J. London Math. Soc. (2) 89 (2014) 762–778
doi:10.1112/jlms/jdt081

Infinite transitivity on universal torsors

Ivan Arzhantsev, Alexander Perepechko and Hendrik Süß

Abstract

Let X be an algebraic variety covered by open charts isomorphic to the affine space and $q : \hat{X} \to X$ be the universal torsor over X. We prove that the special automorphism group of the quasi-affine variety $\hat{X}$ acts on $\hat{X}$ infinitely transitively. Also we find wide classes of varieties X admitting such a covering.

Introduction

The aim of this paper is to investigate the infinite transitivity property for the special automorphism group of universal torsors over smooth algebraic varieties.

Recall that an action of a group G on a set Y is m-transitive if it is transitive on m-tuples of pairwise distinct points in Y, and is infinitely transitive if it is m-transitive for all positive integers m. If Y is an algebraic variety, let us denote by $\mathrm{SAut}(Y)$ the group of special automorphisms of Y, that is, the subgroup of the automorphism group $\mathrm{Aut}(Y)$ generated by all one-parameter unipotent subgroups. Assume that Y is an affine irreducible variety of dimension at least 2. It is proved in [**5**] that transitivity of the action of $\mathrm{SAut}(Y)$ on the smooth locus Y_{reg} implies infinite transitivity of this action. Moreover, these conditions are equivalent to flexibility of Y, which is a local property formulated in terms of velocity vectors to orbits of one-parameter unipotent subgroups.

The study of flexible varieties is important for several reasons. The infinite transitivity property shows that the group of (special) automorphisms is 'large' in this case. It may indicate that these varieties are the most interesting ones from the geometric point of view. In the recent paper by Bogomolov, Karzhemanov and Kuyumzhiyan [**11**] the connection between flexibility and unirationality is investigated. It is proved in [**5**] that every flexible variety is unirational. As a result in the opposite direction, it is conjectured in [**11**] that any unirational variety is stably birational to some infinitely transitive variety. This conjecture is confirmed in [**11**] for several important cases.

It is easy to show that the affine space $\mathbb{A}^n$ is flexible. Kaliman and Zaidenberg [**23**] proved that any hypersurface in $\mathbb{A}^{n+2}$ given by equation $uv = f(x_1, \ldots, x_n)$ with a non-constant polynomial f has the infinite transitivity property. More generally, if X is a flexible affine variety, then the suspension over X, that is, a hypersurface in $\mathbb{A}^2 \times X$ given by equation $uv = f(x)$, is flexible as well [**7**]. Also it is shown in [**7**] that any non-degenerate affine toric variety is flexible. By [**5**], flexibility holds for affine homogeneous spaces of semisimple algebraic groups and total spaces of vector bundles over flexible varieties. In [**28**], flexibility is established for affine cones over some del Pezzo surfaces. Some other examples of flexible varieties can be found in [**5**, **6**].

Received 9 February 2013; revised 8 November 2013; published online 25 March 2014.

2010 *Mathematics Subject Classification* 14M25, 14R20 (primary), 14J50, 14L30 (secondary).

The first and the second authors were partially supported by the Ministry of Education and Science of Russian Federation, project 8214, by RFBR grants 12-01-00704, 12-01-31342 a, and by the Simons-IUM Grant. The first author was also supported by the Dynasty Foundation. The third author was supported by a fellowship of the Alexander von Humboldt Foundation.

In general, one would like to have geometric constructions of flexible varieties. In this paper, we prove that the universal torsor over a variety admitting a covering by affine spaces leads to a flexible variety and thus we obtain a wide class of quasi-affine varieties with the infinite transitivity property.

Let X be a smooth algebraic variety. Assume that the divisor class group $\mathrm{Cl}(X)$ is a lattice of rank r. The universal torsor $q : \hat{X} \to X$ is a locally trivial H-principal bundle with certain characteristic properties, where H is an algebraic torus of dimension r; see [**32**, Section 1]. Here $\hat{X}$ is a smooth quasi-affine algebraic variety.

Universal torsors were introduced by Colliot-Thélène and Sansuc in the framework of arithmetic geometry to investigate rational points on algebraic varieties; see [**13**, **14**, **32**]. In recent years they were used to obtain positive results on Manin's Conjecture. Another source of interest is Cox's paper [**15**], where an explicit description of the universal torsor over a toric variety is given. This approach had an essential impact on toric geometry. For generalizations and relations to Cox rings, see [**4**, **9**, **10**, **18**, **19**].

The paper is organized as follows. In Section 1, we recall basic definitions and facts on Cox rings and universal torsors. The group of special automorphisms $\mathrm{SAut}(Y)$ of an algebraic variety Y is considered in Section 2. It is shown in [**5**] that if Y is affine of dimension at least 2 and the group $\mathrm{SAut}(Y)$ acts transitively on an open subset in Y, then this action is infinitely transitive. In Theorem 2, we extend this result to the case when Y is quasi-affine.

It is observed in [**24**] that open cylindric subsets on a projective variety X give rise to one-parameter unipotent subgroups in the automorphism group of an affine cone over X. This idea is developed further in [**25**, **28**]. In Section 3, we show that if X is a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$, which is transversally covered by cylinders, then the group $\mathrm{SAut}(\hat{X})$ acts on the universal torsor $\hat{X}$ transitively.

As a particular case, in Section 4 we study A-covered varieties, that is, varieties covered by open subsets isomorphic to the affine space. Clearly, any A-covered variety is smooth and rational. We list wide classes of A-covered varieties including smooth complete toric or, more generally, spherical varieties, smooth rational projective surfaces, and some Fano threefolds. It is shown that the condition to be A-covered is preserved under passing to vector bundles and their projectivizations as well as to the blow up in a linear subvariety. In the appendix to this paper, we prove that every smooth complete rational variety with a torus action of complexity 1 is A-covered. This part uses the technique of polyhedral divisors from [**1**, **2**].

In Section 5, we summarize our results on universal torsors and infinite transitivity. Theorem 3 claims that if X is an A-covered algebraic variety of dimension at least 2, then $\mathrm{SAut}(\hat{X})$ acts on the universal torsor $\hat{X}$ infinitely transitively. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then the total coordinate space $\bar{X} := \mathrm{Spec}\,\mathcal{R}(X)$ is a factorial affine variety, the group $\mathrm{SAut}(\bar{X})$ acts on $\bar{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\bar{X})$ on O is infinitely transitive; see Theorem 4. In particular, the Makar–Limanov invariant of $\bar{X}$ is trivial; see Corollary 1.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

1. *Preliminaries on Cox rings and universal torsors*

Let X be a normal algebraic variety with free finitely generated divisor class group $\mathrm{Cl}(X)$. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism. We define the *Cox sheaf* associated to K to be

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \quad \mathcal{R}_{[D]} := \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is given by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasi-coherent sheaf of normal integral K-graded $\mathcal{O}_X$-algebras and, up to isomorphy, it does not depend on the choice of the subgroup $K \subseteq \mathrm{WDiv}(X)$; see [**4**, Construction I.4.1.1]. The *Cox ring* of X is the algebra of global sections

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \quad \mathcal{R}_{[D]}(X) := \Gamma(X, \mathcal{O}_X(D)).$$

Let us assume that X is a smooth variety with only constant invertible functions. Then the sheaf $\mathcal{R}$ is locally of finite type, and the relative spectrum $\mathrm{Spec}_X\, \mathcal{R}$ is a quasi-affine variety $\hat{X}$; see [**4**, Corollary I.3.4.6]. We have $\Gamma(\hat{X}, \mathcal{O}) \cong \mathcal{R}(X)$, and the ring $\mathcal{R}(X)$ is a unique factorization domain with only constant invertible elements; see [**4**, Proposition I.4.1.5]. Since the sheaf $\mathcal{R}$ is K-graded, the variety $\hat{X}$ carries a natural action of the torus $H := \mathrm{Spec}\, \mathbb{K}[K]$. The projection $q\colon \hat{X} \to X$ is called the *universal torsor* over the variety X. By [**4**, Remark I.3.2.7], the morphism $q\colon \hat{X} \to X$ is a locally trivial H-principal bundle. In particular, the torus H acts on $\hat{X}$ freely.

LEMMA 1. *Let X be a normal variety. Assume that there is an open subset U on X which is isomorphic to the affine space $\mathbb{A}^n$. Then any invertible function on X is constant and the group $\mathrm{Cl}(X)$ is freely generated by classes $[D_1], \dots, [D_k]$ of the prime divisors such that*

$$X \setminus U = D_1 \cup \dots \cup D_k.$$

Proof. The restriction of an invertible function to U is constant, so the function is constant. Since U is factorial, any Weil divisor on X is linearly equivalent to a divisor whose support does not intersect U. This shows that the group $\mathrm{Cl}(X)$ is generated by $[D_1], \dots, [D_k]$.

