COLLINEATION GROUP AS A SUBGROUP OF THE SYMMETRIC GROUP

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Abstract. Let Ψ be the projectivization (i.e., the set of one-dimensional vector subspaces) of a vector space of dimension ≥ 3 over a field. Let H be a closed (in the pointwise convergence topology) subgroup of the permutation group $\mathfrak{S}_\Psi$ of the set Ψ. Suppose that H contains the projective group and an arbitrary self-bijection of Ψ transforming a triple of collinear points to a non-collinear triple. It is well-known from [9] that if Ψ is finite then H contains the alternating subgroup $\mathfrak{A}_\Psi$ of $\mathfrak{S}_\Psi$.

We show in Theorem 3.1 below that $H = \mathfrak{S}_\Psi$, if Ψ is infinite.

Let a group G act on a set Ψ. For an integer $N \geq 1$, the G-action on Ψ is called *N-transitive* if G acts transitively on the set of embeddings into Ψ of a set of N elements. This action is called *highly transitive* if for any finite set S the group G acts transitively on the set of all embeddings of S into Ψ. The action is highly transitive if and only if the image of G in the permutation group $\mathfrak{S}_\Psi$ is dense in the pointwise convergence topology, cf. below.

Let Ψ be a projective space of dimension ≥ 2, i.e., the projectivization of a vector space V of dimension ≥ 3 over a field k. The k-linear automorphisms of V induce permutations of Ψ, called *projective transformations*.

Suppose that a group G of permutations of the set Ψ contains all projective transformations and an element which is not a collineation, i.e., transforming a triple of collinear points to a non-collinear triple. The main result of this paper (Theorem 3.1) asserts that under these assumptions G is a dense subgroup of $\mathfrak{S}_\Psi$ if Ψ is infinite.

Somewhat similar results have already appeared in geometric context. We mention only some of them:

- J.Huisman and F.Mangolte have shown in [7, Theorem 1.4] that the group of algebraic diffeomorphisms of a rational nonsingular compact connected real algebraic surface X is dense in the group of all permutations of the set Ψ of points of X;
- J.Kollár and F.Mangolte have found in [10, Theorem 1] a collection of transformations generating, together with the orthogonal group $\mathrm{O}(3,1)$, the group of algebraic diffeomorphisms of the two-dimensional real sphere.

We note, however, that our result allows to work with quite arbitrary algebraically non-closed fields.

Example. Let $K|k$ be a field extension and τ be a self-bijection of the projective space $\mathbb{P}_k(K) := K^\times/k^\times$, satisfying one of the following conditions: (i) $\tau : x \mapsto 1/x$ for all $x \in \mathbb{P}_k(K)$ and $K|k$ is separable of degree > 2,[1] (ii) $\tau : x \mapsto x^n$ for some integer $n > 1$ and all $x \in \mathbb{P}_k(K)$, the subfield k contains all roots of unity of all n-primary degrees in K, and the multiplicative group $K^\times$ is n-divisible (e.g., if the field K is algebraically closed). Then the group generated by the group $\mathrm{PGL}(K)$ of projective transformations (K is considered as a k-vector space) and τ is N-transitive on $\mathbb{P}_k(K)$ for any N. Indeed, it is clear that such τ's are not collineations. Hence our Theorem implies the result.

The proof of Theorem 3.1 consists of verifying the N-transitivity of the group H for all integer $N \geq 1$.

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[1] If $[K : k] = 2$ then τ is projective; if k is of characteristic 2 and $K \subseteq k(\sqrt{k})$ then τ is identical on $\mathbb{P}_k(K)$.

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1. Permutation groups: topology and closed subgroups

Let Ψ be a set and G be a group of its self-bijections. We consider G as a topological group with the base of open subgroups formed by the pointwise stabilizers of finite subsets in Ψ. Then G is a totally disconnected group. In particular, any open subgroup of G is closed.[2]

Denote by $\mathfrak{S}_\Psi$ the group of all permutations of Ψ. Then the above base of open subgroups of $\mathfrak{S}_\Psi$ is formed by the subgroups $\mathfrak{S}_{\Psi|T}$ of permutations of Ψ identical on T, where T runs over all finite subsets of Ψ.

Lemma 1.1. (1) *For any finite non-empty $T \subset \Psi$, $T \neq \Psi$, the normalizer $\mathfrak{S}_{\Psi,T}$ of $\mathfrak{S}_{\Psi|T}$ in $\mathfrak{S}_\Psi$ (i.e., the group of permutations of Ψ, preserving the subset T) is maximal among the proper subgroups of $\mathfrak{S}_\Psi$* ([13], [1]).

(2) *Any proper open subgroup of $\mathfrak{S}_\Psi$ which is maximal among the proper subgroups of $\mathfrak{S}_\Psi$ coincides with $\mathfrak{S}_{\Psi,T}$ for a finite non-empty $T \subset \Psi$, if Ψ is infinite.*

(3) *Any proper open subgroup of $\mathfrak{S}_\Psi$ is contained in a maximal proper subgroup of $\mathfrak{S}_\Psi$.*

Proof. By definition of our topology, any open proper subgroup U in $\mathfrak{S}_\Psi$ contains the subgroup $\mathfrak{S}_{\Psi|T}$ for a non-empty finite subset $T \subset \Psi$. Assume that such T is minimal.

We claim that $\sigma(T) = T$ for all $\sigma \in U$. Indeed, if $\sigma(t) \notin T$ for some $t \in T$ and $\sigma \in \mathfrak{S}_\Psi$ then (i) it is easy to see that the subgroup $\tilde{U}$ generated by σ and $\mathfrak{S}_{\Psi|T}$ meets $\mathfrak{S}_{\Psi|T\smallsetminus\{t\}}$ by a dense subgroup, (ii) $\tilde{U}$ contains $\mathfrak{S}_{\Psi|T\smallsetminus\{t\}}$, since both subgroups, as well as their intersection, are open, and thus, the intersection is closed. This contradicts to the minimality of T, and finally, U is contained in $\mathfrak{S}_{\Psi,T}$.

