

# On a weight system conjecturally related to $\mathfrak{sl}_2$

E. Kulakova, S. Lando<sup>1</sup>, T. Mukhutdinova<sup>2</sup>, G. Rybnikov  
National Research University Higher School of Economics

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We introduce a new series  $R_k$ ,  $k = 2, 3, 4, \dots$ , of integer valued weight systems. The value of the weight system  $R_k$  on a chord diagram is a signed number of cycles of even length  $2k$  in the intersection graph of the diagram. We show that this value depends on the intersection graph only. We check that for small orders of the diagrams, the value of the weight system  $R_k$  on a diagram of order exactly  $2k$  coincides with the coefficient of  $c^k$  in the value of the  $\mathfrak{sl}_2$ -weight system on the projection of the diagram to primitive elements.

## 1. Introduction

Below, we use standard notions from the theory of finite order knot invariants, see, e.g. [2] or [8].

A *chord diagram* of order  $n$  is an oriented circle endowed with  $2n$  pairwise distinct points split into  $n$  disjoint pairs, considered up to orientation-preserving diffeomorphisms of the circle. A *weight system* is a function on chord diagrams satisfying the 4-term relation, see Fig. 1.

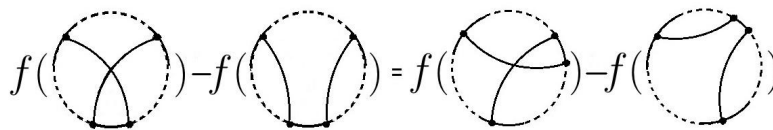


Figure 1: 4-term relation

In figures, the outer circle of the chord diagram is always assumed to be oriented counterclockwise. Dashed arcs may contain ends of arbitrary sets of chords, same for all the four terms in the picture.

A function on chord diagrams satisfying 4-term relations is called a *weight system*. To each weight system, a finite order knot invariant can be associated in a canonical way, which makes studying weight systems an important part of knot theory. There is

a number of approaches to constructing weight systems. In particular, weight systems can be constructed from semisimple Lie algebras, although the result is complicated. The present paper has been motivated by an aspiration for understanding the weight system corresponding to the simplest nontrivial case of the Lie algebra  $\mathfrak{sl}_2$ .

The *intersection graph*  $\gamma(d)$  of a chord diagram  $d$  is the simple graph whose vertices are in one-to-one correspondence with the chords in  $d$ , and two vertices are connected by an edge iff the corresponding chords intersect one another. The *4-term relation for graphs*, introduced in [7], is the graph counterpart of the 4-term relation for chord diagrams. It is defined as follows.

Denote by  $V(\Gamma)$  the set of vertices of a graph  $\Gamma$  and by  $E(\Gamma)$  the set of its edges. Let us associate to each ordered pair of (distinct) vertices  $A, B \in V(\Gamma)$  of a graph  $\Gamma$  two other graphs  $\Gamma'_{AB}$  and  $\tilde{\Gamma}_{AB}$ .

The graph  $\Gamma'_{AB}$  is obtained from  $\Gamma$  by erasing the edge  $AB \in E(\Gamma)$  in the case that this edge exists, and by adding the edge otherwise. In other words, we simply change the adjacency between the vertices  $A$  and  $B$  in  $\Gamma$ . This operation is an analogue of edge deletion, but we prefer to formulate it in a slightly more symmetric way.

The graph  $\tilde{\Gamma}_{AB}$  is obtained from  $\Gamma$  in the following way. For any vertex  $C \in V(\Gamma) \setminus \{A, B\}$  we switch its adjacency with  $A$  to the opposite one if  $C$  is joined with  $B$ , and we do nothing otherwise. All other edges do not change. Note that the graph  $\tilde{\Gamma}_{AB}$  depends not only on the pair  $(A, B)$ , but on the order of vertices in the pair as well. Examples of 4-term relations can be found in Fig. 7.

**Definition 1.1 (4-invariant)** A graph invariant  $f$  is a *4-invariant* if it satisfies the *4-term relation*

$$f(\Gamma) - f(\Gamma'_{AB}) = f(\tilde{\Gamma}_{AB}) - f(\tilde{\Gamma}'_{AB}) \quad (1)$$

for each graph  $\Gamma$  and for any pair  $A, B \in V(\Gamma)$  of its vertices.

Any graph invariant satisfying the 4-term relation (that is, a 4-invariant) determines a weight system [7]: the value of this weight system on a chord diagram is set to be the value of the 4-invariant on the intersection graph of the diagram.

Let  $d$  be a chord diagram. Denote by  $E_l(d)$  the number of circuits of length  $l$  (= edge  $l$ -gons) in the intersection graph of  $d$ . It is well known (see [8], Exercise 6.4.10) that, for  $l \geq 4$ , the parity of the number  $E_l(d)$  is a weight system with values in  $\mathbb{Z}/2\mathbb{Z}$ . However, no integer-valued weight system with the same parity has been known.

