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Quantization Due to Breaking the Commutativity of Symmetries. Wobbling Oscillator and Anharmonic Penning Trap

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Abstract. We discuss two examples of classical mechanical systems which can become quantum either because of degeneracy of an integral of motion or because of tuning parameters at resonance. In both examples, the commutativity of the symmetry algebra is breaking, and noncommutative symmetries arise. Over the new noncommutative algebra, the system can reveal its quantum behavior including the tunneling effect. The important role is played by the creation-annihilation regime for the perturbation or anharmonism. Activation of this regime sometimes needs in an additional resonance deformation (Cartan subalgebra breaking).

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1. INTRODUCTION

All systems are quantum. In this paradigm the term "quantization" can be understood as "making the quantum behavior visible".

Usage of classical mechanics instead of quantum one is possible only due to smallness of the effective Planck scale of the system, when the quantum properties are ignorable.

Let us consider such a classical system and introduce a small smooth distortion into it. Would the system remain to be classical?

If the effective Planck scale is comparable with the scale of distortion, then the answer to this question is, in general, "no". Simple examples (which we demonstrate below) show that the classical behavior of the system can change dramatically and the hidden quantum behavior can reveal itself.

For instance, some trajectories of the system can be damped by the decay of quantum probability or can be translocated by quantum tunneling, the discrete spectrum of some observable can become "visible", etc.

A source (a mechanism) of such a quantum incarnation is the breaking of commutative symmetry and the bearing of a noncommutative symmetry structure. This can happen if the energy of the system degenerates near some values of integrals of motion. In this case, deviations and perturbations along the degeneracy directions decelerate the dynamics (make de Broglie wavelength bigger) and become responsible for quantum transfers due to creation-annihilation process over the symmetry algebra factorized by its Cartan subalgebra.

Another mechanism is switched on if the parameters controlling the system occur to be at resonance values. In this case, the quantum behavior of the system is revealed under an additional symmetry breaking which works similarly to the Higgs boson in the standard model: it breaks the symmetry Cartan subalgebra and turns the anharmonism and perturbations into the creation-annihilation regime.

2. FIRST EXAMPLE: DECELERATION IMPLYING QUANTIZATION

Let us consider a particle with two degrees of freedom described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x) + \varepsilon^2 W(y, x) \tag{2.1}$$

over the phase space with canonical Poisson brackets $\{p_x, x\} = \{p_y, y\} = 1$. The potential V is assumed to be a well so that all trajectories of the Hamiltonian

$$H_0 = \frac{1}{2}p_x^2 + V(x) \tag{2.2}$$

are periodic over a certain interval of energies; for instance, H_0 can be just an oscillator. The perturbation potential W in (2.1) is assumed to be negative and have a compact support. It generates a wobbling of the oscillator trajectories.

Until the values of the momentum p_y are not small, the influence of the perturbation $\varepsilon^2 W$ in (2.1) is insignificant. But in the domain where $p_y = O(\varepsilon)$, the wobbling potential can significantly decelerate the particle in the y-direction making its de Broglie wave length $1/\varepsilon$ times bigger. Thus if the small scale ε correlates with the scale of the effective Planck constant h, then the particle becomes a wave in the y-direction.

Mathematically, we have to consider the quantum version of (2.1) by replacing $p \to \hat{p} = -ih\nabla$, where $\nabla = (\nabla_x, \nabla_y)$, and set $\varepsilon = h$. The Hamiltonian becomes

$$\hat{H} = \hat{H}_0 + \varepsilon^2\Big(-\frac{1}{2}\frac{\partial^2}{\partial y^2} + W(y,x)\Big), \tag{2.3}$$

where

$$\hat{H}_0 = \frac{1}{2}\hat{p}_x^2 + V(x), \qquad \hat{p}_x = -i\varepsilon\frac{\partial}{\partial x}. \tag{2.4}$$

One can see that in y-direction the Hamiltonian does not contain any small parameter at the kinetic energy term $-\frac{1}{2}\frac{\partial^2}{\partial y^2}$. Thus, the properties of this system along the y-direction are purely quantum (or wave-like).

The properties along the x-direction can still be treated as classical because of presence of a small parameter ε in the momentum $\hat{p}$ (2.4).

