Journal of Geometry and Physics 62 (2012) 1175–1188

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Braided Weyl algebras and differential calculus on $U(u(2))$

Dimitri Gurevich [a,*], Pavel Pyatov [b,c], Pavel Saponov [b,d]

[a] LAMAV, Université de Valenciennes, 59313 Valenciennes, France
[b] Faculty of Mathematics, NRU HSE, 101000 Moscow, Russia
[c] Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia
[d] Division of Theoretical Physics, IHEP, 142281 Protvino, Russia

ARTICLE INFO

Article history:
Received 25 September 2011
Received in revised form 30 November 2011
Accepted 2 December 2011
Available online 9 December 2011

MSC:
17B37
81R60

Keywords:
Hecke symmetry
(modified) Reflection equation algebra
Braided Weyl algebra
Permutation relations
Laplace operator
Radial part

ABSTRACT

On any reflection equation algebra corresponding to a skew-invertible Hecke symmetry (i.e., a special type solution of the Quantum Yang–Baxter equation) we define analogs of the partial derivatives. Together with elements of the initial reflection equation algebra they generate a "braided analog" of the Weyl algebra. When $q \to 1$, the braided Weyl algebra corresponding to the Quantum Group $U_q(sl(2))$ goes to the Weyl algebra defined on the algebra $\mathrm{Sym}(u(2))$ or $U(u(2))$ depending on the way of passing to the limit. Thus, we define partial derivatives on the algebra $U(u(2))$, find their "eigenfunctions", and introduce an analog of the Laplace operator on this algebra. Also, we define the "radial part" of this operator, express it in terms of "quantum eigenvalues", and sketch an analog of the de Rham complex on the algebra $U(u(2))$. Eventual applications of our approach are discussed.

1. Introduction

Since the creation of the Quantum Group (QG) theory numerous attempts of developing a quantum version of differential calculus were undertaken. This study was initiated in [1] where the role of a quantum function space was played by a "q-symmetric" algebra of the fundamental space V equipped with an action of the QG $U_q(sl(m))$, and [2] where this role was played by a compact matrix pseudogroup.

Essentially, such a pseudogroup is the famous RTT algebra (see [3]) associated with a given braiding, i.e. an invertible operator $R : V^{\otimes 2} \to V^{\otimes 2}$, satisfying the Quantum Yang–Baxter equation

$$(R\otimes I)\,(I\otimes R)\,(R\otimes I) = (I\otimes R)\,(R\otimes I)\,(I\otimes R).$$

Hereafter V is a vector space over the ground field $\mathbb{K}\ (=\mathbb{R}$ or $\mathbb{C})$ and I is the identity operator.

In [2] a general scheme of defining differential forms and the de Rham complex on a matrix pseudogroup was suggested. Also, the author considered analogs of vector fields, introducing them by duality. In some subsequent publications (see for

* Corresponding author. Tel.: +33 327251469.
E-mail addresses: gurevich@univ-valenciennes.fr, d.gurevich@free.fr (D. Gurevich), pyatov@theor.jinr.ru (P. Pyatov), Pavel.Saponov@ihep.ru (P. Saponov).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2011.12.002

example [4,5]) the algebra generated by such fields in the case related to the QG $U_q(sl(m))$ was identified as a (modified) Reflection Equation (RE) algebra[1] (see below).

Recently, we have (partially[2]) generalized this construction replacing the RTT algebra by other quantum matrix algebras, in particular, by an RE algebra. Namely, taking a copy of the RE algebra (denoted $\mathcal{M}$) as a quantum function algebra, we treated another copy of this algebra (denoted $\mathcal{L}$) as an analog of the one-sided or adjoint differential operators. Below, we deal with the total algebra (denoted $\mathcal{B}(\mathcal{L},\mathcal{M})$) where we assume that $\mathcal{M}$ is equipped with a left (right-invariant) action of $\mathcal{L}$.

Besides, the braiding R coming in the definition of the algebra $\mathcal{B}(\mathcal{L},\mathcal{M})$ is taken to be a Hecke symmetry. This means that it is subject to the equation

$$(R - qI)\,(R + q^{-1}I) = 0,$$

where $q \in \mathbb{K}$ is assumed to be generic. In particular, such a Hecke symmetry comes from the QG $U_q(sl(m))$. This Hecke symmetry and all related objects will be called *standard*. The RTT and RE algebras associated with a standard Hecke symmetry are deformations of the algebra $\mathrm{Sym}(gl(m))$. With the use of other Hecke symmetries we can get analogous deformations of the super-algebra $\mathrm{Sym}(gl(m|n))$.

Note that any RE algebra associated with a Hecke symmetry has another basis in which the permutation relations between basic elements are quadratic-linear. So, the RE algebra in this basis becomes more similar to the enveloping algebra $U(gl(m|n))$. We call this quadratic-linear algebra the *modified* RE (mRE) algebra.[3] Namely in this form the RE algebra comes in constructing quantum analogs of vector fields.

Besides, in [6] we have constructed a representation category of a mRE algebra looking like that for the enveloping algebra $U(gl(m|n))$. Moreover, according to our construction the former category turns into the later one, provided that the mRE algebra goes to $U(gl(m|n))$ in the limit $q \to 1$.

One of the aim of the present paper is to define a braided counterpart of the partial derivatives on the RE algebras (modified or not). First, following [7] we equip an RE algebra $\mathcal{M}$ with a left (i.e. right invariant) action of a mRE algebra $\mathcal{K}$. (The both algebras are defined via the same Hecke symmetry.)

Second, by combining the generating matrices of these two algebras we construct a matrix of "partial derivatives" on $\mathcal{M}$. In the standard case these "derivatives" turn into the usual ones on the algebra $\mathrm{Sym}(gl(m))$ as $q \to 1$. The total algebra generated by the algebra $\mathcal{M}$ and by these partial derivatives on it, is called the *braided Weyl algebra* and is denoted $\mathcal{W}(\mathcal{M})$.

Upon replacing the algebra $\mathcal{M}$ by its modified form $\mathcal{N}$, we get (after a slight rescaling) the partial derivatives on the algebra $\mathcal{N}$. The corresponding Weyl algebra is denoted $\mathcal{W}(\mathcal{N})$.

Observe that in the case related to the QG $U_q(sl(2))$ the algebra $\mathcal{W}(\mathcal{M})$ appeared in [8–10] in construction of a q-analog of the Minkowski space. As for the braided Weyl algebra $\mathcal{W}(\mathcal{N})$, it is, up to our knowledge, an absolutely new object which has a very interesting limit as $q \to 1$. Namely, assuming the initial Hecke symmetry R to be a deformation of a super-flip, we get in the limit $q \to 1$ the partial derivatives on the algebra $U(gl(m|n))$ and consequently the Weyl algebra $\mathcal{W}(U(gl(m|n)))$.

Naturally, the usual Leibniz rule valid for the derivatives on a commutative algebra is modified when we pass to one of the mentioned algebras. On the algebra $U(gl(m))$ this Leibniz rule can be expressed via the following coproduct

$$\Delta(\partial_{n_i^j}) = \partial_{n_i^j} \otimes 1 + 1 \otimes \partial_{n_i^j} + \sum_k \partial_{n_i^k} \otimes \partial_{n_k^j}, \tag{1.1}$$

where n_i^j are entries of the generating matrix of this algebra. This form of the Leibniz rule was first found by S. Meljanac and Z. Škoda. As for the braided Weyl algebras $\mathcal{W}(\mathcal{M})$ (resp., $\mathcal{W}(\mathcal{N})$) such a simple formula for partial derivatives is not yet found, and their action on the algebra $\mathcal{M}$ (resp., $\mathcal{N}$) is defined via the permutations relations between elements of this algebra and derivatives completed via a counit.

We consider the Weyl algebra $\mathcal{W}(U(gl(2)))$ in detail. More precisely, we pass to the compact form of the Lie algebra $gl(2,\mathbb{C})$ and deal with the Weyl algebra $\mathcal{W}(U(u(2)))$. This form is more convenient for defining wave operators in our noncommutative (NC) algebra setting. Namely, they can be introduced in the classical way but with a new meaning of the derivatives.

Also, we describe a way of defining the radial part Δ_{rad} of the Laplace operator Δ. The crucial role in our construction is played by eigenvalues of the generating matrix of the algebra $U(u(2))$ (see (4.1)). These eigenvalues are defined as roots of

[1] In the paper [2] the deformation property of algebra in question was disregarded. Note, that only for the A_n series it is possible to construct a quantum differential algebra with good deformation property. Moreover, in the family of classical simple Lie algebras only for $g = sl(m)$ (and consequently, $gl(m)$) there exist $U_q(g)$-covariant deformations of the algebras $\mathrm{Sym}(g)$ and $\bigwedge(g)$ with classical dimensions of homogeneous components. $U_q(sl(m))$-covariant deformations of the algebra $\mathrm{Sym}(gl(m))$ and $\bigwedge(gl(m))$ can be constructed with the use of some idempotents playing the role of symmetrizers (resp., skew-symmetrizers) (see [6]). Whereas the operators used in [2] (as well as in a number of papers devoted to the so-called Woronowicz–Nichols algebras) in a similar construction are not idempotents and are not motivated by the algebraic structure of the initial braiding.

[2] We disregarded quantum analogs of differential forms.

