A(\mathbf{x}_t) = k^A(\mathbf{p}_t, \mathbf{p}_0) \cdot A_t, \; A_t = \langle \mathbf{p}_0, \mathbf{x}_t \rangle \tag{22}$$

where A_t is the real quantity of consumption

A special case $k(\mathbf{p}_t, \mathbf{p}_0) \equiv 1$ is possible, but it requires the utility function of a special type which compares not the quantities of goods, but their values at base prices.

Deflator index Substituting utility (22) in the optimality condition (12) we get:

$$\mathbf{x}_t^A = k^A \cdot A_t \cdot \nabla \bar{U}^A(\mathbf{p}).$$

Multiplying this equation by base prices $\mathbf{p}_0$ and taking into account second equation in (22) we get, that:

$$k^A(\mathbf{p}, \mathbf{p}_0) = \frac{1}{\langle \mathbf{p}_0, \nabla \bar{U}^A(\mathbf{p}) \rangle},$$

and finally

$$\mathbf{x}_t = A_t \cdot \frac{\nabla \bar{U}^A(\mathbf{p}_t)}{\left\langle \mathbf{p}_0, \nabla \bar{U}^A(\mathbf{p}_t) \right\rangle}. \tag{23}$$

For nominal quantity of purchase from (23):

$$\tilde{A}_t \left\langle \mathbf{p}_t, \mathbf{x}_t \right\rangle = \frac{A_t \left\langle \mathbf{p}_t, \nabla \bar{U}^A(\mathbf{p}_t) \right\rangle}{\left\langle \mathbf{p}_0, \nabla \bar{U}^A(\mathbf{p}_t) \right\rangle} = A_t \frac{\bar{U}^A(\mathbf{p}_t)}{\left\langle \mathbf{p}_0, \nabla \bar{U}^A(\mathbf{p}_t). \right\rangle}$$

So, the agent purchases real product A_t at aggregated price

$$p_t^A = \frac{\bar{U}^A(\mathbf{p}_t)}{\left\langle \mathbf{p}_0, \nabla \bar{U}^A(\mathbf{p}_t) \right\rangle} \tag{24}$$

which is called the **deflator index** for this agent, $A_t = p_t^A \cdot A_t$.

From (24) it follows that

$$p_0^A = 1.$$

2.3.4 Aggregated products

Groups of goods (products) We assumed that all the goods are divided into set N of non-overlapping groups

$$\mathcal{G} = \sum_{\nu \in \mathrm{N}} \mathcal{G}^\nu.$$

We call a group of goods a product. Let us denote:

$$\mathbf{x}_t^A(\mathcal{G}^\nu) = \mathbf{x}_t^A(\nu), \mathbf{p}(\mathcal{G}^\nu) = \mathbf{p}(\nu).$$

Real and nominal indicators are introduced for products:

We do not observe these indicators, but we try to reconstruct them.

Due to (3) the following balances hold for the indicators:

$$Y_t(\nu) + I_t(\nu) = C_t(\nu) + G_t(\nu) + J_t(\nu) + E_t(\nu)$$

$$p_t^Y(\nu) \cdot Y_t(\nu) + p_t^I(\nu) \cdot I_t(\nu) = p_t^C(\nu) \cdot C_t(\nu) +$$
$$+ p_t^G(\nu) \cdot G_t(\nu) + p_t^J(\nu) \cdot J_t(\nu) + p_t^E(\nu) \cdot E_t(\nu)$$

due to (4) – (6):

Name	Real	Nominal	Deflator
Production ν	$X_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^X(\nu) \rangle$	$\tilde{X}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^X(\nu) \rangle$	$\tilde{X}_t(\nu) = p_t^X(\nu) \cdot X_t(\nu)$
Import ν	$I_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^I(\nu) \rangle$	$\tilde{I}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^I(\nu) \rangle$	$\tilde{I}_t(\nu) = p_t^I(\nu) \cdot I_t(\nu)$
Intermediate consumption =Current expenditure ν	$V_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^V(\nu) \rangle$	$\tilde{V}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^V(\nu) \rangle$	$\tilde{V}_t(\nu) = p_t^V(\nu) \cdot V_t(\nu)$
Gross formation ν	$J_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^J(\nu) \rangle$	$\tilde{J}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^J(\nu) \rangle$	$\tilde{J}_t(\nu) = p_t^J(\nu) \cdot J_t(\nu)$
Private consumption ν	$C_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^C(\nu) \rangle$	$\tilde{C}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^C(\nu) \rangle$	$\tilde{C}_t(\nu) = p_t^C(\nu) \cdot C_t(\nu)$
Government (public) consumption ν	$G_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^G(\nu) \rangle$	$\tilde{G}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^G(\nu) \rangle$	$\tilde{G}_t(\nu) = p_t^G(\nu) \cdot G_t(\nu)$
Export ν	$E_t(\nu) = \langle \mathbf{p}_0(\nu), \mathbf{x}_t^E(\nu) \rangle$	$\tilde{E}_t(\nu) = \langle \mathbf{p}(\nu), \mathbf{x}_t^E(\nu) \rangle$	$\tilde{E}_t(\nu) = p_t^E(\nu) \cdot E_t(\nu)$

$$A_t = \sum_{\nu \in \mathrm{N}} A_t(\nu), \quad p_t A_t = \sum_{\nu \in \mathrm{N}} p_t(\nu) A_t(\nu), \quad A \in \{X, V, J, C, G, E, I\}$$

Hierarchical utility We assume that the utility function of all sets of goods depends on homogeneous utilities of separate products, the utility functions of products are individual for each product:

We assume that utility function of all set of goods depends on homogeneous utilities of separate products, utility functions of products are individual for each product:

$$U(\mathbf{x}) = W(\{\mathbf{u}_\nu(\mathbf{x}(\nu))\}_{\nu \in \mathrm{N}}), \mathbf{u}_\nu(\cdot),\, W(\cdot) \in \mathrm{U} \tag{25}$$

then $U(\cdot) \in \mathrm{U}$

For the solution of problem (18) for this utility we get from the first order conditions (11)

$$\partial_\nu W(\mathbf{u}_\mathrm{N}) \nabla \mathbf{u}_\nu(\mathbf{x}^U(\nu)) = \frac{\mathbf{p}(\nu)}{\bar{U}(\mathbf{p})} \tag{26}$$

Here and below we denote a vector of dimension |N| with bold font with index N

Theorem 4. *Conjugated hierarchy*

The conjugated function to the utility function of the whole set of goods can be represented as a conjugated function of the conjugated product utility functions

$$\bar{U}(\mathbf{p}) = \bar{W}(\bar{\mathbf{u}}_{\mathrm{N}}) = \bar{W}\left(\{\bar{\mathbf{u}}_\nu(\mathbf{p}(\nu))\}_{\nu\in\mathrm{N}}\right). \tag{27}$$

Now we can get from (26) and (27), that under hierarchical utility (25)

$$\nabla W(\mathbf{u}_{\mathrm{N}}) = \frac{\bar{\mathbf{u}}_{\mathrm{N}}}{\bar{W}(\bar{\mathbf{u}}_{\mathrm{N}})} \tag{28}$$

$$\nabla \mathbf{u}_\nu(\mathbf{x}^U(\nu)) = \frac{\mathbf{p}(\nu)}{\bar{\mathbf{u}}_\nu(\mathbf{p}(\nu))} \tag{29}$$

If there is a budget constraint

If there is a budget constraint

$$\tilde{U} = \left\langle \mathbf{p}, \mathbf{x}^U \right\rangle$$

then by Euler's theorem from (29) we get

$$\left\langle \mathbf{p}(\nu), \mathbf{x}^U(\nu) \right\rangle = \bar{\mathbf{u}}_\nu \cdot \mathbf{u}_\nu$$

(28) means that an agent-consumer with hierarchical utility determines the quantities of the consumption of products $\mathbf{u}_{\mathrm{N}}$ by their price indices $\bar{\mathbf{u}}_{\mathrm{N}}$, and then determines the quantities of the consumption of separate goods by their individual prices $\mathbf{p}(\nu)$.

