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Abstract

The notion of a *metric modular* on an arbitrary set and the corresponding *modular spaces*, generalizing classical modulars over linear spaces and Orlicz spaces, were recently introduced and studied by the author [Chistyakov: Dokl. Math. 73(1):32–35, 2006 and Nonlinear Anal. 72(1):1–30, 2010]. In this chapter we present yet one more application of the metric modulars theory to the existence of fixed points of modular contractive maps in modular metric spaces. These are related to contracting generalized average velocities rather than metric distances, and the successive approximations of fixed points converge to the fixed points in the modular sense, which is weaker than the metric convergence. We prove the existence of solutions to a Carathéodory-type differential equation with the right-hand side from the Orlicz space.

Metric modular, Modular convergence, Modular contraction, Fixed point, Mapping of finite ϕ-variation, Carathéodory-type differential equation

Modular Contractions and Their Application

Vyacheslav V. Chistyakov

Keywords Metric modular • Modular convergence • Modular contraction • Fixed point • Mapping of finite φ-variation • Carathéodory-type differential equation

1 Introduction

The metric fixed-point theory [14, 18] and its variations [15] are far-reaching developments of Banach's contraction principle, where *metric conditions* on the underlying space and maps under consideration play a fundamental role. This chapter addresses fixed points of nonlinear maps in *modular spaces* introduced recently by the author [3–10] as generalizations of Orlicz spaces and classical modular spaces [19, 20, 22–27], where *modular structures* (involving nonlinearities with more rapid growth than power-like functions), play the crucial role. Under different contractive assumptions and the supplementary Δ_2-condition on modulars fixed-point theorems in classical modular linear spaces were established in [1, 16, 17].

We begin with a certain motivation of the definition of a (metric) *modular*, introduced axiomatically in [7, 9]. A simple and natural way to do it is to turn to physical interpretations. Informally speaking, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ (the absolute value of) an average velocity $w_\lambda(x,y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $w_\lambda(x,y)$. Let us comment on this in more

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detail by exhibiting an appropriate example. If $d(x,y) \geq 0$ is the distance from x to y and a number $\lambda > 0$ is interpreted as time, then the value

$$w_\lambda(x,y) = \frac{d(x,y)}{\lambda} \tag{1}$$

is the average velocity, with which one should move from x to y during time λ, in order to cover the distance $d(x,y)$. The following properties of the quantity from Eq. (1) are quite natural.

1. Two points x and y from X coincide (and $d(x,y) = 0$) if and only if any time $\lambda > 0$ will do to move from x to y with velocity $w_\lambda(x,y) = 0$ (i.e., no movement is needed at any time). Formally, given $x, y \in X$, we have

$$x = y \ \text{ iff } \ w_\lambda(x,y) = 0 \ \text{ for all } \ \lambda > 0 \ \text{(nondegeneracy)}, \tag{2}$$

   where "iff" means as usual "if and only if".
2. Assuming the distance function to be symmetric, $d(x,y) = d(y,x)$, we find that for any time $\lambda > 0$, the average velocity during the movement from x to y is the same as the average velocity in the opposite direction, i.e., for any $x, y \in X$ we have

$$w_\lambda(x,y) = w_\lambda(y,x) \ \text{ for all } \ \lambda > 0 \ \text{(symmetry)}. \tag{3}$$

3. The third property of Eq. (1), which is, in a sense, a counterpart of the triangle inequality (for velocities!), is the most important. Suppose the movement from x to y happens to be made in two different ways, but the *duration of time is the same* in each case: (a) passing through a third point $z \in X$ or (b) straightforward from x to y. If λ is the time needed to get from x to z and μ is the time needed to get from z to y, then the corresponding average velocities are $w_\lambda(x,z)$ (during the movement from x to z) and $w_\mu(z,y)$ (during the movement from z to y). The total time needed for the movement in the case (a) is equal to $\lambda + \mu$. Thus, in order to move from x to y as in the case (b), one has to have the average velocity equal to $w_{\lambda+\mu}(x,y)$. Since (as a rule) the straightforward distance $d(x,y)$ does not exceed the sum of the distances $d(x,z) + d(z,y)$, it becomes clear from the physical intuition that the velocity $w_{\lambda+\mu}(x,y)$ does not exceed at least one of the velocities $w_\lambda(x,z)$ or $w_\mu(z,y)$. Formally, this is expressed as

$$w_{\lambda+\mu}(x,y) \leq \max\{w_\lambda(x,z), w_\mu(z,y)\} \leq w_\lambda(x,z) + w_\mu(z,y) \tag{4}$$

for all points $x, y, z \in X$ and all times $\lambda, \mu > 0$ ("triangle" inequality). In fact, these inequalities can be verified rigorously: if, on the contrary, we assume that $w_\lambda(x,z) < w_{\lambda+\mu}(x,y)$ and $w_\mu(z,y) < w_{\lambda+\mu}(x,y)$, then multiplying the first inequality by λ, the second inequality—by μ, summing the results and taking into account Eq. (1), we find $d(x,z) = \lambda w_\lambda(x,z) < \lambda w_{\lambda+\mu}(x,y)$ and $d(z,y) = \mu w_\mu(z,y) < \mu w_{\lambda+\mu}(x,y)$, and it follows that $d(x,z) + d(z,y) < (\lambda + \mu) w_{\lambda+\mu}(x,y) = d(x,y)$, which contradicts the triangle inequality for d.

Inequality (4) can be obtained in a little bit more general situation. Let $f : (0,\infty) \to (0,\infty)$ be a function from the set of positive reals into itself such that the function $\lambda \mapsto \lambda/f(\lambda)$ is nonincreasing on $(0,\infty)$. Setting $w_\lambda(x,y) = d(x,y)/f(\lambda)$ (note that $f(\lambda) = \lambda$ in Eq. (1)), we have

$$w_{\lambda+\mu}(x,y) = \frac{d(x,y)}{f(\lambda+\mu)} \le \frac{d(x,z)+d(z,y)}{f(\lambda+\mu)} \le \frac{\lambda}{\lambda+\mu}\cdot\frac{d(x,z)}{f(\lambda)} + \frac{\mu}{\lambda+\mu}\cdot\frac{d(z,y)}{f(\mu)}$$
$$\le \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\lambda+\mu} w_\mu(z,y) \le w_\lambda(x,z) + w_\mu(z,y). \tag{5}$$

AQ1 A nonclassical example of "generalized velocities" satisfying Eqs. (2)–(4) is given by $w_\lambda(x,y) = \infty$ if $\lambda \le d(x,y)$ and $w_\lambda(x,y) = 0$ if $\lambda > d(x,y)$.

A (*metric*) *modular* on a set X is any one-parameter family $w = \{w_\lambda\}_{\lambda>0}$ of functions $w_\lambda : X \times X \to [0,\infty]$ satisfying Eqs. (2)–(4). In particular, the family given by Eq. (1) is the canonical (= natural) modular on a metric space (X,d), which can be interpreted as a field of average velocities. For a different interpretation of modulars related to the joint generalized variation of univariate maps and their relationships with classical modulars on linear spaces we refer to [9] (cf. also Sect. 4).

The difference between a metric (= distance function) and a modular on a set is now clearly seen: a modular depends on a positive parameter and may assume infinite values; the latter property means that it is impossible (or prohibited) to move from x to y in time λ, unless one moves with infinite velocity $w_\lambda(x,y) = \infty$. In addition (cf. Eq. (1)), the "velocity" $w_\lambda(x,y)$ is *nonincreasing* as a function of "time" $\lambda > 0$. The knowledge of "average velocities" $w_\lambda(x,y)$ for all $\lambda > 0$ and $x,y \in X$ provides more information than simply the knowledge of distances $d(x,y)$ between x and y: the distance $d(x,y)$ can be recovered as a "limit case" via the formula (again cf. Eq. (1)):

$$d(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le 1\}.$$

Now we describe briefly the main result of this chapter. Given a modular w on a set X, we introduce the *modular space* $X_w^* = X_w^*(x_0)$ around a point $x_0 \in X$ as the set of those $x \in X$, for which $w_\lambda(x,x_0)$ is finite for some $\lambda = \lambda(x) > 0$. A map $T : X_w^* \to X_w^*$ is said to be *modular contractive* if there exists a constant $0 < k < 1$ such that for all small enough $\lambda > 0$ and all $x,y \in X_w^*$ we have $w_{k\lambda}(Tx,Ty) \le w_\lambda(x,y)$. Our main result (Theorem 6) asserts that if w is *convex* and *strict*, X_w^* is *modular complete* (the emphasized notions will be introduced in the main text below) and $T : X_w^* \to X_w^*$ is modular contractive, then T admits a (unique) fixed point: $Tx_* = x_*$ for some $x_* \in X_w^*$. The successive approximations of x_* constructed in the proof of this result converge to x_* in the modular sense, which is weaker than the metric convergence. In particular, Banach's contraction principle follows if we take into account Eq. (1).

This chapter is organized as follows. In Sect. 2 we study modulars and convex modulars and introduce two modular spaces. In Sect. 3 we introduce the notions of modular convergence, modular limit and modular completeness and show that they are "weaker" than the corresponding metric notions. These notions are illustrated in Sect. 4 by examples. Section 5 is devoted to a fixed-point theorem for modular contractions in modular complete modular metric spaces. This theorem is then applied in Sect. 6 to the existence of solutions of a Carathéodory-type ordinary differential equation with the right-hand side from the Orlicz space L^{φ}. Finally, in Sect. 7 some concluding remarks are presented.

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2 Modulars and Modular Spaces

In what follows X is a nonempty set, $\lambda > 0$ is understood in the sense that $\lambda \in (0,\infty)$ and, in view of the disparity of the arguments, functions $w : (0,\infty) \times X \times X \to [0,\infty]$ will be also written as $w_\lambda(x,y) = w(\lambda,x,y)$ for all $\lambda > 0$ and $x, y \in X$, so that $w = \{w_\lambda\}_{\lambda>0}$ with $w_\lambda : X \times X \to [0,\infty]$.

Definition 1 ([7,9]). A function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a (metric) *modular on* X if it satisfies the following three conditions:

(i) Given $x, y \in X$, $x = y$ iff $w_\lambda(x,y) = 0$ for all $\lambda > 0$
(ii) $w_\lambda(x,y) = w_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$
(iii) $w_{\lambda+\mu}(x,y) \le w_\lambda(x,z) + w_\mu(y,z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$

If, instead of (i), the function w satisfies only

(i′) $w_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$

then w is said to be a *pseudomodular* on X, and if w satisfies (i′) and

($\mathrm{i_s}$) given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y, such that $w_\lambda(x,y) = 0$, then $x = y$

the function w is called a *strict modular* on X.

A modular (pseudomodular, strict modular) w on X is said to be *convex* if, instead if (iii), for all $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality:

(iv) $$w_{\lambda+\mu}(x,y) \le \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\lambda+\mu} w_\mu(y,z)$$

A motivation of the notion of *convexity* for modulars, which may look unexpected at first glance, was given in [9, Theorem 3.11], cf. also inequality (5); a further generalization of this notion was presented in [8, Sect. 5].