[3] Note that the passage to this new basis (or in other words, the isomorphism between the RE algebra and its modified form) fails for $q = 1$. Besides, in order to treat any enveloping algebras as a deformation of the corresponding symmetric one we introduce another parameter $\hbar$ in the defining relations of the enveloping algebra (and also in the related mRE algebra).

the Cayley–Hamilton identity for the generating matrix and are treated as elements of the algebraic extension of the centre of this algebra.

Our final formula shows that the operator $\Delta_{\rm rad}$ expressed via symmetric function of these eigenvalues is a second order difference operator. We consider this operator as a first step in the direction of constructing similar analogs of Calogero–Moser operators and of radial parts of Laplace operators on super-algebra in the spirit of [11]. A plan of applying our method to this end is exhibited at the end of the paper.

The paper is organized as follows. In the next section we present a construction of the braided Weyl algebras on the RE algebras (modified or not) and find their $q\to 1$ limits. In Section 3 we consider a two-dimensional example in detail. As a result we get the Weyl algebra $\mathcal{W}(U(u(2)))$. Also, in Section 3 we calculate "eigenfunctions" of the partial derivatives and present an analog of the de Rham complex on the enveloping algebra $U(u(2))$. Besides, in Sections 2 and 3 we calculate Poisson counterparts arising from deformations in question. In Section 4 we define the Laplacian on the algebra $U(u(2))$ and compute its radial part.

2. Braided Weyl algebras and their $q \to 1$ limits

In what follows $R : V^{\otimes 2} \to V^{\otimes 2}$ is assumed to be a skew-invertible Hecke symmetry (see [12,6] for definitions). The space V is called *basic*.

With any skew-invertible Hecke symmetry $R : V^{\otimes 2} \to V^{\otimes 2}$ we associate a quantum matrix algebra which is referred to as the Reflection Equation (RE) algebra.

Definition 1. The RE algebra is a unital associative algebra over the field $\mathbb{K}$ generated by entries l_i^j of a matrix $L = \|l_i^j\|$, $1 \le i, j \le \dim V$, which are subject to the system of commutation relations

$$RL_1RL_1 = L_1RL_1R, \qquad L_1 = L\otimes I. \tag{2.1}$$

In this definition I stands for the unit matrix. Also, we assume a basis $\{x_i\}$ to be fixed in the space V. Then, in the basis $\{x_i\otimes x_j\}$ of $V^{\otimes 2}$ the braiding R is represented by a matrix $\|R_{ij}^{kl}\|$:

$$R(x_i\otimes x_j) = R_{ij}^{kl}x_k\otimes x_l.$$

Hereafter a summation over the repeated indices is assumed. The RE algebra defined above is a particular cases of more general construction of a quantum matrix algebra introduced in [12].

Let us consider two copies of the RE algebra: one of them, generated by entries of $M = \|m_j^i\|$ and denoted $\mathcal{M}$, plays the role of the quantized function algebra $\mathrm{Sym}(gl(m))$. The other one, generated by entries of $L = \|l_j^i\|$ and denoted $\mathcal{L}$, plays the role of quantized right-invariant differential operators on $\mathcal{M}$. Their action is encoded in the following permutation relations between two families of generators

$$RL_1RM_1 = M_1RL_1R^{-1}. \tag{2.2}$$

Thus, the whole algebra $\mathcal{B}(\mathcal{M},\mathcal{L})$ is generated by entries of two matrices M and L which satisfy the RE algebra permutation relations (2.1) and, additionally, subject to (2.2).

Permutation relation (2.2) defines one of the possible doubles of the two reflection equation algebras. This double was intensively studied as an algebra possessing coproduct and coaddition structures (see [13] and references therein) and also in application to a q-deformation of the Poincare algebra [9,10].

To get the action of an element $l\in\mathcal{L}$ to an element $m\in\mathcal{M}$ we proceed as follows. Note, that the permutation relations enable us to reduce any element of the whole algebra $\mathcal{B}(\mathcal{M},\mathcal{L})$ to that of the tensor product $\mathcal{M}\otimes\mathcal{L}$. So, given an element $l\otimes m\in\mathcal{L}\otimes\mathcal{M}$, we reduce it to the form of an element from $\mathcal{M}\otimes\mathcal{L}$, and then apply a counit $\varepsilon : \mathcal{L}\to\mathbb{K}$ to the components from $\mathcal{L}$. Here ε is an ingredient of a braided bi-algebra structure on any RE algebra which was discovered by Majid [14]. On generators of RE algebra the coproduct and counit of the braided bi-algebra are defined as follows

$$\Delta(1_{\mathcal{L}}) = 1_{\mathcal{L}}\otimes 1_{\mathcal{L}}, \qquad \Delta(l_i^j) = l_i^k\otimes l_k^j, \qquad \varepsilon(1_{\mathcal{L}}) = 1, \qquad \varepsilon(l_i^j) = \delta_i^j.$$

(Hereafter, $1_{\mathcal{A}}$ stands for the unit of a unital algebra $\mathcal{A}$.)

Thus, in order to compute the action $l_i^j(m)$ we employ the following chain of transformations

$$l_i^j(m) \equiv l_i^j(m1_{\mathcal{M}}) \xrightarrow{(2.2)} m' l_{i'}^{j'}(1_{\mathcal{M}}) \overset{\rm def}{=} m'\varepsilon(l_{i'}^{j'}) = m'\delta_{i'}^{j'}, \tag{2.3}$$

where we set by definition $l(1_{\mathcal{M}}) = \varepsilon(l)1_{\mathcal{M}}$.

This action can be extended to all polynomials in generators l_i^j in a straightforward way and putting additionally $1_{\mathcal{L}}(m) = m$ for any $m\in\mathcal{M}$, we define an action of any polynomial in l_i^j onto the whole algebra $\mathcal{M}$.

Remark 2. However, we can renormalize the action of the generators l_i^j on $1_{\mathcal{M}}$ by putting $l_i^j(1_{\mathcal{M}}) = \lambda\delta_i^j 1_{\mathcal{M}}$ where λ is a nontrivial numerical factor. Thus, we get a representation of the algebra $\mathcal{L}$ multiple to the previous one. Observe that the algebra $\mathcal{L}$ allows an automorphism $l_i^j \to \lambda, l_i^j$.

Now, let us define a modified version of the RE algebra.

Definition 3. Let $R: V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke symmetry. The modified RE algebra $\mathcal{K}_\hbar$ is a unital associative algebra generated by entries k_i^j of the matrix $K = \|k_i^j\|$, $1 \le i, j \le \dim V$, which are subject to the system of commutation relations

$$RK_1RK_1 - K_1RK_1R = \hbar(RK_1 - K_1R), \qquad K_1 = K \otimes I, \quad \hbar \in \mathbb{K}. \tag{2.4}$$

For the case $\hbar = 1$ we shall omit the subscript in notation $\mathcal{K}_\hbar$.

Unless $q = \pm 1$, the algebras $\mathcal{L}$ and $\mathcal{K}_\hbar$ are isomorphic. The isomorphism is explicitly defined by the following relation between two sets of generators[4]

$$1_{\mathcal{L}} = 1_{\mathcal{K}}, \qquad L = \hbar 1_{\mathcal{K}} I - (q - q^{-1})K. \tag{2.5}$$

Nevertheless, the limits of the algebras $\mathcal{L}$ and $\mathcal{K}_\hbar$ as $q \to 1$ are not isomorphic to each other since the map (2.5) degenerates.

Note that the above coproduct in the RE algebra being rewritten in the generators k_i^j takes the form (here we set $\hbar = 1$)

$$\Delta(1_{\mathcal{K}}) = 1_{\mathcal{K}} \otimes 1_{\mathcal{K}}, \qquad \Delta(k_i^j) = k_i^j \otimes 1_{\mathcal{K}} + 1_{\mathcal{K}} \otimes k_i^j - (q - q^{-1})k_i^s \otimes k_s^j. \tag{2.6}$$

A way of extending this coproduct to the whole algebra $\mathcal{K}$ is described in [6].

The last term of this formula disappears at $q \to 1$ and the coproduct turns into the usual one coming in the Hopf structure of the algebra $U(gl(m|n))$. As for the counit we choose it in the form

$$\varepsilon(1_{\mathcal{K}}) = 1, \qquad \varepsilon(k_i^j) = 0. \tag{2.7}$$

So, we rewrite the defining relations of the algebra $\mathcal{B}(\mathcal{M}, \mathcal{L})$ in terms of the new generators k_i^j (see (2.5)) and denote this algebra as $\mathcal{B}(\mathcal{M}, \mathcal{K})$. The algebra $\mathcal{B}(\mathcal{M}, \mathcal{K})$ contains an RE subalgebra $\mathcal{M}$, a modified RE subalgebra $\mathcal{K}$, while the permutation relation between the generating matrices of these subalgebras becomes

$$RK_1RM_1 = M_1RK_1R^{-1} + RM_1. \tag{2.8}$$

By the same method as above we can define an action of the algebra $\mathcal{K}$ on $\mathcal{M}$: for this end we employ the permutation relations (2.8) and the counit. Thus, we get the braided counterparts of usual right-invariant vector fields. Namely, as $q \to 1$ the elements k_i^j treated as operators turn into such vector fields (see [7] for more detail).