In this paper, we construct, for each  $k = 2, 3, 4, \dots$ , an integer-valued weight system  $R_k$  whose value on each chord diagram has the same parity as  $E_{2k}$ ,  $R_k(d) \equiv E_{2k}(d) \pmod{2}$  for any chord diagram  $d$ . This weight system counts circuits of length  $2k$  in

the intersection graph of the diagram with signs depending on the mutual position of the corresponding chords.

For completeness, let us recall the proof of the fact mentioned above. It is worth to be compared with the proof for the integer-valued analogue below.

**Proposition 1.2** *The value  $E_l \pmod 2$  is a weight system for each  $l \geq 4$ .*

**Proof.** The value  $E_l(d)$  obviously depends on the intersection graph of a chord diagram rather than on the diagram itself. Let us extend the function  $E_l$  to arbitrary graphs (that are not necessarily intersection graphs) in the obvious way: let  $E_l(\Gamma)$  be the number of edge  $l$ -gons in  $\Gamma$  having  $l$  pairwise distinct vertices. We are going to prove that  $E_l(\Gamma) \pmod 2$  satisfies the 4-term relation for graphs; the proposition then follows.

Suppose  $\Gamma$  contains an edge  $AB$ . Then for the two terms on the left-hand side of the 4-term relation we have that  $E_l(\Gamma) - E_l(\Gamma'_{AB})$  is the number of edge  $l$ -gons in  $\Gamma$  passing through the edge  $AB$ . Similarly, for the right-hand side,  $E_l(\tilde{\Gamma}) - E_l(\tilde{\Gamma}'_{AB})$  is the number of edge  $l$ -gons in  $\tilde{\Gamma}_{AB}$  passing through  $AB$ . All the  $l$ -gons in  $\Gamma$  passing through  $AB$  contain a chain  $CABD$  and split into three disjoint classes according to the adjacency of the vertices  $C$  and  $D$  to  $A$  and  $B$ :

1. the vertex  $C$  is adjacent to  $B$  and  $D$  is adjacent to  $A$ ;
2. the vertex  $C$  is adjacent to  $B$ , but  $D$  is not adjacent to  $A$ ;
3. the vertex  $C$  is not adjacent to  $B$ .

All edge  $l$ -gons in  $\tilde{\Gamma}_{AB}$  passing through the four points  $A, B, C, D$  admit a similar classification. Now, the  $l$ -gons in  $\Gamma$  belonging to the second class are in one-to-one correspondence with the  $l$ -gons in  $\tilde{\Gamma}_{AB}$  containing the path  $CBAD$ . The edge  $l$ -gons of the third kind are the same in both graphs  $\Gamma$  and  $\tilde{\Gamma}_{AB}$ . And, finally, the edge  $l$ -gons of the first kind in each of the two graphs come in pairs: the chain  $CABD$  can be replaced with the chain  $CBAD$ . Hence, the number of edge  $l$ -gons of the first type is even for each of the two graphs, and the required assertion follows.  $\square$

The authors are grateful to the participants of the seminar “Combinatorics of finite order knot invariants” at the Department of mathematics NRU HSE for useful discussions.

## 2. Definition of the weight systems $R_k$

### 2.1. Positively and negatively oriented $2k$ -gons

In order to define the weight system  $R_k$ , let us take a chord diagram  $d$  and choose an arbitrary orientation of its chords. This orientation induces an orientation of the

edges of the intersection graph  $\gamma(d)$  in the following way. We orient an edge  $AB$  from  $A$  to  $B$  if the beginning of the chord  $B$  belongs to the arc of the outer circle of  $d$  which starts at the beginning of  $A$  and goes in the positive direction to the end of  $A$ , see Fig. 2. We denote the directed intersection graph of an oriented chord diagram  $d$  by  $\gamma(d)$ , like in the case of ordinary intersection graph of a non-oriented chord diagram, since this convention causes no misunderstanding. Oriented edges in a directed graph will be also called *arrows*.

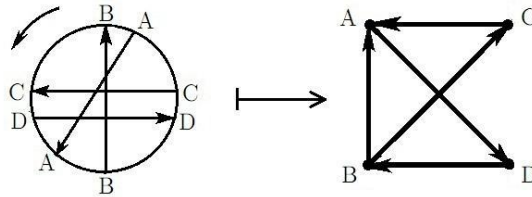


Figure 2: An oriented chord diagram and the corresponding directed intersection graph; the 4-circuit  $ADBC$  is oriented positively

We say that a circuit of even length  $l = 2k$  in a directed graph is *positively oriented* (or its sign is  $+$ ) if the number of arrows in this circuit oriented in either direction is even; otherwise the circuit is *negatively oriented* (or has the sign  $-$ ). Since the total number of arrows is even, the sign is independent of the choice of the direction of the circuit. For example, all the arrows in the 4-circuit  $ADBC$  in the directed intersection graph in Fig. 2 are oriented in the same direction, meaning it is positive.

**Definition 2.1** The function  $R_k$  takes a chord diagram  $d$  to the difference between the number of positively and negatively oriented  $2k$ -gons in the directed intersection graph  $\gamma(d)$  for arbitrarily chosen orientation of the chords in  $d$ .

Now we are going to show that this definition makes sense.