Given a metric space (X,d) with metric d, two *canonical* strict modulars are associated with it: $w_\lambda(x,y) = d(x,y)$ (denoted simply by d), which is independent of the first argument λ and is a (nonconvex) modular on X in the sense of (i)–(iii), and the *convex* modular Eq. (1), which satisfies (i), (ii) and (iv). Both modulars d and Eq. (1) assume only finite values on X.

Clearly, if w is a strict modular, then w is a modular, which in turn implies w is a pseudomodular on X, and similar implications hold for convex w.

The essential property of a pseudomodular w on X (cf. [9, Sect. 2.3]) is that, for any given $x,y \in X$, the function $0<\lambda \mapsto w_\lambda(x,y) \in [0,\infty]$ is *nonincreasing* on $(0,\infty)$, and so, the limit from the right $w_{\lambda+0}(x,y)$ and the limit from the left $w_{\lambda-0}(x,y)$ exist in $[0,\infty]$ and satisfy the inequalities:

$$w_{\lambda+0}(x,y) \le w_\lambda(x,y) \le w_{\lambda-0}(x,y). \tag{6}$$

A *convex* pseudomodular w on X has the following additional property: given $x,y \in X$, we have (cf. [9, Sect. 3.5]):

$$\text{if } 0<\mu\le\lambda, \text{ then } w_\lambda(x,y) \le \frac{\mu}{\lambda} w_\mu(x,y) \le w_\mu(x,y), \tag{7}$$

i.e., functions $\lambda \mapsto w_\lambda(x,y)$ and $\lambda \mapsto \lambda w_\lambda(x,y)$ are *nonincreasing* on $(0,\infty)$.

Throughout this chapter we fix an element $x_0 \in X$ arbitrarily.

Definition 2 ([7, 9]). Given a pseudomodular w on X, the two sets

$$X_w \equiv X_w(x_0) = \{x \in X : w_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_w^* \equiv X_w^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x,x_0) < \infty\}$$

are said to be *modular spaces* (around x_0).

It is clear that $X_w \subset X_w^*$, and it is known (cf. [9, Sects. 3.1 and 3.2]) that this inclusion is proper in general. It follows from [9, Theorem 2.6] that if w is a *modular* on X, then the modular space X_w can be equipped with a (nontrivial) metric d_w, generated by w and given by

$$d_w(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le \lambda\}, \qquad x, y \in X_w. \tag{8}$$

It will be shown later that d_w is a well-defined metric on a larger set X_w^*.

If w is a *convex* modular on X, then according to [9, Sect. 3.5 and Theorem 3.6] the two modular spaces coincide, $X_w = X_w^*$, and this common set can be endowed with a metric d_w^* given by

$$d_w^*(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le 1\}, \qquad x, y \in X_w^*; \tag{9}$$

moreover, d_w^* is *specifically* equivalent to d_w (see [9, Theorem 3.9]). By the convexity of w, the function $\widehat{w}_\lambda(x,y) = \lambda w_\lambda(x,y)$ is a modular on X in the sense of (i)–(iii) and (cf. [9, Formula (3.3)])

$$X_{\widehat{w}}^* = X_w^* = X_w \supset X_{\widehat{w}}, \tag{10}$$

where the last inclusion may be proper; moreover, $d_{\widehat{w}} = d_w^*$ on $X_{\widehat{w}}$.

Even if w is a nonconvex modular on X, the quantity Eq. (9) is also defined for all $x,y \in X_w^*$, but it has only few properties (cf. [9, Theorem 3.6]): $d_w^*(x,x) = 0$ and $d_w^*(x,y) = d_w^*(y,x)$. In this case we have (cf. [9, Theorem 3.9 and Example 3.10]): if $d_w(x,y) < 1$, then $d_w^*(x,y) \le d_w(x,y)$, and if $d_w^*(x,y) \ge 1$, then $d_w(x,y) \le d_w^*(x,y)$.

Let us illustrate the above in the case of a metric space (X,d) with the two canonical modulars d and w from Eq. (1) on it. We have: $X_d = \{x_0\} \subset X_d^* = X_w = X_w^* = X$, and given $x,y \in X$, $d_d(x,y) = d(x,y)$, $d_d^*(x,y) = 0$, $d_w(x,y) = \sqrt{d(x,y)}$, $d_w^*(x,y) = d(x,y)$ and $\widehat{d}(x,y) = \lambda w_\lambda(x,y) = d(x,y)$. Thus, the convex modular w from Eq. (1) plays a more adequate role in restoring the metric space (X,d) from w (cf. $d_w^* = d$ on $X_w = X_w^* = X$, whereas $X_d \subset X_d^* = X$, $d_d = d$ and $d_d^* = 0$), and so, in what follows, any metric space (X,d) will be considered equipped only with the modular Eq. (1). This convention is also justified as follows.

Now we exhibit the relationship between convex and nonconvex modulars and show that d_w is a well-defined metric on X_w^* (and not only on X_w). If w is a (not necessarily convex) modular on X, then the function (cf. Eq. (1) where $d(x,y)$ plays the role of a modular)

$$v_\lambda(x,y) = \frac{w_\lambda(x,y)}{\lambda}, \qquad \lambda > 0, \quad x,y \in X,$$

is always a *convex* modular on X. In fact, conditions (i) and (ii) are clear for v, and, as for (iv), we have, by virtue of (iii) for w,

$$v_{\lambda+\mu}(x,y) = \frac{w_{\lambda+\mu}(x,y)}{\lambda+\mu} \le \frac{w_\lambda(x,z) + w_\mu(y,z)}{\lambda+\mu}$$
$$= \frac{\lambda}{\lambda+\mu} \cdot \frac{w_\lambda(x,z)}{\lambda} + \frac{\mu}{\lambda+\mu} \cdot \frac{w_\mu(y,z)}{\mu} = \frac{\lambda}{\lambda+\mu} v_\lambda(x,z) + \frac{\mu}{\lambda+\mu} v_\mu(y,z).$$

Moreover, because $w = \widehat{v}$, we find from Eq. (10) that $X_w \subset X_w^* = X_v = X_v^*$. Since $d_v^*(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y)/\lambda \le 1\} = d_w(x,y)$ for all $x,y \in X_w^*$, i.e., $d_v^* = d_w$ on X_w^* and d_v^* is a metric on $X_v^* = X_w^*$, then we conclude that d_w is a *well-defined metric on* X_w^* (the same conclusion follows immediately from [8, Theorem 1]) with $X' = X_w^*$). This property distinguishes our theory of modulars from the classical theory: if ρ is a classical modular on a linear space X in the sense of Musielak and Orlicz [22] and $w_\lambda(x,y) = \rho((x-y)/\lambda)$, $\lambda > 0$, $x,y \in X$, then the expression $v_\lambda(x,y) = (1/\lambda)w_\lambda(x,y) = (1/\lambda)\rho((x-y)/\lambda)$ is *not allowed* as a classical modular on X. Since v is convex and $d_v^* = d_w$ on X_w^*, given $x,y \in X_w^*$, by virtue of [9, Theorem 3.9], we have

$$d_w(x,y) < 1 \text{ iff } d_v(x,y) < 1, \text{ and } d_w(x,y) \le d_v(x,y) \le \sqrt{d_w(x,y)};$$

$$d_w(x,y) \ge 1 \text{ iff } d_v(x,y) \ge 1, \text{ and } \sqrt{d_w(x,y)} \le d_v(x,y) \le d_w(x,y).$$

More metrics can be defined on X_w^* for a given modular w on X in the following general way (cf. [8, Theorem 1]): if $\mathbb{R}^+ = [0,\infty)$ and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is superadditive (i.e. $\kappa(\lambda)+\kappa(\mu) \le \kappa(\lambda+\mu)$ for all $\lambda,\mu \ge 0$) and such that $\kappa(u) > 0$ for $u > 0$ and $\kappa(+0) = \lim_{u\to+0}\kappa(u) = 0$, then the function $d_{\kappa,w}(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le \kappa(\lambda)\}$ is a well-defined metric on X_w^*.

Given a pseudomodular (modular, strict modular, convex or not) w on X, $\lambda > 0$ and $x,y \in X$, we define the *left* and *right regularizations* of w by

$$w_\lambda^-(x,y) = w_{\lambda-0}(x,y) \quad \text{and} \quad w_\lambda^+(x,y) = w_{\lambda+0}(x,y).$$

Since, by Eq. (6), $w_\lambda^+(x,y) \le w_\lambda(x,y) \le w_\lambda^-(x,y)$, and

$$w_{\lambda_2}^-(x,y) \le w_\lambda(x,y) \le w_{\lambda_1}^+(x,y) \quad \text{for all} \quad 0<\lambda_1<\lambda<\lambda_2, \tag{11}$$

it is a routine matter to verify that w^- and w^+ are pseudomodulars (modulars, strict modulars, convex or not, respectively) on X, $X_{w^-} = X_w = X_{w^+}$, $X_{w^-}^* = X_w^* = X_{w^+}^*$, $d_{w^-} = d_w = d_{w^+}$ on X_w and $d_{w^-}^* = d_w^* = d_{w^+}^*$ on X_w^*. For instance, let us check the last two equalities for metrics. Given $x,y \in X_w^*$, by virtue of Eq. (6), we find $d_{w^-}^*(x,y) \ge d_w^*(x,y) \ge d_{w^+}^*(x,y)$. In order to see that $d_{w^-}^*(x,y) \le d_w^*(x,y)$, we let $\lambda > d_w^*(x,y)$ be arbitrary and choose μ such that $d_w^*(x,y) < \mu < \lambda$, which, by Eq. (11), gives $w_\lambda^-(x,y) \le w_\mu(x,y) \le 1$, and so, $d_{w^-}^*(x,y) \le \lambda$, and then let $\lambda \to d_w^*(x,y)$. In order to prove that $d_w^*(x,y) \le d_{w^+}^*(x,y)$, we let $\lambda > d_{w^+}^*(x,y)$ be arbitrary and choose μ such that $d_{w^+}^*(x,y) < \mu < \lambda$, which, by Eq. (11), implies $w_\lambda(x,y) \le w_\mu^+(x,y) \le 1$, and so, $d_w^*(x,y) \le \lambda$, and then let $\lambda \to d_{w^+}^*(x,y)$.

In this way we have seen that the regularizations provide no new modular spaces as compared to X_w and X_w^* and no new metrics as compared to d_w and d_w^*. The right regularization will be needed in Sect. 5 for the characterization of metric Lipschitz maps in terms of underlying modulars.

3 Sequences in Modular Spaces and Modular Convergence

The notions of modular convergence, modular limit, modular completeness, etc., which we study in this section, are known in the classical theory of modulars on linear spaces (e.g., [20, 22, 25, 27]). Since the theory of (metric) modulars from [7, 8, 10] is significantly more general than the classical theory, the notions mentioned above do not carry over to metric modulars in a straightforward way and ought to be reintroduced and justified.

Definition 3. Given a pseudomodular w on X, a sequence of elements $\{x_n\} \equiv \{x_n\}_{n=1}^\infty$ from X_w or X_w^* is said to be *modular convergent* (more precisely, *w-convergent*) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} w_\lambda(x_n,x) = 0$. This will be written briefly as

$x_n \xrightarrow{w} x$ (as $n \to \infty$), and any such element x will be called a *modular limit* of the sequence $\{x_n\}$.