Note that the x-dependence of the potential W in (2.3) can be replaced by H_0-dependence using the operator averaging transformation [1]. Namely, let us consider a unitary operator

$$U = \exp(-i\varepsilon\hat{C} - i\varepsilon^2\hat{D}) \tag{2.5}$$

and choose the generators $\hat{C}$, $\hat{D}$ in such a way that $U^{-1}\hat{H}U$ commutes with $\hat{H}_0$ up to $O(\varepsilon^4)$. We can construct $\hat{C}$, $\hat{D}$ in the form

$$\hat{C} = C(x, \hat{p}_x), \qquad \hat{D} = D(x, \hat{p}_x), \tag{2.6}$$

where $C(x, p_x)$ and $D(x, p_x)$ are some operators in the y-space depending on x, p_x as parameters. Let us compute

$$U^{-1}\hat{H}U = \hat{H}_0 + \varepsilon^2\hat{H}_2 + \varepsilon^3\hat{H}_3 + O(\varepsilon^4), \tag{2.7}$$

where

$$H_2 = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + W - \{H_0, C\}_{x,p_x}, \tag{2.8}$$

$$H_3 = i\big[C, -\frac{1}{2}\frac{\partial^2}{\partial y^2} + W\big] - \frac{i}{2}\big[C, \{H_0, C\}_{x,p_x}\big] - \{H_0, D\}_{x,p_x}. \tag{2.9}$$

Here the brackets $\{\cdot,\cdot\}_{x,p_x}$ are taken by x, p_x-coordinates, and the commutators $[\cdot,\cdot]$ are taken over the y-space.

From (2.8) one can see that, by solving the homological equations

$$\{H_0, W^{\#}\}_{x,p_x} = W - \underline{W}, \qquad \{H_0, \underline{W}\}_{x,p_x} = 0, \tag{2.10}$$

the operators H_2 and C become

$$H_2 = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + \underline{W}, \qquad C = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + W^{\#}. \tag{2.11}$$

The function $\underline{W}$ can be defined by averaging along the trajectories of the Hamiltonian flow $\gamma^t_{H_0}(x, p_x)$ in the x, p_x-space:

$$\underline{W}(y, H_0) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T W(y, \gamma^t_{H_0}(x, p_x))\,dt, \tag{2.12}$$

where $T = T(H_0)$ is the period of trajectories for the Hamiltonian H_0. Under this choice, the function $W^{\#}$ in (2.10) is given by

$$W^{\#}(x,p_x) = \frac{1}{T}\int_0^T W(y,\gamma^t_{H_0}(x,p_x))\Big(t-\frac{T}{2}\Big)\,dt. \tag{2.13}$$

In the same way, from (2.9) we choose

$$\begin{aligned} H_3 &= -i\frac{\partial \underline{W}}{\partial y}\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial^2 \underline{W}}{\partial y^2}, \\ D &= -i\frac{\partial W^{\&}}{\partial y}\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial^2 W^{\&}}{\partial y^2}, \qquad W^{\&} \overset{\text{def}}{=} \frac{1}{2}W^{\#} - (W^{\#})^{\#}. \end{aligned} \tag{2.14}$$

Lemma 2.1. *By using the operator U (2.5), (2.6) with C and D given by (2.11), (2.13), and (2.14), one can transform the Hamiltonian (2.3) to the form*

$$U^{-1}\hat{H}U = \hat{H}_0 + \varepsilon^2\Big(-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \underline{W}(y,\hat{H}_0)\Big) + \varepsilon^3\Big(-i\frac{\partial \underline{W}}{\partial y}(y,\hat{H}_0)\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial^2\underline{W}}{\partial y^2}(y,\hat{H}_0)\Big) + O(\varepsilon^4). \tag{2.15}$$

Here the averaged potential $\underline{W}(y,H_0)$ is determined by (2.12).

It follows from (2.15) that the discrete eigenvalues λ^E_k of the operator $-\frac{1}{2}\frac{\partial^2}{\partial y^2} + \underline{W}(y,E)$ generate discrete series of corrections to the (x,p_x)-classical Hamiltonian: $H_0 + \varepsilon^2\lambda^{H_0}_k + O(\varepsilon^3)$, and therefore instead of the oscillator frequency wobbling one obtains discrete frequency jumps: $\frac{2\pi}{T}(1 + \varepsilon^2\partial\lambda^{H_0}_k/\partial H_0 + O(\varepsilon^3))$.