**Proposition 2.2** *The function  $R_k$  is well defined, that is, its value on a chord diagram does not depend on the chosen orientation of the chords.*

**Proof.** It suffices to check that if we switch the direction of a single chord in a chord diagram  $d$  to the opposite one, then the sign of each circuit remains the same. Changing the direction of a chord  $A$  in  $d$  leads to switching directions of all the arrows in  $\gamma(d)$  incident to the vertex  $A$ , preserving the directions of all other arrows. Hence, the sign of any circuit not containing the vertex  $A$  remains the same. Any circuit passing through  $A$  contains exactly two arrows incident to  $A$ . After switching the direction of both of them, the sign also remains the same.  $\square$

**Example 2.3** Let us compute the value of  $R_2$  on the chord diagram  $d$  shown in Fig. 3. Due to the Proposition above, an arbitrary orientation of the chords can be chosen. For the one in the figure, the directed intersection graph  $\gamma(d)$  is shown in Fig. 3. Figure 3 also shows all the oriented 4-gons in the intersection graph. The difference between the number of positively and negatively oriented 4-gons is  $2 - 1 = 1 = R_2(d)$ .

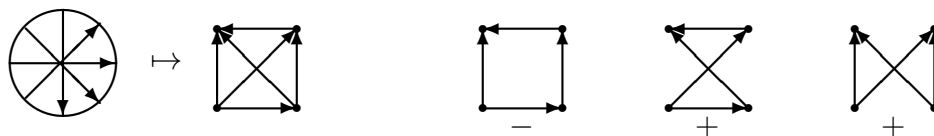


Figure 3: An oriented chord diagram, its directed intersection graph, and all the 4-gons in it, with signs indicated

## 2.2. Proof of the fact that $R_k$ is a weight system

**Theorem 2.4** *The function  $R_k$  on chord diagrams is indeed a weight system, that is, it satisfies the 4-term relation.*

**Proof.** The proof reminds that of Proposition 1.2, but is slightly more complicated. Let  $d$  be a chord diagram, let  $A$  and  $B$  be a pair of chords in  $d$  having neighboring ends and intersecting one another. Pick an orientation of the chords in  $d$  such that the arrow  $AB$  in the directed graph  $\gamma(d)$  of  $d$  is oriented from  $A$  to  $B$ .

The difference  $R_k(d) - R_k(d'_{AB})$  counts the number of signed circuits of length  $2k$  in  $\gamma(d)$  containing the edge  $AB$ .

Each such circuit contains a sequence of vertices  $CABD$ . Let us split the set of such circuits into the following three groups:

1. the vertex  $C$  is adjacent to  $B$  and  $D$  is adjacent to  $A$ ;
2. the vertex  $C$  is adjacent to  $B$ , but  $D$  is not adjacent to  $A$ ;
3. the vertex  $C$  is not adjacent to  $B$ .

All edge  $2k$ -gons in  $\widetilde{\gamma(d)}_{AB}$  passing through the four points  $A, B, C, D$  admit a similar classification.

Now, the  $2k$ -gons in  $\gamma(d)$  belonging to the second class are in one-to-one correspondence with the  $2k$ -gons in  $\widetilde{\gamma(d)}_{AB}$  containing the path  $CBAD$ : just preserve the other edges in each  $2k$ -gon. In order to show that the signs of the corresponding circuits in both directed graphs coincide, it suffices to consider all possible mutual positions of the four chords  $A, B, C, D$  in the chord diagram  $d$ . There are essentially

two different such positions, depending on whether the chords  $C$  and  $D$  intersect one another. In both cases, for a chosen orientation of the chords, we easily check coincidence of the signs of the corresponding  $2k$ -gons.

The edge  $2k$ -gons of the third kind are the same in both graphs  $\gamma(d)$  and  $\tilde{\gamma}(d)_{AB}$ .

Finally, the edge  $2k$ -gons of the first kind in each of the two directed intersection graphs come in pairs: the chain  $CABD$  can be replaced with the chain  $CBAD$ . In order to prove that the signs of the two  $2k$ -gons in a pair are opposite, it suffices to consider all possible mutual positions of the four chords  $A, B, C, D$  in the chord diagram  $d$ .

Hence, the required assertion follows. □

### 3. $R_k$ and intersection graphs

Different chord diagrams can have one and the same intersection graph. Below, we show that the value of the weight system  $R_k$  depends on the intersection graph of the chord diagram rather than on the diagram itself. A natural question then arises, whether  $R_k$  can be extended to a 4-invariant of graphs. We show that this is true for  $k = 2$  and  $k = 3$ ; however, for arbitrary  $k$  the question remains open.

#### 3.1. Proof of the fact that $R_k$ depends on the intersection graph only

**Theorem 3.1** *The weight system  $R_k$  acquires the same value on any two chord diagrams with coinciding intersection graphs.*

Since weight systems taking the same values on chord diagrams with coinciding intersection graphs are in one-to one correspondence with finite order knot invariants not distinguishing mutant knots [3], we conclude that

**Corollary 3.2** *The canonical knot invariant associated to the weight system  $R_k$  does not distinguish mutant knots.*

The proof of the theorem is based on a statement in [3] giving a complete description of the situations where two chord diagrams have the same intersection graph. We start with the definition of a share.