The subgroups $\mathfrak{S}_{\Psi,T}$ are maximal, since they are not embedded to each other for various T. □

The following 'folklore' model-theoretic description of *closed* subgroups of $\mathfrak{S}_\Psi$ is well-known, cf. [2, 4, 5]. Suppose that Ψ is countable. Let $\mathcal{L} = \{R_i\}_{i\in I}$ be a countable relational language and $\mathcal{A} = (\Psi, \{R_i\}_{i\in I})$ be a structure for $\mathcal{L}$ with universe Ψ. Then $\mathrm{Aut}(\mathcal{A})$, the group of automorphisms of $\mathcal{A}$, is a closed subgroup of $\mathfrak{S}_\Psi$. Conversely, let H be a subgroup of $\mathfrak{S}_\Psi$. For each n, let I_n be the set of H-orbits on Ψ^n. Set $I := \coprod_{n\geq 1} I_n$ and consider the structure $\mathcal{A}_H := (\Psi, \{R_i^H\}_{i\in I})$ associated with H, where $R_i^H = i \subset \Psi^{n(i)}$. One easily checks that $\mathrm{Aut}(\mathcal{A}_H)$ is the closure of H.

Apart from that, G.Bergman and S.Shelah prove the following result. Assume that Ψ is countable. Let us say that two subgroups $G_1, G_2 \subseteq \mathfrak{S}_\Psi$ are equivalent if there exists a finite set $U \subseteq \mathfrak{S}_\Psi$ such that G_1 and U generate the same subgroup as G_2 and U. It is shown in [3] that the closed subgroups of $\mathfrak{S}_\Psi$ lie in precisely four equivalence classes under this relation. Which of these classes a closed subgroup G belongs to depends on which of the following statements about open subgroups of G holds:

1. Any open subgroup of G has at least one infinite orbit in Ψ.
2. There exist open subgroups $H \subset G$ such that all the H-orbits are finite, but none such that the cardinalities of these orbits have a common finite bound.
3. The group G is not discrete, but there exist open subgroups $H \subset G$ such that the cardinalities of the H-orbits have a common finite bound.
4. The group G is discrete.

Examples of a set Ψ and a group G acting on it: (1) (a) Ψ is arbitrary and $G = \mathfrak{S}_\Psi$. (b) Ψ is a projective space over a field with non-discrete automorphism group and G is the collineation group. (c) Ψ is an infinite-dimensional projective (or affine, or linear) space and G is the projective (or affine, or linear) group. (d) If Ψ is arbitrary, G contains a transposition ι of some elements

[2] Indeed, the complement to an open subgroup is the union of translations of the subgroup, so it is open.

$p, q \in \Psi$ and G is 2-transitive then G is a dense subgroup in $\mathfrak{S}_\Psi$.[3] (Indeed, as all transpositions generate any finite symmetric group, it suffices to show that G contains all the transpositions. If G is 2-transitive then for any pair of distinct elements $p', q' \in \Psi$ there is $g \in G$ with $g(p) = p', g(q) = q'$, and thus, $g\iota g^{-1}$ is the transposition the elements p' and q'.)

(4) Ψ is the set of closed (or rational) points of a variety and G is the group of points of an algebraic group acting faithfully on this variety; Ψ is the function field of a variety and G is a field automorphism group of Ψ.

2. Dense subgroups of the symmetric groups and the transitivity

It is evident that the G-action on Ψ is highly transitive for any *dense* subgroup $G \subseteq \mathfrak{S}_\Psi$. Conversely, if a group G is highly transitive on Ψ then it is dense in $\mathfrak{S}_\Psi$. Indeed, for any $\sigma \in \mathfrak{S}_\Psi$ any neighborhood of σ contains a subset $\sigma\,\mathfrak{S}_{\Psi|T}$ for a finite subset $T \subset \Psi$, on the other hand, the identity embedding of T into Ψ and the restriction of σ to T belong to a common G-orbit, i.e., $\tau|_T = \sigma|_T$ for some $\tau \in G$.

We use the following terminology: (i) an *N-set* in Ψ is a subset in Ψ of order N; (ii) an *N-configuration* in Ψ is an ordered N-tuple of pairwise distinct points of Ψ. The group G acts naturally on the sets of N-sets and of N-configurations in Ψ. Two configurations or sets are called *G-equivalent* if they belong to the same G-orbit.

The G-action on the set of N-configurations in Ψ commutes with the natural action of the symmetric group $\mathfrak{S}_N$ (given by $\sigma(T) := (p_{\sigma(1)}, \dots, p_{\sigma(N)})$ for all $T = (p_1, \dots, p_N)$ and all $\sigma \in \mathfrak{S}_N$).

Lemma 2.1. *For each $N \geq 1$ consider the following conditions:* $(\mathrm{i})_N$ *G is N-transitive on Ψ,* $(\mathrm{ii})_N$ *any N-configuration in Ψ is G-equivalent to a fixed N-configuration R in Ψ,* $(\mathrm{iii})_N$ *any N-configuration T is G-equivalent to $\sigma(T)$ for any permutation $\sigma \in \mathfrak{S}_N$.*[4]

Then the conditions $(\mathrm{i})_N$ *and* $(\mathrm{ii})_N$ *are equivalent;* $(\mathrm{i})_N$ *implies* $(\mathrm{iii})_N$*;* $(\mathrm{iii})_{N+1}$ *implies* $(\mathrm{i})_N$ *if $|\Psi| > N$. In particular, the group G is highly transitive on Ψ if and only if for all N, all N-configurations T and all permutations $\sigma \in \mathfrak{S}_N$ the N-configurations T and $\sigma(T)$ are G-equivalent.*

Proof. Implications $(\mathrm{iii})_N \Leftarrow(\mathrm{i})_N \Leftrightarrow(\mathrm{ii})_N$ are evident. Assume now the condition $(\mathrm{iii})_{N+1}$.

