
Volume 41, Issue 6, November 2013
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Operations Research Letters 41 (2013) 636–638

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On Pareto optimality conditions in the case of two-dimension non-convex utility space

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ARTICLE INFO

Article history:
Received 4 June 2013
Received in revised form
27 August 2013
Accepted 28 August 2013
Available online 5 September 2013

Keywords:
Pareto optimality
Two-dimension utility space
Scalarization

ABSTRACT

The paper suggests a new — to the best of the author's knowledge — characterization of Pareto-optimal decisions for the case of two-dimensional utility space which is not supposed to be convex. The main idea is to use the angle distances between the bisector of the first quadrant and points of utility space. A necessary and sufficient condition for Pareto optimality in the form of an equation is derived. The first-order necessary condition for optimality in the form of a pair of equations is also obtained.

1. Introduction

Many real-life problems can be properly considered only as multiple criteria decision making. The primary goal of the multi-objective optimization is to find so-called Pareto-optimal decisions that cannot be improved for each criterion without deteriorating the others. For example, the concept of Pareto optimality is used in solving some engineering and finance problems (Steuer [8]), and in analysis of insurance models (Golubin [2]). There is a large variety of methods for the determination of Pareto-optimal solutions, e.g., Miettinen [5], Steuer [8]. Many of them are based on a scalarization approach [5,1,6] which transforms the initial problem into a single-objective optimization problem. Usually it involves some parameters that are changed in order to detect different Pareto-optimal points: weights in the linear scalarization function or norm parameter for L_p-scalarization in [6]. Another group of methods for approximating the Pareto frontier for various decision problems with a small number of objectives (mainly, two) are provided in [7]. In Makela et al. [4], different types of zero-order geometric conditions are considered.

The present paper suggests a new "bisector scalarization" method, making use of the dimensionality (that equals two) particularity of the utility space to be considered. On this basis, a necessary and sufficient condition for weak Pareto optimality is derived in the form of an equation without involving any extra parameters defined by the decision maker. The first-order necessary conditions for weak and locally weak Pareto-optimal points are also investigated.

Mathematically, for our case, the multi-objective optimization problem can be written as:

$$F(a) \equiv (F_1(a), F_2(a)) \to \max \quad \text{s.t. } a \in A,$$

where A is a decision set or a set of admissible points, $F_i(a)$ are scalar functions or utilities defined on A. A point $a^* \in A$ is called *Pareto-optimal* if there is no other $a \in A$ such that $F_i(a) \geq F_i(a^*)$ for $i = 1, 2$ and at least one of the inequalities is strict. A point $a^* \in A$ is *weak Pareto-optimal* if there is no other $a \in A$ such that $F_i(a) > F_i(a^*)$ for $i = 1, 2$.

2. A "bisector" scalarization of the problem

Denote by $\mathcal{P} = \{y \in R^2 : y \geq 0\}$ the positive cone of R^2, and denote by $\mathcal{P}_1 = \{y \in R^2 : y > 0\}$ the strictly positive cone. For a set B and a point x in R^2, $B + x \stackrel{\text{def}}{=} \{y + x : y \in B\}$ denotes the set translated by x.

From the definition of Pareto optimality it is clear (see, e.g., Lin [3] and Fig. 1) that $a^* \in A$ is weak Pareto-optimal if and only if (iff) the open cone $\{\mathcal{P}_1 + F(a^*)\}$ and utility space $\mathcal{F} \stackrel{\text{def}}{=} \{F(a) : a \in A\}$ have no common point, namely,

$$\{\mathcal{P}_1 + F(a^*)\} \cap \mathcal{F} = \emptyset. \quad (1)$$

Analogously, a^* is Pareto-optimal iff $F(a^*)$ is the only common point of the closed cone $\{\mathcal{P} + F(a^*)\}$ and utility space, namely,

$$\{\mathcal{P} + F(a^*)\} \cap \mathcal{F} = \{F(a^*)\}. \quad (2)$$

Note that in our case the utility space $\mathcal{F}$ is not supposed to be convex. Due to the fact that $\mathcal{F} \subseteq R^2$, the positive cone $\mathcal{P}$ can be