Note that if $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$, then by virtue of the monotonicity of the function $\lambda' \mapsto w_{\lambda'}(x_n, x)$, we have $\lim_{n\to\infty} w_\mu(x_n, x) = 0$ for all $\mu \ge \lambda$.

It is clear for a metric space (X, d) and the modular Eq. (1) on it that the metric convergence and the modular convergence in X coincide.

We are going to show that the modular convergence is much weaker than the metric convergence (in the sense to be made more precise below). First, we study to what extent the above definition is correct, and what is the relationship between the modular and metric convergences in X_w and X_w^*.

Theorem 1. *Let w be a pseudomodular on X. We have:*

(a) *The modular spaces X_w and X_w^* are closed with respect to the modular convergence, i.e., if $\{x_n\} \subset X_w$ (or X_w^*), $x \in X$ and $x_n \xrightarrow{w} x$, then $x \in X_w$ (or $x \in X_w^*$, respectively).*

(b) *If w is a strict modular on X, then the modular limit is determined uniquely (if it exists).*

Proof. (a) Since $x_n \xrightarrow{w} x$, there exists a $\lambda_0 = \lambda_0(\{x_n\}, x) > 0$ such that $w_{\lambda_0}(x_n, x) \to 0$ as $n \to \infty$.

1. First we treat the case when $\{x_n\} \subset X_w$. Let $\varepsilon > 0$ be arbitrarily fixed. Then there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $w_{\lambda_0}(x_{n_0}, x) \le \varepsilon/2$. Since $x_{n_0} \in X_w = X_w(x_0)$, we have $w_\lambda(x_{n_0}, x_0) \to 0$ as $\lambda \to \infty$, and so, there exists a $\lambda_1 = \lambda_1(\varepsilon) > 0$ such that $w_{\lambda_1}(x_{n_0}, x_0) \le \varepsilon/2$. Then conditions (iii) and (ii) from Definition 1 imply

$$w_{\lambda_0+\lambda_1}(x, x_0) \le w_{\lambda_0}(x, x_{n_0}) + w_{\lambda_1}(x_0, x_{n_0}) \le \varepsilon.$$

The function $\lambda \mapsto w_\lambda(x, x_0)$ is nonincreasing on $(0, \infty)$, and so,

$$w_\lambda(x, x_0) \le w_{\lambda_0+\lambda_1}(x, x_0) \le \varepsilon \quad \text{for all} \quad \lambda \ge \lambda_0 + \lambda_1,$$

implying $w_\lambda(x, x_0) \to 0$ as $\lambda \to \infty$, i.e., $x \in X_w$.

2. Now suppose that $\{x_n\} \subset X_w^*$. Then there exists an $n_0 \in \mathbb{N}$ such that $w_{\lambda_0}(x_{n_0}, x)$ does not exceed 1. Since $x_{n_0} \in X_w^* = X_w^*(x_0)$, there is a $\lambda_1 > 0$ such that $w_{\lambda_1}(x_{n_0}, x_0) < \infty$. Now it follows from conditions (iii) and (ii) that

$$w_{\lambda_0+\lambda_1}(x, x_0) \le w_{\lambda_0}(x, x_{n_0}) + w_{\lambda_1}(x_0, x_{n_0}) < \infty,$$

and so, $x \in X_w^*$.

(b) Let $\{x_n\} \subset X_w$ or X_w^* and $x, y \in X$ be such that $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. By the definition of the modular convergence, there exist $\lambda = \lambda(\{x_n\}, x) > 0$ and

$\mu = \mu(\{x_n\}, y) > 0$ such that $w_\lambda(x_n, x) \to 0$ and $w_\mu(x_n, y) \to 0$ as $n \to \infty$. By conditions (iii) and (ii),

$$w_{\lambda+\mu}(x,y) \le w_\lambda(x,x_n) + w_\mu(y,x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that $w_{\lambda+\mu}(x,y) = 0$, and so, by condition ($\mathrm{i_s}$) from Definition 1, we get $x = y$. □

It was shown in [9, Theorem 2.13] that if w is a modular on X, then for $\{x_n\} \subset X_w$ and $x \in X_w$ we have

$$\lim_{n\to\infty} d_w(x_n, x) = 0 \quad \text{iff} \quad \lim_{n\to\infty} w_\lambda(x_n, x) = 0 \text{ for all } \lambda > 0, \tag{12}$$

and so, the metric convergence (with respect to the metric d_w) implies the modular convergence (cf. Definition 3), but not vice versa in general. As the proof of [9, Theorem 2.13] suggests, Eq. (12) is also true for $\{x_n\} \subset X_w^*$ and $x \in X_w^*$. An assertion similar to Eq. (12) holds for Cauchy sequences from the modular spaces X_w and X_w^*.

Now we establish a result similar to Eq. (12) for *convex* modulars.

Theorem 2. *Let w be a convex modular on X. Given a sequence $\{x_n\}$ from $X_w^* (= X_w)$ and an element $x \in X_w^*$, we have*

$$\lim_{n\to\infty} d_w^*(x_n, x) = 0 \quad \textit{iff} \quad \lim_{n\to\infty} w_\lambda(x_n, x) = 0 \textit{ for all } \lambda > 0.$$

A similar assertion holds for Cauchy sequences with respect to d_w^*.

Proof. Step 1. *Sufficiency.* Given $\varepsilon > 0$, by the assumption, there exists a number $n_0(\varepsilon) \in \mathbb{N}$ such that $w_\varepsilon(x_n, x) \le 1$ for all $n \ge n_0(\varepsilon)$, and so, the Definition (9) of d_w^* implies $d_w^*(x_n, x) \le \varepsilon$ for all $n \ge n_0(\varepsilon)$.

Necessity. First, suppose that $0 < \lambda \le 1$. Given $\varepsilon > 0$, we have either (a) $\varepsilon < \lambda$ or (b) $\varepsilon \ge \lambda$. In case (a), by the assumption, there is an $n_0(\varepsilon) \in \mathbb{N}$ such that $d_w^*(x_n, x) < \varepsilon^2$ for all $n \ge n_0(\varepsilon)$, and so, by the definition of d_w^*, $w_{\varepsilon^2}(x_n, x) \le 1$ for all $n \ge n_0(\varepsilon)$. Since $\varepsilon^2 < \lambda^2 \le \lambda$ and $\varepsilon < \lambda$, inequality (7) yields

$$w_\lambda(x_n, x) \le \frac{\varepsilon^2}{\lambda} w_{\varepsilon^2}(x_n, x) \le \frac{\varepsilon}{\lambda}\varepsilon < \varepsilon \quad \text{for all} \quad n \ge n_0(\varepsilon).$$

In case (b) we set $n_1(\varepsilon) = n_0(\lambda/2)$, where $n_0(\cdot)$ is as above. Then, as we have just established, $w_\lambda(x_n, x) < \lambda/2 \le \varepsilon/2 < \varepsilon$ for all $n \ge n_1(\varepsilon)$.

Now, assume that $\lambda > 1$. Again, given $\varepsilon > 0$, we have either (a) $\varepsilon < \lambda$ or (b) $\varepsilon \ge \lambda$. In case (a) there is an $N_0(\varepsilon) \in \mathbb{N}$ such that $d_w^*(x_n, x) < \varepsilon$ for all $n \ge N_0(\varepsilon)$, and so, $w_\varepsilon(x_n, x) \le 1$ for all $n \ge N_0(\varepsilon)$. Since $\varepsilon < \lambda$ and $\lambda > 1$, by virtue of Eq. (7), we find

$$w_\lambda(x_n, x) \le \frac{\varepsilon}{\lambda} w_\varepsilon(x_n, x) \le \frac{\varepsilon}{\lambda} < \varepsilon \quad \text{for all} \quad n \ge N_0(\varepsilon).$$

In case (b) we put $N_1(\varepsilon) = N_0(\lambda/2)$, where $N_0(\cdot)$ is as above. Then it follows that $w_\lambda(x_n, x) < \lambda/2 \le \varepsilon/2 < \varepsilon$ for all $n \ge N_1(\varepsilon)$.

Thus, we have shown that $w_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$.

Step 2. The assertion for Cauchy sequences is of the form

$$\lim_{n,m\to\infty} d^*_w(x_n, x_m) = 0 \quad \text{iff} \quad \lim_{n,m\to\infty} w_\lambda(x_n, x_m) = 0 \text{ for all } \lambda > 0;$$

its proof is similar to the one given in Step 1 with suitable modifications. □

Theorem 2 shows, in particular, that in a metric space (X, d) with modular Eq. (1) on it the metric and modular convergences are equivalent.

Definition 4. A pseudomodular w on X is said to *satisfy the* (sequential) Δ_2-*condition* (on X^*_w) if the following condition holds: given a sequence $\{x_n\} \subset X^*_w$ and $x \in X^*_w$, if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$, then $\lim_{n\to\infty} w_{\lambda/2}(x_n, x) = 0$.

A similar definition applies with X^*_w replaced by X_w.

In the case of a metric space (X, d) the modular Eq. (1) clearly satisfies the Δ_2-condition on X.

The following important observation, which generalizes the corresponding result from the theory of classical modulars on linear spaces (cf. [22, I,5.2.IV]), provides a criterion for the metric and modular convergences to coincide.

Theorem 3. *Given a modular w on X, ~~we have~~ the metric convergence on X^*_w (with respect to d_w if w is arbitrary, and with respect to d^*_w if w is convex) coincides with the modular convergence iff w satisfies the Δ_2-condition on X^*_w.*

Proof. Let $\{x_n\} \subset X^*_w$ and $x \in X^*_w$ be given. We know from Eq. (12) and Theorem 2 that the metric convergence (with respect to d_w if w is a modular or with respect to d^*_w if w is a convex modular) of x_n to x is equivalent to

$$\lim_{n\to\infty} w_\lambda(x_n, x) = 0 \quad \text{for all} \quad \lambda > 0. \tag{13}$$

(⇒) Suppose that the metric convergence coincides with the modular convergence on X^*_w. If there exists a $\lambda_0 > 0$ such that $w_{\lambda_0}(x_n, x) \to 0$ as $n \to \infty$, then x_n is modular convergent to x, and so, x_n converges to x in metric (d_w or d^*_w). It follows that Eq. (13) holds implying, in particular, $w_{\lambda_0/2}(x_n, x) \to 0$ as $n \to \infty$, and so, w satisfies the Δ_2-condition.

(⇐) By virtue of Eq. (13), the metric convergence on X^*_w always implies the modular convergence, and so, it suffices to verify the converse assertion, namely: if $x_n \overset{w}{\to} x$, then Eq. (13) holds. In fact, if $x_n \overset{w}{\to} x$, then $w_{\lambda_0}(x_n, x) \to 0$ as $n \to \infty$ for some constant $\lambda_0 = \lambda_0(\{x_n\}, x) > 0$. The Δ_2-condition implies $w_{\lambda_0/2}(x_n, x) \to 0$ as $n \to \infty$, and so, the induction yields $w_{\lambda_0/2^j}(x_n, x) \to 0$ as $n \to \infty$ for all $j \in \mathbb{N}$.