Now, consider a matrix $D = M^{-1}K$. The matrix M^{-1} inverse to M can be defined via the Cayley–Hamilton (CH) identity established in [16] for the generating matrix M of any RE algebra $\mathcal{M}$. For the existence of M^{-1} we should only assume the lowest coefficient (which is always central) of the CH identity to be invertible. More precisely, this operation can be realized as a localization by the mentioned central element.

Remark 4. Emphasize that in the frameworks of the method suggested in [17] of inverting matrices with entries from a NC algebra one requires the invertibility of a large number of elements of this algebra. In contrast with [17], our method can be applied only to the generating matrix of the RE algebra. Say, other matrices with entries from this algebra are not subject to a similar CH identity and our method of inversion is not valid for them.

Following the classical pattern we treat entries of the matrix $D = \|\partial_i^j\|$ as analogs of partial derivatives (momenta) : $\partial_i^j = \partial/\partial m_j^i$. Moreover, we can give a literal sense to this treatment, by defining an action of the partial derivatives onto the algebra $\mathcal{M}$. To this end we explicitly present the permutation relations between the generators m_i^j and ∂_k^l.

Proposition 5. *Consider a unital associative algebra $\mathcal{W}(\mathcal{M})$ generated by entries of the matrices M and D. Then the defining relations of the algebra $\mathcal{B}(\mathcal{M}, \mathcal{K})$ leads to the following relations between generators of $\mathcal{W}(\mathcal{M})$*

$$\begin{aligned} &RM_1RM_1 = M_1RM_1R, \\ &R^{-1}D_1R^{-1}D_1 = D_1R^{-1}D_1R^{-1}, \\ &D_1RM_1R = RM_1R^{-1}D_1 + R. \end{aligned} \tag{2.9}$$

Proof. We only should check the second an third relations. For this purpose we need some additional formulae for commutation relations of entries of M^{-1} with each other and with entries of K. First, the RE for M directly leads to the relation

$$M_1^{-1}R^{-1}M_1^{-1}R^{-1} = R^{-1}M_1^{-1}R^{-1}M_1^{-1}.$$

[4] Since a rescaling $L \to \lambda L$, $\lambda \in \mathbb{K}$, $\lambda \neq 0$ is an automorphism of the algebra $\mathcal{L}$ the map (2.5) can be also rescaled. Thus, in [15] we used another normalization of this map.

Then, we rewrite the defining relation (2.8) in an equivalent form

$$K_1R^{-1}M_1^{-1} = R^{-1}M_1^{-1}RK_1R - R^{-1}M_1^{-1}R.$$

Now, we make the identical transformations:

$$\begin{aligned} R^{-1}D_1R^{-1}D_1 &= R^{-1}M_1^{-1}\underline{K_1R^{-1}M_1^{-1}}K_1 = \underline{R^{-1}M_1^{-1}R^{-1}M_1^{-1}}RK_1RK_1 - \underline{R^{-1}M_1^{-1}R^{-1}M_1^{-1}}RK_1 \\ &= M_1^{-1}R^{-1}M_1^{-1}K_1RK_1 - M_1^{-1}R^{-1}M_1^{-1}K_1. \end{aligned}$$

Here we underline the part of an expression which undergoes an identical transformation at the next step of calculations.

On the other hand we get

$$\begin{aligned} D_1R^{-1}D_1R^{-1} &= M_1^{-1}\underline{K_1R^{-1}M_1^{-1}}K_1R^{-1} = M_1^{-1}R^{-1}M_1^{-1}\underline{RK_1RK_1}R^{-1} - M_1^{-1}R^{-1}M_1^{-1}RK_1R^{-1} \\ &= M_1^{-1}R^{-1}M_1^{-1}K_1RK_1 - M_1^{-1}R^{-1}M_1^{-1}K_1. \end{aligned}$$

Here we used the defining relations of the modified RE algebra $\mathcal{K}$ in order to transform the term RK_1RK_1 in the last step of transformations. Since the right hand sides of the above equalities coincide, we conclude that permutation rules for D is indeed as is claimed in (2.9).

To prove the last line relation of the system (2.9) we use the following auxiliary formulae

$$M_1^{-1}R^{-1}M_1R = RM_1R^{-1}M_1^{-1} \quad \text{and} \quad K_1RM_1 = R^{-1}M_1RK_1R^{-1} + M_1.$$

Now, we have

$$\begin{aligned} D_1RM_1R &= M_1^{-1}\underline{K_1RM_1}R = \underline{M_1^{-1}R^{-1}M_1R}K_1R^{-1} + R \\ &= RM_1R^{-1}M_1^{-1}K_1 + R = RM_1R^{-1}D_1 + R. \end{aligned}$$

The proof is completed. □

Definition 6. We call the algebra $\mathcal{W}(\mathcal{M})$ defined in (2.9) a braided Weyl algebra.

Note, that the subalgebra $\mathcal{D} \subset \mathcal{W}(\mathcal{M})$ generated by ∂_i^j is also an RE algebra associated with the Hecke symmetry R^{-1}.

As we noticed in Introduction, in the case related to the QG $U_q(sl(2))$ the relations (2.9) appeared in [8–10] in a study of the q-Minkowski space.

Now, we define an action of partial derivatives onto elements of the algebra $\mathcal{M}$ via the same method as above with the help of an additional requirement

$$\partial_i^j(1_{\mathcal{W}}) = 0 \tag{2.10}$$

which is a direct consequence of the relation $D = M^{-1}K$ and counit (2.7).

Our next aim is to define similar derivatives on the mRE algebra. Above we noticed that any RE algebra admits a rescaling automorphism. As a consequence, we can get any nonzero factor at the summand R in the last line of the system (2.9). Let us choose this factor to be equal $-(q-q^{-1})$ and pass from the RE algebra $\mathcal{M}$ to its modified version with generating matrix $N = \|n_i^j\|$ by applying a shift similar to (2.5)

$$1_{\mathcal{N}} = 1_{\mathcal{M}}, \qquad M = \hbar 1_{\mathcal{M}} I - (q-q^{-1})N. \tag{2.11}$$

In this way we get a braided Weyl $\mathcal{W}(\mathcal{N}_\hbar)$ (below we omit the subscript $\hbar$) defined by the following relations

$$\begin{aligned} &RN_1RN_1 - N_1RN_1R = \hbar(RN_1 - N_1R), \\ &R^{-1}D_1R^{-1}D_1 = D_1R^{-1}D_1R^{-1}, \\ &D_1RN_1R - RN_1R^{-1}D_1 = R + \hbar D_1R. \end{aligned} \tag{2.12}$$

According to our scheme, in order to treat entries of the matrix D as operators we should complete the above permutation relations with an action of partial derivatives on the unit element of the algebra. Let us define it by the same formula (2.10). The action of the derivatives onto generators n_k^l is the same as on m_k^l.

Now, assume a Hecke symmetry R to be a deformation of a super-flip P: $R \xrightarrow{q\to 1} P$. Here P acts on the tensor square of a superspace V:

$$P : V^{\otimes 2} \to V^{\otimes 2}, \qquad V = V_0 \oplus V_1, \qquad \dim V_0 = m, \qquad \dim V_1 = n,$$

where V_0 (resp., V_1) is the even (resp., odd) component of the space V. Then, passing in (2.12) to the limit $q \to 1$ we get a Weyl algebra $\mathcal{W}(U(gl(m|n)_\hbar))$ on a NC algebra $U(gl(m|n)_\hbar)$.[5] The defining relations of this algebra are

$$\begin{aligned} &PN_1PN_1 - N_1PN_1P = \hbar(PN_1 - N_1P), \\ &PD_1PD_1 = D_1PD_1P, \\ &D_1PN_1P - PN_1PD_1 = P + \hbar D_1P. \end{aligned} \tag{2.13}$$

The first line of the above system is just the defining relations of the algebra $U(gl(m|n)_\hbar)$ (but with a basis slightly different from the usual one formed by the matrix units). The second one means that the partial derivatives form a super-commutative algebra and the third line exhibits the permutation relations between two ingredients of the algebra $\mathcal{W}(U(gl(m|n)_\hbar))$. Note that the algebra $\mathcal{W}(U(gl(m|n)_\hbar))$ is a one-parameter deformation of the super-Weyl algebra $\mathcal{W}(\mathrm{Sym}(gl(m|n)))$, whereas the algebra $\mathcal{W}(\mathcal{N})$ above is a two-parameter deformation of the same algebra (the parameters are $\hbar$ and q).

Now, discuss the Poisson counterparts of the algebra $\mathcal{W}(U(gl(m)))$ (i.e. we restrict ourselves to the even case $n = 0$). Let $\mathcal{W}(\mathcal{M})_0$ be the algebra defined by (2.9) but without the last term R in the third relation. This algebra is a graded quadratic one, i.e. it is defined by quadratic relations on generators. For a generic q the dimensions of its homogeneous components are classical (i.e. they equal those of the space $\mathrm{Sym}(gl(m))^{\otimes 2}$). It is so since this algebra equals (as a set) the tensor product of two RE algebras and for them this property was proved in [6].

Besides, consider the quadratic-linear algebra $\mathcal{W}(\mathcal{N})_0$ which is also defined by cancelling the term R in the third defining relations (2.12) of the algebra $\mathcal{W}(\mathcal{N})$. Since the passage from the algebra $\mathcal{W}(\mathcal{M})_0$ to the algebra $\mathcal{W}(\mathcal{N})_0$ can be done via (2.11), it is easy to see that the graded algebra Gr $\mathcal{W}(\mathcal{N})_0$ associated with $\mathcal{W}(\mathcal{N})_0$ is isomorphic to $\mathcal{W}(\mathcal{M})_0$. This entails the existence of a Poisson pencil on the commutative algebra which is the $q \to 1$ limit of the algebra $\mathcal{W}(\mathcal{M})_0$. In the next section we explicitly write down these Poisson brackets in a two-dimensional example.