But really dramatic consequence is that the eigenstates dispersion or the tunneling translocations in the potential well $\underline{W}(y,H_0)$ completely change the classical behavior of the dynamics making it wave-like or quantum in the y-direction.[1]

Let us look, for instance, at the Heisenberg equation

$$\frac{\partial \hat{F}}{\partial t} = \frac{i}{\hbar}[\hat{H},\hat{F}]. \tag{2.16}$$

In the usual semiclassical approach, the commutator can be replaced by the Poisson brackets $\frac{i}{\hbar}[\cdot,\cdot] \to \{\cdot,\cdot\} + O(h^2)$ (under the Weyl correspondence) and Eq. (2.16) is reduced to

$$\frac{\partial F}{\partial t} = \{H,F\} + O(h^2).$$

Even on the long-time scale $t \sim 1/h$, this equation is well approximated by the classical Liouville equation. The presence of the perturbation W in (2.1) plays no role in these standard arguments.

But actually in the phase space domain where $p_y = O(h)$, the commutator $\frac{i}{h}[W,\hat{F}]$ differs from the Poisson brackets $\widehat{\{W,F\}}$ significantly, and the presence of the perturbation W does not allow one to approximate the quantum system by the classical one. The above Lemma 2.1 implies that, on the $1/h$-time scale, the solution of the Heisenberg equation is not approximated by the solution of the classical Liouville evolution.

Theorem 2.1. *On the long-time scale $t = \tau/h$, the asymptotics as $h = \varepsilon \to 0$ of the solution to equation (2.16) with the Hamiltonian (2.1) and the initial distribution $F|_{t=0} = f_0(x,y;p_x,p_y/h)$, where $f_0 \in C_0^\infty$, is given by*

$$F = f_\tau(x,y;p_x,p_y/h), \qquad f_\tau \overset{\text{def}}{=} \gamma^{\tau/h}_{H_0}\Gamma^\tau_{H_2} f_0 + O(h).$$

Here $\gamma^t_{H_0}$ is the classical evolution in the (x,p_x)-space and $\Gamma^\tau_{H_2}$ is the quantum Heisenberg evolution along y-direction with the Hamiltonian H_2 (2.11).

Note that the geometric source of this quantization effect originates from the degeneracy of the Hamiltonian $\frac{1}{2}(p_x^2 + p_y^2) + V(x)$ on the hypersurface $\{p_y = 0\}$: at small p_y, this Hamiltonian is

[1] Possible physical model: curved wave-guides [11] (in this case, x is the geometrical optics cross-coordinate and y the quantum mechanics longitudinal coordinate, and the potential W is generated by the curvature).

approximated by H_0 up to $O(p_y^2)$. Also remark that in the whole phase 4D-space the Hamiltonian H_0 (2.2) has a noncommutative symmetry algebra in contrast to the original one (2.1) at $\varepsilon = 0$ whose symmetry algebra is commutative.

The quantum behavior manifests itself along symplectic leaves of the symmetry algebra of H_0. These leaves open new degrees of freedom which are a priori quantum, but their quantum properties are indeed activated only if the reduced system over the symmetry algebra is based on a creation-annihilation process.

3. SECOND EXAMPLE: RESONANCE AND SYMMETRY BREAKING

The main mechanism to obtain a noncommutative symmetry is to tune a system to a resonance. The main mechanism to activate the creation-annihilation process is to distort commutative symmetry. Resonance plus distortion trigger classical systems to be quantum.

It is well known that, for Hamiltonians like $H(q,\hat{p})$ (where $\hat{p} = -ih\frac{\partial}{\partial q}$) whose eigenvalues E_k are simple (not degenerate), the asymptotics of E_k as $h \to 0$ can be obtained by the Weyl rule:

$$\frac{1}{(2\pi h)^N}\int_{H\leqslant E_k} dpdq \approx k,$$

where $2N$ is the dimension of the (q,p)-space (see, for instance, in [2], formula (5.28)). In this case, the typical spectral gap is estimated as

$$\Delta E \sim h^N. \tag{3.1}$$

The estimation (3.1) is applicable out of the degeneracies of H, for instance, out of equilibrium points near which the creation-annihilation regime is working and multiplicity of eigenvalues can appear due to resonances.