Now, given $\lambda > 0$, there exists a $j = j(\lambda) \in \mathbb{N}$ such that $\lambda > \lambda_0/2^j$. By the monotonicity of $\lambda \mapsto w_\lambda(x_n, x)$, we have

$$w_\lambda(x_n, x) \le w_{\lambda_0/2^j}(x_n, x) \to 0 \quad \text{as} \quad n \to \infty.$$

By the arbitrariness of $\lambda > 0$, condition (13) follows. $\square$

Definition 5. Given a modular w on X, a sequence $\{x_n\} \subset X_w^*$ is said to be *modular Cauchy* (or w-*Cauchy*) if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $w_\lambda(x_n, x_m) \to 0$ as $n, m \to \infty$, i.e.,

$$\forall \varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N} \text{ such that } \forall n \ge n_0(\varepsilon),\ m \ge n_0(\varepsilon)\text{: } w_\lambda(x_n, x_m) \le \varepsilon.$$

It follows from Theorem 2 (Step 2 in its proof) and Definition 5 that a sequence from X_w^*, which is Cauchy in metric d_w or d_w^*, is modular Cauchy.

Note that a modular convergent sequence is modular Cauchy. In fact, if $x_n \overset{w}{\to} x$, then $w_\lambda(x_n, x) \to 0$ as $n \to \infty$ for some $\lambda > 0$, and so, for each $\varepsilon > 0$, there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that $w_\lambda(x_n, x) \le \varepsilon/2$ for all $n \ge n_0(\varepsilon)$. It follows from (iii) that if $n, m \ge n_0(\varepsilon)$, then $w_{2\lambda}(x_n, x_m) \le w_\lambda(x_n, x) + w_\lambda(x_m, x) \le \varepsilon$, which implies that $\{x_n\}$ is modular Cauchy.

The following definition will play an important role below.

Definition 6. Given a modular w on X, the modular space X_w^* is said to be *modular complete* (or w-*complete*) if each modular Cauchy sequence from X_w^* is modular convergent in the following (more precise) sense: if $\{x_n\} \subset X_w^*$ and there exists a $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{n,m\to\infty} w_\lambda(x_n, x_m) = 0$, then there exists an $x \in X_w^*$ such that $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$.

The notions of modular convergence, modular limit and modular completeness, introduced above, are illustrated by examples in the next section. It is clear from Eq. (1) that for a metric space (X, d) these notions coincide with respective notions in the metric space setting.

4 Examples of Metric and Modular Convergences

We begin with recalling certain properties of φ-functions and convex functions on the set of all nonnegative reals $\mathbb{R}^+ = [0, \infty)$.

A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *φ-function* if it is continuous, nondecreasing and unbounded (and so, $\varphi(\infty) \equiv \lim_{u\to\infty} \varphi(u) = \infty$) and assumes the value zero only at zero: $\varphi(u) = 0$ iff $u = 0$.

If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function such that $\varphi(u) = 0$ iff $u = 0$, then it is (automatically) continuous, strictly increasing and unbounded, and so, it is a convex φ-function. Also, φ is superadditive: $\varphi(u_1) + \varphi(u_2) \le \varphi(u_1 + u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$ (cf. [19, Sect. I.1]). Moreover, φ admits the inverse function $\varphi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$, which is continuous, strictly increasing, $\varphi^{-1}(u) = 0$ iff $u = 0$, $\varphi^{-1}(\infty) = \infty$, and which is subadditive: $\varphi^{-1}(u_1 + u_2) \le \varphi^{-1}(u_1) + \varphi^{-1}(u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$. The function φ is said to *satisfy the* Δ_2*-condition at infinity* (cf. [19, Sect. I.4]) if there exist constants $K > 0$ and $u_0 \ge 0$ such that $\varphi(2u) \le K\varphi(u)$ for all $u \ge u_0$.

4.1. Let the triple $(M, d, +)$ be a *metric semigroup*, i.e., the pair (M, d) is a metric space with metric d, the pair $(M, +)$ is an Abelian semigroup with respect to the operation of addition $+$ and d is translation invariant in the sense that $d(p+r, q+r) = d(p, q)$ for all $p, q, r \in M$. Any normed linear space $(M, |\cdot|)$ is a metric semigroup with the induced metric $d(p, q) = |p - q|$, $p, q \in M$ and the addition operation $+$ from M. If $K \subset M$ is a convex cone (i.e., $p + q, \lambda p \in K$ whenever $p, q \in K$ and $\lambda \ge 0$), then the triple $(K, d, +)$ is also a metric semigroup. A nontrivial example of a metric semigroup is as follows (cf. [12, 26]). Let $(Y, |\cdot|)$ be a real normed space and M be the family of all nonempty closed bounded convex subsets of Y equipped with the Hausdorff metric d given by $d(P, Q) = \max\{\mathrm{e}(P, Q), \mathrm{e}(Q, P)\}$, where $P, Q \in M$ and $\mathrm{e}(P, Q) = \sup_{p \in P} \inf_{q \in Q} |p - q|$. Given $P, Q \in M$, we define $P \oplus Q$ as the closure in Y of the Minkowski sum $P + Q = \{p + q : p \in P, q \in Q\}$. Then the triple $(M, d, \oplus)$ is a metric semigroup (actually, M is an abstract convex cone). For more information on metric semigroups and their special cases, abstract convex cones, including examples, we refer to [5, 6, 9, 10] and references therein.

Given a closed interval $[a, b] \subset \mathbb{R}$ with $a < b$, we denote by $\mathbb{X} = M^{[a,b]}$ the set of all mappings $x : [a, b] \to M$. If φ is a *convex* φ-function on $\mathbb{R}^+$, we define a function $w : (0, \infty) \times \mathbb{X} \times \mathbb{X} \to [0, \infty]$ for all $\lambda > 0$ and $x, y \in \mathbb{X}$ by (note that w depends on φ)

$$w_\lambda(x, y) = \sup_\pi \sum_{i=1}^{m} \varphi\left(\frac{d\big(x(t_i) + y(t_{i-1}), x(t_{i-1}) + y(t_i)\big)}{\lambda \cdot (t_i - t_{i-1})}\right) \cdot (t_i - t_{i-1}), \qquad (14)$$

where the supremum is taken over all partitions $\pi = \{t_i\}_{i=1}^m$ of the interval $[a, b]$, i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = b$. It was shown in [5, Sects. 3 and 4] that w is a *convex pseudomodular* on $\mathbb{X}$. Thus, given $x_0 \in M$, the modular space $\mathbb{X}_w^* = \mathbb{X}_w^*(x_0)$ (here x_0 denotes also the constant mapping $x_0(t) = x_0$ for all $t \in [a, b]$), which was denoted in [5, Eq. (3.20) and Sect. 4.1] by $\mathrm{GV}_\varphi([a, b]; M)$ and called the *space of mappings of bounded generalized* φ*-variation*, is well defined and, by the translation invariance of d on M, we have $x \in \mathbb{X}_w^* = \mathrm{GV}_\varphi([a, b]; M)$ iff $x : [a, b] \to M$ and there exists a constant $\lambda = \lambda(x) > 0$ such that

$$w_\lambda(x, x_0) = \sup_\pi \sum_{i=1}^{m} \varphi\left(\frac{d\big(x(t_i), x(t_{i-1})\big)}{\lambda\,(t_i - t_{i-1})}\right)(t_i - t_{i-1}) < \infty. \qquad (15)$$

Note that $w_\lambda(x,x_0)$ from Eq. (15) is independent of $x_0 \in M$; this value is called the *generalized* φ_λ*-variation* of x, where $\varphi_\lambda(u) = \varphi(u/\lambda)$, $u \in \mathbb{R}^+$. Since w satisfies on $\mathbb{X}$ conditions (i′), (ii) and (iv) (and not (i) in general) from Definition 1, the quantity d_w^* from Eq. (9) is only a pseudometric on $\mathbb{X}_w^*$ and, in particular, only $d_w^*(x,x) = 0$ holds for $x \in \mathbb{X}_w^*$ (note that $d_w^*(x,y)$ was denoted by $\Delta_\varphi(x,y)$ in [5, Equality (4.5)]).

4.2. In order to "turn" Eq. (14) into a modular, we fix an $x_0 \in M$ and set $X = \{x : [a,b] \to M \mid x(a) = x_0\} \subset \mathbb{X}$. We assert that w from Eq. (14) is a *strict* convex modular on X. In fact, given $x, y \in X$ and $t, s \in [a,b]$ with $t \neq s$, it follows from Eq. (14) that

$$\varphi\left(\frac{d\big(x(t)+y(s),x(s)+y(t)\big)}{\lambda\,|t-s|)}\right)|t-s| \le w_\lambda(x,y),$$

and so, by the translation invariance of d and the triangle inequality,

$$\begin{aligned}|d(x(t),y(t)) - d(x(s),y(s))| &\le d(x(t)+y(s),x(s)+y(t)) \\ &\le \lambda\,|t-s|\,\varphi^{-1}\left(\frac{w_\lambda(x,y)}{|t-s|}\right). \qquad (16)\end{aligned}$$

Now, if we suppose that $w_\lambda(x,y) = 0$ for some $\lambda > 0$, then for all $t \in [a,b]$, $t \neq s = a$, we get (note that $x(a) = y(a) = x_0$)

$$d(x(t),y(t)) = |d(x(t),y(t)) - d(x(a),y(a))| \le 0.$$

Thus, $x(t) = y(t)$ for all $t \in [a,b]$, and so, $x = y$ as elements of X.

It is clear for the modular space $X_w^* = X_w^*(x_0)$ that

$$X_w^* = \mathbb{X}_w^* \cap X = \mathrm{GV}([a,b];M) \cap X, \qquad (17)$$

i.e., $x \in X_w^*$ iff $x : [a,b] \to M$, $x(a) = x_0$ and Eq. (15) holds for some $\lambda > 0$. Moreover, the function d_w^* from Eq. (9) is a *metric* on X_w^*.

4.3. In this section we show that if $(M,d,+)$ is a *complete* metric semigroup (i.e. (M,d) is complete as a metric space), then the modular space X_w^* from Eq. (17) is *modular complete* in the sense of Definition 6.