For an arbitrary pair N-configurations $T = (p_1, \dots, p_N)$ and $T' = (q_1, \dots, q_N)$ denote by s, $0 \leq s \leq N$, the only integer such that $p_1, \dots, p_s, q_1, \dots, q_s$ are pairwise distinct and $p_i = q_i$ for all i, $s < i \leq N$. To show that T and T' are G-equivalent, we proceed by induction on $s \geq 0$, the case $s = 0$ being trivial. For $s > 0$, the $(N+1)$-configurations $(p_1, \dots, p_N, q_1)$ and $(q_1, p_2, \dots, p_N, p_1)$ are G-equivalent, so the N-configurations T and $(q_1, p_2, \dots, p_N)$ are also G-equivalent.

On the other hand, the sets $\{q_1, p_2, \dots, p_N\}$ and $\{q_1, \dots, q_N\}$ have $N - s + 1$ common elements, so the N-configurations T' and $(q_1, p_2, \dots, p_N)$ are G-equivalent by the induction assumption. □

It is shown by H.D.Macpherson and P.M.Neumann in the usual framework of Zermelo–Fraenkel set theory with the axiom of choice (cf. [11, Observation 6.1]) that any maximal proper non-open subgroup of $\mathfrak{S}_\Psi$ is dense in $\mathfrak{S}_\Psi$. In particular, if Ψ is a projective space then the collineation group (which is obviously closed) is not maximal. However, it will follow from Theorem 3.1 below that the collineation group is maximal among proper *closed* subgroups (in dimension > 1).

3. Projective group as a subgroup of the symmetric group

In this section we prove the following

Theorem 3.1. *Let k be a field, V be a k-vector space of dimension > 2 and $\Psi := \mathbb{P}_k(V) = (V \smallsetminus \{0\})/k^\times$ be the projectivization of V. Let H be a subgroup of $\mathfrak{S}_\Psi$ containing* $\mathrm{PGL}(V)$ *and an arbitrary self-bijection of Ψ which is not a collineation. Then H is dense in $\mathfrak{S}_\Psi$, if Ψ is infinite.*

[3] The transitivity does not suffices: if $\Psi = \mathbb{Z}$ and G is generated by the transposition (01) and by the shift $n \mapsto n+2$ then the G-orbit of the pair $(0, 1)$ is $\{(a, a + (-1)^a) \mid a \in \mathbb{Z}\}$, so G is not dense in $\mathfrak{S}_\Psi$.

[4] Instead of all permutations one can equivalently consider only a generating system of the group $\mathfrak{S}_N$. E.g., the transpositions (involutions interchanging only a pair of elements of $\{1, \dots, N\}$).

Proof. By induction on N, we are going to show that H is N-transitive on Ψ for any $N \geq 1$. The cases $N = 1, 2$ are clear, since even the group $\mathrm{PGL}(V)$ is 2-transitive on Ψ. Though $\mathrm{PGL}(V)$ (and even the bigger group of all the collineations of Ψ) is not 3-transitive on Ψ if $\dim_k V \geq 3$, the 3-transitivity of H on Ψ is evident (whenever $\dim_k V \geq 2$): all general 3-configurations, as well as all collinear 3-configurations, are $\mathrm{PGL}(V)$-equivalent, while any non-collineation sends some collinear 3-configuration to a general 3-configuration. In other words, the case $N = 3$ is also trivial.

There are two possibilities for the group H:

A. There exist hyperplanes P, P' in Ψ and an element $h \in H$ such that $h(\Psi \smallsetminus P) \subseteq P'$. (This can happen only if Ψ is infinite, i.e., either the field k is infinite or Ψ is of infinite dimension.)

B. For any element $h \in H$ and any hyperplane P in Ψ the set $h(\Psi \smallsetminus P)$ is not contained in a hyperplane.

Lemma 3.2. *In the case A, all N-configurations in Ψ are H-equivalent for all N.*

Proof proceeds by induction on N, the cases $N = 1, 2$ being trivial. Let $T = (p_1, \dots, p_N)$ and $T' = (p'_1, \dots, p'_N)$ be a pair of N-configurations. We need to show that $\xi(T) = T'$ for some $\xi \in H$. By the induction assumption, we may assume that $(p_2, \dots, p_N) = (p'_2, \dots, p'_N)$.

It remains to show that for any point $p \in \Psi$ lying on neither of the lines passing through p_1 and one of the points of the set $\{p_2, \dots, p_N\}$ there exists an element $\xi_1 \in H$ with $\xi_1(T) = (p, p_2, \dots, p_N) =: T''$. (Indeed, the assumption A implies that $|\Psi| \geq 2N|k|$, so we can choose a point p outside the union of the $2N-2$ lines joining the points of the set $\{p'_1, p_1\}$ and the points of the set $\{p_2, \dots, p_N\}$. Then the point p'_1 lies on neither of the lines passing through p and one of the points of the set $\{p_2, \dots, p_N\}$. Therefore, $\xi_2(T'') = T'$ for some $\xi_2 \in H$, and thus, $\xi_2\xi_1(T) = T'$.)

By the hypothesis A, there exist hyperplanes P, P' in Ψ and an element $h \in H$ such that $h(\Psi \smallsetminus P) \subseteq P'$. As h is surjective (and even bijective), $h(r), h(q) \notin P'$ for a pair of distinct points $r, q \in P$. First, we can find a projective transformation $g_2 \in \mathrm{PGL}(V)$ sending the pair (p_1, p) to the pair (r, q) such that the support of $g_2(T)$ meets P only at r. Next, we can find a projective involution $g_1 \in \mathrm{PGL}(V)$ identical on P' and interchanging the points $h(q)$ and $h(r)$. Then $g_2^{-1}h^{-1}g_1hg_2(T) = T''$. □

From now on we assume the hypothesis B.