Due to theorem 1, (28) and (29) are equivalent to (see (12)):

$$\frac{\mathbf{u}_\nu}{W(\mathbf{u}_{\mathrm{N}})} = \partial_\nu \bar{W}(\bar{\mathbf{u}}_{\mathrm{N}}) \tag{30}$$

$$\frac{\mathbf{x}^U(\nu)}{\mathbf{u}_{\mathrm{N}}(\mathbf{x}^U(\nu))} = \nabla \bar{\mathbf{u}}_{\mathrm{N}}(\mathbf{p}(\nu)) \tag{31}$$

Deflator indices and the real quantities of products Now we apply techniques, proposed in paragraph 2.3.3 to the deflator indices, assuming that every agent has its own hierarchical utility function of the form:

$$U^A(\mathbf{x}) = W^A\left(\{\mathbf{u}_\nu(\mathbf{x}(\nu))\}_{\nu\in\mathrm{N}}\right), \quad A \in \mathcal{C}$$

We assume that aggregating functions of goods $\mathbf{u}_\nu$ are the same for all agents $A \in \mathcal{P}$, though it is not necessary.

As in paragraph 2.3.3, equation (23) was derived from optimality condition (12), from (31) we can get:

$$\mathbf{x}_t^A(\nu) = A_t(\nu)\frac{\nabla\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))}{\langle\mathbf{p}_0(\nu), \nabla\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))\rangle}, \Rightarrow \mathbf{u}_\nu\left(\mathbf{x}_t^A(\nu)\right) = \frac{A_t(\nu)}{\langle\mathbf{p}_0(\nu), \nabla\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))\rangle} \tag{32}$$

$$\tilde{A}_t(\nu) = p_t(\nu)\cdot A_t(\nu), p_t(\nu) = \frac{\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))}{\langle\mathbf{p}_0(\nu), \nabla\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))\rangle} \tag{33}$$

The last relation shows that all agents buy the aggregated product at the same price $p_t(\nu)$.

Now, applying the same logic to (30), we get the relations for macroindicators. To do so, we need to understand what the value of utility in base prices is. So we repeat the logic of paragraph 2.3.3

For optimal value (30) we want the following conditions to hold:

$$A_t = \sum_{\nu\in\mathrm{N}} A_t(\nu)\left\langle\mathbf{p}_0, \mathbf{x}_t^A\right\rangle = \mathrm{K}^A(\mathbf{p},\mathbf{p}_0)\cdot W^A(\mathbf{u}_\mathrm{N}) \tag{34}$$

$$\mathbf{u}_\nu = W^A(\mathbf{u}_\mathrm{N})\partial_\nu\bar{W}^A(\bar{\mathbf{u}}_\mathrm{N}). \tag{35}$$

Substituting $W^A(\mathbf{u}_\mathrm{N})$ from (34) into (35), we get for optimal values:

$$\mathbf{u}_\nu = \frac{A_t}{\mathrm{K}^A(\mathbf{p},\mathbf{p}_0)}\partial_\nu\bar{W}^A(\bar{\mathbf{u}}_\mathrm{N}), \tag{36}$$

then we substitute optimal $\mathbf{u}_\nu$ from (32)

$$A_t(\nu) = \frac{A_t\cdot\langle\mathbf{p}_0(\nu), \nabla\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))\rangle}{K^A(\mathbf{p},\mathbf{p}_0)}\partial_\nu\bar{W}^A(\bar{\mathbf{u}}_\mathrm{N}), \tag{37}$$

sum over ν, take into account the first equality in (34) and get:

$$K^A(\mathbf{p}, \mathbf{p}_0) = \sum_{\nu \in \mathrm{N}} \langle \mathbf{p}_0(\nu), \nabla \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \rangle \cdot \partial_\nu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N}).$$

The first factor under the sum can be expressed with the deflator of product from (33)

$$K(\mathbf{p}, \mathbf{p}_0) = \sum_{\nu \in \mathrm{N}} \frac{\bar{\mathbf{u}}_\nu \cdot \partial_\nu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})}{p_t(\nu)}$$

and, finally, from (37), (32) we get

$$W^A(\mathbf{u}_\mathrm{N}) = \frac{A_t}{\sum_{\nu \in \mathrm{N}} \langle \mathbf{p}_0(\nu), \nabla \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \rangle \cdot \partial_\nu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})} \tag{38}$$

and demand on real quantities of products:

$$A_t(\nu) = \frac{A_t}{\sum_{\mu \in \mathrm{N}} \frac{\bar{\mathbf{u}}_\mu \cdot \partial_\mu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})}{p_t(\mu)}} \frac{\bar{\mathbf{u}}_\nu \cdot \partial_\nu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})}{p_t(\nu)} \tag{39}$$

where $\bar{\mathbf{u}}_\nu = \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))$.

To check whether the financial balance holds, we can use the Euler's theorem and the deflators from (33) to get from (39):

$$\sum_{\nu \in \mathrm{N}} p_t(\nu) \cdot A_t(\nu) = \frac{A_t \cdot \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})}{\sum_{\mu \in \mathrm{N}} \langle \mathbf{p}_0(\mu), \nabla \bar{\mathbf{u}}_\mu(\mathbf{p}_t(\mu)) \rangle \cdot \partial_\mu \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})}$$

Given (38), this is equal to

$$W^A(\bar{\mathbf{u}}_\mathrm{N}) \cdot \bar{W}^A(\bar{\mathbf{u}}_\mathrm{N})$$

and given (25), (27) and (20) we get

$$U^A(\mathbf{x}_t^A) \cdot \bar{U}^A(\mathbf{p}_t) = \tilde{A}_t$$

2.3.5 The closure of the deflators system on the meso-level

The formulation of the problem We want demand (39) to depend only on the deflators, not on the prices of goods. For this, it is sufficient that $\bar{\mathbf{u}}_\nu = \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))$ depends only on the deflators, but because $\bar{\mathbf{u}}_\nu$ depends only on its prices, the demand must depend only on its deflator (see (33)).

$$p_t(\nu) = \frac{\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))}{\langle \mathbf{p}_0(\nu), \nabla \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \rangle} \tag{40}$$

so

$$\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \equiv f(p_t(\nu)),\ \forall \mathbf{p}_t(\nu). \tag{41}$$

Condition on the conjugate utility As $\bar{\mathbf{u}}_\nu(\cdot)$ is homogeneous and deflator (40) is also homogeneous, in (41) we can replace $\mathbf{p}_t(\nu)$ with $\alpha \cdot \mathbf{p}_t(\nu)$, and get $\alpha \cdot \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \equiv f(\alpha \cdot p_t(\nu))$. Substituting here $\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu))$ from (41),

$$\alpha \cdot f(p_t(\nu)) \equiv f(\alpha \cdot p_t(\nu))$$

But a homogeneous function of one variable $p_t(\nu)$ can be only linear. So:

$$\bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \equiv \kappa_\nu(\mathbf{p}_0(\nu)) \cdot p_t(\nu)$$

Constant κ can depend on the parameters of the problem, among which we are interested in base prices, which sometimes have to be changed.