Let $\{x_n\} \subset X_w^*$ be a w-Cauchy sequence, so that $w_\lambda(x_n,x_m) \to 0$ as $n,m \to \infty$ for some constant $\lambda = \lambda(\{x_n\}) > 0$. Given $n,m \in \mathbb{N}$ and $t \in [a,b]$, $t \neq a$, it follows from Eq. (16) with $x = x_n$, $y = x_m$ and $s = a$ that (again note that $x_n(a) = x_0$ for all $n \in \mathbb{N}$)

$$d(x_n(t),x_m(t)) \le \lambda\,(t-a)\,\varphi^{-1}\left(\frac{w_\lambda(x_n,x_m)}{t-a}\right).$$

This estimate, the modular Cauchy property of $\{x_n\}$, the continuity of φ^{-1} and the completeness of $(M,d,+)$ imply the existence of an $x:[a,b]\to M$, $x(a)=x_0$ (and so, $x\in X$), such that the sequence $\{x_n\}$ converges pointwise on $[a,b]$ to x, i.e., $\lim_{n\to\infty} d(x_n(t),x(t))=0$ for all $t\in[a,b]$. We assert that $\lim_{n\to\infty} w_\lambda(x_n,x)=0$. By the (sequential) lower semicontinuity of the functional $w_\lambda(\cdot,\cdot)$ from Eq. (14) (cf. [5, Assertion (4.8) on p. 27]), we get

$$w_\lambda(x_n,x)\le \liminf_{m\to\infty} w_\lambda(x_n,x_m) \quad \text{for all} \quad n\in\mathbb{N}. \tag{18}$$

Now, given $\varepsilon>0$, by the modular Cauchy condition for $\{x_n\}$, there is an $n_0(\varepsilon)\in\mathbb{N}$ such that $w_\lambda(x_n,x_m)\le\varepsilon$ for all $n\ge n_0(\varepsilon)$ and $m\ge n_0(\varepsilon)$, and so,

$$\limsup_{m\to\infty} w_\lambda(x_n,x_m)\le \sup_{m\ge n_0(\varepsilon)} w_\lambda(x_n,x_m)\le\varepsilon \quad \text{for all} \quad n\ge n_0(\varepsilon).$$

Since the limit inferior does not exceed the limit superior (for any real sequences), it follows from the last displayed line and Eq. (18) that $w_\lambda(x_n,x)\le\varepsilon$ for all $n\ge n_0(\varepsilon)$, i.e., $w_\lambda(x_n,x)\to 0$ as $n\to\infty$. Finally, since, by Theorem 1(a), X_w^* is closed with respect to the modular convergence, we infer that $x\in X_w^*$, which was to be proved.

4.4. In order to be able to calculate explicitly, for the sake of simplicity we assume furthermore that $M=\mathbb{R}$ with $d(p,q)=|p-q|$, $p,q\in\mathbb{R}$, and the function φ satisfies the *Orlicz condition at infinity*: $\varphi(u)/u\to\infty$ as $u\to\infty$. In this case the value $w_1(x,0)$ (cf. Eq. (15) with $\lambda=1$) is known as the *φ-variation* of the function $x:[a,b]\to\mathbb{R}$ (in the sense of F. Riesz, Yu. T. Medvedev and W. Orlicz), the function x with $w_1(x,0)<\infty$ is said to be *of bounded φ-variation* on $[a,b]$, and we have

$$w_\lambda(x,y)=w_\lambda(x-y,0)=w_1\left(\frac{x-y}{\lambda},0\right), \quad \lambda>0, \quad x,y\in\mathbb{X}=\mathbb{R}^{[a,b]}. \tag{19}$$

Denote by $\mathrm{AC}[a,b]$ the space of all absolutely continuous real-valued functions on $[a,b]$ and by $\mathrm{L}^1[a,b]$ the space of all (equivalence classes of) Lebesgue summable functions on $[a,b]$.

The following criterion is known for functions $x:[a,b]\to\mathbb{R}$ to be in the space $\mathrm{GV}_\varphi[a,b]=\mathbb{X}_w^*$ (for more details see [2], [5, Sects. 3 and 4], [11], [20, Sect. 2.4], [21]): $x\in\mathrm{GV}_\varphi[a,b]$ iff $w_\lambda(x,0)=w_1(x/\lambda,0)<\infty$ for some $\lambda=\lambda(x)>0$ (i.e., x/λ is of bounded φ-variation on $[a,b]$) iff $x\in\mathrm{AC}[a,b]$ and its derivative $x'\in\mathrm{L}^1[a,b]$ (defined almost everywhere on $[a,b]$) satisfies the condition:

$$w_\lambda(x,x_0)=w_\lambda(x,0)=\int_a^b\varphi\left(\frac{|x'(t)|}{\lambda}\right)dt<\infty, \quad x_0\in\mathbb{R}. \tag{20}$$

Given $x_0 \in \mathbb{R}$, we set $X = \{x : [a,b] \to \mathbb{R} \mid x(a) = x_0\}$, and so (cf. Eq. (17)),

$$X_w^* = X_w^*(x_0) = \{x \in \mathrm{GV}_\varphi[a,b] : x(a) = x_0\}. \tag{21}$$

Thus, the modular w is strict and convex on X and the modular space Eq. (21) is modular complete. Note that X_w^* is *not* a linear subspace of $\mathrm{GV}_\varphi[a,b]$, which is a normed Banach algebra (cf. [3, Theorem 3.6]).

4.5. Here we present an example when the metric and modular convergences coincide. This example is a modification of Example 3.5(c) from [5]. We set $[a,b] = [0,1]$, $M = \mathbb{R}$ and $\varphi(u) = e^u - 1$ for $u \in \mathbb{R}^+$. Clearly, φ satisfies the Orlicz condition but does not satisfy the Δ_2-condition at infinity.

Given a number $\alpha > 0$, we define a function $x_\alpha : [0,1] \to \mathbb{R}$ by

$$x_\alpha(t) = \alpha t(1 - \log t) \quad \text{if} \quad 0 < t \le 1 \quad \text{and} \quad x_\alpha(0) = 0.$$

Since $x'_\alpha(t) = -\alpha \log t$ for $0 < t \le 1$, by Eq. (20), for any number $\lambda > 0$ we find

$$w_\lambda(x_\alpha, 0) = \int_0^1 \varphi\left(\frac{|x'_\alpha(t)|}{\lambda}\right) dt = \int_0^1 \frac{dt}{t^{\alpha/\lambda}} - 1 = \begin{cases} \infty & \text{if } 0 < \lambda \le \alpha, \\ \dfrac{\alpha}{\lambda - \alpha} & \text{if } \quad \lambda > \alpha. \end{cases}$$

It follows that the modular w can take infinite values (although it is strict) and that $x_\alpha \in X_w^* = X_w^*(0)$ for all $\alpha > 0$. Also, we have

$$d_w^*(x_\alpha, 0) = \inf\{\lambda > 0 : w_\lambda(x_\alpha, 0) \le 1\} = 2\alpha.$$

Thus, if we set $\alpha = \alpha(n) = 1/n$ and $x_n = x_{\alpha(n)}$ for $n \in \mathbb{N}$, then we find that $d_w^*(x_n, 0) \to 0$ as $n \to \infty$ and $w_\lambda(x_n, 0) \to 0$ as $n \to \infty$ for all $\lambda > 0$, and, in accordance with Theorem 2, these two convergences are equivalent.

4.6. Here we expose an example when the modular convergence is weaker than the metric convergence. Let $[a,b]$, M and φ be as in Example 4.5.

Given $0 \le \beta \le 1$, we define a function $x_\beta : [0,1] \to \mathbb{R}$ as follows:

$$x_\beta(t) = t - (t + \beta)\log(t + \beta) + \beta \log \beta \quad \text{if} \quad \beta > 0 \quad \text{and} \quad 0 \le t \le 1$$

and

$$x_0(t) = t - t\log t \quad \text{if} \quad 0 < t \le 1 \quad \text{and} \quad x_0(0) = 0.$$

Since $x'_\beta(t) = -\log(t + \beta)$ for $\beta > 0$ and $t \in [0,1]$, we have

$$|x'_\beta(t)| = -\log(t + \beta) \text{ if } 0 \le t \le 1 - \beta, \text{ and } |x'_\beta(t)| = \log(t + \beta) \text{ if } 1 - \beta \le t \le 1,$$

and so, by virtue of Eq. (20), given $\lambda > 0$, we find

$$w_\lambda(x_\beta, 0) = \int_0^1 \varphi(|x'_\beta(t)|/\lambda)\, dt = I_1 + I_2 - 1, \qquad \beta > 0,$$

where

$$I_1 = \int_0^{1-\beta} \frac{dt}{(t+\beta)^{1/\lambda}} = \begin{cases} \dfrac{\lambda}{\lambda - 1}\left(1 - \beta^{(\lambda-1)/\lambda}\right) & \text{if } 0 < \lambda \neq 1, \\ -\log\beta & \text{if } \quad \lambda = 1, \end{cases}$$

and

$$I_2 = \int_{1-\beta}^1 (t+\beta)^{1/\lambda}\, dt = \frac{\lambda}{\lambda+1}\left((1+\beta)^{(\lambda+1)/\lambda} - 1\right) \quad \text{for all} \quad \lambda > 0.$$

Also, $w_\lambda(x_0, 0) = \infty$ if $0 < \lambda \le 1$, and $w_\lambda(x_0, 0) = 1/(\lambda - 1)$ if $\lambda > 1$ (cf. Example 4.5 with $\alpha = 1$). Thus, $x_\beta \in X_w^* = X_w^*(0)$ for all $0 \le \beta \le 1$.

Clearly, x_β converges pointwise on $[0, 1]$ to x_0 as $\beta \to +0$ (actually, the first inequality in the proof of [5, Lemma 4.1(a)] shows that the convergence is uniform on $[0, 1]$).

Now we calculate the values $w_\lambda(x_\beta, x_0)$ for $\lambda > 0$ and $d_w^*(x_\beta, x_0)$ and investigate their convergence to zero as $\beta \to +0$. Since

$$(x_\beta - x_0)'(t) = -\log(t+\beta) + \log t \quad \text{for} \quad 0 < t \le 1,$$

we have

$$\frac{|(x_\beta - x_0)'(t)|}{\lambda} = \frac{\log(t+\beta) - \log t}{\lambda} = \log\left(1 + \frac{\beta}{t}\right)^{1/\lambda},$$

and so, by virtue of Eqs. (19) and (20),

$$w_\lambda(x_\beta, x_0) = \int_0^1 \varphi\left(\frac{|(x_\beta - x_0)'(t)|}{\lambda}\right) dt = -1 + \int_0^1 \left(1 + \frac{\beta}{t}\right)^{1/\lambda} dt.$$

If $0 < \lambda \le 1$, we have

$$\left(1 + \frac{\beta}{t}\right)^{1/\lambda} \ge 1 + \frac{\beta}{t} \quad \text{and} \quad \int_0^1 \left(1 + \frac{\beta}{t}\right) dt = \infty,$$

and so, $w_\lambda(x_\beta, x_0) = \infty$ for all $0 < \beta \le 1$ and $0 < \lambda \le 1$.

Now suppose that $\lambda > 1$. Then

$$w_\lambda(x_\beta, x_0) = -1 + \int_0^\beta \left(1 + \frac{\beta}{t}\right)^{1/\lambda} dt + \int_\beta^1 \left(1 + \frac{\beta}{t}\right)^{1/\lambda} dt \equiv -1 + II_1 + II_2,$$

where

$$II_1 \le \int_0^\beta \left(\frac{2\beta}{t}\right)^{1/\lambda} dt = (2\beta)^{1/\lambda} \int_0^\beta t^{-1/\lambda} dt = (2\beta)^{1/\lambda} \cdot \frac{\lambda}{\lambda-1} \cdot \beta^{1-(1/\lambda)} =$$
$$= 2^{1/\lambda} \cdot \frac{\lambda\beta}{\lambda-1} \to 0 \quad \text{as} \quad \beta \to +0$$

and

$$II_2 \le \int_\beta^1 \left(1+\frac{\beta}{t}\right) dt = (1-\beta) - \beta\log\beta \to 1 \quad \text{as} \quad \beta \to +0.$$

It follows that $w_\lambda(x_\beta, x_0) \to 0$ as $\beta \to +0$ for all $\lambda > 1$.