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http://dx.doi.org/10.1016/j.orl.2013.08.012

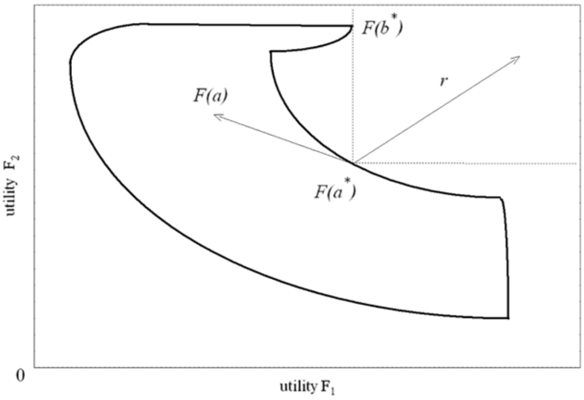

Fig. 1. A non-convex utility space in R^2 with a weak Pareto-optimal vector $F(a^*)$.

represented as a set of vectors y such that the angle between the bisector $r = (1/\sqrt{2}, 1/\sqrt{2})$ of the first quadrant and each y is not greater than $\pi/4$ in absolute value. Then condition (1) is translated as follows: the cosine of the angle between $F(a) - F(a^*)$ and r is not greater than $\cos|\pi/4| = 1/\sqrt{2}$ (see Fig. 1), that is,

$$\frac{(F_1(a) - F_1(a^*))/\sqrt{2} + (F_2(a) - F_2(a^*))/\sqrt{2}}{\|F(a) - F(a^*)\|} \le \frac{1}{\sqrt{2}}, \tag{3}$$

where $F(a) \neq F(a^*)$ and $\|x\|$ denotes the Euclidean norm in R^2, $\|x\| = \sqrt{x_1^2 + x_2^2}$.

Define a scalar function on A,

$$G(a) = \sup_{b \in A} F_1(b) - F_1(a) + F_2(b) - F_2(a) - \|F(b) - F(a)\|. \tag{4}$$

By construction, $G(a) \ge 0$ for any $a \in A$ and takes values in the extended real half-line $R_+ \cup \{\infty\}$. Now the necessary and sufficient condition (3) for weak Pareto optimality of a^* can be rewritten as $G(a^*) \le 0$. Taking into account that supremum in the right-hand side of (4) is attained, in particular, at $b = a^*$, the latter inequality is equivalent to $G(a^*) = 0$. Thus, we have proved the following proposition

Proposition 1. *A point a^* is weak Pareto-optimal iff a^* is a root of the equation*

$$G(a) = 0, \tag{5}$$

where $G(a)$ is defined in (4).

To find the Pareto-optimal (strong) solutions, return to condition (2). This means that the cone $\mathcal{P} + F(a^*)$ has no common point with $\mathcal{F}$, except $F(a^*)$. So, all we need is to find a weak Pareto-optimal point and to exclude the situation depicted in Fig. 1.

Proposition 2. *A point a^* is Pareto-optimal iff a^* is a root of (5), and maximum in the problem*

$$\max_{b \in A} F_1(b) - F_1(a^*) + F_2(b) - F_2(a^*) - \|F(b) - F(a^*)\| \tag{6}$$

is attained at a "unique" point in the sense that if b^* gives maximum in (6) then $F(b^*) = F(a^*)$.

Remark 1. The "bisector" scalarization introduced in (4) and providing necessary and sufficient conditions for Pareto optimality is different from the known (see, e.g., [8]) non-smooth maximin scalarization of n-criterion problem, where the equation corresponding (5) has the form $\sup_{b \in A} \min\{F_i(b) - F_i(a),\ i = 1, \ldots, n\} = 0$. However, an extension of the "bisector" scalarization to the case $n > 2$ encounters difficulties arising from impossibility of representing the cone $\mathcal{P}$ in an appropriate form.