At the same time, for systems with periodic flow, we have the Planck discretization rule

$$\frac{1}{2\pi h}\oint_{H=E_k} p\,dq \approx k,$$

where the integral is taken along a closed trajectory belonging to the energy level E_k. In this case, the spectral gap is estimated as

$$\Delta E \sim h \tag{3.2}$$

for any dimension N. Of course, the difference in asymptotic estimations (3.1) and (3.2) appears due to the strong degeneracy of eigenvalues in the periodic case. The degeneracy is related to the volumes of symplectic leaves of the noncommutative symmetry algebra, So, for systems with $N > 1$ degrees of freedom, the periodicity of trajectories becomes a mechanism needed to make the spectral gaps bigger (against the chaotic case), and thus to make the system “more quantum”. In the opposite way, small spectral gaps are negligible in experiments and the corresponding system can be considered as classical, not quantum.

For multidimensional harmonic (oscillator) systems, the periodicity of trajectories appears due to a resonance between frequencies. Thus, *the resonance can trigger the quantum behavior of systems* which are treated as classical out of resonance.

In the situation of real systems which are, in general, not harmonic, their anharmonic parts, after averaging, play the role of effective quantum Hamiltonians over noncommutative symmetry algebras related to the resonance harmonic parts. But one has to keep in mind that additional symmetries are able to erase the resonance effect by arranging the effective Hamiltonians just on the (commutative) Cartan symmetry subalgebra. In this situation, the noncommutativity can be saved by specific distortion or symmetry breaking which creates a secondary resonance and places the effective Hamiltonians to the creation-annihilation regime.

Let us consider, as an example, the electron confined in the planar Penning trap [3–6]. The trap is formed by a homogeneous magnetic field and by a saddle electric potential generated by three concentric electrodes placed on a plane. In the usual constructions, the magnetic field is directed along the perpendicular to the plane. The Hamiltonian has the form of a 3D-harmonic oscillator plus an anharmonic part of the electric potential. The values of the voltage W on the band (ring-like) electrode of the trap and the magnetic field strength B determine the frequencies of the oscillator.

Out of frequency resonance, the energy gaps of the electron in the Penning trap are estimated as $\Delta E \sim \mu B \cdot h^2$. Here $\mu = e\hbar/mc$ is the Bohr magneton and h is the effective Planck constant related to the magnetic length $\rho_0 = \sqrt{\hbar c/eB}$ and the inner scale ρ_1 of the band electrode as follows:

$$h = (\rho_0/\rho_1)^2, \qquad h \ll 1.$$

Even for strong enough magnetic field $B \sim 0.6T$ (i.e., $\rho_0 \sim 30\,\mathrm{nm}$) and even for very micro scale of the trap $\rho_1 \sim 300\,\mathrm{nm}$ (i.e., $h \sim 10^{-2}$), the spectral gaps in the nonresonance case

$$\Delta E \sim 0.6 \cdot 10^{-8}\,\mathrm{eV} \tag{3.3}$$

are "invisible". Therefore, the nonresonance Penning traps are classical systems.

In the papers [7–9], we have suggested to use the basic hyperbolic resonance $2 : (-1) : 2$ in the Penning oscillator. This means the following relation between parameters of the trap: $(\rho_1\rho_2^2)^{-1/3}\sqrt{\frac{mc^2W}{eB^2}} = \frac{2}{3\sqrt{3}}$.

The Hamiltonian of such a resonance trap looks as

$$\mu B\big(L_0/3 + \varepsilon \cdot \text{cubic} + \varepsilon^2 \cdot \text{quartic} + O(\varepsilon^3)\big). \tag{3.4}$$

Here L_0 is an action operator (oscillator) with spectrum $n_0 + 1/2$ ($n_0 \in \mathbb{Z}$), and the terms "cubic", "quartic", and the higher ones are obtained from the anharmonic part of the electric potential near the trap center. The small parameter $\varepsilon = (\rho_1/\rho_2)^{1/3}$ in (3.4) is determined by the ratio of the inner to outer scales of the band electrode. In order to keep the micro scale of the trap, we assume that $\rho_2 \sim 10\,\mu\mathrm{m}$, $\varepsilon^2 \sim 10^{-1}$.

By applying the averaging transformation similar to (2.5), the Hamiltonian (3.4) can be reduced to

$$\mu B\Big(L_0/3 + \varepsilon^2 h \sum_{j,k} c_{jk} L_j L_k + O(\varepsilon^4)\Big) \tag{3.5}$$

with mutually commuting action operators L_j and explicitly given coefficients c_{jk} without resonances. The spectral gaps are now estimated as

$$\Delta E \sim \mu B \cdot \varepsilon^2 h \sim 0.6 \cdot 10^{-7}\,\mathrm{eV}. \tag{3.6}$$

They are still too small and this system can be treated as classical.