Let $N \geq i \geq 1$ be integers and $T = (p_1, \dots, p_N)$ be an N-configuration $T = (p_1, \dots, p_N)$ in Ψ. Denote by $P_T^{\{i\}}$ the projective envelope of $p_1, \dots, \widehat{p_i}, \dots, p_N$.

We say that T is *i-disjoint* if $p_i \notin P_T^{\{i\}}$, and that T is *disjoint* if T is i-disjoint for some i.

Lemma 3.3. *Let $N \geq i \geq 1$ be integers. Assume that H is $(N-1)$-transitive on Ψ.*

Then, in the case B, all i-disjoint N-configurations are H-equivalent.

Proof. Let $T = (p_1, \dots, p_N)$ and $T' = (p'_1, \dots, p'_N)$ be i-disjoint N-configurations for some i (so $\dim \Psi \geq 2$ if $N \geq 3$).

As H is $(N-1)$-transitive on Ψ, we can choose an element $h \in H$ with $h(p_1, \dots, \widehat{p_i}, \dots, p_N) = (p'_1, \dots, \widehat{p'_i}, \dots, p'_N)$. As $h(\Psi \smallsetminus P_T^{\{i\}})$ is not contained in $P_{T'}^{\{i\}}$, there exist: (i) a point $p \in \Psi \smallsetminus P_T^{\{i\}}$ such that $hp \notin P_{T'}^{\{i\}}$, (ii) $g_2 \in \mathrm{PGL}(V)$ identical on $P_T^{\{i\}}$ and sending p_i to p, (iii) $g_1 \in \mathrm{PGL}(V)$ identical on $P_{T'}^{\{i\}}$ and sending $h(p)$ to p'_i. Therefore, $g_1hg_2(T) = T'$, as desired. □

Lemma 3.4. *Assume that H is $(N-1)$-transitive on Ψ for an integer $N \geq 1$.*

Then, in the case B, all disjoint N-configurations are H-equivalent. In particular, the permutation group $\mathfrak{S}_N$ preserves the H-equivalence class of any disjoint N-configuration.

Proof. Fix some pair $i \neq j$ with $1 \leq i, j \leq N$. Let us show that any i-disjoint N-configuration $T = (p_1, \dots, p_N)$ in Ψ is H-equivalent to a j-disjoint N-configuration. As H is $(N-1)$-transitive on Ψ, we can choose an element $\xi \in H$ such that $\xi(p_s) = p_s$ for all $s \neq i, j$ and $\xi(p_j) = p_i$. If $\xi(p_i) \in P_T^{\{i\}}$ then $\xi(T)$ is j-disjoint. If $\xi(p_i) \notin P_T^{\{i\}}$ then there exists a projective involution ι fixing $P_T^{\{i\}}$ and interchanging $\xi(p_i)$ and p_i, i.e., $\xi^{-1}\iota\xi(T)$ (having the same support as T) is j-disjoint. In both cases T is H-equivalent to a j-disjoint N-configuration, while all j-disjoint N-configurations are H-equivalent. □

The following Lemma reduces verification of the N-transitivity to checking of the H-equivalence of all N-sets.

Lemma 3.5. *Let P be a hyperplane in Ψ. Suppose that H is $(N-1)$-transitive on Ψ.*

Then, in the case B, the permutation group $\mathfrak{S}_N$ preserves the H-equivalence class of any N-configuration in Ψ, if $|P| \geq N-2$.

Proof. Suppose that an N-configuration T in Ψ is not H-equivalent to a disjoint one. By $(N-1)$-transitivity of H, we may assume that $T = (q_1, \dots, q_N)$, where $(q_1, \dots, \widehat{q_i}, \dots, \widehat{q_j}, \dots, q_N)$ is a fixed $(N-2)$-configuration in P for some pair $i \neq j$ with $1 \leq i, j \leq N$, and $q_i \in \Psi$ is a fixed point outside of P. Then, as T is not H-equivalent to a disjoint configuration, $q_j \in \Psi \smallsetminus P$.

To show that T is H-equivalent to $\sigma(T) := (q_{\sigma(1)}, \dots, q_{\sigma(N)})$ for any permutation $\sigma \in \mathfrak{S}_N$ it suffices to verify that T is H-equivalent to $\sigma_{ij}(T)$ for the transposition σ_{ij} of any pair $1 \leq i < j \leq N$. But this is clear, since there exists a projective involution ι_{ij} fixing P and interchanging q_i and q_j. □

For any pair of subsets $\Pi_1, \Pi_2 \subset \Psi$ we introduce the subset $H_{\Pi_1,\Pi_2} := \{h \in H \mid h(\Pi_1) \subseteq \Pi_2\}$ in H. There is a natural composition law: $H_{\Pi_1,\Pi_2} \times H_{\Pi_2,\Pi_3} \to H_{\Pi_1,\Pi_3}$. One has $gH_{\Pi_1,\Pi_2} = H_{\Pi_1,g\Pi_2}$ and $H_{\Pi_1,\Pi_2}g = H_{g^{-1}\Pi_1,\Pi_2}$ for any $g \in H$.

Define a binary relation $\succ_{\Pi_1,\Pi_2}$ on Ψ by the condition $x \succ_{\Pi_1,\Pi_2} y$ if and only if one has $h(y) \in \Pi_2$ for any $h \in H_{\Pi_1,\Pi_2}$ such that $h(x) \in \Pi_2$. Clearly, $g(x) \succ_{g(\Pi_1),\Pi_2} g(y)$ for any $g \in H$ if $x \succ_{\Pi_1,\Pi_2} y$.