Remark 7. Observe that if $n = 0$ and $\hbar = 0$ the permutation relations from (2.13) completed with the counit are equivalent to the usual Leibniz rule. It is interesting to find a similar rule in a general case. In the case $n = 0$ and $\hbar \neq 0$ the Leibniz rule takes the form

$$\Delta(\partial_i^j) = \partial_i^j \otimes 1 + 1 \otimes \partial_i^j + \hbar \partial_k^j \otimes \partial_i^k \tag{2.14}$$

found S. Meljanac and Z. Škoda. (For $\hbar = 1$ this formula turns into (1.1) since $\partial_i^j = \partial_{n_j^i}$.)

This form together with the rules $\partial_i^j(1_{\mathcal{W}}) = 0$ and $\partial_i^j(n_k^l) = \delta_k^j\,\delta_i^l$ enables us to compute the action of partial derivatives on any element from $U(gl(m)_\hbar)$ via the usual formula

$$\partial_i^j(f \cdot g) = \cdot \Delta(\partial_i^j)(f \otimes g)$$

where $\cdot$ stand for the product in the algebra $U(gl(m)_\hbar)$.

Emphasize the similarity of the coproducts (2.6) and (2.14).

It is also interesting to compare the Leibniz rule (2.14) and that valid for the momenta on the κ-Minkowski space (see [18]).

3. Example: $m = 2$, $n = 0$

Let V be a two-dimensional vector space with a fixed basis $\{x, y\}$. Consider a Hecke symmetry, which in the basis $\{x \otimes x,\ x \otimes y,\ y \otimes x,\ y \otimes y\}$ of $V^{\otimes 2}$ is represented by the matrix

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

This matrix is just the product of the usual flip and the image of the universal $U_q(sl(2))R$-matrix in the space $V^{\otimes 2}$. Let

$$N = \begin{pmatrix} n_1^1 & n_1^2 \\ n_2^1 & n_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.1}$$

be the generating matrix of the modified RE algebra $\mathcal{N}$. The commutation relations (2.4) in this case are as follows

$$\begin{aligned} &qab - q^{-2}ba = \hbar b \qquad q(bc - cb) = ((q - q^{-1})a - \hbar)(d - a) \\ &qca - q^{-2}ac = \hbar c \qquad q(db - bd) = ((q - q^{-1})a - \hbar)b \\ &ad - da = 0 \qquad q(cd - dc) = c((q - q^{-1})a - \hbar) \end{aligned}$$

where we omit the symbol of the unit element in the linear combination $(q - q^{-1})\,a - \hbar$.

[5] Though in [7] we dealt with even Hecke symmetries, all results of that paper are valid in the general case.

Upon setting $\hbar = 0$ we get the defining relations of the corresponding non-modified RE algebra $\mathcal{M}$. We keep the same letters for entries of the corresponding generating matrix M.

Turn now to the algebra $\mathcal{D}$, which is generated by entries of the matrix

$$D = \begin{pmatrix} \partial_a & \partial_c \\ \partial_b & \partial_d \end{pmatrix}. \tag{3.2}$$

The relations between these partial derivatives are the same in both algebras $\mathcal{W}(\mathcal{M})$ and $\mathcal{W}(\mathcal{N})$. Their explicit form is

$$\begin{aligned}
&\partial_a\partial_b - \partial_b\partial_a = -(q^2-1)\partial_d\partial_b \qquad \partial_b\partial_c - \partial_c\partial_b = (q^2-1)(\partial_d - \partial_a)\partial_d \\
&\partial_a\partial_c - \partial_c\partial_a = (q^2-1)\partial_c\partial_d \qquad q^2\partial_b\partial_d - \partial_d\partial_b = 0 \\
&\partial_a\partial_d - \partial_d\partial_a = 0 \qquad \partial_c\partial_d - q^2\partial_d\partial_c = 0.
\end{aligned} \tag{3.3}$$

The permutation relations between derivatives and generators of the algebra $\mathcal{N}$ are as follows

$$\begin{aligned}
&\partial_a a = q^{-1} + q^{-2}a\partial_a - (1-q^{-2})b\partial_b + q^{-1}\hbar\partial_a \\
&\partial_a b = b\partial_a - (1-q^{-2})a\partial_c + (q-q^{-1})^2 b\partial_d + q^{-1}\hbar\partial_c \\
&\partial_a c = q^{-2}c\partial_a + (1-q^{-2})(a-d)\partial_b \\
&\partial_a d = d\partial_a + (1-q^{-2})(b\partial_b - c\partial_c) - (q-q^{-1})^2(a-d)\partial_d + q(1-q^{-2})^2 + q(1-q^{-2})^2\hbar\partial_a \\
&\partial_b a = q^{-2}a\partial_b + q^{-1}\hbar\partial_b \\
&\partial_b b = q^{-1} + q^{-2}b\partial_b - (1-q^{-2})a\partial_d + q^{-1}\hbar\partial_d \\
&\partial_b c = c\partial_b \\
&\partial_b d = d\partial_b - (q^2-1)c\partial_d \\
&\partial_c a = a\partial_c - (q^2-1)b\partial_d \\
&\partial_c b = b\partial_c \\
&\partial_c c = q^{-3} + q^{-2}c\partial_c + (1-q^{-2})(a-d)\partial_d - (q^{-2}-q^{-4})a\partial_a + (1-q^{-2})^2 b\partial_b + q^{-3}\hbar\partial_a \\
&\partial_c d = q^{-2}d\partial_c + (1-q^{-2})(2-q^2)b\partial_d - (1-q^{-2})b\partial_a + (1-q^{-2})^2 a\partial_c + q^{-3}\hbar\partial_c \\
&\partial_d a = a\partial_d \\
&\partial_d b = q^{-2}b\partial_d \\
&\partial_d c = c\partial_d - (q^{-2}-q^{-4})a\partial_b + q^{-3}\hbar\partial_b \\
&\partial_d d = q^{-3} + q^{-2}d\partial_d + (1-q^{-2})^2 a\partial_d - (q^{-2}-q^{-4})b\partial_b + q^{-3}\hbar\partial_d.
\end{aligned}$$

Thus, the result of applying the partial derivatives to the generators of the algebra $\mathcal{W}(\mathcal{N})$ is (we only exhibit nontrivial terms)

$$\partial_a(a) = q^{-1}, \qquad \partial_b(b) = q^{-1}, \qquad \partial_c(c) = q^{-3}, \qquad \partial_d(d) = q^{-3}.$$

This system becomes much more simple in the limit $q \to 1$:

$$\begin{aligned}
&\partial_a a - a\partial_a = 1 + \hbar\partial_a \qquad \partial_a b - b\partial_a = \hbar\partial_c \\
&\partial_a c - c\partial_a = 0 \qquad \partial_a d - d\partial_a = 0 \\
&\partial_b a - a\partial_b = \hbar\partial_b \qquad \partial_b b - b\partial_b = 1 + \hbar\partial_d \\
&\partial_b c - c\partial_b = 0 \qquad \partial_b d - d\partial_b = 0 \\
&\partial_c a - a\partial_c = 0 \qquad \partial_c b - b\partial_c = 0 \\
&\partial_c c - c\partial_c = 1 + \hbar\partial_a \qquad \partial_c d - d\partial_c = \hbar\partial_c \\
&\partial_d a - a\partial_d = 0 \qquad \partial_d b - b\partial_d = 0 \\
&\partial_d c - c\partial_d = \hbar\partial_b \qquad \partial_d d - d\partial_d = 1 + \hbar\partial_d.
\end{aligned}$$

These are the permutation relations of the partial derivatives with generators of the algebra $U(gl(2)_\hbar)$, which are subject to relations

$$[a,b] = \hbar b, \qquad [a,c] = -\hbar c, \qquad [a,d] = 0, \qquad [b,c] = \hbar(a-d), \qquad [b,d] = \hbar b, \qquad [c,d] = -\hbar c.$$

Also, it follows from (3.3) that all partial derivatives commute with each other. Consequently, they form a commutative subalgebra $\mathcal{D}$ of the algebra $\mathcal{W}(U(gl(2)_\hbar))$.