On the other hand, since $w_\lambda(x_\beta, x_0) = \infty$ for all $0 < \beta \le 1$ and $0 < \lambda \le 1$ (as noticed above), we get $d_w^*(x_\beta, x_0) = \inf\{\lambda > 0 : w_\lambda(x_\beta, x_0) \le 1\} \ge 1$, and so, $d_w^*(x_\beta, x_0)$ cannot converge to zero as $\beta \to +0$.

Thus, if we set $\beta = \beta(n) = 1/n$ and $x_n = x_{\beta(n)}$ for $n \in \mathbb{N}$, then we find $d_w^*(x_n, x_0) \not\to 0$ as $n \to \infty$, whereas $w_\lambda(x_n, x_0) \to 0$ as $n \to \infty$ only for $\lambda > 1$.

5 A Fixed-Point Theorem for Modular Contractions

Since convex modulars play the central role in this section, we concentrate mainly on them. We begin with a characterization of d_w^*-Lipschitz maps on the modular space X_w^* in terms of their generating convex modulars w.

Theorem 4. *Let w be a convex modular on X and $k > 0$ be a constant. Given a map $T : X_w^* \to X_w^*$ and $x, y \in X_w^*$, the Lipschitz condition $d_w^*(Tx, Ty) \le k\, d_w^*(x, y)$ is equivalent to the following: $w_{k\lambda+0}(Tx, Ty) \le 1$ for all $\lambda > 0$ such that $w_\lambda(x, y) \le 1$.*

Proof. First, note that, given $c > 0$, the function, defined by $\overline{w}_\lambda(x, y) = w_{c\lambda}(x, y)$, $\lambda > 0$, $x, y \in X$, is also a convex modular on X and $d_{\overline{w}}^* = \frac{1}{c} d_w^*$:

$$d_{\overline{w}}^*(x,y) = \inf\{\lambda > 0 : w_{c\lambda}(x,y) \le 1\} = \inf\{\mu/c > 0 : w_\mu(x,y) \le 1\} =$$
$$= \frac{1}{c}\, d_w^*(x,y) \quad \text{for all } x, y \in X_{\overline{w}}^* = X_w^*. \tag{22}$$

Necessity. We may suppose that $x \ne y$. For any $c > k$, by the assumption, we find $d_w^*(Tx, Ty) \le k\, d_w^*(x, y) < c\, d_w^*(x, y)$, whence $d_w^*(Tx, Ty)/c < d_w^*(x, y)$. It follows that if $\lambda > 0$ is such that $w_\lambda(x, y) \le 1$, then, by Eq. (9), $d_w^*(x, y) \le \lambda$ implying, in view of Eq. (22),

$$\lambda > \frac{1}{c}\, d_w^*(Tx, Ty) = \inf\{\mu > 0 : w_{c\mu}(Tx, Ty) \le 1\},$$

and so, $w_{c\lambda}(Tx,Ty) \le 1$. Passing to the limit as $c \to k+0$, we arrive at the desired inequality $w_{k\lambda+0}(Tx,Ty) \le 1$.

Sufficiency. By the assumption, the set $\{\lambda > 0 : w_\lambda(x,y) \le 1\}$ is contained in the set $\{\lambda > 0 : w^+_{k\lambda}(Tx,Ty) = w_{k\lambda+0}(Tx,Ty) \le 1\}$, and so, taking the infima, by virtue of Eqs. (9), (22) and the equality $d^*_{w^+} = d^*_w$, we get

$$d^*_w(x,y) \ge \frac{1}{k} d^*_{w^+}(Tx,Ty) = \frac{1}{k} d^*_w(Tx,Ty),$$

which implies that T satisfies the Lipschitz condition with constant k. □

Theorem 4 can be reformulated as follows. Since (cf. [9, Theorem 3.8(a)] and Eq. (9)), for $\lambda^* = d^*_w(x,y)$,

$$(\lambda^*,\infty) \subset \{\lambda > 0 : w_\lambda(x,y) < 1\} \subset \{\lambda > 0 : w_\lambda(x,y) \le 1\} \subset [\lambda^*,\infty),$$

we have $d^*_w(Tx,Ty) \le k\, d^*_w(x,y)$ iff $w_{k\lambda}(Tx,Ty) \le 1$ for all $\lambda > \lambda^* = d^*_w(x,y)$.

For a metric space (X,d) and the modular w from Eq. (1) on it, Theorem 4 gives the usual Lipschitz condition: $d(Tx,Ty)/(k\lambda) = w_{k\lambda}(Tx,Ty) \le 1$ for all $\lambda > 0$ such that $d(x,y)/\lambda = w_\lambda(x,y) \le 1$, i.e., $d(Tx,Ty) \le k\lambda$ for all $\lambda \ge d(x,y)$, and so, $d(Tx,Ty) \le kd(x,y)$.

As a corollary of Theorem 4, we find that

$$\text{if } w_{k\lambda}(Tx,Ty) \le w_\lambda(x,y) \text{ for all } \lambda > 0, \text{ then } d^*_w(Tx,Ty) \le k\, d^*_w(x,y); \tag{23}$$

in fact, it suffices to note only that if $\lambda > 0$ is such that $w_\lambda(x,y) \le 1$, then, by Eq. (6), $w_{k\lambda+0}(Tx,Ty) \le w_{k\lambda}(Tx,Ty) \le w_\lambda(x,y) \le 1$, and apply Theorem 4.

Now we briefly comment on d_w-Lipschitz maps on X^*_w, where w is a general modular on X and d_w is the metric from Eq. (8). Note that, given $c > 0$, the function $\overline{w}_\lambda(x,y) = \frac{1}{c} w_{c\lambda}(x,y)$ is also a modular on X and $d_{\overline{w}} = \frac{1}{c} d_w$ on $X^*_{\overline{w}} = X^*_w$. Following the lines of the proof of Theorem 4, we get

Theorem 5. *If w is a modular on X and $k > 0$, given $T : X^*_w \to X^*_w$ and $x,y \in X^*_w$, we have $d_w(Tx,Ty) \le k\, d_w(x,y)$ iff $w_{k\lambda+0}(Tx,Ty) \le k\lambda$ for all $\lambda > 0$ such that $w_\lambda(x,y) \le \lambda$.*

The following assertion is a corollary of Theorem 5:

$$\text{if } w_{k\lambda}(Tx,Ty) \le k\, w_\lambda(x,y) \text{ for all } \lambda > 0, \text{ then } d_w(Tx,Ty) \le k\, d_w(x,y).$$

Definition 7. Given a (convex) modular w on X, a map $T : X^*_w \to X^*_w$ is said to be *modular contractive* (or a *w-contraction*) provided there exist numbers $0 < k < 1$ and $\lambda_0 > 0$, possibly depending on k, such that

$$w_{k\lambda}(Tx,Ty) \le w_\lambda(x,y) \quad \text{for all } 0 < \lambda \le \lambda_0 \text{ and } x,y \in X^*_w. \tag{24}$$

A few remarks are in order. First, by virtue of Eq. (1), for a metric space (X,d), condition (24) is equivalent to the usual one: $d(Tx,Ty) \le kd(x,y)$. Second, condition (24) is a *local* one with respect to λ as compared to the assumption on the left in Eq. (23), and the principal inequality in it may be of the form $\infty \le \infty$. Third, if, in addition, w is *strict* and if we set $\infty/\infty = 1$, then Eq. (24) is a consequence of the following: there exists a number $0<h<1$ such that

$$\limsup_{\lambda\to+0}\left(\sup_{x\neq y}\frac{w_{h\lambda}(Tx,Ty)}{w_\lambda(x,y)}\right) \le 1, \tag{25}$$

where the supremum is taken over all $x,y \in X_w^*$ such that $x \neq y$. In order to see this, we first note that the left-hand side in Eq. (25) is well defined in the sense that, by virtue of ($\mathrm{i_s}$) from Definition 1, $w_\lambda(x,y) \neq 0$ for all $\lambda > 0$ and $x \neq y$. Choose any k such that $h < k < 1$. It follows from Eq. (25) that

$$\lim_{\mu\to+0}\sup_{\lambda\in(0,\mu]}\left(\sup_{x\neq y}\frac{w_{h\lambda}(Tx,Ty)}{w_\lambda(x,y)}\right) \le 1 < \frac{k}{h},$$

and so, there exists a $\mu_0 = \mu_0(k) > 0$ such that

$$\sup_{x\neq y}\frac{w_{h\lambda}(Tx,Ty)}{w_\lambda(x,y)} < \frac{k}{h} \quad \text{for all} \quad 0<\lambda\le\mu_0,$$

whence

$$w_{h\lambda}(Tx,Ty) \le \frac{k}{h}w_\lambda(x,y), \qquad 0<\lambda\le\mu_0, \qquad x,y\in X_w^*.$$

Taking into account inequalities (7) and $(h/k)\lambda < \lambda$, we get

$$w_\lambda(x,y) \le \frac{(h/k)\lambda}{\lambda}w_{(h/k)\lambda}(x,y) = \frac{h}{k}w_{(h/k)\lambda}(x,y),$$

which together with the previous inequality gives

$$w_{h\lambda}(Tx,Ty) \le w_{(h/k)\lambda}(x,y) \quad \text{for all} \quad 0<\lambda\le\mu_0 \quad \text{and} \quad x,y\in X_w^*.$$

Setting $\lambda' = (h/k)\lambda$ and $\lambda_0 = (h/k)\mu_0$ and noting that $0<\lambda'\le\lambda_0$ and $h\lambda = k\lambda'$, the last inequality implies $w_{k\lambda'}(Tx,Ty) \le w_{\lambda'}(x,y)$ for all $0<\lambda'\le\lambda_0$ and $x,y\in X_w^*$, which is exactly Eq. (24).

The main result of this chapter is the following fixed-point theorem for modular contractions in modular metric spaces X_w^*.

Theorem 6. *Let w be a strict convex modular on X such that the modular space X_w^* is w-complete and $T: X_w^* \to X_w^*$ be a w-contractive map such that*

$$\textit{for each } \lambda>0 \textit{ there exists an } x = x(\lambda)\in X_w^* \textit{ such that } w_\lambda(x,Tx)<\infty. \tag{26}$$

Then T has a fixed point, i.e., $Tx_* = x_*$ for some $x_* \in X_w^*$. If, in addition, the modular w assumes only finite values on X_w^*, then condition (26) is redundant, the fixed point x_* of T is unique and for each $\bar{x} \in X_w^*$ the sequence of iterates $\{T^n\bar{x}\}$ is modular convergent to x_*.