**Definition 3.3** *A share is a part of a chord diagram consisting of two arcs of the outer circle possessing the following property: each chord one of whose ends belongs to these arcs has both ends on these arcs.*

The complement of a share also is a share. The whole chord diagram is its own share whose complement contains no chords.

**Definition 3.4** A *mutation of a chord diagram* is another chord diagram obtained by a rotation/reflection of a share, see Fig. 4.

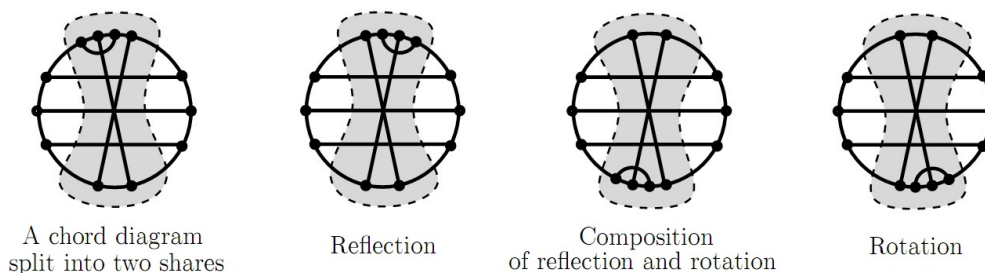


Figure 4: Mutations of a chord diagram

Obviously, mutations preserve the intersection graphs of chord diagrams. We call the subgraph of the intersection graph induced by the vertices corresponding to chords forming a given share also a share in the intersection graph.

**Theorem 3.5** [3] *Two chord diagrams have the same intersection graph if and only if they are related by a sequence of mutations.*

**Proof** of Theorem 3.1. We are going to prove that a mutation of a chord diagram does not change the value of the weight system  $R_k$ , for arbitrary  $k = 2, 3, 4, \dots$ . It suffices to consider only reflection and rotation of a share; for the composition of reflection and rotation, the result will follow automatically.

Pick an arbitrary orientation of chords in a chord diagram. A mutation of a chord diagram with oriented chords produces a chord diagram with oriented chords in a natural way. For a given share in the intersection graph, its reflection results in inverting the orientation of all the arrows belonging to the corresponding share in the intersection graph. Similarly, rotation of a share results in inverting all the arrows between the two shares in the intersection graph.

Denote the two sets of vertices belonging to the complementary shares in the intersection graph by  $U$  and  $W$ , respectively. The set  $U$  contains a subset  $u \subset U$  and the set  $W$  contains a subset  $w \subset W$  such that

- any vertex from  $u$  is connected to all vertices of  $w$  (and vice versa);

- each edge connecting vertices from  $U$  and  $W$  connects, in fact, a vertex from  $u$  to a vertex from  $w$ .

Denote the set of all edges in the intersection graph connecting a vertex from  $u$  to a vertex from  $w$  by  $K(u, w)$ . The graph with the set of vertices  $u \sqcup w$  and the set of edges  $K(u, w)$  is the complete bipartite graph with the parts  $u, w$ .

Any circuit in the intersection graph contains an even number of edges from  $K(u, w)$ . Indeed, any path starting in  $U$  switches between  $U$  and  $W$  after each passing through an edge in  $K(u, w)$ . Since a circuit returns to the original vertex, the number of such passings must be even.

Pick an arbitrary orientation of the chord diagram such that all the arrows in  $K(u, w)$  are oriented from  $U$  to  $W$ . Such an orientation always exists. Indeed, if the share  $U$  consists of two arcs  $U_1, U_2$  and the share  $W$  consists of two arcs  $W_1, W_2$  following along positive direction of the outer circle in the alternating order,  $U_1, W_1, U_2, W_2$ , see Fig. 5, then it suffices to orient all the chords connecting  $U_1, U_2$  and  $W_1, W_2$  from  $U_1$  to  $U_2$  and from  $W_1$  to  $W_2$ , respectively.

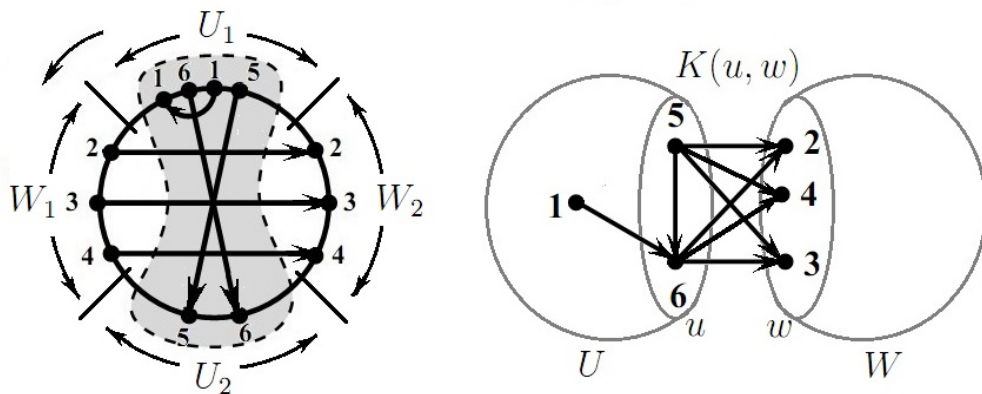


Figure 5: Distinguished orientation of chords allowing for a specific orientation of share connecting arrows in the directed intersection graph

Now, the rotation mutation changes the orientation of all arrows in  $K(u, w)$ , whence of an even number of arrows in any circuit. In particular, it changes orientation of an even number of arrows in any  $2k$ -gon, hence preserving its sign.