Lemma 3.6. *The binary relation $\succ_{\Pi_1,\Pi_2}$ is reflexive and transitive. Moreover, if Π_1 is contained in a proper projective subspace P in Ψ then* (i) *the restriction of $\succ_{\Pi_1,\Pi_2}$ to $\Psi \smallsetminus P$ is an equivalence relation;* (ii) *the equivalence classes in $\Psi \smallsetminus P$ are complements to P of projective subspaces in Ψ.*

Proof. The reflexivity and the transitivity of $\succ_{\Pi_1,\Pi_2}$ are trivial. (i) If Π_1 is contained in a subspace $P \subset \Psi$ then for any pair of points $x, y \in \Psi \smallsetminus P$ there is a projective involution ι identical on P and interchanging x and y. Now, if $x \succ_{\Pi_1,\Pi_2} y$ then $h(y) \in \Pi_2$ for any $h \in H_{\Pi_1,\Pi_2}$ such that $h(x) \in \Pi_2$. As $H_{\Pi_1,\Pi_2} = H_{\Pi_1,\Pi_2}\iota$, one has $h'\iota(y) \in \Pi_2$ for any $h' \in H_{\Pi_1,\Pi_2}$ such that $h'\iota(x) \in \Pi_2$, i.e., $y \succ_{\Pi_1,\Pi_2} x$.

(ii) A subset in $\Psi \smallsetminus P$ is a complement to P of a projective subspace in Ψ if and only if together with a pair of points $x, y \in \Psi \smallsetminus P$ it contains the line $\overline{xy}$ passing through them (eventually, punctured at the meeting point with P). Thus, we need to show that for any triple of pairwise distinct collinear points $x, y, z \in \Psi \smallsetminus P$ such that $x \succ_{\Pi_1,\Pi_2} y$ one has $x \succ_{\Pi_1,\Pi_2} z$. Indeed, there is a projective transformation α identical on $P \cup \{x\}$ and sending y to z. As $H_{\Pi_1,\Pi_2} = H_{\Pi_1,\Pi_2}\alpha$, one has $h'\alpha(y) \in \Pi_2$ for any $h' \in H_{\Pi_1,\Pi_2}$ such that $h'\alpha(x) \in \Pi_2$, i.e., $x \succ_{\Pi_1,\Pi_2} z$. □

Lemma 3.7. *Suppose that H is $(N-1)$-transitive on Ψ. Then, in the case B, any N-set is H-equivalent to a disjoint one,*[5] *if $N \leq |k| + 2$.*

Proof. By $(N-1)$-transitivity of H, if $N \leq |k| + 2 = \#\mathbb{P}^1(k) + 1$ then any N-set in Ψ is H-equivalent to $T = \{q_1, q_2\} \cup R$, where R is a fixed $(N-2)$-subset of a projective line $l \subset \Psi$ and $q_2 \in \Psi$ is a fixed point. If $N = |k| + 2 = \#\mathbb{P}^1(k) + 1$ then taking $q_2 \in l$ we get a disjoint T.

From now on $N \leq |k| + 1$. Then fixing $q_2 \in \Psi \smallsetminus l$ we may assume that $q_1 \in \Psi \smallsetminus l$, that $q_2 \succ_{R,l} q_1$ and that T is coplanar, as otherwise T is H-equivalent to a disjoint N-set.

Let q_0 be the intersection point of l and the line $\overline{q_1q_2}$ passing through q_1 and q_2.

Suppose first that $q_0 \notin R$. Then $\{q_0, q_2\} \cup R$ is a disjoint N-set and Lemma 3.4 implies that there is $\xi \in H$ interchanging q_0 and q_2 and identical on R. Then $\xi(q_0) \notin l$. By Lemma 3.6 (ii), where we take $\Pi_1 = R$ and $\Pi_2 = l$, one has $\xi(\overline{q_1q_2} \smallsetminus \{q_0\}) \subset l$. We can choose such $\psi \in H$ that $\psi(R) \subset \overline{q_1q_2}$ and $\psi(q_2) = q_0$. If $\psi(T)$ is not a disjoint set then $\psi(q_1) \in \overline{q_1q_2}$, but then $\xi\psi(T)$ is a disjoint set.

This settles the case $q_0 \notin R$, so let us now suppose that $q_0 \in R$ and that T is not H-equivalent to a disjoint N-set. If we change q_2 in its $\succ_{R,l}$-equivalence class then the resulting new T is not

[5]Naturally, a set is called *disjoint* if one of its points is not in the projective envelope of the others.

H-equivalent to a disjoint N-set. Then $\overline{q_1 q_2} \setminus \{q_0\}$ is precisely the $\succ_{R,l}$-equivalence class of q_1. (Otherwise, by Lemma 3.6 (ii), we can choose a new q_2 in its equivalence class so that the new q_0 is not in R, but then T is H-equivalent to a disjoint N-set.) We can find an element $\gamma \in H$ such that all the points in the support of $\gamma(T)$ are on the line $\overline{q_1 q_2}$, and $\gamma(q_1) = q_0 \in R \subset l$.

Then any $\beta \in H$ fixing $\gamma(q_2)$ and inducing a cyclic permutation of R induces an automorphism of the $\succ_{R,l}$-equivalence class $\overline{q_1 q_2} \setminus \{q_0\}$, so $\beta\gamma(T)$ is disjoint. □

4. Case of finite field k and the end of the proof

Lemma 4.1. *Let V be a vector space over a finite field k, $\tilde{h}$ be a self-embedding of $V \setminus \{0\}$, $J \subset V$ be a finite set such that J and $\tilde{h}(J)$ consist of independent vectors and $J \subset P_0 \subset P_1 \subset P_2 \subset \cdots \subset V = \bigcup_i P_i$, $\dim P_i = |J| + i$, be a flag of vector subspaces. Then there exists a basis $\mathcal{B} = J \cup \{e_1, e_2, \dots\}$ of V such that $e_i \in P_i \setminus P_{i-1}$ and $\tilde{h}(\mathcal{B})$ consists of independent vectors.*

Proof. It is possible to choose such e_i inductively, since $\#\tilde{h}(P_i \setminus P_{i-1}) = \#(P_i \setminus P_{i-1}) = (\#k - 1)(\#k)^{|J|+i-1} > \#[\langle \tilde{h}(J), \tilde{h}(e_1), \dots, \tilde{h}(e_{i-1})\rangle \setminus \{0\}] = \#(P'_{i-1} \setminus \{0\}) = (\#k)^{|J|+i-1} - 1$. □