Assume that $a_1D_1 + \dots + a_kD_k = \mathrm{div}(f)$ for some $f \in \mathbb{K}(X)$. Then f is a regular invertible function on U and thus f is a constant. This shows that the classes $[D_1], \dots, [D_k]$ generate the group $\mathrm{Cl}(X)$ freely. □

The Cox ring $\mathcal{R}(X)$ and the relative spectrum $q\colon \hat{X} \to X$ can be defined and studied under weaker assumptions on the variety X; see [**4**, Chapter I]. But in this paper we are interested in smooth varieties with free finitely generated divisor class group.

Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we may consider the *total coordinate space* $\bar{X} := \mathrm{Spec}\, \mathcal{R}(X)$. This is a factorial affine H-variety. By [**4**, Construction I.6.3.1], there is a natural open H-equivariant embedding $\hat{X} \hookrightarrow \bar{X}$ such that the complement $\bar{X} \setminus \hat{X}$ is of codimension at least 2.

2. *Special automorphisms and infinite transitivity*

An action of a group G on a set A is said to be *m-transitive* if for every two tuples of pairwise distinct points $(a_1, \dots, a_m)$ and $(a'_1, \dots, a'_m)$ in A there exists $g \in G$ such that $g \cdot a_i = a'_i$ for $i = 1, \dots, m$. An action which is m-transitive for all $m \in \mathbb{Z}_{>0}$ is called *infinitely transitive*.

Let Y be an algebraic variety. Consider a regular action $\mathbb{G}_a \times Y \to Y$ of the additive group $\mathbb{G}_a = (\mathbb{K}, +)$ of the ground field on Y. The image L of $\mathbb{G}_a$ in the automorphism group $\mathrm{Aut}(Y)$ is a 1-parameter unipotent subgroup. We let $\mathrm{SAut}(Y)$ denote the subgroup of $\mathrm{Aut}(Y)$ generated by all its one-parameter unipotent subgroups. Automorphisms from the group $\mathrm{SAut}(Y)$ are called *special*. In general, $\mathrm{SAut}(Y)$ is a normal subgroup of $\mathrm{Aut}(Y)$.

Denote by Y_{reg} the smooth locus of a variety Y. We say that a point $y \in Y_{\mathrm{reg}}$ is *flexible* if the tangent space T_yY is spanned by the tangent vectors to the orbits $L \cdot y$ over all one-parameter unipotent subgroups L in $\operatorname{Aut}(Y)$. The variety Y is *flexible* if every point $y \in Y_{\mathrm{reg}}$ is. Clearly, Y is flexible if one point of Y_{reg} is and the group $\operatorname{Aut}(Y)$ acts transitively on Y_{reg}.

The following result is proved in [**5**, Theorem 0.1].

THEOREM 1. *Let Y be an irreducible affine variety of dimension at least 2. Then the following conditions are equivalent.*

(1) *The group* $\operatorname{SAut}(Y)$ *acts transitively on* Y_{reg}.
(2) *The group* $\operatorname{SAut}(Y)$ *acts infinitely transitively on* Y_{reg}.
(3) *The variety Y is flexible.*

A more general version of implication $1 \Rightarrow 2$ is given in [**5**, Theorem 2.2]. In this section, we obtain an analog of this result for quasi-affine varieties; see Theorem 2.

Let Y be an algebraic variety. A regular action $\mathbb{G}_a \times Y \to Y$ defines a structure of a rational $\mathbb{G}_a$-algebra on $\Gamma(Y, \mathcal{O})$. The differential of this action is a locally nilpotent derivation D on $\Gamma(Y, \mathcal{O})$. Elements in $\operatorname{Ker} D$ are precisely the functions invariant under $\mathbb{G}_a$. The structure of a $\mathbb{G}_a$-module on $\Gamma(Y, \mathcal{O})$ can be reconstructed from D via the exponential map.

Assume that Y is quasi-affine. Then regular functions separate points on Y. In particular, any automorphism of Y is uniquely defined by the induced automorphism of the algebra $\Gamma(Y, \mathcal{O})$. Hence, a regular $\mathbb{G}_a$-action on Y can be reconstructed from the corresponding locally nilpotent derivation D. At the same time, if Y is not affine, then not every locally nilpotent derivation on $\Gamma(Y, \mathcal{O})$ gives rise to a regular $\mathbb{G}_a$-action on Y. For example, the derivation $\partial/\partial x_1$ does not define a regular $\mathbb{G}_a$-action on $\mathbb{A}^2 \setminus \{(0,0)\}$, while $x_2(\partial/\partial x_1)$ does.

If D is a locally nilpotent derivation assigned to a $\mathbb{G}_a$-action on a quasi-affine variety Y and $f \in \operatorname{Ker} D$, then the derivation fD is locally nilpotent and it corresponds to a $\mathbb{G}_a$-action on Y with the same orbits on $Y \setminus \operatorname{div}(f)$, which fixes all points on the divisor $\operatorname{div}(f)$. The one-parameter subgroup of $\operatorname{SAut}(Y)$ defined by fD is called a *replica* of the subgroup given by D.

We say that a subgroup G of $\operatorname{Aut}(Y)$ is *algebraically generated* if it is generated as an abstract group by a family $\mathcal{G}$ of connected algebraic (not necessarily affine) subgroups of $\operatorname{Aut}(Y)$.

PROPOSITION 1 [**5**, Proposition 1.5]. *There are* (*not necessarily distinct*) *subgroups $H_1, \ldots, H_s \in \mathcal{G}$ such that*

$$G.x = (H_1 \cdot H_2 \cdot \cdots \cdot H_s) \cdot x \quad \forall x \in X. \tag{1}$$

A sequence $\mathcal{H} = (H_1, \ldots, H_s)$ satisfying the condition (1) of Proposition 1 is called *complete*.

Let us say that a subgroup $G \subseteq \operatorname{SAut}(Y)$ is *saturated* if it is generated by one-parameter unipotent subgroups and there is a complete sequence $(H_1, \ldots, H_s)$ of one-parameter unipotent subgroups in G such that G contains all replicas of $H_1, \ldots, H_s$. In particular, $G = \operatorname{SAut}(X)$ is a saturated subgroup.

THEOREM 2. *Let Y be an irreducible quasi-affine algebraic variety of dimension at least 2 and let $G \subseteq \operatorname{SAut}(Y)$ be a saturated subgroup, which acts with an open orbit $O \subseteq Y$. Then G acts on O infinitely transitively.*

REMARK 1. Let H be a one-parameter unipotent subgroup of G. According to [**29**, Theorem 3.3], the field of rational invariants $\mathbb{K}(Y)^H$ is the field of fractions of the algebra $\mathbb{K}[Y]^H$ of regular invariants. Hence, by Rosenlicht's Theorem (see [**29**, Proposition 3.4]), regular invariants separate orbits on an H-invariant open dense subset $U(H)$ in Y. Furthermore, $U(H)$ can be chosen to be contained in O and to consist of 1-dimensional H-orbits.

For the remaining part of this section we fix the following notation. Let $H_1, \ldots, H_s$ be a complete sequence of one-parameter unipotent subgroups in G. We choose subsets $U(H_1), \ldots, U(H_s) \subseteq O$ as in Remark 1 and let

$$V = \bigcap_{k=1}^{s} U(H_k).$$

In particular, V is open and dense in O. We say that a set of points $x_1, \ldots, x_m$ in Y is *regular* if $x_1, \ldots, x_m \in V$ and $H_k \cdot x_i \neq H_k \cdot x_j$ for all $i, j = 1, \ldots, m$, $i \neq j$, and all $k = 1, \ldots, s$.

REMARK 2. For any H_k, any 1-dimensional H_k-orbit $O_1, \ldots, O_r$ intersecting V and any $p = 1, \ldots, s$ we may choose a replica $H_{k,p}$ such that all O_q except O_p are pointwise $H_{k,p}$-fixed. To this end, we find H_k-invariant functions $f_{k,p,p'}$ such that $f_{k,p,p'}|_{O_p} = 1$, $f_{k,p,p'}|_{O_{p'}} = 0$. Then we take

$$H_{k,p} = \left\{ \exp\left(t \left(\prod_{p' \neq p} f_{k,p,p'} \right) D_k \right) ;\ t \in \mathbb{K} \right\},$$

where D_k is a locally nilpotent derivation corresponding to H_k.

LEMMA 2. *For all points $x_1, \ldots, x_m \in O$ there exists an element $g \in G$ such that the set $g \cdot x_1, \ldots, g \cdot x_m$ is regular.*

Proof. For any x_i it is the case that $V \subset O = H_1 \cdots H_s \cdot x_i$. The condition $h_1 \cdots h_s \cdot x \in V$ is open and non-empty, hence we obtain an open subset $W \subset H_1 \times \cdots \times H_s$ such that $h_1 \cdots h_s \cdot x_i \in V$ for all $(h_1, \ldots, h_s) \in W$ and all x_i.

So we may suppose that $x_1, \ldots, x_m \in V$. Let N be the number of triples (i, j, k) such that $i \neq j$ and $H_k \cdot x_i = H_k \cdot x_j$. If $N = 0$ then the lemma is proved. Assume that $N \geqslant 1$ and fix such a triple (i, j, k).