Substituting here (40) we obtain the expression for $\bar{\mathbf{u}}_\nu(\cdot)$

$$\langle \mathbf{p}_0(\nu), \nabla \bar{\mathbf{u}}_\nu(\mathbf{p}_t(\nu)) \rangle = \kappa_\nu(\mathbf{p}_0(\nu)) \tag{42}$$

If this is the case, from (39) we get:

$$A_t(\nu) = A_t \cdot \frac{\kappa_\nu(\mathbf{p}_0(\nu)) \cdot \partial_\nu \bar{W}\left(\{\kappa_\lambda(\mathbf{p}_0(\lambda)) \cdot p_t(\lambda)\}_{\lambda \in \mathrm{N}}\right)}{\sum\limits_{\mu \in \mathrm{N}} \kappa_\mu(\mathbf{p}_0(\mu)) \cdot \partial_\mu \bar{W}\left(\{\kappa_\lambda(\mathbf{p}_0(\lambda)) \cdot p_t(\lambda)\}_{\lambda \in \mathrm{N}}\right)} \tag{43}$$

Nontrivial homogeneous concave functions, satisfying (43) exist under any $\kappa_\nu(\mathbf{p}_0(\nu))$, though they can have strange form.

If we set all $\kappa_\nu(\mathbf{p}_0(\nu))$ equal to 1, we get the equality of the optimal utility to real consumption.

Unfortunately, in the general case we get the conjugated function and, hence, the utility function which depends on base prices.

2.3.6 Independence of utility from base prices

The dependence of utility on base prices, from the theoretical viewpoint, looks unnatural and, from the practical viewpoint, makes the transition to other base prices impossible.

The only possibility for (42) to be fulfilled for the smooth function $\bar{\mathbf{u}}_\nu(\cdot)$, which does not depend on parameters $\mathbf{p}_0(\nu)$, is linear function $\bar{\mathbf{u}}_\nu(\cdot)$. THis motivates the following theorem:

Theorem 5. *About the basket*

If (42) is fulfilled for all $\mathbf{p}(\nu)$ and $\mathbf{p}_0(\nu)$, $\bar{\mathbf{u}}_\nu$ (or $\mathbf{u}_\nu$) does not depend on $\mathbf{p}_0(\nu)$, then under some $\mathbf{b}(\nu) > 0$

$$\bar{\mathbf{u}}_\nu(\mathbf{p}(\nu)) = \langle \mathbf{p}(\nu), \mathbf{b}(\nu) \rangle \tag{44}$$

$$u_\nu(x(\nu)) = \underset{i \in G^N}{min}\{\frac{x(i)}{b(i)}\} \tag{45}$$

$$\kappa_\nu(\mathbf{p}_0(\nu)) = \langle \mathbf{p}_0(\nu), \mathbf{b}(\nu) \rangle \tag{46}$$

Interestingly, a similar result can be obtained for nonsmooth functions. Interestingly, a similar result can be obtained for nonsmooth functions. The only possibility for (42) to be fulfilled for the function $\bar{\mathbf{u}}_\nu(\cdot)$, which does not depend on the parameters $\mathbf{p}_0(\nu)$ in the case of nonsmooth functions, is a function that looks like a quarter of an octagon (see figure 2), which motivates the next theorem.

Theorem 6. *The utility function in the case of nonsmooth functions*

If (42) is fulfilled for all $p(\nu)$, the level hyperplane of function $\bar{u}_\nu$ consists of a simplex and hyperplanes, parallel to coordinate axis.

Hereinafter we assume (44) – (46) to hold.

Then the deflator index (33) will be:

$$p_t(\nu) = \frac{\langle \mathbf{p}_t(\nu), \mathbf{b}(\nu) \rangle}{\langle \mathbf{p}_0(\nu), \mathbf{b}(\nu) \rangle} \tag{47}$$

It turns out to be a simple basket price index. But the basket $\mathbf{b}(\nu)$ is full (including all the goods, not a sample of them, as it usually is in practice).

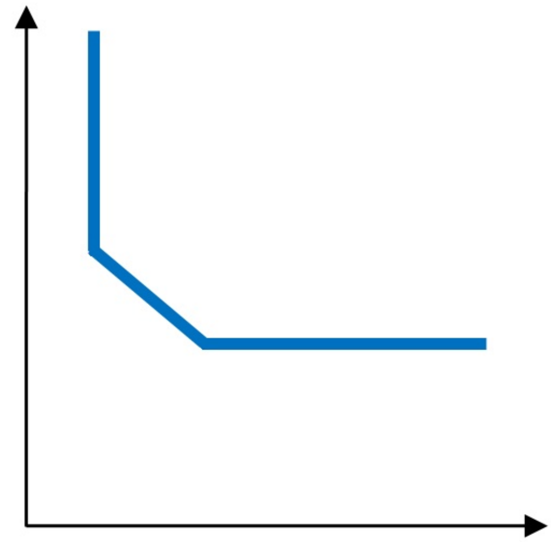

Figure 2: The independent of prices utility function in the case of nonsmooth functions

For the demand on real quantities of products from (39) under the conditions (44) – (46) we get:

$$A_t(\nu) = A_t \frac{\langle \mathbf{p}_0(\nu), \mathbf{b}(\nu)\rangle \cdot \partial_\nu \bar{W}^A\left(\{\langle \mathbf{p}(\lambda), \mathbf{b}(\lambda)\rangle\}_{\lambda\in \mathrm{N}}\right)}{\sum\limits_{\mu\in \mathrm{N}} \langle \mathbf{p}_0(\mu), \mathbf{b}(\mu\rangle \cdot \partial_\mu \bar{W}^A\left(\{\langle \mathbf{p}(\lambda), \mathbf{b}(\lambda)\rangle\}_{\lambda\in \mathrm{N}}\right)} \tag{48}$$

Generators of demand To facilitate the work with (48), it is convenient to introduce the following functions instead of conjugated utilities:

$$\hat{W}^A\left(p(\mathrm{N})\right) = \bar{W}^A\left(\{p(\lambda)\cdot\langle \mathbf{p}_0(\lambda), \mathbf{b}(\lambda)\rangle\}_{\lambda\in \mathrm{N}}\right), p(\mathrm{N})\left\{p(\lambda)\right\}_{\lambda\in \mathrm{N}} \tag{49}$$

which we will call the **generators** of demand.