Lemma 4.2. *Suppose that H is $(N-1)$-transitive on Ψ, the field $k \cong \mathbb{F}_q$ is finite of order q and either Ψ is infinite-dimensional, or $\dim \Psi \geq N - 1 \geq 3$ and $q > 2$. Then, in the case B, any N-set in Ψ is H-equivalent to a general N-set in Ψ.*

Proof. For any subset $A \subset \Psi$ denote by P_A the projective envelope of A. Let P be an $(N-2)$-dimensional subspace in Ψ. There is a disjoint N-set in P: $N \leq \#\mathbb{P}^{N-3}(\mathbb{F}_q) + 1 = \frac{q^{N-2}-1}{q-1} + 1$. Then, by Lemma 3.4, there exists $h \in H$ such that $h(P)$ is contained in no $(N-2)$-dimensional subspace. Fix such h. Let S be a maximal independent subset in P such that $h(S)$ is also independent.

By Lemma 4.1, one has $P_S = P$ (in other words, $|S| = N - 1$).

By $(N-1)$-transitivity, any N-set is H-equivalent to an N-set $T = \{p_1\} \cup S$ in Ψ. If either $p_1 \in P$ and p_1 is not in general position with respect to S, or $p_1 \notin P$ then T is disjoint, so by Lemma 3.4, T is H-equivalent to a general N-set. Suppose therefore that T is a general N-set in P.

As all general N-sets in P are $\mathrm{PGL}(V)$-equivalent, we may assume that $h(p') \in P_{h(S)}$ and that $h(p')$ is in general position with respect to $h(S)$ for any point $p' \in P$ in general position with respect to S. Note, however, that there is only one such p' in the case $q = 2$.

Fix some $p \in P$ such that $h(p) \notin P_{h(S)}$. In particular, the set $\{p\} \cup S$ is H-equivalent to a general set. Then p is a point of P_I in general position with respect to I for some subset $I \subsetneq S$ with $|I| \geq 2$.

Fix some $s \in I$ and choose homogeneous coordinates $X_2, \dots, X_N$ on P such that the elements of S are given by $X_2 = \cdots = \widehat{X_i} = \cdots = X_N = 0$ for $2 \leq i \leq N$, the elements of I correspond to $2 \leq i \leq |I|+1$, the point s corresponds to $i = |I|+1$ and the point p is given by $X_2 = \cdots = X_{|I|+1} \neq 0$ & $X_{|I|+2} = \cdots = \widehat{X_i} = \cdots = X_N = 0$.

In the case $q = 2$, our choice of h contradicts to the conclusion of Lemma 4.3 below, and thus, to the assumption that T is not H-equivalent to a general set.

In the case of $q > 2$, we are looking for a point $s' \in P$ in general position with respect to S and in general position with respect to $\{p\} \cup (S \setminus \{s\})$. In coordinates: $X_2 \cdots X_N \neq 0$ and $\prod_{i=1}^{|I|}(X_i - X_{|I|+1}) \neq 0$. As $q \geq 3$, we can find such a point.

Then $\{p, s'\} \cup S \setminus \{s\}$ is a general N-set in P and its image under h is disjoint, i.e., it is H-equivalent to a general N-set. □

Lemma 4.3. *Let $m, n \geq 1$ be integers, and P be an m-dimensional vector subspace of an infinite-dimensional $\mathbb{F}_2$-vector space V. Let h be a self-bijection of V such that $h(0) = 0$. Suppose that the image under h of any general $(n+1)$-set in any n-dimensional vector subspace is not general.*

Then $h(P)$ is a vector subspace of V isomorphic to P.

Proof. Set $m = \dim P$ and consider an $(m+1)$-dimensional vector subspace $\tilde{P}$ in V containing P. By Lemma 4.1, there exist a basis $\mathcal{B}$ of V such that (i) the vectors of $\tilde{h}(\mathcal{B})$ are independent, (ii) $\mathcal{B}$ contains bases of P and of $\tilde{P}$. If x is the unique element of $\mathcal{B} \cap (\tilde{P} \setminus P)$, we write also $\mathcal{B} = \mathcal{B}_x$.

Denote by V_x^+ the hyperplane in V of sums of even number of elements of $\mathcal{B}_x$. Denote by P^+ (resp., by $\tilde{P}_x^+$) the corresponding hyperplane in P (resp., in $\tilde{P}$, $\tilde{P}_x^+$ does not contain x). There are precisely three hyperplanes in $\tilde{P}$ containing P^+: (i) P, (ii) a hyperplane containing x, (iii) $\tilde{P}_x^+$.

Set $V_x^- := P \smallsetminus V_x^+$, $P^- := P \smallsetminus P^+$ and $\tilde{P}_x^- := \tilde{P} \smallsetminus \tilde{P}_x^+$. Suppose that h does not transform a general $(n+1)$-set in any n-dimensional vector subspace to a general one. Then for any general set $I \subset V$ of order n with general $h(I)$ one has $h(\sum_{v\in I} v) = \sum_{v\in I} h(v)$. By Lemma 4.4, $h(V_x^-)$ is an affine subspace of the span of $h(\mathcal{B}_x)$ and $h|_{V_x^-}$ is the restriction of a linear endomorphism of V.

As $P \cap \tilde{P}_x^+ = P^+$, one has $\#(\tilde{P} \smallsetminus (P \cup \tilde{P}_x^+)) = 2^m - 2^{m-1} = 2^{m-1} > \#(P \smallsetminus P^-) = 2^{m-1} - 1$, and thus, there is $y \in \tilde{P} \smallsetminus (P \cup \tilde{P}_x^+)$ with $h(y) \notin P$. Therefore, $\tilde{P} = P \cup \tilde{P}_x^+ \cup \tilde{P}_y^+$.