There exists l such that $H_k \cdot x_i$ has at most finite intersection with H_l-orbits; otherwise $H_k \cdot x_i$ is invariant with respect to all $H_1, \ldots, H_s$, a contradiction with the condition $\dim O \geqslant 2$.

We claim that there is a one-parameter subgroup H in G such that

$$H_k \cdot (h \cdot x_i) \neq H_k \cdot (h \cdot x_j) \quad \text{for all but finitely many elements } h \in H. \tag{2}$$

Let us take first $H = H_l$. Condition (2) is determined by a finite set of H_k-invariant functions. So, either it holds or $H_k \cdot (h \cdot x_i) = H_k \cdot (h \cdot x_j)$ for all $h \in H$.

Assume that $H_l \cdot x_i \neq H_l \cdot x_j$. By Remark 2 there exists a replica H_l' such that $H_l' \cdot x_i = x_i$, but $H_l' \cdot x_j = H_l \cdot x_j$. We take $H = H_l'$, and condition (2) is fulfilled.

Assume now the contrary. Then there exists $h_l \in H_l$ such that $h_l \cdot x_i = x_j$. Then the set $\{h_l^n \cdot x_i | n \in \mathbb{Z}_{>0}\}$ has finite intersection with any H_k-orbit, and $h_l^n \cdot x_j = h_l^{n+1} \cdot x_i$ lie in different H_k-orbits for an infinite set of $n \in \mathbb{Z}_{>0}$. Therefore, this holds for an open subset of H_l, and condition (2) is again fulfilled.

Finally, the following conditions are open and non-empty on H:

(C1) $h \cdot x_1, \ldots, h \cdot x_m \in V$;
(C2) if $H_p \cdot x_{i'} \neq H_p \cdot x_{j'}$ for some p and $i' \neq j'$, then $H_p \cdot (h \cdot x_{i'}) \neq H_p \cdot (h \cdot x_{j'})$.

Hence there exists $h \in H$ satisfying (C1), (C2), and condition (2). We conclude that for the set $(h \cdot x_1, \ldots, h \cdot x_m)$ the value of N is smaller, and proceed by induction. □

Lemma 3. *Let $x_1, \ldots, x_m$ be a regular set and $G(x_1, \ldots, x_{m-1})$ be the intersection of the stabilizers of the points $x_1, \ldots, x_{m-1}$ in G. Then the orbit $G(x_1, \ldots, x_{m-1}) \cdot x_m$ contains an open subset in O.*

Proof. We claim that there is a non-empty open subset $U \subseteq H_1 \times \cdots \times H_s$ such that for every $(h_1, \ldots, h_s) \in U$, we have

$$h_1 \cdots h_s \cdot x_m = g \cdot x_m \quad \text{for some } g \in G(x_1, \ldots, x_{m-1}).$$

Indeed, let Z be the union of orbits $H_k \cdot x_i$, $k = 1, \ldots, s$, $i = 1, \ldots, m-1$. The set $V \setminus Z$ is open and contains x_m. Let U be the set of all $(h_1, \ldots, h_s)$ such that $h_r \cdots h_s \cdot x_m \in V \setminus Z$ for any $r = 1, \ldots, s$. Then U is open and non-empty. Let us show that for any $(h_1, \ldots, h_s) \in U$ and any $r = 1, \ldots, s$ the point $h_r \cdots h_s \cdot x_m$ is in the orbit $G(x_1, \ldots, x_{m-1}) \cdot x_m$. Assume that $h_{r+1} \cdots h_s \cdot x_m \in G(x_1, \ldots, x_{m-1}) \cdot x_m$. By Remark 2, there is a replica H'_r of the subgroup H_r which fixes $x_1, \ldots, x_{m-1}$ and such that the orbits

$$H_r \cdot (h_{r+1} \cdots h_s \cdot x_m) \quad \text{and} \quad H'_r \cdot (h_{r+1} \cdots h_s \cdot x_m)$$

coincide. Then H'_r is contained in $G(x_1, \ldots, x_{m-1})$ and the point $h_r h_{r+1} \cdots h_s \cdot x_m$ is in the orbit $G(x_1, \ldots, x_{m-1}) \cdot x_m$ for any $h_r \in H_r$. The claim is proved.

Now the image of the dominant morphism

$$U \longrightarrow O, \quad (h_1, \ldots, h_s) \longmapsto h_1 \cdots h_s \cdot x_m$$

contains an open subset in O. □

Proof of Theorem 2. Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ be two sets of pairwise distinct points in O. We have to show that there is an element $g \in G$ such that $g \cdot x_1 = y_1, \ldots, g \cdot x_m = y_m$.

We argue by induction on m. If $m = 1$, then the claim is obvious. If $m > 1$, then by inductive hypothesis there exists $g' \in G$ such that $g' \cdot x_1 = y_1, \ldots, g' \cdot x_{m-1} = y_{m-1}$. If $g' \cdot x_m = y_m$, the assertion is proved. Assume that $g' \cdot x_m \neq y_m$. By Lemma 2, there exists $g'' \in G$ such that the set

$$g'' \cdot y_1, \ldots, g'' \cdot y_{m-1}, g'' \cdot y_m, g''g' \cdot x_m$$

is regular. Lemma 3 implies that the orbits

$$G(g'' \cdot y_1, \ldots, g'' \cdot y_{m-1}) \cdot (g'' \cdot y_m) \quad \text{and} \quad G(g'' \cdot y_1, \ldots, g'' \cdot y_{m-1}) \cdot (g''g' \cdot x_m)$$

intersect, so there is $g''' \in G(g'' \cdot y_1, \ldots, g'' \cdot y_{m-1})$ such that $g'''g''g'x_m = g''y_m$. Then the element $g = (g'')^{-1}g'''g''g'$ is as desired. □

3. *Cylinders and $\mathbb{G}_a$-actions*

The following definition is taken from [**24**], see also [**25**].

Definition 1. Let X be an algebraic variety and U be an open subset of X. We say that U is a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is an irreducible affine variety with $\mathrm{Cl}(Z) = 0$.

Proposition 2. *Let X be a smooth algebraic variety with a free finitely generated divisor class group $\mathrm{Cl}(X)$, $q\colon \hat{X} \to X$ be the universal torsor, and $U \cong Z \times \mathbb{A}^1$ be a cylinder in X. Then there is an action $\mathbb{G}_a \times \hat{X} \to \hat{X}$ such that*

(i) *the set of $\mathbb{G}_a$-fixed points is $\hat{X} \setminus q^{-1}(U)$;*

(ii) *for any point* $y \in q^{-1}(U)$ *the isomorphism* $U \cong Z \times \mathbb{A}^1$ *identifies the subset* $q(\mathbb{G}_a \cdot y)$ *with a fiber of the projection* $Z \times \mathbb{A}^1 \to Z$.

Proof. Since $\mathrm{Cl}(U) \cong \mathrm{Cl}(Z) = 0$, we have an isomorphism $q^{-1}(U) \cong Z \times \mathbb{A}^1 \times H$ compatible with the projection q, see [**4**, Remark I.3.2.7]. Thus, the subset $q^{-1}(U)$ admits a $\mathbb{G}_a$-action

$$a \cdot (z, t, h) = (z, t + a, h), \quad z \in Z, \ t \in \mathbb{A}^1, \ h \in H,$$

with property (ii). Denote by D the locally nilpotent derivation on $\Gamma(U, \mathcal{O})$ corresponding to this action.

Our aim is to extend the action to $\hat{X}$. Since the open subset $q^{-1}(U)$ is affine, its complement $\hat{X} \setminus q^{-1}(U)$ is a divisor Δ in $\hat{X}$. We can find a function $f \in \Gamma(\hat{X}, \mathcal{O})$ such that $\Delta = \mathrm{div}(f)$. In particular,

$$\Gamma(q^{-1}(U), \mathcal{O}) = \Gamma(\hat{X}, \mathcal{O})[1/f].$$

Since f has no zero on any $\mathbb{G}_a$-orbit on $q^{-1}(U)$, it is constant along orbits, and f lies in $\mathrm{Ker}\, D$.

LEMMA 4. *Let Y be an irreducible quasi-affine variety,*

$$Y = \bigcup_{i=1}^{s} Y_{g_i}, \quad g_i \in \Gamma(Y, \mathcal{O}),$$

be an open covering by principal affine subsets, and let

$$\Gamma(Y_{g_i}, \mathcal{O}) = \mathbb{K}[c_{i1}, \ldots, c_{ir_i}][1/g_i]$$

for some $c_{ij} \in \Gamma(Y, \mathcal{O})$. Consider a finitely generated subalgebra C in $\Gamma(Y, \mathcal{O})$ containing all the functions g_i and c_{ij}. Then the natural morphism $Y \to \mathrm{Spec}\, C$ is an open embedding.