In contrast to the conjugated functions, the generators need to be recalculated when we change the base prices, which is not a problem if deflators are observed (the arguments should be renormalized by the ratio of basket values at the old and new base prices).

From (49) we have:

$$\partial_\nu \hat{W}^A\left(p(\mathrm{N})\right) = \langle \mathbf{p}_0(\lambda), \mathbf{b}(\lambda)\rangle \cdot \partial_\nu \bar{W}^A\left(\{p(\lambda)\cdot\langle \mathbf{p}_0(\lambda), \mathbf{b}(\lambda)\rangle\}_{\lambda\in \mathrm{N}}\right)$$

and due to (47) with $p(\lambda) = p_t(\lambda)$

$$\langle \mathbf{p}_0(\lambda), \mathbf{b}(\lambda)\rangle \cdot \partial_\nu \bar{W}^A \left(\{\langle \mathbf{p}(\lambda), \mathbf{b}(\lambda)\rangle\}_{\lambda\in\mathrm{N}}\right) = \partial_\nu \hat{W}^A\left(p(\mathrm{N})\right).$$

Hence, from (48)

$$A_t(\nu) = A_t \frac{\partial_\nu \hat{W}^A\left(p_t(\mathrm{N})\right)}{\sum_{\mu\in\mathrm{N}} \partial_\mu \hat{W}^A\left(p_t(\mathrm{N})\right)}. \tag{50}$$

If we do not change the base prices, we work with real quantities and deflators.

Generators $\hat{W}^A(\cdot) \in \mathrm{U}$.

From (50) it follows that

$$A_t = \sum_{\nu\in\mathrm{N}} A_t(\nu)$$

3 Model estimation

The estimation of the decomposition model is based on equation (48). It allows us to get an explicit formula for unobserved model products by specifying the utility function W^A (and, hence, the conjugated function $\bar{W}^A$). Assume we have data on GDP and its components by expenditure (consumption, investment, etc.) of length T. We have real and nominal series, as well as deflators. Then, considering the set of (macro)products N, for each time t, for each series $A \in \{C, J, G, E, I\}$[6] we have the following relations:

$$A_t(\nu) = A_t \frac{k^A_\nu(\mathbf{p}_0) \cdot \partial_\nu \bar{W}^A\left(\mathbf{p}_1, ..., \mathbf{p}_N\right)}{\sum_{\mu\in\mathrm{N}} k^A_\mu(\mathbf{p}_0) \cdot \partial_\mu \bar{W}^A\left(\mathbf{p}_1, ..., \mathbf{p}_N\right)},\ \nu \in \mathrm{N} \tag{51}$$

$$p^A_t A_t = \sum_{\nu\in\mathrm{N}} p_t(\nu) A_t(\nu) \tag{52}$$

(51) is a simplified version of (48). $k^A_\nu(\mathbf{p}_0)$ here is a constant that changes only when we change the base year, it is needed to correct on the ratio of the deflators of new and old base years. If we treat prices $\mathbf{p}_1, ..., \mathbf{p}_N$ as deflators, they should be recalculated when we change the base year. Then we have to correct them in the utility functions to avoid the influence of the change of the base year on the results of the decomposition.

[6] We neglect the intermediate consumption as it is an almost constant share in GDP, so there is no sense to consider it separately

(52) is the balance in current prices. The fulfillment of the analogous balance in constant prices: $A_t = \sum_{\nu \in \mathrm{N}} A_t(\nu)$ follows from (51), which can be proved by summation over ν.

We get $|N| + 1$ equations for each of 5 equations, for a total of $5(|N| + 1)$ on $6|N|$ unknown values for each moment of time ($|N|$ unknown prices and 5 times $|N|$ unknown components of each of the GDP components). We should also add the unknown parameters of utility functions, but the number of them depends only on the type of selected utility functions, not on the length of series, hence it is possible to estimate them on a series of sufficient length even if we leave only one degree of freedom for each time t. To preserve the opportunity to estimate the model, it is necessary that the number of equations be not less then the number of unknown values, i.e. $5(|N| + 1) > 6|N|$, hence, $|N| < 5$ (a strict inequality for each time t is needed to preserve at least one degree of freedom for the estimation of utility function parameters). So, in our case if we have a time series of sufficient length we can make a two-, three- of four-product decompositions of the data. This paper focuses on a two-product decomposition, as this model is simpler, easier to understand, and demonstrates sufficient accuracy.

Another problem is that in the majority of situations the number of equations is higher than the number of unknowns, so the precise execution of the system of equations is impossible and we have to introduce errors in the equations (and find the solution by minimizing them). Here we propose preserving the strict equality in (52) and introducing the errors in (51). There are several reasons for this. First, (52) is the balance in current prices and it is usually the most precise of all the balances as it is computed directly from the observed data, whereas balances in constant prices are calculated with the use of deflators, which considerably affects the quality of data. Second, (51) represents the rationality conditions that sum to the balance in constant prices, so it seems natural that both of these conditions may not hold strictly. Agents can deviate from rational behaviour for different reasons and the computation of the balance in constant prices is not very precise.

The last issue we mention before the actual computation is the utility function specification. In this paper we propose one of the most popular alternatives – the CES function. For convenience, we will specify the conjugated function as CES and start to work with it directly.

3.1 Two-product decomposition

In this paper we present a two-product decomposition (a and b are the products) of the main macroeconomic balance. For each of the GDP components by expenditure two rationality conditions of corresponding agents (or, equivalently, one rationality condition and the balance in constant prices) and the balance in current prices are introduced. The conjugated utility function in this case:

$$\bar{W}(p_a, p_b) = (\alpha p_a^\rho + (1-\alpha) p_b^\rho)^{1/\rho}\,, \tag{53}$$

where p_a and p_b are prices of products a and b, α and ρ are parameters of the utility function. Then the partial derivatives with respect to prices p_a and p_b are:

$$\partial_a \bar{W} = \alpha p_a^{\rho-1} \left(\alpha p_a^\rho + (1-\alpha) p_b^\rho\right)^{\frac{1-\rho}{\rho}}\,,$$
$$\partial_b \bar{W} = (1-\alpha) p_b^{\rho-1} \left(\alpha p_a^\rho + (1-\alpha) p_b^\rho\right)^{\frac{1-\rho}{\rho}}\,.$$

So, we get explicit expressions for the a and b components of series $A \in \{C, J, G, E, I\}$:

$$A_a = A \frac{k_a^A \alpha_A p_a^{\rho_A - 1}}{k_a^A \alpha_A p_a^{\rho_A - 1} + k_b^A (1-\alpha_A) p_b^{\rho_A - 1}} \tag{54}$$

$$A_b = A \frac{k_b^A (1-\alpha_A) p_b^{\rho_A - 1}}{k_a^A \alpha_A p_a^{\rho_A - 1} + k_b^A (1-\alpha_A) p_b^{\rho_A - 1}}$$