3. Weak Pareto optimality conditions

The next statement deals with a "zero-order" condition for weak Pareto optimality of some point a^*, i.e., a condition for solvability of Eq. (5) with respect to a^*. Denote by $\Delta_i^*(a) = F_i(a) - F_i(a^*)$, $i = 1, 2$, where $a^* \in A$ and $a \in A$.

Proposition 3. *A point a^* is weak Pareto-optimal iff for all $a \in A$ such that $\Delta_1^*(a) + \Delta_2^*(a) > 0$, if any, it holds that $\Delta_1^*(a)\,\Delta_2^*(a) \le 0$.*

Proof. A point $b = a^*$ is a maximizer in problem (6) iff

$$\sqrt{(\Delta_1^*(a))^2 + (\Delta_2^*(a))^2} \ge \Delta_1^*(a) + \Delta_2^*(a) \tag{7}$$

for any $a \in A$. If a is such that $\Delta_1^*(a) + \Delta_2^*(a) \le 0$ then (7) holds. If $\Delta_1^*(a) + \Delta_2^*(a) > 0$ then, after squaring both parts of (7), we have that (7) holds iff $\Delta_1^*(a)\,\Delta_2^*(a) \le 0$. □

Let a^* be a weak Pareto-optimal point, i.e., a root of (5). Denote by b^* a maximum point in (6). It is easily seen that b^* is also weak Pareto-optimal. We will call such a pair (a^*, b^*) a *weak Pareto-optimal pair*. Of course, b^* can always be taken equal to a^*, so (a^*, a^*) is always a weak Pareto-optimal pair. A more interesting situation is that where a^* is not a unique solution to maximization problem (6). In the sequel of the paper we suppose that the decision set $A \subseteq R^k$ and utility functions $F_i(a)$, $i = 1, 2$, are differentiable on R^k.

Proposition 4. *Let (a^*, b^*) be a weak Pareto-optimal pair and b^* be an internal point of A. Then*

$$F_1'(b^*)\,\Delta_2^*(b^*) + F_2'(b^*)\,\Delta_1^*(b^*) = 0, \tag{8}$$

$$\Delta_1^*(b^*) + \Delta_2^*(b^*) - \|F(b^*) - F(a^*)\| = 0. \tag{9}$$

Proof. Suppose, at first, that $F(b^*) \neq F(a^*)$. Since b^* solves problem (6), the first-order optimality condition is

$$F_1'(b^*) + F_2'(b^*) - \frac{F_1'(b^*)\,\Delta_1^*(b^*) + F_2'(b^*)\,\Delta_2^*(b^*)}{\sqrt{[\Delta_1^*(b^*)]^2 + [\Delta_2^*(b^*)]^2}} = 0, \tag{10}$$

where, recall, $\Delta_i^*(b^*) = F_i(b^*) - F_i(a^*)$. From Proposition 1 it follows that

$$F_1(b^*) - F_1(a^*) + F_2(b^*) - F_2(a^*) = \|F(b^*) - F(a^*)\|. \tag{11}$$

Then (10) converts into

$$F_1'(b^*)\frac{\Delta_2^*(b^*)}{\Delta_1^*(b^*) + \Delta_2^*(b^*)} + F_2'(b^*)\frac{\Delta_1^*(b^*)}{\Delta_1^*(b^*) + \Delta_2^*(b^*)} = 0$$

or

$$F_1'(b^*)\,\Delta_2^*(b^*) + F_2'(b^*)\,\Delta_1^*(b^*) = 0.$$

The latter relation admits the degenerated case $F(b^*) = F(a^*)$ also. Taking (11) into account, we complete the proof. □

Remark 2. The statement of Proposition 4 becomes trivial if a maximum point b^* in (6) corresponds to the same point in utility space as a^*, $F(b^*) = F(a^*)$. Nevertheless, Proposition 4 provides an informative necessary condition in the case where a^* is just weak Pareto-optimal, but not strong Pareto-optimal, as shown in Fig. 1.