Proof. Note that $\Gamma(Y_{g_i}, \mathcal{O}) = \Gamma(Y, \mathcal{O})[1/g_i] = C[1/g_i]$. This shows that the morphism $Y \to \mathrm{Spec}\, C$ induces isomorphisms $Y_{g_i} \cong (\mathrm{Spec}\, C)_{g_i}$. □

Let $Y = \hat{X}$ and $\hat{X} \hookrightarrow \mathrm{Spec}\, C$ be an affine embedding as in Lemma 4 with $f \in C$. A finite generating set of the algebra C is contained in a finite-dimensional D-invariant subspace W of $\Gamma(q^{-1}(U), \mathcal{O})$. Replacing D with $f^m D$ we may assume that W is contained in $\Gamma(\hat{X}, \mathcal{O})$. We enlarge C and assume that it is generated by W. Then C is an $(f^m D)$-invariant finitely generated subalgebra in $\Gamma(\hat{X}, \mathcal{O})$ and we have an open embedding $\hat{X} \hookrightarrow \mathrm{Spec}\, C =: \tilde{X}$.

Replacing $f^m D$ with $D' := f^{m+1} D$, we obtain a locally nilpotent derivation D' on C such that $D'(C)$ is contained in fC. The corresponding $\mathbb{G}_a$-action on $\tilde{X}$ fixes all points on $\mathrm{div}(f)$ and has the same orbits on $q^{-1}(U)$. Hence, the subset $\hat{X} \subseteq \tilde{X}$ is $\mathbb{G}_a$-invariant and the restriction of the action to $\hat{X}$ has the desired properties. The proof of Proposition 2 is completed. □

REMARK 3. Under the assumption that the algebra $\Gamma(\tilde{X}, \mathcal{O})$ is finitely generated the proof of Proposition 2 is much simpler.

The following definitions appeared in [**28**].

DEFINITION 2. Let X be a variety and $U \cong Z \times \mathbb{A}^1$ be a cylinder in X. A subset W of X is said to be *U-invariant* if $W \cap U = p_1^{-1}(p_1(W \cap U))$, where $p_1 \colon U \to Z$ is the projection to the first factor. In other words, every $\mathbb{A}^1$-fiber of the cylinder either is contained in W or does not meet W.

DEFINITION 3. We say that a variety X is *transversally covered* by cylinders U_i, $i = 1, \ldots, s$, if $X = \bigcup_{i=1}^{s} U_i$ and there is no proper subset $W \subset X$ invariant under all U_i.

PROPOSITION 3. *Let X be a smooth algebraic variety with a free, finitely generated divisor class group $\mathrm{Cl}(X)$ and $q\colon \hat{X} \to X$ be the universal torsor. Assume that X is transversally covered by cylinders. Then the group $\mathrm{SAut}(\hat{X})$ acts on $\hat{X}$ transitively.*

Proof. Consider a $\mathbb{G}_a$-action on $\hat{X}$ associated with the cylinder U_i as in Proposition 2. Let L_i be the corresponding $\mathbb{G}_a$-subgroup in $\mathrm{SAut}(\hat{X})$ and G be the subgroup of $\mathrm{SAut}(\hat{X})$ generated by all the L_i.

By Proposition 2, the projection of any G-orbit on $\hat{X}$ to X is invariant under all the cylinders U_i, and thus this projection coincides with X. In particular, every $\mathrm{SAut}(\hat{X})$-orbit S on $\hat{X}$ projects to X surjectively.

Let H_S be the stabilizer of the subset S in H. The subgroup $\mathrm{SAut}(\hat{X})$ is normalized by H. This yields that if for some $x \in \hat{X}$ and $h \in H$ the point $h \cdot x$ lies in S, then h is contained in H_S. In other words, the orbit S intersects every fiber of the torsor $q\colon \hat{X} \to X$ in an H_S-orbit.

By [**5**, Proposition 1.3], any $\mathrm{SAut}(\hat{X})$-orbit is locally closed in $\hat{X}$. Since the torus H permutes G-orbits, all of them are closed in $\hat{X}$. This yields that H_S is a closed subgroup of H.

Assume that H_S is a proper subgroup of H. Then there is a non-zero character $\chi \in \mathbb{X}(H)$ such that $\chi|_{H_S} = 1$. Consider a trivialization covering $X = U_1 \cup \cdots \cup U_r$ of the bundle $q\colon \hat{X} \to X$, that is, $q^{-1}(U_i) \cong U_i \times H$. Let

$$\psi_{ij}\colon U_i \cap U_j \longrightarrow H$$

be the transition functions of this bundle. We define a locally trivial $\mathbb{K}^\times$-bundle X_χ over X by gluing the covering $\{U_i \times \mathbb{K}^\times\}$ with the transition functions $U_i \cap U_j \to \mathbb{K}^\times$, $x \mapsto \chi(\psi_{ij}(x))$. Then the maps

$$U_i \times H \longrightarrow U_i \times \mathbb{K}^\times, \quad (x, h) \longmapsto (x, \chi(h))$$

define a surjective morphism $\hat{X} \to X_\chi$. The image of S under this morphism intersects every fiber of $X_\chi \to X$ in one point. This shows that the $\mathbb{K}^\times$-bundle X_χ is trivial. Then the pull back of the coordinate function along a fiber of X_χ is an invertible function on $\hat{X}$. Since $\hat{X}$ has only constant invertible functions, we conclude that $H_S = H$ and thus $S = \hat{X}$. This shows that $\mathrm{SAut}(\hat{X})$ acts on $\hat{X}$ transitively. $\square$

4. *A-covered varieties*

The affine space $\mathbb{A}^n$ admits n coordinate cylinder structures $\mathbb{A}^{n-1} \times \mathbb{A}^1$, and the covering of $\mathbb{A}^n$ by these cylinders is transversal. This elementary observation motivates the following definition.

DEFINITION 4. An irreducible algebraic variety X is said to be *A-covered* if there is an open covering $X = U_1 \cup \cdots \cup U_r$, where every chart U_i is isomorphic to the affine space $\mathbb{A}^n$.

A choice of such a covering together with isomorphisms $U_i \cong \mathbb{A}^n$ is called an *A-atlas* of X. A subvariety Z of an A-covered variety X is called *linear* with respect to an A-atlas if it is linear in all charts, that is, $Z \cap U_i$ is a linear subspace in $U_i \cong \mathbb{A}^n$. Any A-covered variety is rational, smooth, and by Lemma 1 the group $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ is finitely generated and free.

Clearly, the projective space $\mathbb{P}^n$ is A-covered. This fact can be generalized in several ways.

(1) Every smooth complete toric variety X is A-covered.

(2) Every smooth rational complete variety with a torus action of complexity 1 is A-covered; see the appendix to this paper.

(3) Let G be a semisimple algebraic group and be P a parabolic subgroup of G. Then the flag variety G/P is A-covered. Indeed, a maximal unipotent subgroup N of G acts on G/P with an open orbit U isomorphic to an affine space. Since G acts on G/P transitively, we obtain the desired covering.

(4) More generally, every smooth complete spherical variety is A-covered; see [**12**, Corollary 1.5].

(5) The product of two A-covered varieties is again A-covered.

(6) Every vector bundle over $\mathbb{A}^n$ trivializes, and total spaces of vector bundles over A-covered varieties are A-covered. The same holds for their projectivizations.

(7) If a variety X is A-covered and X' is a blow-up of X at some point $p \in X$, then X' is A-covered.

(8) All smooth projective rational surfaces are obtained either from $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or from the Hirzebruch surfaces F_n by a sequence of blow-ups of points, and thus they are A-covered by (7).

(9) We may generalize the blow-up example as follows. The blow-up of X in a linear subvariety Z is A-covered. Moreover, the strict transforms of linear subvarieties, which either contain Z or do not intersect with it, are linear again (with the choice of an appropriate A-atlas). Hence, we may iterate this procedure.

Proof of statement (9). We consider one chart U of the covering on X. We may assume that we blow up $\mathbb{A}^n = U$ in the linear subspace given by $x_1 = \cdots = x_k = 0$. By definition, the blow-up X' is given in the product $\mathbb{A}^n \times \mathbb{P}^{k-1}$ by equations $x_i z_j = x_j z_i$, where $1 \leqslant i, j \leqslant k$. If the homogeneous coordinate z_j equals 1 for some $j = 1, \ldots, k$, then $x_i = x_j z_i$, and we are in the open chart V_j with independent coordinates x_j, x_s with $s > k$ and z_i, $i \neq j$. So the variety X' is covered by k such charts.

Let L be a linear subspace in U containing $[x_1 = \cdots = x_k = 0]$ and given by linear equations $f_i(x_1, \ldots, x_k) = 0$. The strict transform of L is given in V_j by the equations $f_i(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_k) = 0$. After a change of variables $x_j \mapsto x_j - 1$ these equations become linear.

Finally, if a linear subvariety Z' does not meet the linear subvariety Z, then Z' does not intersect charts of our atlas that intersect Z, and the assertion follows. □

EXAMPLE 1. Consider the quadric threefold Q. Choose two points and a conic passing through them. Then these are linear subvarieties of Q with respect to an appropriate atlas. Hence, the iterated blow-up in the points, first, and then in the strict transform of the conic, is A-covered. This variety has number 4.4 in the classification of Fano threefolds; see Proposition 4.