Proof. Since w is convex, the following inequality follows by induction from condition (iv) of Definition 1:

$$(\lambda_1+\lambda_2+\cdots+\lambda_N)w_{\lambda_1+\lambda_2+\cdots+\lambda_N}(x_1,x_{N+1}) \le \sum_{i=1}^{N} \lambda_i w_{\lambda_i}(x_i,x_{i+1}), \tag{27}$$

where $N \in \mathbb{N}$, $\lambda_1,\lambda_2,\ldots,\lambda_N \in (0,\infty)$ and $x_1,x_2,\ldots,x_{N+1} \in X$. In the proof below we will need a variant of this inequality. Let $n,m \in \mathbb{N}$, $n > m$, $\lambda_m,\lambda_{m+1},\ldots,\lambda_{n-1} \in (0,\infty)$ and $x_m,x_{m+1},\ldots,x_n \in X$. Setting $N = n-m$, $\lambda'_j = \lambda_{j+m-1}$ for $j = 1,2,\ldots,N$, and $x'_j = x_{j+m-1}$ for $j = 1,2,\ldots,N+1$ and applying Eq. (27) to the primed lambda's and x's, we get

$$(\lambda_m+\lambda_{m+1}+\cdots+\lambda_{n-1})w_{\lambda_m+\lambda_{m+1}+\cdots+\lambda_{n-1}}(x_m,x_n) \le \sum_{i=m}^{n-1} \lambda_i w_{\lambda_i}(x_i,x_{i+1}). \tag{28}$$

By the w-contractivity of T, there exist two numbers $0<k<1$ and $\lambda_0 = \lambda_0(k) > 0$ such that condition (24) holds. Setting $\lambda_1 = (1-k)\lambda_0$, the assumption (26) implies the existence of an element $\bar{x} = \bar{x}(\lambda_1) \in X_w^*$ such that $C = w_{\lambda_1}(\bar{x},T\bar{x})$ is finite. We set $x_1 = T\bar{x}$ and $x_n = Tx_{n-1}$ for all integer $n \ge 2$, and so, $\{x_n\} \subset X_w^*$ and $x_n = T^n\bar{x}$, where T^n designates the nth iterate of T. We are going to show that the sequence $\{x_n\}$ is w-Cauchy. Since $k^i\lambda_1 < \lambda_1 < \lambda_0$ for all $i \in \mathbb{N}$, inequality (24) yields

$$w_{k^i\lambda_1}(x_i,x_{i+1}) = w_{k(k^{i-1}\lambda_1)}(Tx_{i-1},Tx_i) \le w_{k^{i-1}\lambda_1}(x_{i-1},x_i),$$

and it follows by induction that

$$w_{k^i\lambda_1}(x_i,x_{i+1}) \le w_{\lambda_1}(\bar{x},x_1) = C \quad \text{for all} \quad i \in \mathbb{N}. \tag{29}$$

Let integers n and m be such that $n > m$. We set

$$\lambda = \lambda(n,m) = k^m\lambda_1 + k^{m+1}\lambda_1 + \cdots + k^{n-1}\lambda_1 = k^m\frac{1-k^{n-m}}{1-k}\lambda_1.$$

By virtue of Eq. (28) with $\lambda_i = k^i\lambda_1$ and Eq. (29), we find

$$w_\lambda(x_m,x_n) \le \sum_{i=m}^{n-1}\frac{k^i\lambda_1}{\lambda}w_{k^i\lambda_1}(x_i,x_{i+1}) \le \frac{1}{\lambda}\left(\sum_{i=m}^{n-1}k^i\lambda_1\right)C = C, \quad n > m.$$

Taking into account that

$$\lambda_0 = \frac{\lambda_1}{1-k} > k^m \frac{1-k^{n-m}}{1-k} \lambda_1 = \lambda(n,m) = \lambda \quad \text{for all} \quad n > m,$$

and applying Eq. (7), we get

$$w_{\lambda_0}(x_m, x_n) \le \frac{\lambda}{\lambda_0} w_\lambda(x_m, x_n) \le k^m \frac{1-k^{n-m}}{1-k} \cdot \frac{\lambda_1}{\lambda_0} C \le k^m C \to 0 \text{ as } m \to \infty.$$

Thus, the sequence $\{x_n\}$ is modular Cauchy, and so, by the w-completeness of X_w^*, there exists an $x_* \in X_w^*$ such that

$$w_{\lambda_0}(x_n, x_*) \to 0 \quad \text{as} \quad n \to \infty.$$

Since w is strict, by Theorem 1(b), the modular limit x_* of the sequence $\{x_n\}$ is determined uniquely.

Let us show that x_* is a fixed point of T, i.e., $Tx_* = x_*$. In fact, by property (iii) of Definition 1 and Eq. (24), we have (note that $Tx_n = x_{n+1}$)

$$w_{(k+1)\lambda_0}(Tx_*, x_*) \le w_{k\lambda_0}(Tx_*, Tx_n) + w_{\lambda_0}(x_*, x_{n+1}) \le$$
$$\le w_{\lambda_0}(x_*, x_n) + w_{\lambda_0}(x_*, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty,$$

and so, $w_{(k+1)\lambda_0}(Tx_*, x_*) = 0$. By the strictness of w, $Tx_* = x_*$.

Finally, assuming w to be finite valued on X_w^*, we show that the fixed point of T is unique. Suppose $x_*, y_* \in X_w^*$ are such that $Tx_* = x_*$ and $Ty_* = y_*$. Then the convexity of w and inequalities $k\lambda_0 < \lambda_0$ and Eq. (24) imply

$$w_{\lambda_0}(x_*, y_*) \le \frac{k\lambda_0}{\lambda_0} w_{k\lambda_0}(x_*, y_*) = k w_{k\lambda_0}(Tx_*, Ty_*) \le k w_{\lambda_0}(x_*, y_*),$$

and since $w_{\lambda_0}(x_*, y_*)$ is finite, $(1-k) w_{\lambda_0}(x_*, y_*) \le 0$. Thus, $w_{\lambda_0}(x_*, y_*) = 0$, and by the strictness of w, we get $x_* = y_*$. The last assertion is clear. □

It is to be noted that assumption (26) in Theorem 6 is (probably) too strong, and what we actually need for the iterative procedure to work in the proof of Theorem 6 is only the existence of an $\bar{x} \in X_w^*$ such that $w_{(1-k)\lambda_0}(\bar{x}, T\bar{x}) < \infty$, where λ_0 is the constant from Eq. (24).

A standard corollary of Theorem 6 is as follows: if w is finite valued on X_w^* and an nth iterate T^n of $T: X_w^* \to X_w^*$ satisfies the assumptions of Theorem 6, then T has a unique fixed point. In fact, by Theorem 6 applied to T^n, $T^n x_* = x_*$ for some $x_* \in X_w^*$. Since $T^n(Tx_*) = T(T^n x_*) = Tx_*$, the point Tx_* is also a fixed point of T^n, and so, the uniqueness of a fixed point of T^n implies $Tx_* = x_*$. We infer that x_* is a unique fixed point of T: if $y_* \in X_w^*$ and $Ty_* = y_*$, then $T^n y_* = T^{n-1}(Ty_*) =$

$T^{n-1}y_* = \cdots = y_*$, i.e., y_* is yet another fixed point of T^n, and again the uniqueness of a fixed point of T^n yields $y_* = x_*$.

Another corollary of Theorem 6 concerns general (nonconvex) modulars w on X (cf. Theorem 7). Taking into account Theorem 5 and its corollary, we have

Definition 8. Given a modular w on X, a map $T : X_w^* \to X_w^*$ is said to be *strongly modular contractive* (or a *strong w-contraction*) if there exist numbers $0 < k < 1$ and $\lambda_0 = \lambda_0(k) > 0$ such that

$$w_{k\lambda}(Tx, Ty) \le kw_\lambda(x,y) \quad \text{for all } 0 < \lambda \le \lambda_0 \text{ and } x, y \in X_w^*. \tag{30}$$

Clearly, condition (30) implies condition (24).

Theorem 7. *Let w be a strict modular on X such that X_w^* is w-complete and $T : X_w^* \to X_w^*$ be a strongly w-contractive map such that condition (26) holds. Then T admits a fixed point. If, in addition, w is finite valued on X_w^*, then Eq. (26) is redundant, the fixed point x_* of T is unique and for each $\overline{x} \in X_w^*$ the sequence of iterates $\{T^n\overline{x}\}$ is modular convergent to x_*.*

Proof. We set $v_\lambda(x,y) = w_\lambda(x,y)/\lambda$ for all $\lambda > 0$ and $x, y \in X$. It was observed in Sect. 2 that v is a convex modular on X. It is also clear that v is strict and the modular space $X_v^* = X_w^*$ is v-complete. Moreover, condition (30) for w implies condition (24) for v, and Eq. (26) is satisfied with w replaced by v. By Theorem 6, applied to X and v, there exists an $x_* \in X_v^* = X_w^*$ such that $Tx_* = x_*$. The remaining assertions are obvious. □

6 An Application of the Fixed-Point Theorem

In this section we present a rather standard application of Theorem 6 to the Carathéodory-type ordinary differential equations. The key interest will be in obtaining the inequality (24).

Given a convex φ-function φ on $\mathbb{R}^+$ satisfying the Orlicz condition at infinity, we denote by $\mathrm{L}^\varphi[a,b]$ the Orlicz space of real-valued functions on $[a,b]$ (cf. [22, Chap. II]), i.e., a function $z : [a,b] \to \mathbb{R}$ (or an almost everywhere finite-valued function z on $[a,b]$) belongs to $\mathrm{L}^\varphi[a,b]$ provided z is measurable and $\rho(z/\lambda) < \infty$ for some number $\lambda = \lambda(z) > 0$, where $\rho(z) = \int_a^b \varphi(|z(t)|)dt$ is the classical Orlicz modular.

Suppose $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a (Carathéodory-type) function, which satisfies the following two conditions:

(C.1) For each $x \in \mathbb{R}$ the function $f(\cdot, x) = [t \mapsto f(t,x)]$ is measurable on $[a,b]$ and there exists a point $y_0 \in \mathbb{R}$ such that $f(\cdot, y_0) \in \mathrm{L}^\varphi[a,b]$.

(C.2) There exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$ for almost all $t \in [a,b]$ and all $x, y \in \mathbb{R}$.

Given $x_0 \in \mathbb{R}$, we let X_w^* be the modular space Eq. (21) generated by the modular w from Eq. (14) under the assumptions from Example 4.4.

Consider the following integral operator:

$$(Tx)(t) = x_0 + \int_a^t f(s, x(s))ds, \qquad x \in X_w^*, \quad t \in [a,b]. \tag{31}$$

Theorem 8. *Under the assumptions (C.1) and (C.2), the operator T maps X_w^* into itself, and the following inequality holds in $[0,\infty]$:*

$$w_{L(b-a)\lambda}(Tx, Ty) \le w_\lambda(x,y) \quad \textit{for all } \lambda > 0 \textit{ and } x,y \in X_w^*. \tag{32}$$

Proof. We will apply the *Jensen integral inequality* with the convex φ-function φ (e.g. [24, X.5.6]) several times:

$$\varphi\left(\frac{1}{b-a}\int_a^b |x(t)|dt\right) \le \frac{1}{b-a}\int_a^b \varphi\big(|x(t)|\big)dt, \quad x \in \mathrm{L}^1[a,b], \tag{33}$$

where the intergral in the right-hand side is well defined in the sense that it takes values in $[0,\infty]$.