A_a is calculated from these conditions, but instead of the second condition we use the balance in current prices (otherwise, if we calculate A_b from rationality conditions, only the balance in constant prices will hold)

$$A_b = \frac{p_A A - p_a A_a}{p_b} \tag{55}$$

When series A_a and A_b are calculated directly, we can compute the balance in constant prices (with errors ε_A):

$$A(1+\varepsilon_A) = A_a + A_b$$
$$\varepsilon_A = \frac{A_a + A_b - A}{A}$$

If the balance is fulfilled precisely, the error $\varepsilon_A = 0$. There are 5 errors (one for each of the GDP components). They all are normalized on the magnitude of the corresponding indicator, hence we can compare them. The unknowns in the problem (two series of prices and five sets of utility function parameters) are chosen to minimize the errors. We use the sum of squared errors as a criterion, so that the problem takes the form:

$$\sum_{t=1}^{T} \sum_{A \in \{C,J,G,E,I\}} \varepsilon_A(t)^2 \rightarrow \min_{p_a(t), p_b(t), \alpha_A, \rho_A, t=1,\ldots,T, A \in \{C,J,G,E,I\}}$$

3.1.1 The data

The model was estimated on official Rosstat data. We used quarterly data on GDP and its components by expenditure for the period from the beginning of 2001 until the 3rd quarter of 2013 (earlier observations are not included in order to preserve the homogeneity of data):

- Consumption of households (variable C)
- Government expenditure (variable G)
- Gross accumulation of capital (variable J)
- Exports (variable E)
- Imports (variable I)

We used series in current prices, constant prices of 2008 and their deflators.

All series were seasonally adjusted with a procedure, invariant to deflation and the decomposition was performed on seasonally adjusted data.

3.1.2 Decomposition results

Due to the very high complexity of the problem, it is impossible to find the solution analytically and we had to do it numerically. Nevertheless, if the starting values are selected properly, it is possible to find the solution, which turns out to be quite stable to change in starting values. The results look quite reasonable.

We also recalculated the data to base prices of a year outside the period of observation (in our case to 2000 prices), estimated the decomposition, and then returned to 2008 prices. The problem is that when the base year is inside the period of observation, all deflators intersect at the same point (in the base year, when they all equal to 1), and the system of balances at this point is:

$$\frac{A_a}{A}p_a + \frac{A_b}{A}p_b = 1, \quad A \in \{C, J, G, E, I\}$$

If all $\frac{A_a}{A}$ and all $\frac{A_b}{A}$ are not equal to each other (and there is no reason to think so), the system has no solutions. In the actual data deflators do not equal 1 precisely, so some solution can still be found, but it turns out to be very unstable. The shift of the base year outside the period of observation eliminates this problem and allows us to obtain precise estimates of the model. The return to the old base prices is easy: both the seasonal adjustment procedure and decomposition allow it.

To begin with, we demonstrate the preciseness of the reproduction of the statistics on disaggregated data (figures 3 – 7).

Model data were obtained by summing the series of products a and b estimated by (54) and (55) for each of the GDP components. The preciseness of the model is very high. Divergences (though very small) can be noticed only in government expenditures. The prices of products p_a and p_b are presented in figure 8.

The price of product b is rising quickly, the price of product a is changing slowly. So, we can state that the model products are in some sense "fast" and "slow" products, where the "speed" of the product is the speed of price changes. This result was expected, as from the fulfilment of the balances in current prices it follows that all observed deflators can be represented by a combination of p_a and p_b with nonnegative weights, summing to 1. All deflators in every moment of time should lie between p_a and p_b. One of them should always be the highest before the base year and the lowest after it and vice versa. And it is exactly what we see.

The drop of price p_b in 2012, which is not observed for p_a also looks interesting. The influence of 2008 crisis on p_a is not strong, and there is almost no influence on p_b.

The coefficients of the utility functions are also of interest (table 1, coefficients estimated with (53) parametrization).

We can see that, for example, two functions out of five – investment and government expenditure – are Cobb-Douglas functions (CES with $\rho = 0$[7] is Cobb-Douglas function). Functions

[7]We should also notice that, of course, coefficients are not 0 precisely, but due to rounding the difference can