We may use the above observations to take a closer look at Fano threefolds.

PROPOSITION 4. *In the classification of Iskovskikh* [**22**] *and Mori–Mukai* [**27**] *we have the following* (*possibly not complete*) *list of A-covered Fano threefolds*:

(a) $\mathbb{P}^3$, Q, V_5 *and the Mukai–Umemura threefold* V_{22}';
(b) 2.33–2.36, 3.26–3.31, 4.9–4.11, 5.2, 5.3;

(c) 2.29, 2.30, 2.31, 2.32, 3.18–3.23, 3.24, 4.4, 4.7, 4.8 *and* (*at least*) *one element of the family* 2.24, 3.8 *and* 3.10;
(d) 5.3–5.8;
(e) 2.26.

Proof. Existence of an A-covering for varieties in (a) can be seen directly from defining equations. List (b) are exactly the toric Fano threefolds.

The varieties in (c) are precisely non-toric Fano threefolds admitting a complexity one torus action; this is more or less straightforward to check via the description in [**27**]; see also [**34**]. Thus the claim follows from Theorem A.1. In families 2.24 and 3.8 we find a 2-torus action on the hypersurface $V(x_1y_1^2 + x_2y_2^2 + x_3y_3^2) \subset \mathbb{P}^2 \times \mathbb{P}^2$ and on the blow-up of this hypersurface in the curve $(* : 1 : 0, 0 : 0 : 1)$, respectively. Moreover, we have a 2-torus action on the blow-up of the quadric $Q = V(x_1x_2 + x_3x_4 + x_5^2)$ in the conics $C_1 = Q \cap [x_1 = x_2 = 0]$ and $C_2 = Q \cap [x_3 = x_4 = 0]$. This is an element of family 3.10.

The varieties in (d) are precisely the products of del Pezzo surfaces and $\mathbb{P}^1$. Finally, the varieties 2.26 are obtained from V_5 by blow-up in a linear subvariety as explained in (9). □

Remark 4. For Fano threefolds of Picard rank 1 the list of A-covered ones in Proposition 4 (a) is almost complete. Indeed, by [**17**] the varieties $\mathbb{P}^3$, Q, V_5 and V_{22} are the only possible compactifications of $\mathbb{A}^3$. In particular, the Fano threefolds V_{12}, V_{16}, V_{18} and V_4 from Iskovskikh's classification [**22**] are rational but not A-covered. The situation remains unclear only for members of family V_{22} different from the Mukai–Umemura threefold V_{22}'.

For higher Picard rank we do not expect the list to be complete, but our arguments (1)–(9) do not apply for other than the given examples. Consider, for example, the threefold 4.6. This is a blow-up of $\mathbb{P}^3$ in three disjoint lines. Here, we cannot apply (9) directly, since the three lines are not linear with respect to the same A-atlas.

5. *Main results*

The following theorem summarizes our results on universal torsors and infinite transitivity.

Theorem 3. *Let X be an A-covered algebraic variety of dimension at least* 2 *and* $q\colon \hat{X} \to X$ *be the universal torsor. Then the group* $\mathrm{SAut}(\hat{X})$ *acts on the quasi-affine variety* $\hat{X}$ *infinitely transitively.*

Proof. If X is covered by m open charts isomorphic to $\mathbb{A}^n$, and every chart is equipped with n transversal cylinder structures, then the covering of X by these mn cylinders is transversal. By Proposition 3, the group $\mathrm{SAut}(\hat{X})$ acts on $\hat{X}$ transitively. Theorem 2 yields that the action is infinitely transitive. □

Theorem 3 provides many examples of quasi-affine varieties with rich symmetries. In particular, if X is a del Pezzo surface, a description of the universal torsor $q\colon \hat{X} \to X$ may be found in [**8**, **30**, **31**]. It follows from Theorem 3 that the group $\mathrm{SAut}(\hat{X})$ acts on $\hat{X}$ infinitely transitively.

If X is the blow-up of nine points in general position on $\mathbb{P}^2$, then it is well known that the Cox ring $\mathcal{R}(X)$ is not finitely generated, and thus $\hat{X}$ is a quasi-affine variety with a non-finitely generated algebra of regular functions $\Gamma(\hat{X}, \mathcal{O})$. Theorem 3 works in this case as well.

THEOREM 4. *Let X be an A-covered algebraic variety of dimension at least 2. Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then the total coordinate space $\bar{X} := \operatorname{Spec}\mathcal{R}(X)$ is an affine factorial variety, the group $\mathrm{SAut}(\bar{X})$ acts on $\bar{X}$ with an open orbit O, and the action of $\mathrm{SAut}(\bar{X})$ on O is infinitely transitive.*

Proof. Lemma 1 shows that the group $\mathrm{Cl}(X)$ is finitely generated and free, hence the ring $\mathcal{R}(X)$ is a unique factorization domain; see [**4**, Proposition I.4.1.5]. Since

$$\Gamma(\bar{X}, \mathcal{O}) = \mathcal{R}(X) \cong \Gamma(\hat{X}, \mathcal{O}),$$

any $\mathbb{G}_a$-action on $\hat{X}$ extends to $\bar{X}$. We conclude that $\hat{X}$ is contained in one $\mathrm{SAut}(\bar{X})$-orbit O on $\bar{X}$, the action of $\mathrm{SAut}(\bar{X})$ on O is infinitely transitive, and by [**5**, Proposition 1.3] the orbit O is open in $\bar{X}$. □

Recall from [**16**] that the *Makar–Limanov invariant* $\mathrm{ML}(Y)$ of an affine variety Y is the intersection of the kernels of all locally nilpotent derivations on $\Gamma(Y, \mathcal{O})$. In other words, $\mathrm{ML}(Y)$ is the subalgebra of all $\mathrm{SAut}(Y)$-invariants in $\Gamma(Y, \mathcal{O})$. In a similar way to that in [**26**] the *field Makar–Limanov invariant* $\mathrm{FML}(Y)$ is the subfield of $\mathbb{K}(Y)$ which consists of all rational $\mathrm{SAut}(Y)$-invariants. If the field Makar–Limanov invariant is trivial, that is, if $\mathrm{FML}(Y) = \mathbb{K}$, then so is $\mathrm{ML}(Y)$, but the converse is not true in general.

COROLLARY 1. *Under the assumptions of Theorem 4 the field Makar–Limanov invariant $FML(\bar{X})$ is trivial.*

Proof. By Theorem 4, the group $\mathrm{SAut}(\bar{X})$ acts on $\bar{X}$ with an open orbit. So any rational $\mathrm{SAut}(\bar{X})$-invariant is constant. □

Appendix: Rational T-varieties of complexity 1

By a T-variety we mean a normal variety equipped with an effective action of an algebraic torus T. The difference of dimensions $\dim X - \dim T$ is called the *complexity* of a T-variety. Hence, toric varieties are T-varieties of complexity zero. For the case of complexity 1 we are going to prove the following theorem.

THEOREM A.1. *Any smooth complete rational T-variety of complexity 1 is A-covered.*

Owing to [**1**–**3**] T-varieties can be described and studied in the language of polyhedral divisors. Here, we restrict ourselves to the case of rational T-varieties of complexity 1. This means that the divisors live on $\mathbb{P}^1$. This allows us to simplify some of the definitions.

The affine case We consider a lattice M of rank n, the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ and the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $T = N \otimes_{\mathbb{Z}} \mathbb{K}^*$ be the algebraic torus of dimension n with character lattice M.

Every polyhedron $\Delta \subset N_{\mathbb{Q}}$ has a Minkowski decomposition $\Delta = P + \sigma$, where P is a (compact) polytope and σ is a polyhedral cone. We call σ the *tail cone* of Δ and denote it by $\mathrm{tail}(\Delta)$. A *polyhedral divisor* on $\mathbb{P}^1$ over N is a formal sum

$$\mathcal{D} = \sum_{y \in \mathbb{P}^1} \mathcal{D}_y \cdot y,$$

where $\mathcal{D}_y$ are polyhedra with common pointed tail cone σ and only finitely many coefficients differ from σ itself. Note that we allow empty coefficients.

We call $\mathcal{D}$ a *proper* polyhedral divisor, or a *p-divisor* for short, if

$$\deg \mathcal{D} := \sum_{y \in \mathbb{P}^1} \mathcal{D}_y \subsetneq \sigma. \tag{A.1}$$

Here, $\deg \mathcal{D} = \emptyset$ if and only if $D_y = \emptyset$ for some $y \in \mathbb{P}^1$.

By [**1**, Theorems 3.1, 3.4], there is a functor X associating to a p-divisor $\mathcal{D}$ on $\mathbb{P}^1$ a rational complexity-1 T-variety $X(\mathcal{D})$ of dimension $n+1$, and every such variety arises this way.