What happens if we reflect the share  $W$ ? If the  $2k$ -gon we consider has even number of arrows in the share  $W$ , then its sign remains the same. The circuits with odd number of arrows in  $W$  split into pairs in the following way. The circuit intersects the set  $K(u, w)$  by even number of arrows. Let us split these arrows into pairs: two



edges belong to one and the same pair iff their ends in the share  $W$  are connected by a path that is a part of the circuit totally contained in  $W$ . For such pair of arrows, let the first one be  $u_1w_1$  and the second one be  $u_2w_2$ ,  $u_i \in U$ ,  $w_i \in W$ ,  $i = 1, 2$  (it may well happen that either  $w_1 = w_2$  or  $u_1 = u_2$ , but not both).

By *switching pairs* we mean replacing each pair of edges  $u_1w_1, u_2w_2$ , with the pair  $u_1w_2, u_2w_1$ , see Fig. 6.

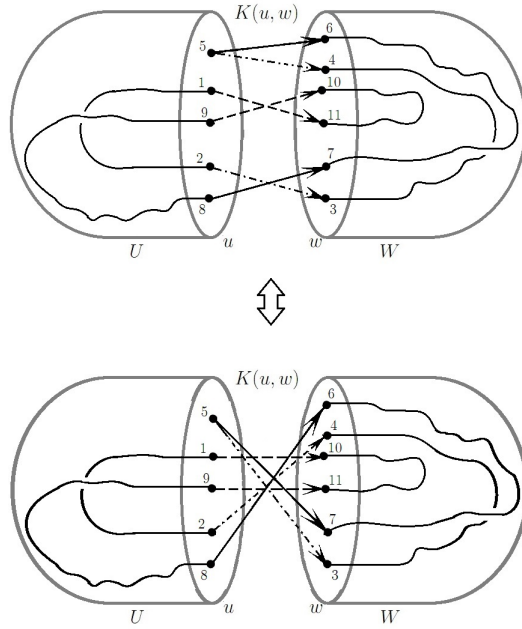


Figure 6: Two circuits obtained from one another by switching pairs

Switching all pairs for a circuit produces a new circuit. This transformation is an involution: switching all pairs in this new circuit restores the original one. Thus, all circuits are split into pairs. We are going to show that any two circuits of even length  $2k$  having odd number of arrows in  $W$  and belonging to the same pair have opposite signs. Indeed, if we choose directions in both circuits that coincide inside the share  $U$ , then they are opposite inside the share  $W$ . Both circuits have the same number of edges inside  $K(u, w)$  in either direction. Therefore, the only difference in the orientation is inside the share  $W$ , where each arrow has opposite directions in the two circuits. Since the number of arrows in  $W$  is odd, the signs of the two circuits are opposite.

Thus, both kinds of mutation preserve the value of  $R_k$ .  $\square$

### 3.2. Extending $R_k$ to arbitrary graphs

Theorem 3.1 above shows that each weight system  $R_k$  defines a function on intersection graphs; we denote this function by  $R_k$  as well. It remains unclear, however, whether this function can be extended to arbitrary graphs in such a way that the extension is a 4-invariant of graphs.

In this section we prove the following statement.

**Proposition 3.6** *The functions  $R_2$  and  $R_3$  can be extended to a 4-invariant of graphs.*

**Proof.** In order to prove that a function  $R_k$  can be extended to a 4-invariant of graphs, it suffices to prove this for graphs with exactly  $2k$  vertices. Indeed, denote by  $\overline{R}_k$  the function whose values on graphs with  $2k$  vertices coincides with the constructed extension, and which vanishes on all other graphs. Then the convolution  $\overline{R}_k * U$  provides us with the desired extension to arbitrary graphs; here  $U$  is the function whose value on arbitrary graph is equal to 1. It is obvious that when restricted to intersection graphs,  $\overline{R}_k * U$  coincides with  $R_k$ . Here we make use of the fact that the vector space spanned by graphs modulo 4-term relations carries a natural graded bialgebra structure; the convolution product is induced on the dual space of 4-invariants from the product on the space of graphs [7].

If  $k = 2$ , then each graph with  $2k = 4$  vertices is an intersection graph, and any 4-term relation for intersection graphs has a chord diagram counterpart, meaning we are done.