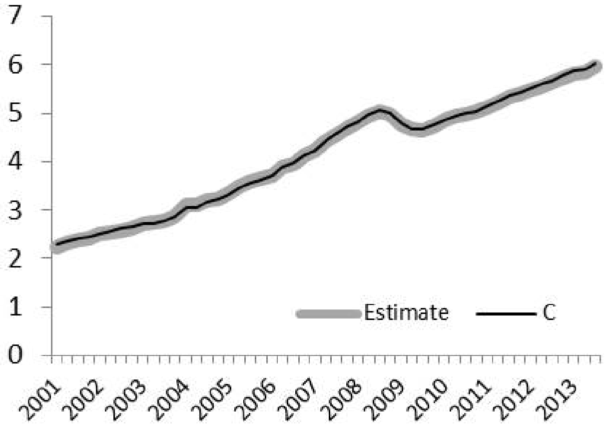

Figure 3: Dynamics of real and model consumption, trillion roubles

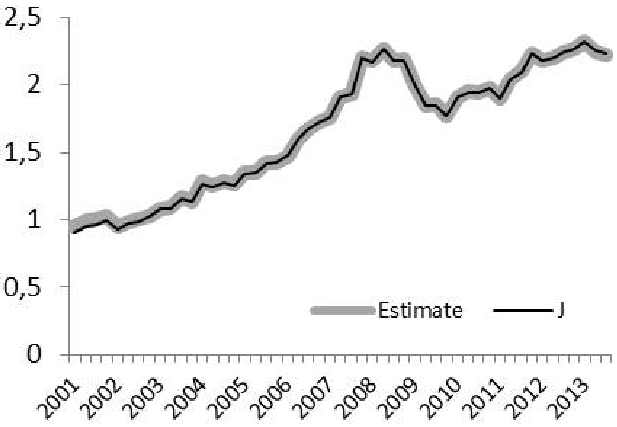

Figure 4: Dynamics of real and model investment, trillion roubles

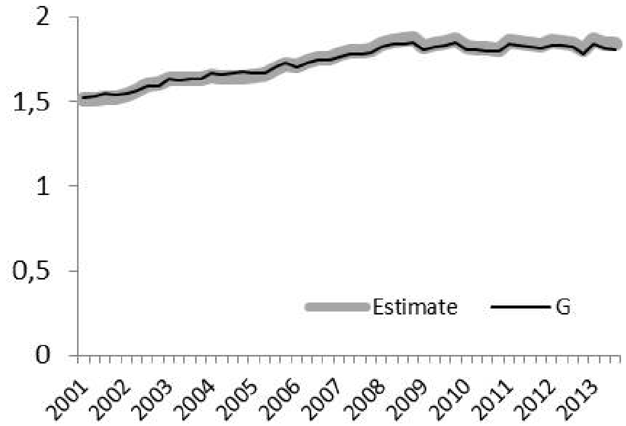

Figure 5: Dynamics of real and model government expenditure, trillion roubles

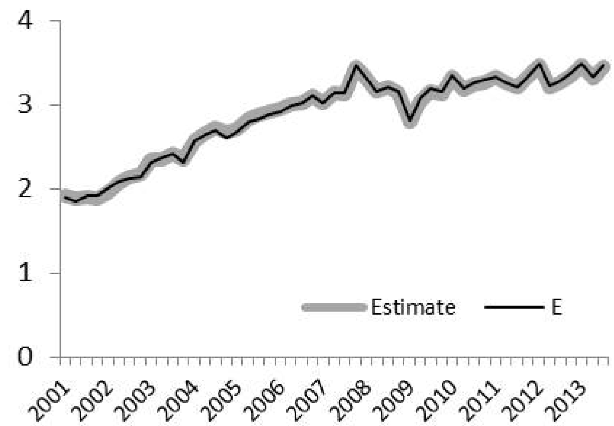

Figure 6: Dynamics of real and model export, trillion roubles

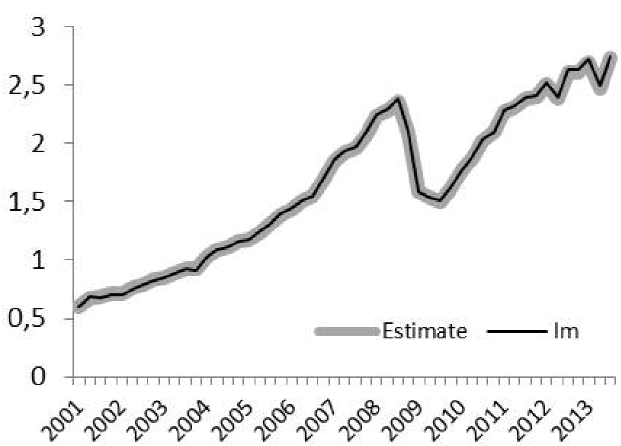

Figure 7: Dynamics of real and model import, trillion roubles

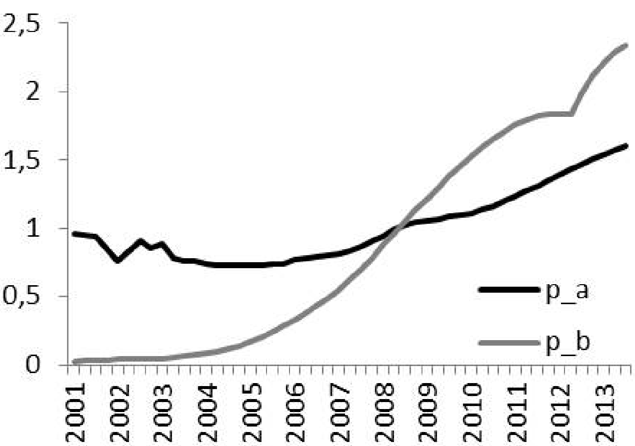

Figure 8: Prices of model products

Table 1: Utility function coefficients estimates

	C	J	G	E	I
α	0.286	0.267	0.153	0.602	0.491
ρ	0.017	0.000	0.000	0.258	0.004

for consumption and imports are also close to the Cobb-Douglas form, and only the export function differs significantly. In general, the export function is quite different from the other functions, which can also be seen in values of α coefficient (imports also have α values different from other indicators). We can say that "domestic" functions (consumption, investment and government expenditure) are quite similar, imports have the same functional form, but weighsvthe two products differently, and exports differ both in functional form and weighting. We can conclude that the mechanisms of the export of products is different from the mechanism of their consumption, which seems reasonable. Differences in α can be explained by different structures of consumption inside and outside the country.

The results of the decomposition (in current prices, which are more precise) are presented in figures 9 – 13.

In most cases the fast product b dominates the slow a, the latter dominates only in imports and sometimes in exports, which seems reasonable in the light of the above-mentioned arguments about different nature of domestic and external indicators. The behaviour of the products also differs for different indicators: in investment, government expenditure and exports the fast product was the first to react to the crisis, in imports, the slow one, in consumption the reaction of both products was nearly the same (probably the preservation of structure is more important for consumption, which makes sense). Interestingly, the fast components of investment and exports declined the final quarters. The same series in constant prices (though they are less interesting than the series in current prices) are presented in figures 14 – 18.

3.1.3 Change in inventories

Another important and interesting question is the decomposition of the change in the inventories in terms of the proposed scheme. It is not possible to use the same logic because the indicator

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not be seen

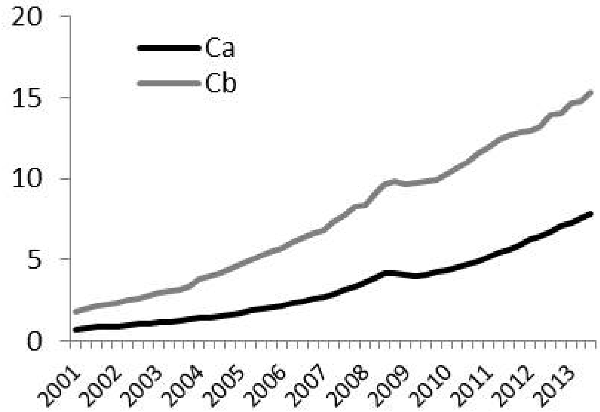

Figure 9: Dynamics of consumption components, trillion roubles in current prices

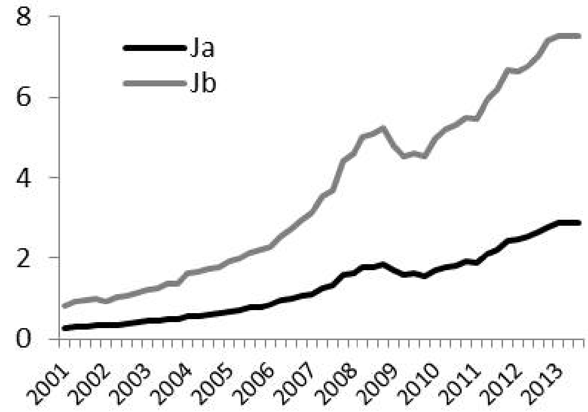

Figure 10: Dynamics of investment components, trillion roubles in current prices

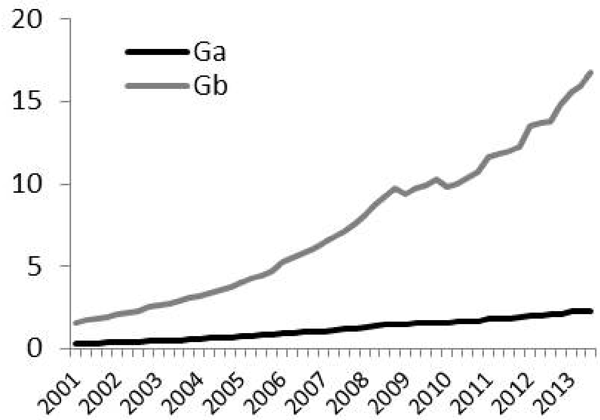

Figure 11: Dynamics of government expenditure components, trillion roubles in current prices