It is convenient to pass from $gl(2)_\hbar$ to the compact form $u(2)_\hbar$. Introducing new generators as follows

$$t = \frac{1}{2}(a+d), \qquad x = \frac{i}{2}(b+c), \qquad y = \frac{1}{2}(c-b), \qquad z = \frac{i}{2}(a-d)$$

we get the defining relations of the algebra $u(2)_\hbar$

$$[x,y]=\hbar z,\qquad [y,z]=\hbar x,\qquad [z,x]=\hbar y,\qquad [t,x]=[t,y]=[t,z]=0.$$

The change of generators in the partial derivatives is usual: $\partial_t=\partial_a+\partial_d$, etc. However, for the future convenience we prefer using the "shifted" derivative $\tilde{\partial}_t=\partial_t+\frac{2}{\hbar}\mathrm{id}$ instead of ∂_t. The partial derivatives remain commutative, while the permutation relations become

$$\begin{array}{llll}
\tilde{\partial}_t t - t\tilde{\partial}_t=\dfrac{\hbar}{2}\tilde{\partial}_t & \tilde{\partial}_t x - x\tilde{\partial}_t=-\dfrac{\hbar}{2}\partial_x & \tilde{\partial}_t y - y\tilde{\partial}_t=-\dfrac{\hbar}{2}\partial_y & \tilde{\partial}_t z - z\tilde{\partial}_t=-\dfrac{\hbar}{2}\partial_z\\
\partial_x t - t\partial_x=\dfrac{\hbar}{2}\partial_x & \partial_x x - x\partial_x=\dfrac{\hbar}{2}\tilde{\partial}_t & \partial_x y - y\partial_x=\dfrac{\hbar}{2}\partial_z & \partial_x z - z\partial_x=-\dfrac{\hbar}{2}\partial_y\\
\partial_y t - t\partial_y=\dfrac{\hbar}{2}\partial_y & \partial_y x - x\partial_y=-\dfrac{\hbar}{2}\partial_z & \partial_y y - y\partial_y=\dfrac{\hbar}{2}\tilde{\partial}_t & \partial_y z - z\partial_y=\dfrac{\hbar}{2}\partial_x\\
\partial_z t - t\partial_z=\dfrac{\hbar}{2}\partial_z & \partial_z x - x\partial_z=\dfrac{\hbar}{2}\partial_y & \partial_z y - y\partial_z=-\dfrac{\hbar}{2}\partial_x & \partial_z z - z\partial_z=\dfrac{\hbar}{2}\tilde{\partial}_t.
\end{array}\tag{3.4}$$

In the generators $t,x,y,z,\tilde{\partial}_t,\partial_x,\partial_z,\partial_z$, the algebra $\mathcal{W}(U(u(2)_\hbar))$ can be treated as the enveloping algebra of a semi-direct product of the commutative Lie algebra generated by the partial derivatives $\tilde{\partial}_t,\partial_x,\partial_z,\partial_z$ and that $u(2)_\hbar$: the latter algebra acts onto the former one in accordance with formulae (3.4). Then, from the Poincaré–Birkhoff–Witt theorem it follows that the graded algebra $\mathrm{Gr}\mathcal{W}(U(u(2)_\hbar))$ associated with the Weyl algebra $\mathcal{W}(U(u(2)_\hbar))$ is canonically isomorphic to the commutative algebra $\mathrm{Sym}(W)$ of the vector space $W=\mathrm{span}(t,x,y,z,\partial_t,\partial_x,\partial_z,\partial_z)$.

It is not difficult to compute the Poisson brackets arising on the algebra $\mathrm{Sym}(W)$ in the limit $\hbar\to 0$ of the algebra $\mathcal{W}(U(gl(2)_\hbar))$ (with the generator $\tilde{\partial}_t$): it is sufficient to set $\hbar=1$ in formulae (3.4). We denote this Poisson bracket $\{\,,\,\}_1$. Thus we have

$$\{p_t,t\}_1=\frac{p_t}{2},\qquad \{p_x,t\}_1=\frac{p_t}{2}\ \text{etc.}$$

Here we replaced the symbols of the partial derivatives $\{\partial_t,\partial_x,\partial_y,\partial_z\}$ by the corresponding momenta $\{p_t,p_x,p_y,p_z\}$.

Besides, on the algebra $\mathrm{Sym}(W)$ there exists the usual Darboux bracket (denoted $\{\,,\,\}_0$):

$$\{p_t,t\}_0=1,\qquad \{p_t,x\}_0=0\ \text{etc.}$$

The brackets $\{\,,\,\}_i$, $i=0,1$, are compatible with each other. The algebra $\mathcal{W}(U(gl(2)_\hbar))$ (with the generator ∂_t) can be treated as a quantization of their sum.

One more bracket on the algebra $\mathrm{Sym}(W)$ is the semi-classical counterpart of the algebra $\mathcal{W}(\mathcal{M})$. We denote this bracket $\{\,,\,\}_2$. In the coordinates $\{l=a+d,h=a-d,b,c\}$ and the corresponding momenta $\{p_l,p_h,p_b,p_c\}$ the Poisson structure $\{\,,\,\}_2$ is given by the following table

$$\begin{array}{lll}
\{h,b\}_2=-2b(h+l) & \{h,c\}_2=2c(h+l) & \{b,c\}_2=-h(h+l)\\
\{l,h\}_2=0 & \{l,b\}_2=0 & \{l,b\}_2=0\\
\{p_h,p_b\}_2=2p_b(p_h-p_l) & \{p_h,p_c\}_2=-2p_c(p_h-p_l) & \{p_b,p_c\}_2=4p_h(p_h-p_l)\\
\{p_l,p_h\}_2=0 & \{p_l,p_b\}_2=0 & \{p_l,p_c\}_2=0
\end{array}$$

$$\begin{array}{ll}
\{l,p_l\}_2=2+lp_l+hp_h+bp_b+cp_c & \{h,p_l\}_2=lp_h+hp_l+bp_b-cp_c\\
\{b,p_l\}_2=b(p_h-p_l)+\dfrac{1}{2}p_c(h+l) & \{c,p_l\}_2=c(p_h+p_l)+\dfrac{1}{2}p_b(l-h)\\
\{l,p_h\}_2=lp_h+hp_l-bp_b+cp_c & \{h,p_h\}_2=2+hp_h+lp_l+3bp_b-cp_c\\
\{b,p_h\}_2=-b(p_h-p_l)+\dfrac{1}{2}p_c(h+l) & \{c,p_h\}_2=c(p_h+p_l)-\dfrac{1}{2}lp_b-\dfrac{3}{2}hp_b\\
\{l,p_b\}_2=p_b(h+l)-2c(p_h-p_l) & \{h,p_b\}_2=p_b(h+l)+2c(p_h-p_l)\\
\{b,p_b\}_2=2+2bp_b+(h+l)(p_h-p_l) & \{c,p_b\}_2=0\\
\{l,p_c\}_2=p_c(l-h)+2b(p_l+p_h) & \{h,p_c\}_2=2b(p_l-3p_h)-p_c(l-h)\\
\{b,p_c\}_2=0 & \{c,p_c\}_2=2+2cp_c+l(p_l+p_h)-hp_l+3hp_h.
\end{array}$$

Note, that on the complexification of the algebra $\mathrm{Sym}(W)$ the bracket $\{\,,\,\}_2$ is compatible with those $\{\,,\,\}_0$ and $\{\,,\,\}_1$. Thus, we get a pencil generated by three Poisson brackets $\{\,,\,\}_i$, $i=0,1,2$. Whereas, the algebra $\mathcal{W}(\mathcal{N})$ corresponding to the standard Hecke symmetry is a quantum counterpart of this Poisson pencil.

Now, let us turn to the permutation relations (3.4). They enable us to compute similar relations between the partial derivatives and any power of a generator of the algebra $U(u(2)_\hbar)$.

Proposition 8. *For all $k = 0, 1, 2, \ldots$ the following relations hold true*

$$\tilde{\partial}_t t^k = \left(t + \frac{\hbar}{2}\right)^k \tilde{\partial}_t \qquad \tilde{\partial}_t x^k = A_k(x)\tilde{\partial}_t - B_k(x)\partial_x$$
$$\tilde{\partial}_t y^k = A_k(y)\tilde{\partial}_t - B_k(y)\partial_y \qquad \tilde{\partial}_t z^k = A_k(z)\tilde{\partial}_t - B_k(z)\partial_z$$
$$\partial_x t^k = \left(t + \frac{\hbar}{2}\right)^k \partial_x \qquad \partial_x x^k = A_k(x)\partial_x + B_k(x)\tilde{\partial}_t$$
$$\partial_x y^k = A_k(y)\partial_x + B_k(y)\partial_z \qquad \partial_x z^k = A_k(z)\partial_x - B_k(z)\partial_y$$
$$\partial_y t^k = \left(t + \frac{\hbar}{2}\right)^k \partial_y \qquad \partial_y x^k = A_k(x)\partial_y - B_k(x)\partial_z$$
$$\partial_y y^k = A_k(y)\partial_y + B_k(y)\tilde{\partial}_t \qquad \partial_y z^k = A_k(z)\partial_y + B_k(z)\partial_x$$
$$\partial_z t^k = \left(t + \frac{\hbar}{2}\right)^k \partial_z \qquad \partial_z x^k = A_k(x)\partial_z + B_k(x)\partial_y$$
$$\partial_z y^k = A_k(y)\partial_z - B_k(y)\partial_x \qquad \partial_z z^k = A_k(z)\partial_z + B_k(z)\tilde{\partial}_t,$$

where

$$A_k(v) = \frac{1}{2}\left(\left(v - i\frac{\hbar}{2}\right)^k + \left(v + i\frac{\hbar}{2}\right)^k\right), \qquad B_k(v) = \frac{i}{2}\left(\left(v - i\frac{\hbar}{2}\right)^k - \left(v + i\frac{\hbar}{2}\right)^k\right).$$

Proof. The proof can be done by induction in the power k of a monomial. □

The above formulae enable us to compute the permutation relations between partial derivatives and any polynomial or a formal series in one variable.