Remark A.1. [**21**, Remark 1.8.] Let us fix two points $y_0, y_\infty \in \mathbb{P}^1$. For $y \in \mathbb{P}^1 \setminus \{y_0, y_\infty\}$ we consider lattice points $v_y \in N$ such that only finitely many of them are different from 0. We denote the sum $\sum_{y \neq y_0, y_\infty} v_y$ by v and choose $w_0, w_\infty \in N$ with $w_0 + w_\infty = v$.

A polyhedral divisor $\mathcal{D}$ of the form:

$$\mathcal{D}_0 \cdot y_0 + \mathcal{D}_\infty \cdot y_\infty + \sum_y (v_y + \sigma) \cdot y \tag{A.2}$$

on $\mathbb{P}^1$ corresponds to the affine toric variety of the cone

$$\begin{aligned} \operatorname{cone}(w_0 + \mathcal{D}_0, w_\infty + \mathcal{D}_\infty) := \mathbb{Q}_{\geqslant 0} \cdot ((w_0 + \mathcal{D}_0) \times \{1\} \cup \sigma \\ \times \{0\} \cup (w_\infty + \mathcal{D}_\infty) \times \{-1\}) \subset N_\mathbb{Q} \oplus \mathbb{Q} \end{aligned}$$

together with the subtorus action given by the lattice embedding $N \hookrightarrow N \oplus \mathbb{Z}$. Here, we allow $\mathcal{D}_0 = \emptyset$ or $\mathcal{D}_\infty = \emptyset$. Different choices of w_0 and w_∞ lead to cones which can be transformed into each other by a lattice automorphism of $N \times \mathbb{Z}$. Hence, the corresponding toric varieties are isomorphic and the above statement indeed makes sense. If the affine toric variety is assumed to be smooth, the cone has to be regular. In this case, if $\mathcal{D}_0$ or $\mathcal{D}_\infty$ has dimension n, then the constructed cone has dimension $n+1$ and the variety $X(\mathcal{D})$ is an affine space.

It is not hard to exhibit the extremal rays of the cone constructed in Remark A.1.

Lemma A.1. *There are three types of extremal rays in* $C := \operatorname{cone}(w_0 + \mathcal{D}_0, w_\infty + \mathcal{D}_\infty)$:

(1) $\rho \times \{0\}$ *for every* $\rho \in \sigma(1)$, *where* $\deg \mathcal{D} \cap \rho = (w_0 + w_\infty + \mathcal{D}_0 + \mathcal{D}_\infty) \cap \rho = \emptyset$;
(2) $\mathbb{Q}_{\geqslant 0} \cdot (w_0 + v, 1)$, *where* $v \in \mathcal{D}_0$ *is a vertex*;
(3) $\mathbb{Q}_{\geqslant 0} \cdot (w_\infty + v, -1)$, *where* $v \in \mathcal{D}_\infty$ *is a vertex*.

Proposition A.1 [**33**, Proposition 3.1 and Theorem 3.3.]. *Let* $\mathcal{D}$ *be a p-divisor on* $\mathbb{P}^1$. *Then* $X(\mathcal{D})$ *is smooth if and only if*

(i) *either* $\deg \mathcal{D} \neq \emptyset$, $\mathcal{D}$ *is of the form* (A.2), *and the cone* C *is regular, or*
(ii) $\deg \mathcal{D} = \emptyset$ *and* $\operatorname{cone}(\mathcal{D}_y) := \operatorname{cone}(\mathcal{D}_y, \emptyset)$ *is regular for every* $y \in \mathbb{P}^1$.

Polyhedral divisors of the second type do not necessarily correspond to affine spaces. This is only the case if at most two coefficients are not lattice translates of the tail cone, see Remark A.1.

As a consequence of Lemma A.1 and Proposition A.1, we easily obtain that for two special cases all coefficients of $\mathcal{D}$ have to be translated cones in order to obtain a smooth affine variety.

Corollary A.1. *Assume that $X(\mathcal{D})$ is smooth. If $\mathcal{D}$ has a tail cone σ of maximal dimension and $\deg \mathcal{D} \cap \tau = \emptyset$ for some facet $\tau \prec \sigma$, then all the coefficients are translates of σ and all but two are even lattice translates.*

Corollary A.2. *If $\deg \mathcal{D} = \emptyset$ and $X(\mathcal{D})$ is smooth, then the tail cone σ has to be regular. Moreover, if σ is maximal, then $\mathcal{D}_y$ is either empty or a lattice translate of σ for every $y \in \mathbb{P}^1$.*

The complete case and affine coverings Consider two p-divisors $\mathcal{D}$ and $\mathcal{D}'$ on $\mathbb{P}^1$ such that $\mathcal{D}'_y$ is a face of $\mathcal{D}_y$ for every $y \in \mathbb{P}^1$ and $\deg \mathcal{D}' = \deg \mathcal{D} \cap \operatorname{tail} \mathcal{D}'$. Then by [**20**, Proposition 1.1], we obtain an open embedding $X(\mathcal{D}') \hookrightarrow X(\mathcal{D})$. For two p-divisors $\mathcal{D}$ and $\mathcal{D}'$ we define their *intersection* by $\mathcal{D} \cap \mathcal{D}' := \sum_y (\mathcal{D}_y \cap \mathcal{D}'_y) \cdot y$.

For a given complete T-variety we consider an open covering by affine torus invariant subsets X_i, $i = 1, \ldots, m$, and let $X_{ij} = X_i \cap X_j$. Every such subset corresponds to a polyhedral divisor $\mathcal{D}^i$ or $\mathcal{D}^{ij}$. We obtain a finite set $\mathcal{S} = \{\mathcal{D}^1, \ldots, \mathcal{D}^m\}$. By [**2**, Theorem 5.6, Remark 7.4(iv)], we may assume that $\mathcal{D}^{ij} = \mathcal{D}^i \cap \mathcal{D}^j$ holds and the set $\mathcal{S}$ satisfies the following compatibility conditions.

> *Slice rule*. The *slices* $\mathcal{S}_y = \{\mathcal{D}_y \mid \mathcal{D} \in \mathcal{S}\}$ are complete polyhedral subdivisions of $N_{\mathbb{Q}}$, that is, they cover $N_{\mathbb{Q}}$ and the intersection of every two polyhedra is a face of both of them.
> *Degree rule*. For $\tau = (\operatorname{tail} \mathcal{D}) \cap (\operatorname{tail} \mathcal{D}')$ one has $\tau \cap (\deg \mathcal{D}) = \tau \cap (\deg \mathcal{D}')$.

Note that $\operatorname{tail} \mathcal{S} := \{\operatorname{tail} \mathcal{D} \mid \mathcal{D} \in \mathcal{S}\}$ generates a fan and all but finitely many slices $\mathcal{S}_y$ just equal $\operatorname{tail} \mathcal{S}$. Consider a maximal tail cone σ in $\operatorname{tail} \mathcal{S}$. Then for every y there is a unique polyhedron $\mathcal{S}_y(\sigma)$ in $\mathcal{S}_y$ having this tail.

A maximal cone $\sigma \in \operatorname{tail} \mathcal{S}$ is called *marked* if the corresponding polyhedral divisor $\mathcal{D}$ with $\sigma = \operatorname{tail} \mathcal{D}$ fulfills $\deg \mathcal{D} \neq \emptyset$. We denote the set of all marked cones by $\operatorname{tail}^m(\mathcal{S}) \subset \operatorname{tail}(\mathcal{S})$.

In general, there are many torus-invariant affine coverings of X. But by [**20**, Proposition 1.6] every rational complete T-variety of complexity 1 is uniquely determined by the slices $\mathcal{S}_y$ and the markings in $\operatorname{tail} \mathcal{S}$. Hence, another set $\mathcal{S}'$ of p-divisors with $\mathcal{S}_y = \mathcal{S}'_y$ for all $y \in \mathbb{P}^1$ and $\operatorname{tail}^m(\mathcal{S}) = \operatorname{tail}^m(\mathcal{S}')$ corresponds to another invariant affine covering of the same variety. From now on we assume that X is a rational complete *smooth* T-variety of complexity 1 and we consider an affine covering given by the p-divisors in $\mathcal{S}$. By Proposition A.1, we have the following lemma:

Lemma A.2. *Given a maximal cone σ in $\operatorname{tail} \mathcal{S}$, there are two possible cases*:

(1) *σ is marked and all but two coefficients of $\mathcal{S}_y(\sigma)$ are lattice translates of σ, or*
(2) *σ is not marked; then it has to be regular and $\mathcal{S}_y(\sigma)$ has to be a lattice translate of σ for every $y \in \mathbb{P}^1$.*

In the slices $\mathcal{S}_y$ there might occur maximal polyhedra with non-maximal tail cones. Here, Lemma A.2 does not apply. Instead we need the following crucial fact.