For  $k = 3$ , there are two graphs that are not intersection graphs, namely, the 5-wheel and the 3-prism, see Fig. 7. There are three ways to express the 5-wheel as a linear combination of intersection graphs through the 4-term relation, and for the 3-prism there are two such ways, see Fig. 7. A direct computation shows that in both cases all the different representations provide the same value, which one can admit for the value of the extended graph invariant  $R_3$ : for the 5-wheel it is  $-3$ , and for the 3-prism it is  $-1$ . Now, a direct verification, taking into account those 4-term relations for graphs that do not have chord diagram counterparts, ensures that thus extended invariant of graphs with 6 vertices is indeed a 4-invariant.  $\square$

## 4. $R_k$ and the $\mathfrak{sl}_2$ -weight system

It is well known that for an arbitrary Lie algebra  $\mathfrak{g}$  endowed with a nondegenerate invariant scalar product  $(\cdot, \cdot)$  one can construct a weight system  $b^\mathfrak{g}$  with values in the center  $ZU(\mathfrak{g})$  of the universal enveloping algebra  $U(\mathfrak{g})$ . Invariance of the scalar product means that  $(x, [y, z]) = ([x, y], z)$  for any three elements  $x, y, z \in \mathfrak{g}$ .

The 3-dimensional Lie algebra  $\mathfrak{sl}_2$  (taken, for definiteness, over the field  $\mathbb{C}$  of complex numbers) provides the first nontrivial example of this construction. Since the center  $ZU(\mathfrak{sl}_2)$  of the universal enveloping algebra  $U(\mathfrak{sl}_2)$  is isomorphic to the ring of polynomials in the Casimir element  $c$  of  $\mathfrak{sl}_2$ , we obtain a weight system with values in the ring  $\mathbb{C}[c]$  of polynomials in a single variable  $c$ . We denote this weight system by  $b^{\mathfrak{sl}_2} : \mathcal{M} \rightarrow \mathbb{C}[c]$ . Here  $\mathcal{M}$  denotes the graded vector space spanned by chord diagrams modulo the 4-term relations;

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots,$$

where the finite dimensional vector space  $\mathcal{M}_n$  is spanned by chord diagrams with  $n$  chords, modulo the 4-term relations. The space  $\mathcal{M}$  is a graded commutative cocommutative Hopf algebra, with a multiplication  $m : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}$ , and a comultiplication  $\mu : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}$ . The weight system  $b^{\mathfrak{sl}_2}$  is an algebra homomorphism. It takes an arbitrary chord diagram with  $n$  chords to a polynomial of degree  $n$  in  $c$  with the leading coefficient 1.

The invariant nondegenerate scalar product  $(\cdot, \cdot)$  on  $\mathfrak{sl}_2$  can be chosen in a unique way up to a nonzero multiplicative constant. The choice of the multiplicative constant affects non-leading coefficients of the value of  $b^{\mathfrak{sl}_2}$  on a chord diagram. Below, we use the normalization chosen in [8], which differs from that in [4] and [2].

Being a graded commutative cocommutative Hopf algebra,  $\mathcal{M}$  is generated by its primitive elements. Recall that an element  $p$  of a Hopf algebra is said to be *primitive* if  $\mu(p) = 1 \otimes p + p \otimes 1$ . Primitive elements form a graded vector subspace  $\mathcal{PM}$  of  $\mathcal{M}$ ,

$$\mathcal{PM} = \mathcal{PM}_1 \oplus \mathcal{PM}_2 \oplus \mathcal{PM}_3 \oplus \dots, \quad \mathcal{PM}_n \subset \mathcal{M}_n.$$

For each  $n = 1, 2, 3, \dots$ , there exists a natural projection  $\pi_n : \mathcal{M}_n \rightarrow \mathcal{PM}_n$  of the space of chord diagrams of order  $n$  to the subspace  $\mathcal{PM}_n$  of primitive elements along the subspace  $\mathcal{DM}_n$  spanned by decomposable chord diagrams. A chord diagram with  $n$  chords is said to be *decomposable* if it can be represented as a product of two chord diagrams, each with less than  $n$  of chords.

The following statement is an immediate consequence of a result obtained in [4] for an arbitrary Lie algebra.

**Proposition 4.1** *The value of the weight system  $b^{\mathfrak{sl}_2}$  on an arbitrary element in  $\mathcal{PM}_n$  is a polynomial of degree at most  $n/2$  in  $c$ .*

Now we are able to formulate the conjecture mentioned in the introduction, which relates the weight systems  $R_k$  and  $b^{\mathfrak{sl}_2}$ .

**Conjecture 4.2** *For any chord diagram  $d$  with  $2k$  chords, the coefficient of  $c^k$  in the value  $b^{\mathfrak{sl}_2}(\pi_{2k}(d))$  is  $2R_k(d)$ .*

We did not manage to prove the Conjecture, but the computer experiments show that it is true up to  $k = 4$ . Below, we present a table, which contains the values of the weight systems  $R_k(\cdot)$  and  $b^{\mathfrak{sl}_2}(\pi_{2k}(\cdot))$  for some chord diagrams with  $2k$  chords.