Corollary 9. *Let $f(v)$ be a polynomial or a formal series in one variable v and A and B be difference operators defined by*

$$A(f(v)) = \frac{f\left(v - i\frac{\hbar}{2}\right) + f\left(v + i\frac{\hbar}{2}\right)}{2}, \qquad B(f(v)) = \frac{if\left(v - i\frac{\hbar}{2}\right) - if\left(v + i\frac{\hbar}{2}\right)}{2}.$$

Then all formulae in Proposition 8 remain valid if we respectively replace the powers v^k where $v \in \{t, x, y, z\}$ for $f(v)$ and the factors $(v + \frac{\hbar}{2})^k$, $A_k(v)$ and $B_k(v)$ by $f(v + \frac{\hbar}{2})$, $A(f(v))$ and $B(f(v))$.

The permutation relations described above enable us to find the result $\partial_v(f)$ of applying a partial derivative ∂_v, $v \in \{t, x, y, z\}$, to any polynomial or a formal series f. According to our general recipe (2.3), we should move the symbol of the partial derivative to the most right position and then apply the relation

$$\partial_v(1_{\mathcal{W}}) = 0$$

motivated by (2.10). For example, we can find

$$\partial_x(x^k) = \frac{2}{\hbar}B_k(x),$$

since $\tilde{\partial}_t(1_{\mathcal{W}}) = 2/\hbar$, etc.

Now, using this method we compute the result of applying a partial derivative to a decomposable element $f = f_0(t)f_1(x)f_2(y)f_3(z)$ of the algebra $U(u(2)_\hbar)$ or its completion $U(u(2)_\hbar)[[\hbar]]$. Here f_i are polynomials or formal series in one variable.

Proposition 10. *The following relations hold true for the element $f = f_0(t)f_1(x)f_2(y)f_3(z)$*

$$\tilde{\partial}_t(f) = 2\,\hbar^{-1} f_0\left(t + \frac{\hbar}{2}\right)(A(f_1)A(f_2)A(f_3) - B(f_1)B(f_2)B(f_3)),$$
$$\partial_x(f) = 2\,\hbar^{-1} f_0\left(t + \frac{\hbar}{2}\right)(B(f_1)A(f_2)A(f_3) + A(f_1)B(f_2)B(f_3)),$$
$$\partial_y(f) = 2\,\hbar^{-1} f_0\left(t + \frac{\hbar}{2}\right)(A(f_1)B(f_2)A(f_3) - B(f_1)A(f_2)B(f_3)),$$
$$\partial_z(f) = 2\,\hbar^{-1} f_0\left(t + \frac{\hbar}{2}\right)(A(f_1)A(f_2)B(f_3) + B(f_1)B(f_2)A(f_3)).$$

Consequently,

$$\partial_t(f) = 2\,\hbar^{-1}\left(f_0\left(t+\frac{\hbar}{2}\right)(A(f_1)A(f_2)A(f_3) - B(f_1)B(f_2)B(f_3)) - f\right).$$

It is interesting to find "eigenfunctions" of all partial derivatives. In the classical case such eigenfunctions are exponential functions. Consider an element

$$f_{\overline{\alpha}} = \exp(\alpha_0 t)\exp(\alpha_1 x)\exp(\alpha_2 y)\exp(\alpha_3 z), \quad \overline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \hbar\mathbb{K}^4, \tag{3.5}$$

belonging to the completion $U(u(2)_\hbar)[[\hbar]]$.

By using the last proposition we get the following.

Proposition 11.

$$\tilde{\partial}_t(f_{\overline{\alpha}}) = 2\,\hbar^{-1}\exp\left(\alpha_0\frac{\hbar}{2}\right)\left(\cos\left(\alpha_1\frac{\hbar}{2}\right)\cos\left(\alpha_2\frac{\hbar}{2}\right)\cos\left(\alpha_3\frac{\hbar}{2}\right) - \sin\left(\alpha_1\frac{\hbar}{2}\right)\sin\left(\alpha_2\frac{\hbar}{2}\right)\sin\left(\alpha_3\frac{\hbar}{2}\right)\right)f_{\overline{\alpha}},$$
$$\partial_x(f_{\overline{\alpha}}) = 2\,\hbar^{-1}\exp\left(\alpha_0\frac{\hbar}{2}\right)\left(\sin\left(\alpha_1\frac{\hbar}{2}\right)\cos\left(\alpha_2\frac{\hbar}{2}\right)\cos\left(\alpha_3\frac{\hbar}{2}\right) + \cos\left(\alpha_1\frac{\hbar}{2}\right)\sin\left(\alpha_2\frac{\hbar}{2}\right)\sin\left(\alpha_3\frac{\hbar}{2}\right)\right)f_{\overline{\alpha}},$$
$$\partial_y(f_{\overline{\alpha}}) = 2\,\hbar^{-1}\exp\left(\alpha_0\frac{\hbar}{2}\right)\left(\cos\left(\alpha_1\frac{\hbar}{2}\right)\sin\left(\alpha_2\frac{\hbar}{2}\right)\cos\left(\alpha_3\frac{\hbar}{2}\right) - \sin\left(\alpha_1\frac{\hbar}{2}\right)\cos\left(\alpha_2\frac{\hbar}{2}\right)\sin\left(\alpha_3\frac{\hbar}{2}\right)\right)f_{\overline{\alpha}},$$
$$\partial_z(f_{\overline{\alpha}}) = 2\,\hbar^{-1}\exp\left(\alpha_0\frac{\hbar}{2}\right)\left(\cos\left(\alpha_1\frac{\hbar}{2}\right)\cos\left(\alpha_2\frac{\hbar}{2}\right)\sin\left(\alpha_3\frac{\hbar}{2}\right) + \sin\left(\alpha_1\frac{\hbar}{2}\right)\sin\left(\alpha_2\frac{\hbar}{2}\right)\cos\left(\alpha_3\frac{\hbar}{2}\right)\right)f_{\overline{\alpha}}.$$

Consequently, in our NC setting the ordered exponents (3.5) are still "eigenfunctions" of all partial derivatives but the corresponding eigenvalues are modified with respect to the classical case.

Now, we are able to define an analog of the de Rham operator on the algebra $U(u(2)_\hbar)$. Let $\bigwedge(u(2))$ be the usual skew-symmetric algebra with four generators dt, dx, dy, dz and $\bigwedge^k(u(2))$ be its degree k homogeneous component. Introduce the space of k-differential forms on $U(u(2)_\hbar)$ by $\bigwedge^k(u(2)) \otimes U(u(2)_\hbar)$. Then define an analog of the de Rham operator on the algebra $U(u(2)_\hbar)$ as follows

$$d(f) = dt\,\partial_t(f) + dx\,\partial_x(f) + dy\,\partial_y(f) + dz\,\partial_z(f),$$

where f be an arbitrary element of this algebra. On the higher order differential forms the action of d is extended in the usual way

$$d(\omega f) = \omega d(f), \qquad \omega \in \bigwedge(u(2)), \qquad f \in U(u(2)_\hbar).$$

Note, that we do not transpose elements of $U(u(2)_\hbar)$ and those of $\bigwedge(u(2))$. Nevertheless, the de Rham operator d is well defined and due to the commutativity of the partial derivatives it is easy to see that $d^2 = 0$, i.e. d is a differential indeed. Emphasize that this way of defining the de Rham operator d does not imply any Leibniz rule for it looking like the classical one: $d(fg) = d(f)g + fd(g)$.

In the same way it is possible to define analogs of differential forms and the de Rham operators for any enveloping algebra $U(u(m|n)_\hbar)$.

Remark 12. As for the de Rham complex over the RE algebras there are different approaches to its construction. One of them based on Koszul type complexes was considered in [15] (also, see the references therein). Another one was developed for the RE algebra playing the role of q-Minkowski space in [8–10]. This approach involves some permutation rules between "functions" and "differentials" and the classical form of the Leibniz rule for the de Rham differential.

This approach fails provided the RE algebra $\mathcal{M}$ is replaced by its modified counterpart $\mathcal{N}$. Hopefully, the method above of constructing the de Rham differential on the algebra $U(u(m|n)_\hbar)$ can be applied mutatis mutandis to the algebra $\mathcal{N}$. We plan to go back to this construction in our subsequent publications.

4. Laplace operator on $U(u(2))$ and its radial part

Consider the algebra $\mathcal{W}(U(u(2)_\hbar))$ in more detail. This algebra is covariant with respect to spacial rotations, i.e. actions of elements of the group $SO(3)$ (we treat x, y, z as "space generators" and t plays the role of the time). Indeed, rewriting the generating matrix (3.1) through the generators t, x, y, z

$$N = \begin{pmatrix} t - iz & -ix - y \\ -ix + y & t + iz \end{pmatrix} \tag{4.1}$$

we can realize an action of the group $SO(3)$ as a similarity transformation

$$N \longrightarrow g^{-1}Ng, \tag{4.2}$$

where g is an element of the group $SU(2)$ arising from the spinor representation of the group $SO(3)$ (though this representation is two-fold it does not affect the result). In the same way the group $SO(3)$ acts on the matrix of partial derivatives (3.2) which in terms of ∂_t, ∂_x, ∂_y, ∂_z is as follows

$$D = \frac{1}{2}\begin{pmatrix} \partial_t + i\partial_z & i\partial_x + \partial_y \\ i\partial_x - \partial_y & \partial_t - i\partial_z \end{pmatrix}.$$

Remark 13. Note that the algebra $\mathcal{W}(U(u(2)_\hbar))$ is not covariant with respect to the boosts.[6] However, since the subalgebra $\mathcal{D} \subset \mathcal{W}(U(u(2)_\hbar))$ formed by partial derivatives is commutative, it can be equipped with any metric, for example, Euclidean or Lorentzian ones. Consequently, we can define the Laplace operator of any signature, for instance (4, 0) or (1, 3) (in the latter case it is called d'Alembertian and is denoted $\Box$). Thus, we can define the Klein–Gordon equation

$$(\Box - m^2)f = 0, \qquad \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

in the classical way but with f being an element of the algebra $U(u(2)_\hbar)$ or its completion and the partial derivatives defined above.