Then the restriction of h to $\tilde{P}_x^- = P^- \cup (\tilde{P}_y^+ \smallsetminus P^+)$ coincides with the restriction of a linear map, and hence, if $t \in \tilde{P}_x^-$ and $h(t)$ is in the linear span L of $h(P \cap \mathcal{B}_x)$ then t is in the linear envelope of P^-, i.e., $t \in P$. Similarly, if $t \in \tilde{P}_y^-$ and $h(t)$ is in L then $t \in P$. As $\tilde{P} = P \cup \tilde{P}_x^- \cup \tilde{P}_y^-$, we get that $h(x_0)$ is in L a point $x_0 \in \Psi$ only if $x_0 \in P$. We conclude that $h(P)$ contains L, since h is surjective (and even bijective), and thus, $h(P) = L$, since P is finite and $L \cong P$. □

Lemma 4.4. *Let $\mathcal{B}$ be a basis in an infinite-dimensional $\mathbb{F}_2$-vector V and $n \geq 2$ be an integer. Denote by V^+ the hyperplane in V consisting of all sums of even number of elements of $\mathcal{B}$. A subset of V is called* n-closed *if it contains the sums of all collections of its n independent elements.*

1. *Let $\mathcal{C}$ be the minimal n-closed subset of V containing all one-element subsets of $\mathcal{B}$. Then $\mathcal{C} = V \smallsetminus \{0\}$ if n is even, $\mathcal{C}$ consists of all sums of odd number of elements of $\mathcal{B}$ if n is odd.*[6]
2. *Let $\tilde{\mathcal{C}}$ be the minimal n-closed subset of V containing all sums of odd number of elements of $\mathcal{B}$ and a non-zero vector $I \in V^+$. Then $\tilde{\mathcal{C}} = V \smallsetminus \{0\}$.*

Proof. Note, the map $J \mapsto \sum_{v\in J} v$ induces an isomorphism of the group of the finite subsets in $\mathcal{B}$ (with the operation Δ of symmetric difference) onto V. We have to show that $\mathcal{C}$ contains the sum of all elements of an m-subset $J \subset \mathcal{B}$ for any $m \geq 1$, odd in the case of odd n. Such a J is constructed as $I_1\Delta \dots \Delta I_n$ for some independent sets $I_1, \dots, I_n \in \mathcal{C}$, uniquely (modulo $\mathfrak{S}_{\mathcal{B}}$-action) determined by the numbers $|I_1|$, $|I_1 \cap (I_2 \cup \dots \cup I_n)| = a$ and some extra conditions described below. For any $I_1, \dots, I_n \in \mathcal{C}$ one has $|J| \equiv \sum_{j=1}^n |I_j| \pmod 2$. In particular, $|J| \equiv n \equiv 1 \pmod 2$ if n is odd. In all cases we impose the conditions $|I_2| = \dots = |I_n| = 1$ and $|I_1| + 1 \equiv n + m \pmod 2$.

If $m \leq 2n-1$ is odd, it suffices to impose the conditions $|I_1| = n$, $a = n - \frac{m+1}{2}$.

If $m \leq 2n-2$ is even and n is even, it suffices to ask that $|I_1| = n-1$ (is odd, so $I_1 \in \mathcal{C}$), $a = n - 1 - \frac{m}{2}$.

If $m \geq n$ (m is odd, if n is odd), we proceed by induction on (odd, if n is odd) m and take some disjoint sets $I_1, \dots, I_n \in \mathcal{C}$, where $|I_1| = m - n + 1$.

Suppose that n is odd. To show that $\tilde{\mathcal{C}}$ contains the sum of all elements of an m-set for any even $m \geq 2$, we take some $(m + |I| + n - 2)$-set J containing I and choose pairwise distinct one-element subsets $J_3, \dots, J_n \subset J \smallsetminus I$. Then $|J\Delta I \Delta J_3 \Delta \dots \Delta J_n| = m$. □

The proof of Theorem 3.1 is concluded by combining Lemma 3.4 either with Lemma 3.7 (in the case of infinite k), or with Lemma 4.2 (in the case of finite k). □

5. Remarks on the case of finite Ψ

Theorem 5.1 (Kantor–McDonough, [9])**.** *Let k be a finite field and V be a k-vector space of a finite dimension > 2. Set $\Psi := \mathbb{P}_k(V) = (V \smallsetminus \{0\})/k^\times$. Let H be a subgroup of $\mathfrak{S}_\Psi$ containing* $\mathrm{PGL}(V)$ *and an arbitrary self-bijection of Ψ which is not a collineation. Then H contains the alternating subgroup $\mathfrak{A}_\Psi$.*

For the sake of completeness we mention some details of the proof of Theorem 5.1. Let G be a group acting on a projective space $\mathbb{P}^d(\mathbb{F}_q)$. Suppose that G contains the special projective group, but its action is not m-transitive for $m = \#\mathbb{P}^{d-1}(\mathbb{F}_q)$ then [8, Theorem 1.1 (iii)] gives a choice of 3 groups

[6] This is an affine space over V^+.

of possibilities for G, one is being excluded by the principal theorem of projective geometry, while the remaining 2 being excluded for arithmetical reasons. Then it remains to prove the following

Proposition 5.2. *Let $d \geq 2$ be an integer and $\mathbb{F}_q$ be a finite field. Suppose that a group G acts m-transitively on the projective space $\mathbb{P}^d(\mathbb{F}_q)$, where $m = \#\mathbb{P}^{d-1}(\mathbb{F}_q)$. Then G contains the alternating group $\mathfrak{A}_{\mathbb{P}^d(\mathbb{F}_q)}$.*

Proof. A theorem of H.Wielandt from [14] asserts that *if a group G permuting M elements is m-transitive then $m < 3\log(M-m)$, unless G contains $\mathfrak{A}_M$.*

The following calculation lists all 9 cases where the condition $m < 3\log(M-m)$ of Wielandt's theorem is not satisfied.