Below we will need a notion of local Pareto optimum. A point a^* is called local weak Pareto-optimal if there exists an ε-neighborhood $O_\varepsilon(a^*)$ of this point such that a^* is weak Pareto-optimal with respect to a smaller decision set $O_\varepsilon(a^*) \cap A$. The next statement actually repeats the known result on the first-order necessary condition of local weak Pareto optimality for n-criterion problem (see, e.g., Lin [3, Corollary 7.1]), but the proof in our case turns out to be much simpler.

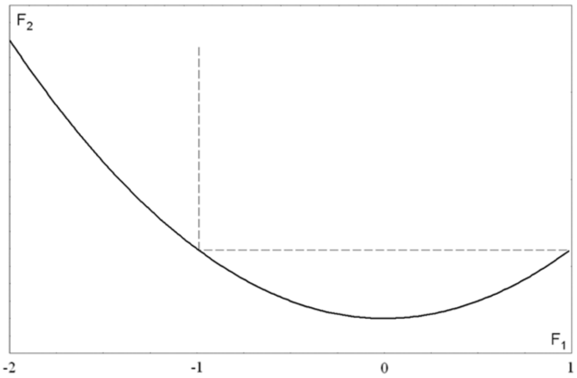

Fig. 2. The utility space for $F_1(a) = a$, $F_2(a) = a^2$, and $a \in [-2, 1]$.

Proposition 5. *Let a^* be a local weak Pareto-optimal point and an internal point of A. Then there exists $\lambda \in [0, 1]$ such that*

$$\lambda F_1'(a^*) + (1 - \lambda) F_2'(a^*) = 0. \quad (12)$$

Proof. Let $F_i'(a^*) \neq 0$ for $i = 1, 2$,—otherwise (12) trivially holds. Suppose the contrary to (12): there is no $\lambda > 0$: $F_1'(a^*) = -\lambda F_2'(a^*)$. Then there exists a vector c with $\|c\| = 1$ such that the scalar products $(F_i'(a^*), c) > 0,\ i = 1, 2$. These inequalities mean positiveness of the directional derivatives at point a^* with respect to direction c. Thus, $F_i(a^* + tc) > F_i(a^*)$ for sufficiently small $t > 0$, which contradicts to the above-supposed local weak Pareto optimality of a^*. □

Example. Consider a simple illustrative example, where utilities $F_1(a) = a$ and $F_2(a) = a^2$, and the decision set $A = [-2, 1]$. At first, employ Proposition 5, making use of an open decision set $(-2, 1)$. Since $F_1'(a) = 1$ and $F_2'(a) = 2a$, the first-order necessary condition (12) yields a set $(-2, 0]$. Clearly, (see Fig. 2) the set of locally weak Pareto-optimal points is $[-2, 0) \cup \{1\}$. So, condition (12) provides an extra point 0 and ignores the boundary point 1.

Now determine the set of all weak Pareto-optimal points via application of Proposition 3. The necessary and sufficient conditions are translated here into the following:

$\triangle_1^*(a) + \triangle_2^*(a) = (a - a^*)(a + a^* + 1) > 0 \Leftrightarrow$ (i) if $a^* > -1/2$ then $a \in [-2, -a^* - 1) \cup (a^*, 1]$, (ii) if $a^* \leq -1/2$ then $a \in [-2, a^*) \cup (-a^* - 1, 1]$;
$\triangle_1^*(a)\, \triangle_2^*(a) = (a - a^*)^2(a + a^*) \leq 0 \Leftrightarrow a \in [-2, -a^*] \cup \{a^*\}$.
This results in the set of weak Pareto-optimal points: $a^* \in [-2, -1] \cup \{1\}$.

Finally, according to Proposition 2, we obtain the set of all Pareto-optimal (strong) decisions by eliminating the weak optimal point $a^* = -1$ (see Fig. 2), which yields the set $[-2, -1) \cup \{1\}$.

Acknowledgments

This study was carried out within "The National Research University Higher School of Economics" Academic Fund Program in 2013–2014, research grant No. 12-01-0090. The work was also supported by Russian Fund for Basic Research, project 12-01-00078.

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