The original definition of a weight system  $b^{\mathfrak{g}}$  is too cumbersome from the computational point of view, which makes it difficult to be computed even for chord diagrams with very few chords. The computations we made used the recurrence relation for  $b^{\mathfrak{sl}_2}$  obtained in [4]:

**Proposition 4.3** *If a chord diagram contains a leaf, that is, a chord intersecting only one other chord, then the value of the  $\mathfrak{sl}_2$  universal weight system on the diagram is  $(c - 1)$  times its value on the result of deleting the leaf. In addition, the value of the weight system  $b^{\mathfrak{sl}_2}$  on a chord diagram satisfies the recurrence relations shown in Fig. 8.*

The projection to the subspace of primitive elements along the subspace of decomposable elements has the following form.

**Proposition 4.4** [6] *The projection  $\pi_n$  takes a chord diagram  $d$  with  $n$  chords to the linear combination*

$$\pi_n : d \mapsto d - 1! \sum_{I \sqcup J = V(d)} d|_I d|_J + 2! \sum_{I \sqcup J \sqcup K = V(d)} d|_I d|_J d|_K - \dots,$$

where the summations run over all unordered partitions of the set  $V(d)$  of the chords in  $d$  into disjoint union of nonempty subsets, and  $d|_I$  is the chord subdiagram induced from  $d$  by the subset  $I \subset V(d)$  of chords.

Here is more evidence supporting the conjecture:

1. the weight system  $b^{\mathfrak{sl}_2}$ , similarly to the weight system  $R_k$ , depends on the intersection graphs of the chord diagram rather than on the diagram itself, see [3];
2. the value  $R_k(d_1 d_2)$  on a product of two nontrivial diagrams with  $2k$  chords in total is 0, since there are no  $2k$ -gons in the intersection graph; on the other hand,  $\pi_{2k}(d_1 d_2) = 0$ , since the diagram  $d_1 d_2$  is decomposable;
3. for a chord diagram  $d$  with  $2k$  chords having a leaf (a chord intersecting only one other chord),  $R_k(d) = 0$ ; it can also be easily proved that the degree of the polynomial  $b^{\mathfrak{sl}_2}(\pi_{2k}(d))$  is less than  $k$ .




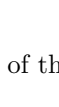



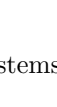

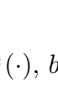

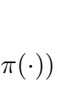

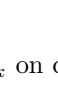
$k$	$d \in \mathcal{M}_{2k}$	$b^{sl_2}(d), b^{sl_2}(\pi_{2k}(d))$	$\gamma(d)$	$R_k(d)$
2		$\frac{c(c-1)^2(c-2)}{-2c}$		0
2		$\frac{c^4 - 4c^3 + 8c^2 - 4c}{2c^2 - 4c}$		1
2		$\frac{c^4 - 5c^3 + 10c^2 - 5c}{2c^2 - 5c}$		1
2		$\frac{c^4 - 6c^3 + 13c^2 - 7c}{2c^2 - 7c}$		1
3		$\frac{c^6 - 6c^5 + 15c^4 - 18c^3 + 18c^2 - 8c}{2c^3 + 3c^2 - 8c}$		1
3		$\frac{c^6 - 8c^5 + 28c^4 - 57c^3 + 66c^2 - 27c}{20c^2 - 27c}$		0
3		$\frac{c^6 - 15c^5 + 115c^4 - 430c^3 + 657c^2 - 295c}{16c^3 - 284c^2 + 295c}$		8

Table 1: Values of the weight systems  $b^{sl_2}(\cdot)$ ,  $b^{sl_2}(\pi(\cdot))$ ,  $R_k$  on certain chord diagrams

**Remark 4.5** The same argument as in the proof of Proposition 3.6 allows one to extend the value of the  $b^{sl_2}$  weight system to a 4-invariant of graphs with up to 6 edges. Namely, we can set the value of this invariant on the 5-wheel to be equal to  $c^6 - 10c^5 + 50c^4 - 139c^3 + 176c^2 - 72c$ , with the projection to the subspace of primitive elements  $-6c^3 + 70c^2 - 72c$ ; and on the 3-prism to be equal to  $c^6 - 9c^5 + 40c^4 - 108c^3 + 146c^2 - 63c$ , with the projection to the subspace of primitive elements  $-2c^3 + 58c^2 - 63c$ .

- [1] Bar-Natan D., *On Vassiliev knot invariants*, Topology, 1995, vol. **34**, no. 2, 423–472.
- [2] S. Chmutov, S. Duzhin, Y. Mostovoy, *Introduction to Vassiliev Knot Invariants*, Cambridge University Press, 2012 <http://www.pdmi.ras.ru/~duzhin>
- [3] S. V. Chmutov, S. K. Lando, *Mutant knots and intersection graphs*, Algebraic and Geometric Topology, vol. 7, 101–120 (2007)
- [4] Chmutov S.V., Varchenko A.N., *Remarks on the Vassiliev knot invariants coming from  $sl_2$* , Topology, 1997, vol. **36**, 153–178.
- [5] Kontsevich M., *Vassiliev knot invariants*, in: Adv. in Soviet Math., 1993, vol. **16**, part 2, 137–150.
- [6] Lando S. K., *On primitive elements in the bialgebra of chord diagrams*, in: Amer. Math. Soc. Transl. Ser. 2, AMS, Providence RI, 1997, vol. **180**, 167–174.
- [7] Lando S. K., *On a Hopf algebra in graph theory*, J. Comb. Theory, Ser. B, 2000, vol. **80**, 104–121
- [8] S. Lando, A.Zvonkin, *Graphs on surfaces and their applications* (Chapter 6), Springer, 2004