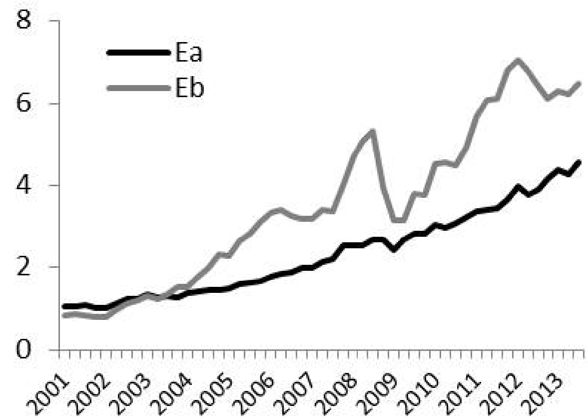

Figure 12: Dynamics of export components, trillion roubles in current prices

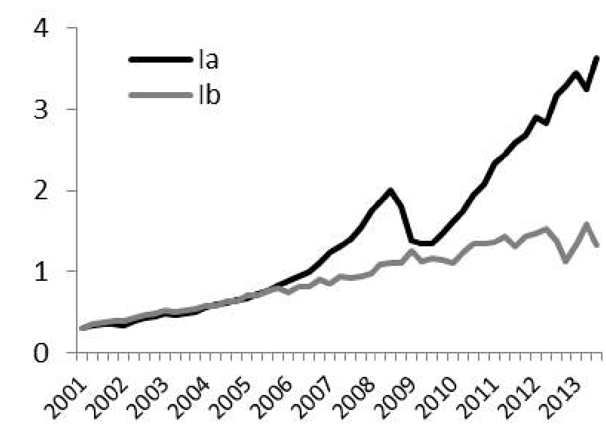

Figure 13: Dynamics of import components, trillion roubles in current prices

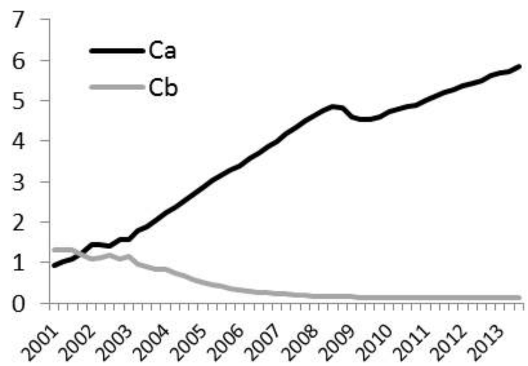

Figure 14: Dynamics of consumption components, trillion roubles in constant prices of 2008

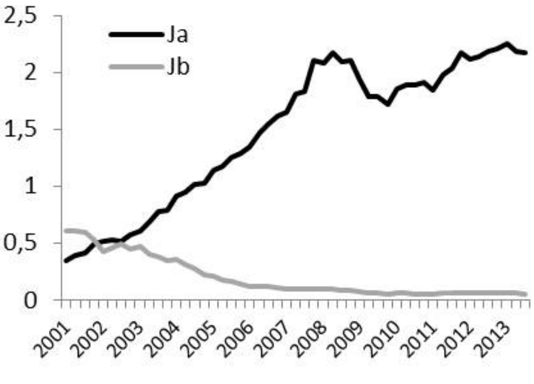

Figure 15: Dynamics of investment components, trillion roubles in constant prices of 2008

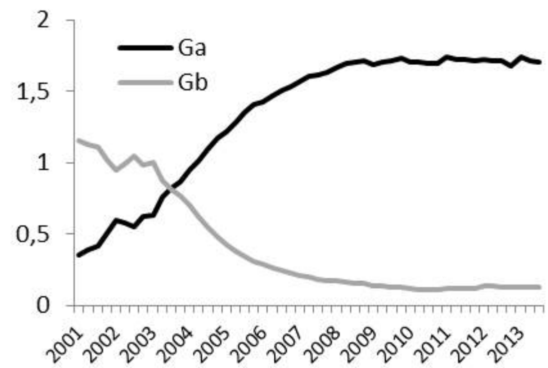

Figure 16: Dynamics of government expenditure components, trillion roubles in constant prices of 2008

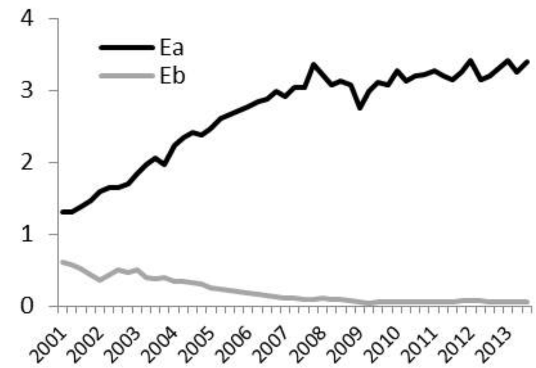

Figure 17: Dynamics of export components, trillion roubles in constant prices of 2008

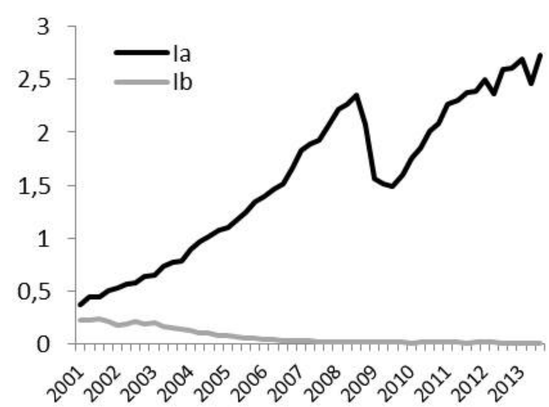

Figure 18: Dynamics of import components, trillion roubles in constant prices of 2008

is of another nature (not flow but change in stock) and it is hard to represent the indicator as a result of optimizing the behaviour of some agent. So we derive the decomposition on other considerations. Let the balances in current and constant prices hold for change in inventories:

$$dS_a p_a + dS_b p_b = pdS$$

$$dS_a + dS_b = dS,$$

where pdS and dS are series of change in inventories in current and constant prices respectively (both can be found in statistics), dS_a and dS_b are the components of the change in inventories. If both relations hold and prices p_a and p_b are known (we have found them earlier), we get the system of two equations with two unknowns.Therefore the components of the change in inventories are:

$$dS_b = \frac{pdS - dSp_a}{p_b - p_a}$$

$$dS_a = \frac{dSp_b - pdS}{p_b - p_a}$$

As a result we get the components of change in inventories, presented in figure 19.