In a similar way we can define analogs of the Dirac and Maxwell operators. Namely, we introduce them by the classical formulae but with a new meaning of the partial derivatives.

In what follows we concentrate ourselves to the Laplace operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

which is invariant with respect to the spacial rotations. In the remaining part of the paper we define and compute its "radial part".

Consider the centre $Z = Z(U(u(2)_h))$ of the algebra $U(u(2)_h)$. It is generated by the elements $\mathrm{Tr}N$, $\mathrm{Tr}N^2$, or, equivalently, by

$$t = \frac{\mathrm{Tr}N}{2} \quad \text{and} \quad \mathrm{Cas} = x^2 + y^2 + z^2 = \frac{1}{4}((\mathrm{Tr}N)^2 - 2\mathrm{Tr}N^2).$$

The latter element is the quadratic Casimir of the algebra $U(su(2)_h)$. More precisely, any element $f \in Z$ is a polynomial in t and Cas. Below we show that

$$\forall f \in Z \Rightarrow \Delta(f) \in Z.$$

This fact motivates the following definition.

Definition 14. The operator Δ restricted to the centre Z is called the radial part of the Laplacian Δ and is denoted Δ_{rad}.

In a similar way we can define the radial part of any linear combination of operators $t^k\mathrm{Cas}^p\partial_t^m\Delta^n$ invariant with respect to rotations.

Note that in the classical case the radial part of the Laplace operator on the vector space $\mathrm{Mat}(m)$ is realized via eigenvalues of symmetric or Hermitian matrices. Fortunately, in the RE algebras (modified or not) and their $q = 1$ counterparts the generating matrices satisfy an analog of the Cayley–Hamilton identity enabling us to introduce quantum version of eigenvalues. Let us exhibit their construction.

The matrix N defined in (4.1) satisfies the Cayley–Hamilton (CH) identity

$$N^2 - (2t + \hbar)N + (t^2 + x^2 + y^2 + z^2 + \hbar t)I = 0,$$

which can be checked by direct calculations.

Let μ_1, μ_2 be the roots of the equation

$$\mu^2 - (2t + \hbar)\mu + (t^2 + x^2 + y^2 + z^2 + \hbar t) = 0.$$

This means that μ_1 and μ_2 satisfy the relations

$$\mu_1 + \mu_2 = 2t + \hbar, \qquad \mu_1\mu_2 = t^2 + x^2 + y^2 + z^2 + \hbar t.$$

Since the coefficients of the CH identity are central, μ_1 and μ_2 belong to the algebraic extension of the centre Z.

6 And we do not know any deformation of the Poincaré group in the spirit of the κ-Poincaré one (see [18]).

Then the quantity $\mathrm{Tr}N^k$, $k \in \mathbb{Z}_+$, can be expressed via these roots by the formula[7]

$$\mathrm{Tr}N^k = \frac{\mu_1 - \mu_2 - \hbar}{\mu_1 - \mu_2}\mu_1^k + \frac{\mu_2 - \mu_1 - \hbar}{\mu_2 - \mu_1}\mu_2^k.$$

In particular,

$$\mathrm{Tr}N = \mu_1 + \mu_2 - \hbar, \qquad \mathrm{Tr}N^2 = \mu_1^2 + \mu_2^2 - \hbar(\mu_1 + \mu_2).$$

Since, on the other hand, $\mathrm{Tr}N = 2t$ and $\mathrm{Tr}N^2 = 2(t^2 - \mathrm{Cas})$, we get

$$t = \frac{\mu_1 + \mu_2 - \hbar}{2}, \qquad \mathrm{Cas} = -\frac{(\mu_1 - \mu_2)^2 - \hbar^2}{4}. \tag{4.3}$$

So, any element of Z, being a polynomial in t and Cas, can be expressed as a *symmetric polynomial* in the roots μ_1 and μ_2.

Our next step consists in computing the quantities

$$\Delta(t^k \mathrm{Cas}^p), \quad k, p \in \mathbb{Z}_+$$

and presenting the result through the roots μ_1, μ_2. To this end we introduce a first order operator $Q = x\partial_x + y\partial_y + z\partial_z$ and consider four second order operators

$$\Delta_0 = \tilde{\partial}_t^2, \qquad \Delta_1 = \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \qquad \Delta_2 = Q\,\tilde{\partial}_t, \qquad \Delta_3 = Q^2.$$

Our immediate aim is to compute permutation relations of all these operators and the elements t^k and Cas^p.

It is not difficult to see that

$$\Delta_i t^k = (t + \hbar)^k \Delta_i, \quad \forall i = 0, 1, 2, 3. \tag{4.4}$$

Thus, we have only to compute the permutation relations of the operators Δ_i and the Casimir element Cas.

Proposition 15. *The following permutation relations hold true:*

$$\Delta_i \mathrm{Cas} = \sum_{j=0}^{3} \Pi_{ij} \Delta_j, \quad 0 \le i \le 3,$$

where the matrix $\Pi = \|\Pi_{ij}\|$ *reads*

$$\Pi = \begin{pmatrix} \mathrm{Cas} - \frac{3}{2}\hbar^2 & \frac{\hbar^2}{2} & -2\hbar & 0 \\ \frac{3}{2}\hbar^2 & \mathrm{Cas} - \frac{\hbar^2}{2} & 2\hbar & 0 \\ \hbar \mathrm{Cas} & 0 & \mathrm{Cas} - \frac{\hbar^2}{2} & -\hbar \\ \hbar^2\,\mathrm{Cas} & -\frac{\hbar^2}{2}\mathrm{Cas} & \hbar\left(2\mathrm{Cas} + \frac{\hbar^2}{4}\right) & \mathrm{Cas} + \frac{\hbar^2}{2} \end{pmatrix}.$$

Proof. can be done by straightforward computations on the base of (3.4). □

Now, we immediately get

$$\Delta_i \mathrm{Cas}^p = \sum_{j=0}^{3} (\Pi^p)_{ij} \Delta_j.$$

In order to get the action of the operators Δ_i we apply the rule of action of the partial derivatives on the unit element $1_{\mathcal{W}}$

$$\Delta_0(1_{\mathcal{W}}) = 4\,\hbar^{-2}, \qquad \Delta_i(1_{\mathcal{W}}) = 0, \quad i = 1, 2, 3.$$

So, we arrive to the following result

$$\Delta_i(\mathrm{Cas}^p) = 4\,\hbar^{-2}(\Pi^p)_{i0}. \tag{4.5}$$

To calculate the p-th power of the matrix Π we substitute the parametrization (4.3) for Cas and calculate the spectrum of Π. By a direct calculation one can verify that the matrix Π is semisimple:

$$\Pi \sim \mathrm{diag}(\lambda_0, \lambda_0, \lambda_+, \lambda_-),$$

[7] This formula can be obtained from its q-analog established in [19] in the limit $q \to 1$.

where the eigenvalues are

$$\lambda_0 = \frac{1}{4}(\hbar^2 - (\mu_1 - \mu_2)^2), \qquad \lambda_\pm = \frac{1}{4}(\hbar^2 - (\mu_1 - \mu_2 \pm 2\hbar)^2).$$

Now, the matrix value $f(\Pi)$ of a function $f(x)$ (provided that $f(x)$ is defined on the spectrum of Π) can be found in terms of the Lagrange–Sylvester polynomial (see [20])

$$f(\Pi) = \sum_{a=0,\pm} f(\lambda_a) \prod_{\substack{b=0,\pm \\ b\neq a}} \frac{(\Pi - \lambda_b I)}{(\lambda_a - \lambda_b)}. \tag{4.6}$$

Taking, in particular, $f(x) = x^p$ we get the following explicit formulae from (4.5)

$$\Delta_0(\mathrm{Cas}^p) = \frac{1}{\hbar^2(\mu_1 - \mu_2)}\Big(2(\mu_1 - \mu_2)\lambda_0^p + (\mu_1 - \mu_2 + 2\hbar)\lambda_+^p + (\mu_1 - \mu_2 - 2\hbar)\lambda_-^p\Big),$$

$$\Delta_1(\mathrm{Cas}^p) = \frac{1}{\hbar^2(\mu_1 - \mu_2)}\Big(2(\mu_1 - \mu_2)\lambda_0^p - (\mu_1 - \mu_2 + 2\hbar)\lambda_+^p - (\mu_1 - \mu_2 - 2\hbar)\lambda_-^p\Big),$$

$$\begin{aligned}\Delta_2(\mathrm{Cas}^p) \;=\; & \frac{1}{2\,\hbar^2(\mu_1 - \mu_2)}\Big(-2\hbar(\mu_1 - \mu_2)\lambda_0^p + ((\mu_1 - \mu_2)^2 + \hbar(\mu_1 - \mu_2) - 2\,\hbar^2)\lambda_+^p \\ & - ((\mu_1 - \mu_2)^2 - \hbar(\mu_1 - \mu_2) - 2\,\hbar^2)\lambda_-^p\Big),\end{aligned}$$

$$\begin{aligned}\Delta_3(\mathrm{Cas}^p) \;=\; & \frac{1}{4\,\hbar^2}\Big((2\,\hbar^2 - (\mu_1 - \mu_2)^2)\lambda 3_0^p + ((\mu_1 - \mu_2)^2 + \hbar(\mu_1 - \mu_2) - 2\,\hbar^2)\lambda_+^p \\ & + ((\mu_1 - \mu_2)^2 - \hbar(\mu_1 - \mu_2) - 2\,\hbar^2)\lambda_-^p\Big).\end{aligned}$$