National Research University Higher School of Economics  
7 Vavilova Moscow 117312 Russia

E. Kulakova	lenaetopena@gmail.com
S. Lando	lando@hse.ru
T. Mukhutdinova	kassalanche@gmail.com
G. Rybnikov	grigory.rybnikov@gmail.com

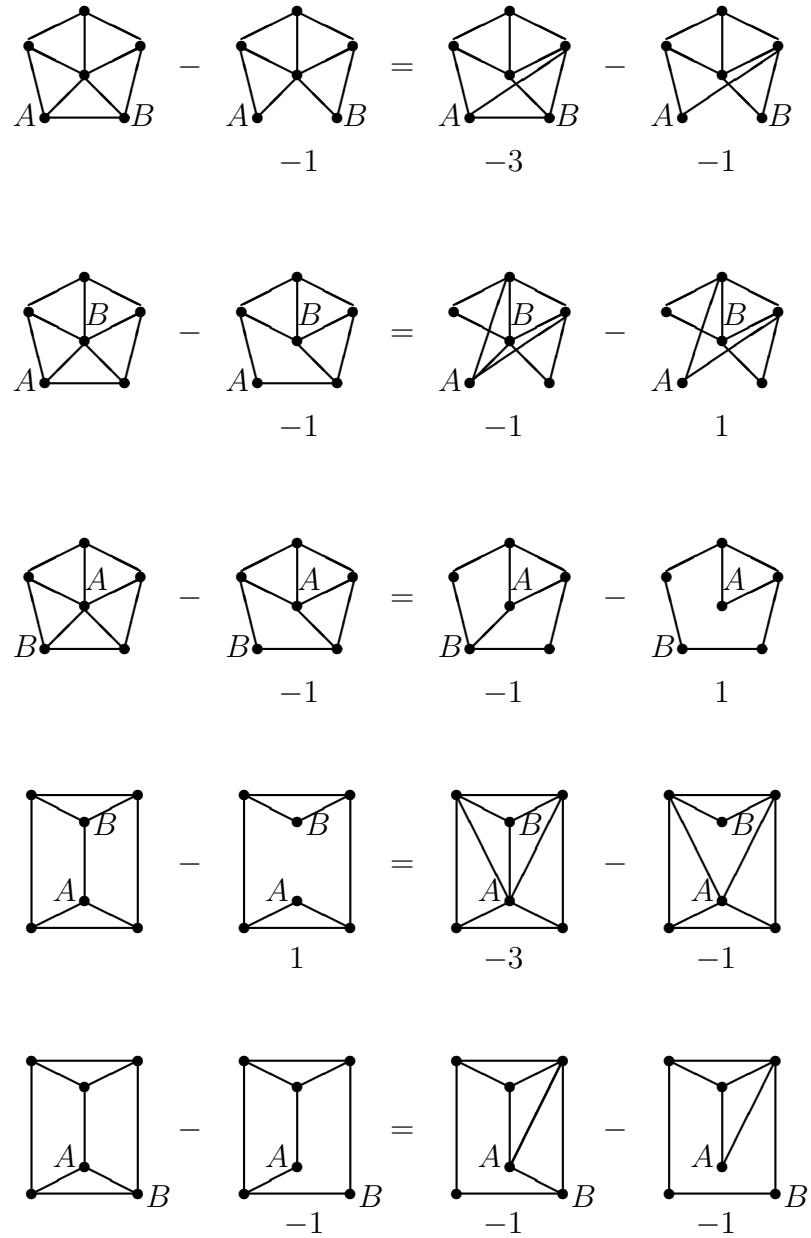


Figure 7: 4-term relations for the 5-wheel and the prism expressing them in terms of intersection graphs; the value of the weight system  $R_3$  is indicated

$$\begin{aligned}
& b^{\mathfrak{sl}(2)} \left( \text{Diagram 1} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 2} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 3} \right) + b^{\mathfrak{sl}(2)} \left( \text{Diagram 4} \right) \\
&= b^{\mathfrak{sl}(2)} \left( \text{Diagram 5} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 6} \right); \\
& b^{\mathfrak{sl}(2)} \left( \text{Diagram 7} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 8} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 9} \right) + b^{\mathfrak{sl}(2)} \left( \text{Diagram 10} \right) \\
&= b^{\mathfrak{sl}(2)} \left( \text{Diagram 11} \right) - b^{\mathfrak{sl}(2)} \left( \text{Diagram 12} \right).
\end{aligned}$$

Figure 8: The recurrence relation for the complete  $\mathfrak{sl}(2)$  weight system