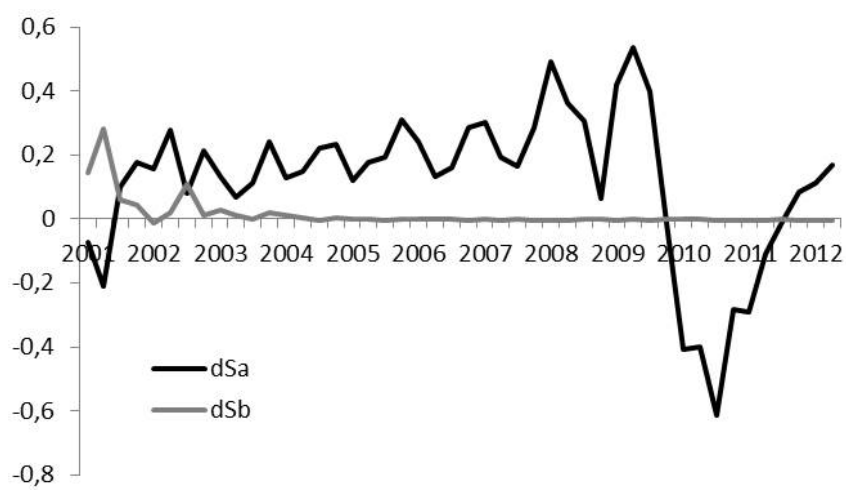

Figure 19: Dynamics of change in inventories components, trillion roubles

We notice, for example, that the change in inventories takes place only due to the slow product. This is reasonable from a purely technical point of view (looking at dS_b and dS_a formula), as well as from the simple logic: there is no sense in storing a product that quickly loses its value.

Decomposition of change in inventories is important, because in previous versions of the procedure there were significant difficulties with this indicator.

4 Conclusion

In this paper we present a method for a multiproduct model decomposition of the components of GDP. The key feature of the proposed model is that it does not require the linking of one of the products to the observed series of imports and exports. Thus, it gives a system of equations which always has a solution. We describe it theoretically and derived the conditions that allow the separation of the data into several products. We propose a methodology for the estimation of the decomposition of the data and perform a two-product decomposition on actual data on Russian GDP.

5 Appendix: proofs of the theorems

Proof of theorem 1 As the function, minimized in (9) is homogeneous, its value in a not trivial maximum can be only zero, i.e.:

$$q \cdot U(\mathbf{x}^U) = \left\langle \mathbf{p}, \mathbf{x}^U \right\rangle, q \cdot U(\mathbf{x}) \leq \langle \mathbf{p}, \mathbf{x} \rangle, \forall \mathbf{x} \tag{56}$$

but due to (14) this is equivalent to (10), thus from the first equation in (56) follows (13).

Equality (11) is a first order condition, that is necessary and sufficient for a smooth concave problem.

As first derivatives of a linearly homogeneous function are homogeneous of degree zero, the solution of (11) is not unique, it can be multiplied by any factor.

As $\mathbf{x}^U = \mathbf{x}^U(\mathbf{p})$ and differentiating first equation in (13) with respect to $\mathbf{p}$ we get that

$$\mathbf{x}^U + \mathbf{p}\frac{\partial \mathbf{x}^U}{\partial \mathbf{p}} = U(\mathbf{x}^U) \cdot \nabla \bar{U}(\mathbf{p}) + \bar{U}(\mathbf{p}) \cdot \nabla U(\mathbf{x}^U) \cdot \mathbf{p}\frac{\partial \mathbf{x}^U}{\partial \mathbf{p}} \tag{57}$$

but because of (11) terms $\partial \mathbf{x}^U / \partial \mathbf{p}$ in (57) cancel out and we get (12).

■

Proof of theorem 2 The expression under minimum does not depend on scale $\mathbf{x}$, thus we can search for minimum on a compact simplex. The function is continuous and has a finite value there, hence the minimum exists and is positive. Function $\bar{U}(\mathbf{p})$ is homogeneous and concave as the lower envelope of the family of linear functions. This proves (15)

The equation (16) is following from (14): $\bar{U}(\mathbf{p}) \leq \frac{\langle \mathbf{p}, \mathbf{x} \rangle}{U(\mathbf{x})} \quad \forall \mathbf{p}, \mathbf{x} \Rightarrow \bar{U}(\mathbf{x}) \leq \frac{\langle \mathbf{p}, \mathbf{x} \rangle}{U(\mathbf{p})} \forall \mathbf{p}, \mathbf{x}$ and at least for one pair $\mathbf{p}, \mathbf{x}$ the equality holds.

As it was proven above, the minimum point in (14) is the solution of the problem $\mathbf{x}^U \in \underset{x \in R^n_+}{Argmax} \{\bar{U}(\mathbf{p})U(x) - < \mathbf{p}, \mathbf{x} >\}$. It satisfies the condition (11) with any $\mathbf{p} > 0$. Applying to the both sides of (11) the homogeneous function $U(\cdot)$ we get the first equality in (17). We get the second from the first and (16)

■

Proof of theorem 3 As the minimization space in (18) is compact, the solution exists. As the Slater condition holds, the Lagrange multiplier also exists.

As $U(\cdot) \in$ U is monotone, the budget constraint is active and the Lagrange multiplier is positive. Denoting its reciprocal as q we get the problem (9) and from (10) we get (19). The activity of the budget constraint alongside with (13) gives (20).

■

Proof of theorem 4 As $\mathbf{u}_\nu(\cdot) \in$ U, can be applied to both sides of (26) $\bar{\mathbf{u}}_\nu(\cdot)$ we can get

$$\bar{U}(\mathbf{p}) \cdot \partial_\nu W(\mathbf{u}) = \bar{\mathbf{u}}_\nu(\mathbf{p}(\nu)) \tag{58}$$

or, in vectorized form:

$$\bar{U}(\mathbf{p}) \cdot \nabla W(\mathbf{u}) = \bar{\mathbf{u}}$$

but $W(\cdot)$ is a function U. Applying to both sides of (58) $\bar{W}(\cdot)$, we get (27)

■

Proof of theorem 5 Let $\nabla \bar{\mathbf{u}}_\nu$ does not depend on $\mathbf{p}_0(\nu)$. So, differentiating (42) with respect to $\mathbf{p}_0(\nu)$, we get

$$\nabla \bar{\mathbf{u}}_\nu(\mathbf{p}(\nu)) = \nabla \kappa_\nu(\mathbf{p}_0(\nu))$$

On the left side is the function of $\mathbf{p}(\nu)$, on the right - only of $\mathbf{p}_0(\nu)$. This equality can hold only if left and right sides equal to the same (vector) constant $\mathbf{b}(\nu)$. The homogeneous function with constant gradient has the form (44).

It is widely known that Leontief function (45) is conjugated to linear (44).

(46) follows from (44), (42).

■

Proof of theorem 6 When the $p(\nu)$ is changed, vector $n^\nu(p(\nu))$ should either remain on the same place (Leontief function) or move along the hyperplane (42). But in the same time it should move along the hyperplane $u_\nu\left(n^\nu(p(\nu))\right) = 1$. The latter, due to its convexity, can not contain unconnected parts of parallel hyperplanes. Hence, it contains not more that one part of hyperplane (42).

An example of aggregating function that allows for the closed system of deflators:

$$\bar{u}(p) = \min\left\{\langle p, b\rangle, \min_i \left\{\frac{p(i)}{q(i)}\right\}\right\}, b, q \geq 0$$

■

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