1. First, we show that T is well defined on X_w^*. Let $x \in X_w^*$, i.e., $x \in \mathrm{GV}_\varphi[a,b]$ and $x(a) = x_0$. Since (cf. Example 4.4) $x \in \mathrm{AC}[a,b]$, by virtue of (C.1) and (C.2), the composed function $t \mapsto f(t,x(t))$ is measurable on $[a,b]$. Let us prove that this function belongs to $\mathrm{L}^1[a,b]$. By Lebesgue's theorem, $x(t) = x_0 + \int_a^t x'(s)ds$ for all $t \in [a,b]$, and so, (C.2) yields

$$\begin{aligned} |f(t,x(t))| &\le |f(t,x(t)) - f(t,y_0)| + |f(t,y_0)| \\ &\le L|x(t) - y_0| + |f(t,y_0)| \\ &\le L\int_a^b |x'(s)|ds + L|x_0 - y_0| + |f(t,y_0)| \end{aligned} \tag{34}$$

for almost all $t \in [a,b]$. Since $x \in X_w^*$, and so, $x \in \mathrm{GV}_\varphi[a,b]$, there exists a constant $\lambda_1 = \lambda_1(x) > 0$ such that (cf. Eq. (20))

$$C_1 \equiv w_{\lambda_1}(x, x_0) = \int_a^b \varphi\left(\frac{|x'(s)|}{\lambda_1}\right)ds < \infty,$$

and since, by (C.1), $f(\cdot,y_0) \in \mathrm{L}^\varphi[a,b]$, there exists a constant $\lambda_2 = \lambda_2(f(\cdot,y_0)) > 0$ such that

$$C_2 \equiv \rho\big(f(\cdot,y_0)/\lambda_2\big) = \int_a^b \varphi\left(\frac{|f(t,y_0)|}{\lambda_2}\right)dt < \infty.$$

Setting $\lambda_0 = L\lambda_1(b-a) + 1 + \lambda_2$ and noting that

$$\frac{L\lambda_1(b-a)}{\lambda_0} + \frac{1}{\lambda_0} + \frac{\lambda_2}{\lambda_0} = 1,$$

by the convexity of φ, we find (see Eq. (34))

$$\varphi\left(\frac{1}{\lambda_0}\left[L\int_a^b |x'(s)|ds + L|x_0 - y_0| + |f(t,y_0)|\right]\right)$$

$$\leq \frac{L\lambda_1(b-a)}{\lambda_0}\,\varphi\left(\frac{1}{b-a}\int_a^b \frac{|x'(s)|}{\lambda_1}\,ds\right) + \frac{1}{\lambda_0}\,\varphi\big(L|x_0-y_0|\big) + \frac{\lambda_2}{\lambda_0}\,\varphi\left(\frac{|f(\cdot,y_0)|}{\lambda_2}\right),$$

and so, Eq. (34) and Jensen's integral inequality yield

$$\int_a^b \varphi\left(\frac{|f(t,x(t))|}{\lambda_0}\right)dt \leq \frac{L\lambda_1(b-a)}{\lambda_0}C_1 + \frac{b-a}{\lambda_0}\varphi\big(L|x_0-y_0|\big) + \frac{\lambda_2}{\lambda_0}C_2 \equiv C_0 < \infty. \tag{35}$$

Now, it follows from Eq. (33) that

$$\varphi\left(\frac{1}{\lambda_0(b-a)}\int_a^b |f(t,x(t))|dt\right) \leq \frac{1}{b-a}\int_a^b \varphi\left(\frac{|f(t,x(t))|}{\lambda_0}\right)dt \leq \frac{C_0}{b-a}$$

implying

$$\int_a^b |f(t,x(t))|dt \leq \lambda_0(b-a)\varphi^{-1}\left(\frac{C_0}{b-a}\right) < \infty.$$

Thus, $[t \mapsto f(t,x(t))] \in \mathrm{L}^1[a,b]$. As a consequence, the operator T is well defined on X_w^*, and, by Eq. (31), $Tx \in \mathrm{AC}[a,b]$ for all $x \in X_w^*$, which implies that the almost everywhere derivative $(Tx)'$ belongs to $\mathrm{L}^1[a,b]$ and satisfies

$$(Tx)'(t) = f(t,x(t)) \quad \text{for almost all} \quad t \in [a,b]. \tag{36}$$

2. It is clear from Eq. (31) that, given $x \in X_w^*$, $(Tx)(a) = x_0$, and so, $Tx \in X = \{y : [a,b] \to \mathbb{R} \mid y(a) = x_0\}$. Now we show that $Tx \in X_w^*$. In fact, by virtue of Eqs. (20), (36) and (35), we have

$$w_{\lambda_0}(Tx,x_0) = \int_a^b \varphi\left(\frac{|(Tx)'(t)|}{\lambda_0}\right)dt = \int_a^b \varphi\left(\frac{|f(t,x(t))|}{\lambda_0}\right)dt \leq C_0, \tag{37}$$

and so, T maps X_w^* into itself.

3. In order to obtain inequality (32), let $\lambda > 0$ and $x, y \in X_w^*$. Taking into account Eqs. (19), (20) and (36), we find

$$w_{L(b-a)\lambda}(Tx,Ty) = w_{L(b-a)\lambda}(Tx-Ty,x_0) = \int_a^b \varphi\left(\frac{|(Tx-Ty)'(t)|}{L(b-a)\lambda}\right)dt$$
$$= \int_a^b \varphi\left(\frac{|f(t,x(t))-f(t,y(t))|}{L(b-a)\lambda}\right)dt. \tag{38}$$

Applying (C.2) and Lebesgue's theorem, we get, for almost all $t \in [a,b]$ (note that $x(a) = y(a) = x_0$),

$$|f(t,x(t))-f(t,y(t))| \le L|x(t)-y(t)| \le L\int_a^b |(x-y)'(s)|ds,$$

and so, by Eq. (33), the monotonicity of φ, Eqs. (20) and (19),

$$\varphi\left(\frac{|f(t,x(t))-f(t,y(t))|}{L(b-a)\lambda}\right) \le \varphi\left(\frac{1}{b-a}\int_a^b \frac{|(x-y)'(s)|}{\lambda}ds\right)$$
$$\le \frac{1}{b-a}\int_a^b \varphi\left(\frac{|(x-y)'(s)|}{\lambda}\right)ds$$
$$= \frac{1}{b-a}w_\lambda(x,y).$$

Now, inequality (32) follows from Eq. (38).

□

As a corollary of Theorems 6 and 8, we have

Theorem 9. *Under the conditions (C.1) and (C.2), given $x_0 \in \mathbb{R}$, the initial value problem*

$$x'(t) = f(t,x(t)) \ \ \textit{for almost all} \ \ t \in [a,b_1] \ \ \textit{and} \ \ x(a) = x_0 \tag{39}$$

admits a solution $x \in \mathrm{GV}_\varphi[a,b_1]$ with $a < b_1 \in \mathbb{R}$ such that $L(b_1 - a) < 1$.

Proof. We know from Example 4.4 that w is a strict convex modular on the set $X = \{x : [a,b_1] \to \mathbb{R} \mid x(a) = x_0\}$ and that the modular space $X_w^* = \mathrm{GV}_\varphi[a,b_1] \cap X$ is w-complete. By Theorem 8, the operator T from Eq. (31) maps X_w^* into itself and is w-contractive. Since the inequality $w_{k\lambda}(Tx,Ty) \le w_\lambda(x,y)$ with $0 < k = L(b_1-a) < 1$ holds for all $\lambda > 0$, in the iterative procedure in the proof of Theorem 6, it suffices to choose any $\overline{x} \in X_w^*$ such that $w_{\overline{\lambda}}(\overline{x},T\overline{x}) < \infty$ for some $\overline{\lambda} > 0$. Since $(x_0)' = 0$, by virtue of Eqs. (37) and (35), we find

$$w_{\lambda_0}(Tx_0,x_0) \le C_0 = \frac{b_1-a}{\lambda_0}\varphi(L|x_0-y_0|) + \frac{\lambda_2}{\lambda_0}C_2 < \infty$$

(the constants λ_2 and C_2 being evaluated on the interval $[a,b_1]$) with $\overline{\lambda} = \lambda_0 = L(b_1 - a) + 1 + \lambda_2$, and so, we may set $\bar{x} = x_0$. Now, by Theorem 6, the integral operator T admits a fixed point: the equality $Tx = x$ on $[a,b_1]$ for some $x \in X_w^*$ is, by virtue of Eqs. (31) and (36), equivalent to Eq. (39). □

7 Concluding Remarks

7.1. It is not our intention in this chapter to study the properties of solutions to Eq. (39) in detail: after Theorem 9 on local solutions of Eq. (39) has been established, the questions of uniqueness, extensions, etc. of solutions can be studied following the same pattern as in, e.g., [13]. Theorems 8 and 9 are valid (with the same proofs) for mappings $x : [a,b] \to M$ and $f : [a,b] \times M \to M$ satisfying (C.1) and (C.2), where $(M, |\cdot|)$ is a reflexive Banach space; the details concerning the equality (20) in this case can be found in [2–5].

7.2. In the theory of the Carathéodory differential equations (39) (cf. [13]) the usual assumption on the right-hand side is of the form $|f(t,x)| \le g(t)$ for almost all $t \in [a,b]$ and all $x \in \mathbb{R}$, where $g \in \mathrm{L}^1[a,b]$, and the resulting solution belongs to $\mathrm{AC}[a,b_1]$ for some $a < b_1 < b$. However, it is known from [19, II.8] that $\mathrm{L}^1[a,b] = \bigcup_{\varphi \in \mathscr{N}} \mathrm{L}^{\varphi}[a,b]$, where $\mathscr{N}$ is the set of all φ-functions satisfying the Orlicz condition at infinity. Also, it follows from [2, Corollary 11] that $\mathrm{AC}[a,b] = \bigcup_{\varphi \in \mathscr{N}} \mathrm{GV}_{\varphi}[a,b]$. Thus, Theorem 9 reflects the *regularity* property of solutions of Eq. (39). Note that, in contrast with functions from $\mathrm{AC}[a,b]$, functions x from $\mathrm{GV}_{\varphi}[a,b]$ have the "qualified" modulus of continuity [5, Lemma 3.9(a)]: $|x(t) - x(s)| \le C_x \cdot \omega_{\varphi}(|t-s|)$ for all $t,s \in [a,b]$, where $C_x = d_w^*(x,0)$ and $\omega_{\varphi} : \mathbb{R}^+ \to \mathbb{R}^+$ is a subadditive function given by $\omega_{\varphi}(u) = u\varphi^{-1}(1/u)$ for $u > 0$ and $\omega_{\varphi}(+0) = \omega_{\varphi}(0) = 0$.

7.3. Theorem 8 does not reflect all the flavour of Theorem 6, namely, the *locality* of condition (24) and the *modular convergence* of the successive approximations of the fixed points, and so, an appropriate example is yet to be found; however, one may try to adjust Example 2.15 from [16] (note that Proposition 2.14 from [16] is similar to our assertion (23) with $k = 1$).

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