Proposition A.2. *Let P be a maximal polyhedron with non-maximal tail in $\mathcal{S}_z$ for some $z \in \mathbb{P}^1$. Then up to one exception $z' \in \mathbb{P}^1$ there is a lattice translate of $\operatorname{tail}(P)$ in $\mathcal{S}_y$, for every $y \neq z$.*

Proof. We denote the tail cone of P by τ. Consider the part R of $\mathcal{S}_z$ consisting of all maximal polyhedra with tail τ. We are looking at the boundary facets of this part. There is a facet having tail τ; it corresponds to a primitive lattice element $u \in \tau^\perp$, which is minimized on this facet. On the other side of the facet, we have a neighboring full-dimensional polyhedron P' having a tail cone $\tau' \succ \tau$. Replacing P with P' and iterating this procedure, we end up with a maximal polyhedron P, a non-maximal tail cone $\tau = \operatorname{tail} P$, a region R of $\mathcal{S}_z$, and a facet of R minimizing some $u \in \tau^\perp$ (which necessarily has tail cone τ) such that the neighboring polyhedron Δ^+ has full-dimensional tail σ^+. Now, we treat two cases separately: (1) $\dim \tau < n-1$ and (2) $\dim \tau = n-1$.

In the first case, the common facet of Δ^+ and R has dimension $n-1$, but tail cone τ of dimension less than $n-1$. This implies that the facet and, hence, Δ^+ has at least $n - \dim \tau > 1$ vertices. In particular, it is not a lattice translate of a cone and by Lemma A.2 the tail cone σ^+ has to be marked. Again by Lemma A.2 for $y \neq z$ all but one of the $\mathcal{S}_y(\sigma^+)$ are lattice translates of σ^+. Hence, the faces of these $\mathcal{S}_y(\sigma^+)$ with tail cone τ are indeed lattice translates of τ and the claim is proved.

In the second case, $-u$ is minimized on another facet of R. For the neighboring full-dimensional polyhedron Δ^- we have $\tau \prec \sigma^- := \operatorname{tail} \Delta^-$. Since τ is of dimension $n-1$, the cone σ^- must be full-dimensional. By construction $\sigma^+ \cap \sigma^- = \tau$. Assume that σ^+ is not marked. Then all polyhedra $\mathcal{S}_y(\sigma^+) \in \mathcal{S}_y$ are lattice translations of σ^+. As before, we infer that the claim is fulfilled in this case. The same applies if σ^- is not marked.

Now assume that both σ^+ and σ^- are marked. There are p-divisors in $\mathcal{D}^+, \mathcal{D}^- \in \mathcal{S}$ with $\operatorname{tail} \mathcal{D}^\pm = \sigma^\pm$ and $\deg \mathcal{D}^\pm \neq \emptyset$. If $\Delta^\pm = \mathcal{D}_z^\pm$ is not a lattice translate, then we know that all other polyhedra $\mathcal{D}_y^\pm$ are lattice translates of $\sigma^\pm$ up to one exception. Hence, every $\mathcal{D}_y^\pm$ up to one exception contains a lattice translation of $\tau \prec \sigma^\pm$ and the claim follows. Hence, we may assume that $\mathcal{D}_z^+$, $\mathcal{D}_z^-$ are just lattice translates of the cones σ^+ and σ^-, respectively.

Remember that we have a maximal polyhedron $P \in \mathcal{S}_z$ with non-maximal tail cone τ. Hence, there is some p-divisor $\mathcal{D}(P) \in \mathcal{S}$ with $\mathcal{D}(P)_z = P$. By the properness condition (A.1) we have $\deg \mathcal{D}(P) = \emptyset$ and by the degree rule we have $\tau \cap \deg \mathcal{D}^\pm = \emptyset$. Now, by Corollary A.1 we know that all $\mathcal{D}_y^\pm$ are just translated cones $(v_y^\pm + \sigma^\pm)$. Moreover, up to two exceptions $\mathcal{D}_{y_0}^\pm = (v_0^\pm + \sigma)$ and $\mathcal{D}_{y_\infty}^\pm = (v_\infty^\pm + \sigma)$, they are even lattice translates, that is, $v_y^\pm \in N$.

Corollary A.2 ensures that τ is a regular cone. Hence, the primitive ray generators $e_1, \ldots, e_{n-1}$ of τ form a part of a basis $e_1, \ldots, e_n$ of N. Since $u \in \tau^\perp$ we have $\langle u, e_n \rangle = 1$. Now, the elements $(e_i, 0)$ together with $(0, 1)$ form a basis of $N \times \mathbb{Z}$. We use this basis for an identification $N \times \mathbb{Z} \cong \mathbb{Z}^{n+1}$. In particular, $\langle u, \cdot \rangle$ equals the nth coordinate on this basis.

By Lemma A.1, the primitive ray generators of $\operatorname{cone}(w_0^\pm + \mathcal{D}_{y_0}^\pm, w_\infty^\pm + \mathcal{D}_{y_\infty}^\pm)$ (as in Remark A.1) are given by the columns of the following matrix. Owing to the smoothness condition these matrices have to be unimodular. The first $n-1$ columns correspond to the rays of τ and the last two columns to the vertices in $\mathcal{D}_{y_0}$ and $\mathcal{D}_{y_\infty}$, respectively.

Here, $\mu_0^\pm$ and $\mu_\infty^\pm$ are minimal positive integers such that $\mu_0^\pm \cdot v_0^\pm$ and $\mu_\infty^\pm \cdot v_\infty^\pm$ are lattice elements. By the slice rule, we have $\langle u, v_y^+ \rangle \geqslant \langle u, v_y^- \rangle$ (otherwise $(v_y^+ + \sigma^+)$ and $(v_y^- + \sigma^-)$ would intersect in a non-face, since $\tau = \sigma^+ \cap u^\perp = \sigma^- \cap u^\perp$ is a common facet). Moreover, $\langle v_z^+, u \rangle > \langle v_z^-, u \rangle$ holds, since $\Delta^+ = (v_z^+ + \sigma^+)$ and $\Delta^- = (v_z^- + \sigma^-)$ are separated by the full-dimensional region R. Note that the compared values are integers. Let us set $\Sigma^\pm = \sum_y v_y^\pm$. By definition, we have $v^\pm = \Sigma^\pm - v_0^\pm - v_\infty^\pm$. We obtain $\langle \Sigma^+, u \rangle \geqslant \langle \Sigma^-, u \rangle + 1$.

$$M^\pm = \begin{pmatrix} 1 & & & * & * \\ & \ddots & & \vdots & \vdots \\ & & 1 & * & * \\ 0 & \cdots & 0 & \langle v_0^\pm + w_0^\pm, u \rangle & \langle v_\infty^\pm + w_\infty^\pm, u \rangle \\ 0 & \cdots & 0 & \mu_0^\pm & -\mu_\infty^\pm \end{pmatrix}.$$

We choose w_0^+ in a way such that $0 \leqslant \langle v_0^+ + w_0^+, u\rangle < 1$ holds and set $w_\infty^+ = v^+ - w_0^+$, $w_\infty^- = w_\infty^+ - \lfloor v_\infty^- - v_\infty^+ \rfloor$ (component-wise rounding) and $w_0^- = v^- - w_\infty^-$. Hence, we obtain $\langle v_\infty^- + w_\infty^-, u\rangle \leqslant \langle v_\infty^+ + w_\infty^+, u\rangle$ and

$$\begin{aligned} v_0^- + w_0^- &= v_0^- + v^- - w_\infty^- = \Sigma^- - v_\infty^- - w_\infty^- \\ &= \Sigma^- - v_\infty^- - w_\infty^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\ &= \Sigma^- - v_\infty^- - v^+ + w_0^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\ &= \Sigma^- - v_\infty^- - \Sigma^+ + v_0^+ + v_\infty^+ + w_0^+ + \lfloor v_\infty^- - v_\infty^+ \rfloor \\ &= w_0^+ + v_0^+ + (\Sigma^- - \Sigma^+) + (\lfloor v_\infty^- - v_\infty^+ \rfloor - (v_\infty^- - v_\infty^+)). \end{aligned}$$

After pairing with u we obtain $\langle v_0^- + w_0^-, u\rangle \leqslant \langle w_0^+ + v_0^+, u\rangle - 1 < 0$. Hence, either $\langle v_0^+ + w_0^+, u\rangle, \langle v_\infty^+ + w_\infty^+, u\rangle \geqslant 0$ or $\langle v_0^- + w_0^-, u\rangle, \langle v_\infty^- + w_\infty^-, u\rangle \leqslant 0$. In both cases, we need to have either $\mu_0^\pm = 1$ or $\mu_\infty^\pm = 1$ in order to obtain $|\det M^\pm| = 1$. All but one coefficient of $\mathcal{D}^+$ or $\mathcal{D}^-$, respectively, are lattice translates. Since τ is a face of $\sigma^\pm$ we will always find a lattice translate of τ as well, and Proposition A.2 is proved. □

Proof of Theorem A.1. Consider a set $\mathcal{S}$ of p-divisors giving rise to a covering of X as above. We construct another set of p-divisors $\mathcal{S}'$ giving rise to an A-covering of X.