The second formula of this list allows us to explicitly calculate the radial part Δ_{rad} (recall that $\Delta = \Delta_1$), the radial parts of other operators above can be calculated in the same way. Indeed, as follows from Definition 14, we have to know the result of action of the operator Δ_1 on any symmetric function in μ_1 and μ_2. It is convenient to fix the following set of generators for the ring of such functions:

$$\mu_1 + \mu_2 \quad \text{and} \quad (\mu_1 - \mu_2)^2 = \hbar^2 - 4\mathrm{Cas}.$$

Now, extend the second formula from the list above to any function $g(\mathrm{Cas})$ where g is a polynomial or a formal series in one variable. We have

$$\Delta(g(\mathrm{Cas})) = \frac{1}{\hbar^2(\mu_1 - \mu_2)}\Big(2(\mu_1 - \mu_2)g(\lambda_0) - (\mu_1 - \mu_2 + 2\hbar)g(\lambda_+) - (\mu_1 - \mu_2 - 2\hbar)g(\lambda_-)\Big).$$

Then, by applying this formula to the function $f(\hbar^2 - 4\mathrm{Cas})$ and using (4.5) and (4.6) we finally get

$$\begin{aligned}\Delta_{\mathrm{rad}}\left(f((\mu_1 - \mu_2)^2)\right) \;=\; & \frac{1}{\hbar^2}\Big(2f((\mu_1 - \mu_2)^2) - f((\mu_1 - \mu_2 - 2\hbar)^2) - f((\mu_1 - \mu_2 + 2\hbar)^2)\Big) \\ & + \frac{2}{\mu_1 - \mu_2}\frac{1}{\hbar}\left(f((\mu_1 - \mu_2 - 2\hbar)^2) - f((\mu_1 - \mu_2 + 2\hbar)^2)\right).\end{aligned} \tag{4.7}$$

The action on a function depending on the both generators $\mu_1 + \mu_2$ and $(\mu_1 - \mu_2)^2$ are easily obtained with the help of (4.4). In order to better visualize the final formula we set $\lambda = \mu_1 + \mu_2$ and $\mu = (\mu_1 - \mu_2)^2$. Then we have

$$\begin{aligned}\Delta_{\mathrm{rad}}\,(f(\lambda, \mu)) \;=\; & \frac{1}{\hbar^2}\Big(2f(\lambda + 2\hbar, \mu) - f(\lambda + 2\hbar, \mu + 4\,\hbar^2 + 4\hbar\sqrt{\mu}) - f(\lambda + 2\hbar, \mu + 4\,\hbar^2 - 4\hbar\sqrt{\mu})\Big) \\ & + \frac{2}{\sqrt{\mu}}\frac{1}{\hbar}\left(f(\lambda + 2\hbar, \mu + 4\,\hbar^2 - 4\hbar\sqrt{\mu}) - f(\lambda + 2\hbar, \mu + 4\,\hbar^2 + 4\hbar\sqrt{\mu})\right).\end{aligned}$$

Note that although the function $\mu \to \sqrt{\mu}$ is twofold it does not affect the result (here we are dealing with functions $f(\lambda, \mu)$ which are polynomials or formal series in commutative variables λ and μ).

In the limit $\hbar \to 0$ the difference operator Δ_{rad} turns into the following second order differential operator

$$-16\mu\frac{\partial^2}{\partial\mu^2} - 24\frac{\partial}{\partial\mu}.$$

Being rewritten via the variable r such that $\mu = -4r^2$ we get the usual radial part of the classical Laplacian on $\mathbb{R}^3$

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

By completing the paper we want to design a plan of a possible application of our method to constructing two-parameter deformations of the rational Calogero–Moser models.

Let $\mathcal{N}$ be the standard mRE algebra (it is a braided deformation of the enveloping algebra $U(gl(m)_\hbar)$). Also, let D be the matrix of the partial derivatives on this mRE algebra as introduced in the Section 2. Consider the operators $Tr_{R^{-1}}D^k$, $k = 0, 1, 2, \dots, m$ acting on the algebra $\mathcal{N}$. (Recall that D is the generating matrix of the RE algebra corresponding to the Hecke symmetry R^{-1}.) They commute with each other. Besides, they map the centre $Z = Z(U(gl(m)_\hbar))$ of the algebra $U(gl(m)_\hbar)$ into itself. Consequently, the restriction of the operators $Tr_{R^{-1}}D^k$ to Z is well defined. By expressing these restricted operators via the eigenvalues of the generating matrix N of the algebra $\mathcal{N}$ we get a family of operators in involution. Hopefully, these operators are difference ones and they are two-parameter deformations of the corresponding classical differential operators which are gauge equivalent to the rational Calogero–Moser operator and its higher counterparts respectively. However, computations in higher dimensional case become much harder.

It would be also interesting to apply this method to algebras which are deformations of the super-algebra $U(gl(m|n)_\hbar)$. Note that radial parts of Laplace operators on supergroups have been studied by Berezin (see [11]).

Acknowledgements

This work was partially supported by the joint RFBR and CNRS grant 09-01-93107-NCNIL-a. The work of P.P. and P.S. was partially supported by the RFBR grant 11-01-00980-a and by the Higher School of Economics Academic Fund grants 11-09-0038, 10-01-0013 and 11-01-0042 respectively. D.G. is thankful to S. Meljanac and Z. Škoda for valuable discussions and the franco-croatian cooperation programm Egide PHC Cogito 24829NH for a financial support of his visit to Zagreb University.

References

[1] J. Wess, B. Zumino, Covariant differential calculus on the quantum hyperplane, Nuclear Phys. B Proc. Suppl. 18B (1990) 302–312. 1991.
[2] S. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), Comm. Math. Phys. 122 (1989) 125–170.
[3] L. Faddeev, N. Reshetikhin, L. Takhtadzhyan, Quantization of Lie groups and Lie algebras, Leningrad Math J. 1 (1990) 193–225 (English translation).
[4] A. Isaev, P. Pyatov, Covariant differential calculus complexes on quantum linear groups, J. Phys. A 28 (8) (1995) 2227–2246.
[5] L. Faddeev, P. Pyatov, The differential calculus on quantum linear groups, Amer. Math. Soc. Transl. Ser. 2 175 (1996) 35–47.
[6] D. Gurevich, P. Pyatov, P. Saponov, Representation theory of (modified) reflection equation algebra of the $GL(m \mid n)$ type, St. Petersburg Math. J. 20 (2009) 213–253 (English translation).
[7] D. Gurevich, P. Pyatov, P. Saponov, Braided differential operators on quantum algebras, J. Geom. Phys. 61 (2011) 1485–1501.
[8] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, q-deformed Poincare' algebra, Comm. Math. Phys. 150 (1992) 495–518.
[9] P.P. Kulish, Representations of q-Minkowski space algebra, Algebra Anal. 6 (2) (1994) 195–205 (in Russian); English transl.: St. Petersburg Math. J. 6 (2) (1995), 365–374.
[10] J. de Azcárraga, P. Kulish, F. Ródenas, Reflection equation and q-Minkowski space algebras, Lett. Math. Phys. 32 (3) (1994) 173–182.
[11] F. Berezin, Introduction to Super-Analysis, D. Reider Publishing Company, 1987.
[12] A. Isaev, O. Ogievetsky, P. Pyatov, On quantum matrix algebras satisfying the Cayley–Hamilton–Newton identities, J. Phys. A: Math. Gen. 32 (1999) L115–L121.
[13] A. Isaev, A. Vladimirov, $GL_q(N)$-covariant braided differential bialgebras, Lett. Math. Phys. 33 (1995) 297–302.
[14] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[15] D. Gurevich, P. Saponov, Braided affine geometry and q-analogs of wave operators, J. Phys. A 42 (31) (2009) 51 pp.
[16] D. Gurevich, P. Pyatov, P. Saponov, The Cayley–Hamilton theorem for quantum matrix algebras of $GL(m \mid n)$ type, St. Petersburg Math. J. 17 (2006) 119–135 (English translation).
[17] I. Gelfand, V. Retakh, Quansideterminants. I, Selecta Math. (NS) 3 (1997) 517–546.
[18] S. Majid, H. Ruegg, Bicrossproduct structure of κ-Poincare' group and non-commutative geometry, Phys. Lett. B 334 (1994) 348–354.
[19] D. Gurevich, P. Saponov, Quantum line bundles via Cayley–Hamilton identity, J. Phys. A 34 (2001) 4553–4569.
[20] Gantmacher, The Theory of Matrices, vol. 1, AMS Chelsea Publ., Providence, Rode Island, 2000.