Lemma 5.3. *Let $d \geq 2$ be an integer and $\mathbb{F}_q$ be a finite field. Suppose that $\#\mathbb{P}^{d-1}(\mathbb{F}_q) < 3\log\#\mathbb{A}^d(\mathbb{F}_q)$. Then $d = 2$ and $q \leq 13$.*

Proof. Consider the function $\xi(d) = \#\mathbb{P}^{d-1}(\mathbb{F}_q) - 3\log\#\mathbb{A}^d(\mathbb{F}_q) = \frac{q^d-1}{q-1} - 3d\log q$. Its derivative $\xi'(d) = \frac{q^d\log q}{q-1} - 3\log q$ vanishes only at $\frac{\log(3(q-1))}{\log q}$. We see from $(q-3/2)^2 + 3/4 > 0$ that $3(q-1) < q^2$, and thus, $\frac{\log(3(q-1))}{\log q} < 2$. In other words, ξ increases in $d \geq 2$.

Now consider the (same) function $\varphi(q) = \#\mathbb{P}^{d-1}(\mathbb{F}_q) - 3\log\#\mathbb{A}^d(\mathbb{F}_q) = \frac{q^d-1}{q-1} - 3d\log q$.

If $d = 3$ then $\varphi(q) = q^2 + q + 1 - 9\log q$. As $\varphi'(q) = 2q + 1 - 9/q$, the critical point of φ is at $\frac{\sqrt{73}-1}{4} < 2$, so φ increases in $q \geq 2$. As $\varphi(2) = 7 - 9\log 2 > 0$, the function φ is positive.

If $d = 2$ then $\varphi(q) = q + 1 - 6\log q$. As $\varphi'(q) = 1 - 6/q$, the critical point of φ is at 6. One has $\varphi(2) = 3 - 6\log 2 < 0$, $\varphi(15) = 16 - 6\log 15 < 0$, $\varphi(16) = 17 - 6\log 16 > 0.3 > 0$, so the function φ is positive for $q \geq 16$. This means that $\varphi(q)$ is negative only if q is one of $2, 3, 4, 5, 7, 8, 9, 11, 13$. $\square$

In the cases $q \neq 2, 4$ we apply the following theorem of G.Miller from [12]: *If $M = qp + m$, where p is a prime, $p > q > 1$ and $m > q$, then a group permuting M elements can be at most m-transitive, unless it contains the alternating group.* Namely, setting m_0 for a lower bound of transitivity,

q	M	m_0	$M = qp + m$
3	13	4	$13 = 2 \cdot 5 + 3$
5	31	6	$31 = 2 \cdot 13 + 5$
7	57	8	$57 = 3 \cdot 17 + 6$
8	73	9	$73 = 5 \cdot 13 + 8$
9	91	10	$91 = 2 \cdot 41 + 9$
11	133	12	$133 = 2 \cdot 61 + 11$
13	183	14	$183 = 10 \cdot 17 + 13$

In the case $q = 2$, i.e., of the projective plane over $k = \mathbb{F}_2$ (with $(M, m_0) = (7, 3)$), one has $\#\mathrm{PGL}_3(\mathbb{F}_2) = 7 \cdot 6 \cdot 4$, so $[\mathfrak{A}_7 : \mathrm{PGL}_3(\mathbb{F}_2)] = 5 \cdot 3 = 15$. By Lemma 3.7, G is 3-transitive. There are precisely $\binom{7}{3} = 35$ 3-sets, so $\#G$ is divisible by the least common multiple $7 \cdot 6 \cdot 5 \cdot 4$ of $7 \cdot 6 \cdot 4$ and 35, and thus, $[\mathfrak{S}_7 : G]|6$. The only such subgroups are $\mathfrak{A}_7$ and $\mathfrak{S}_7$.

In the remaining case $q = 4$ (with $(M, m_0) = (21, 5)$) we apply the following theorem of C.Jordan: *If a primitive group contains a cycle of length p and permutes $M = p + m$ elements, where p is a prime and $m > 2$, then it contains the alternating group.*

Remarks. 1. The collineation group of $\mathbb{P}_k(V)$ is maximal among proper *closed* subgroups of $\mathfrak{S}_\Psi$.

2. *Parity of projective transformations.* Let $\mathbb{F}_q$ be a finite field and $n \geq 1$ be an integer. The projective special linear group $\mathrm{PSL}_{n+1}(\mathbb{F}_q)$ is simple, with two solvable exceptions: $\mathrm{PSL}_2(\mathbb{F}_2) \cong \mathfrak{S}_3$ and $\mathrm{PSL}_2(\mathbb{F}_3) \cong \mathfrak{A}_4$.

In any non-exceptional case, any element of the maximal abelian quotient of the projective group $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$ is presented by a diagonal matrix $g_\lambda := \mathrm{diag}(\lambda, 1 \dots, 1) \in \mathrm{PGL}_{n+1}(\mathbb{F}_q)$ for some

$\lambda \in \mathbb{F}_q^\times$. Let $s \geq 1$ be minimal with $\lambda^s = 1$. Then g_λ acts on an n-dimensional projective space over $\mathbb{F}_q$ with a fixed hyperplane and an extra fixed point. Other orbits consist of s elements, and therefore, the parity of g_λ coincides with the parity of $\frac{q^n-1}{s}(s-1) = q^n - 1 - \frac{q^n-1}{s} \equiv q + 1 - \frac{q-1}{s}(1 + (n-1)q)$ (mod 2). Finally, $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$ contains an odd permutation if and only if qn is odd.

3. Let G be a finite permutation group, which is neither symmetric nor alternating group. As mentioned in [6, §7.3], it can be deduced from the classification of finite simple groups that G is at most 5-transitive; moreover, if G is 4- or 5-transitive then G is one of the Mathieu groups M_{11}, M_{12}, M_{23}, M_{24}.

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