Let σ be a marked maximal cone in $\operatorname{tail}\mathcal{S}$. There is a $\mathcal{D} \in \mathcal{S}$ with $\deg \mathcal{D} \neq \emptyset$ and $\operatorname{tail}\mathcal{D} = \sigma$. We simply add it to $\mathcal{S}'$. By Lemma A.1, $X(\mathcal{D})$ is an affine space. If σ is maximal but not marked, then by Lemma A.2 the polyhedra $\mathcal{S}_y(\sigma)$ are just lattice translates of σ. Now, we add the following two polyhedral divisors to $\mathcal{S}'$:

$$\mathcal{D}_0 = \emptyset \cdot 0 + \sum_{y \neq 0} \mathcal{S}_y(\sigma) \cdot y \quad \text{and} \quad \mathcal{D}_\infty = \emptyset \cdot \infty + \sum_{y \neq \infty} \mathcal{S}_y(\sigma) \cdot y.$$

From Remark A.1, we know that $X(\mathcal{D}_0)$ and $X(\mathcal{D}_\infty)$ are both affine spaces.

By these considerations $\mathcal{S}'_y$ covers all polyhedra from $\mathcal{S}_y$ having maximal tail cones. Moreover, the markings are the same as for $\mathcal{S}$. It remains to consider maximal polyhedra P having non-maximal tail τ. We consider such a polyhedron living in some slice $\mathcal{S}_z$. By Proposition A.2, we have a lattice translate $(v_y + \tau)$ in every slice except for $\mathcal{S}_z$ and $\mathcal{S}_{z'}$. Having this, we can add the p-divisor $\mathcal{D}(P) = \emptyset \cdot z' + P \cdot z + \sum_{y \neq z, z'} (v_y + \tau) \cdot y$ to $\mathcal{S}'$. Thus, for all maximal polyhedra with non-maximal tail we obtain $\mathcal{S}_y = \mathcal{S}'_y$ for all $y \in \mathbb{P}^1$. From Remark A.1, we know that $X(\mathcal{D}(P))$ are affine spaces. Hence, we obtain an A-covering of X. □

Remark A.2. By Remark 4, for complexity 3 Theorem A.1 does not hold. For complexity 2, Theorem A.1 holds at least for surfaces and the threefolds V_5 and V'_{22} carrying a $\mathbb{K}^\times$-action.

Acknowledgement. The authors are grateful to Mikhail Zaidenberg for useful comments and remarks. Special thanks are due to the referee for careful reading, constructive criticism and important suggestions.

References

1. K. Altmann and J. Hausen, 'Polyhedral divisors and algebraic torus actions', *Math. Ann.* 334 (2006) 557–607.
2. K. Altmann, J. Hausen and H. Süss, 'Gluing affine torus actions via divisorial fans', *Transform. Groups* 13 (2008) 215–242.
3. K. Altmann, N. Ilten, L. Petersen, H. Süss and R. Vollmert, 'The geometry of T-varieties', *Contributions to algebraic geometry, IMPANGA lecture notes* (ed. P. Pragacz; European Mathematical Society, Zurich, 2012) 17–69.
4. I. V. Arzhantsev, U. Derenthal, J. Hausen and A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, to appear).

5. I. V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg, 'Flexible varieties and automorphism groups', *Duke Math. J.* 162 (2013) 767–823.

6. I. V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg, 'Infinite transitivity on affine varieties', *Birational geometry, rational curves, and arithmetic* (eds F. Bogomolov, B. Hassett and Yu. Tschinkel; Springer, New York, 2013), 1–13.

7. I. V. Arzhantsev, K. Kuyumzhiyan and M. Zaidenberg, 'Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity', *Sbornik Math.* 203 (2012) 923–949.

8. V. V. Batyrev and O. N. Popov, 'The Cox ring of a Del Pezzo surface', *Arithmetic of higher-dimensional algebraic varieties*, Progress in Mathematics 226 (eds B. Poonen and Yu. Tschinkel, Birkhauser, Cambridge, MA, 2004) 85–103.

9. F. Berchtold and J. Hausen, 'Homogeneous coordinates for algebraic varieties', *J. Algebra* 266 (2003) 636–670.

10. F. Berchtold and J. Hausen, 'Cox rings and combinatorics', *Trans. Amer. Math. Soc.* 359 (2007) 1205–1252.

11. F. Bogomolov, I. Karzhemanov and K. Kuyumzhiyan, 'Unirationality and existence of infinitely transitive models', *Birational geometry, rational curves, and arithmetic* (eds F. Bogomolov, B. Hassett, and Yu. Tschinkel, Springer, New York, 2013) 77–92.

12. M. Brion, D. Luna and Th. Vust, 'Expaces homogénes sphériques', *Invent. Math.* 84 (1986) 617–632.

13. J.-L. Colliot-Thélène and J.-J. Sansuc, 'Torseurs sous des groupes de type multiplicatif; applications à l'étude des points rationnels de certaines variétés algébriques', *C. R. Acad. Sci. Paris Sér. A-B* 282 (1976) 1113–1116.

14. J.-L. Colliot-Thélène and J.-J. Sansuc, 'La descente sur les variétés rationnelles. II', *Duke Math. J.* 54 (1987) 375–492.

15. D. A. Cox, 'The homogeneous coordinate ring of a toric variety', *J. Alg. Geometry* 4 (1995) 17–50.

16. G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia of Mathematical Sciences 136 (Springer, Berlin, 2006).

17. M. Furushima, 'The structure of compactifications of $\mathbb{C}^3$', *Proc. Japan Acad. Ser. A Math. Sci.* 68 (1992) 33–36.

18. J. Hausen, 'Cox rings and combinatorics II', *Moscow Math. J.* 8 (2008) 711–757.

19. Y. Hu and S. Keel, 'Mori dream spaces and GIT', *Michigan Math. J.* 48 (2000) 331–348.

20. N. Ilten and H. Süss, 'Polarized complexity-1 T-varieties', *Michigan Math. J.* 60 (2011) 561–578.

21. N. Ilten and R. Vollmert, 'Deformations of rational T-varieties', *J. Alg. Geometry* 21 (2012) 473–493.

22. V. A. Iskovskikh, 'Fano threefolds. II', *Math. USSR Izv.* 12 (1978) 469–506.

23. S. Kaliman and M. Zaidenberg, 'Affine modifications and affine hypersurfaces with a very transitive automorphism group', *Transform. Groups* 4 (1999) 53–95.

24. T. Kishimoto, Yu. Prokhorov and M. Zaidenberg, 'Group actions on affine cones', *Amer. Math. Soc., CRM Proceedings and Lecture Notes* 54 (2011) 123–164.

25. T. Kishimoto, Yu. Prokhorov and M. Zaidenberg, 'Affine cones over Fano threefolds and additive group actions', *Osaka J. Math.*, Preprint, 2011, arXiv:1106.1312.

26. A. Liendo, '$\mathbb{G}_a$-actions of fiber type on affine $\mathbb{T}$-varieties', *J. Algebra* 324 (2010) 3653–3665.

27. S. Mori and S. Mukai, 'Classification of Fano 3-folds with $B_2 \geqslant 2$', *Manuscripta Math.* 36 (1981) 147–162.

28. A. Yu. Perepechko, 'Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5', *Funct. Anal. Appl.* 47 (2013) 45–52.

29. V. L. Popov and E. B. Vinberg, 'Invariant theory', *Algebraic geometry IV* (eds A. N. Parshin and I. R. Shafarevich; Springer, Berlin, Heidelberg, New York, 1994).

30. V. V. Serganova and A. N. Skorobogatov, 'Del Pezzo surfaces and representation theory', *Algebra Number Theory* 1 (2007) 393–419.

31. V. V. Serganova and A. N. Skorobogatov, 'On the equations for universal torsors over del Pezzo surfaces', *J. Inst. Math. Jussieu* 9 (2010) 203–223.

32. A. N. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics 144 (Cambridge University Press, Cambridge, 2001).

33. H. Süss, 'Canonical divisors on T-varieties', Preprint, 2008, arXiv:0811.0626v1 [math.AG].

34. H. Süss, 'Fano threefolds with 2-torus action – a picture book', Preprint, 2008, arXiv:1308.2379 [math.AG].

Ivan Arzhantsev
Department of Higher Algebra
Faculty of Mechanics and Mathematics
Lomonosov Moscow State University
Leninskie Gory 1, GSP-1
Moscow 119991
Russia

and

National Research University Higher School of Economics
School of Applied Mathematics and Information Science
Bolshoi Trekhsvyatitelskiy 3
Moscow 109028
Russia

arjantse@mccme.ru

Hendrik Süß
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
The King's Buildings
Mayfield Road
Edinburgh
EH9 3JZ
United Kingdom

suess@sdf-eu.org

Alexander Perepechko
Faculty of Mechanics and Mathematics
St Petersburg State University
Universitetsky pr. 28
Stary Peterhof 198504
Russia

and

Institut Fourier
Université Grenoble I,
UMR 5582 CNRS-UJF BP 74
34802 St Martin d'Hères cedex
France

perepeal@gmail.com