Note that the leading resonance action L_0 in (3.5) determines the quantum system whose de Broglie wave length $\rho_1/3$ is comparable with the scale ρ_1 of the trap. But this quantum behavior is erased under interaction with anharmonic parts of the electric potential. Due to this interaction, the de Broglie wave length becomes $\varepsilon\sqrt{h}\rho_1$ which is much less than the scale ρ_1.

The ε^2-term in (3.5) is given on the Cartan subalgebra of the symmetry algebra for L_0. This fact places the ε^2-term out of the creation-annihilation regime which could decelerate the electron making its wave length to be $\varepsilon\rho_1 \sim \rho_1/3$ comparable with the scale ρ_1.

Note that the commutative structure like (3.5) appears in the Hamiltonian thanks to the presence of axial symmetry.

Now, following [7–9], let us break this symmetry and deviate the magnetic field at a small angle $\theta \sim \varepsilon$ from the perpendicular to the plane of electrodes. Then the Hamiltonian (3.4) is changed as follows:

$$\mu B\big(L_0/3 + \varepsilon \cdot \text{quadratic} + \varepsilon \cdot \text{cubic} + \varepsilon^2 \cdot \text{quartic} + O(\varepsilon^3)\big). \tag{3.7}$$

Here the additional "quadratic" term at the ε-order appeared due to contribution of ε-deviation of the magnetic field.

By averaging the perturbation terms in (3.7), one obtains

$$\mu B\big(L_0/3 + \varepsilon f_1(M) + \varepsilon^2 f_2(M) + O(\varepsilon^3)\big), \tag{3.8}$$

where $M = (M_1, \dots, M_5)$ are generators of the symmetry algebra for L_0, i.e., $[M_j, L_0] = 0$. Explicit formulas for M_j and f_1, f_2 can be found in [9].

Now one could again apply the averaging procedure to (3.8) using $f_1(M)$ as the leading term. But to do this, one needs a resonance in the term $f_1(M)$. Such a secondary resonance can be obtained by a specific choice of the deviation angle $\theta = s \cdot \varepsilon$ of the magnetic field. By keeping the notation B for the magnetic field strength, we can take into account the deviation angle just by perturbing the primary resonance condition as follows:

$$(\rho_1\rho_2^2)^{-1/3}\sqrt{\frac{mc^2W}{eB^2}} = \frac{2}{3\sqrt{3}}\Big(1 - \frac{s\varepsilon}{4} - \frac{2s^2-1}{4}\varepsilon^2\Big). \tag{3.9}$$

Under this relation, the ε-term in (3.8) reads $f_1(M) = L_1/4$ with an action like operator L_1 whose spectrum is $6n_1 - n_0 + 5/2 - s/3$, $n_1 \in \mathbb{Z}_+$.

By applying the averaging at ε^2-terms in (3.8), we transform the Hamiltonian to

$$\mu B\Big(\frac{1}{3}L_0 + \frac{\varepsilon}{4}L_1 + \frac{\varepsilon^2}{h}\hat{E} + O(\varepsilon^3)\Big),$$

where the operator $\hat{E}$ belongs to the joint symmetry algebra of L_0 and L_1 (see details in [7–9]). For fixed quantum numbers n_0 and n_1, this Hamiltonian is reduced to

$$\mu B\Big(\frac{1}{3}(n_0+\frac{1}{2}) + \frac{\varepsilon}{4}(6n_1 - n_0 + \frac{5}{2} - \frac{s}{3}) + \frac{\varepsilon^2}{h}(\alpha\hat{X} - \beta\hat{A}^2 - \gamma_n\hat{A} + \delta_n) + O(\varepsilon^3)\Big) \tag{3.10}$$

over the algebra with three generators obeying the commutation relations

$$[\hat{A},\hat{X}] = 2ih\hat{Y}, \qquad [\hat{A},\hat{Y}] = -2ih\hat{X},$$
$$[\hat{X},\hat{Y}] = -ih(3\hat{A}^2 + (2d_n+4h)\hat{A} + hd_n + 3h^2). \tag{3.11}$$

All constants in (3.10) are explicitly known, $\alpha = 8/3s$, $\beta = 1289/1152s^2$, the constants γ_n and δ_n are linear and quadratic in $n = (n_0, n_1)$. The constant d_n in (3.11) is given by $d_n = (n_0 - 2n_1 + 1)h$.

The algebra (3.11) has the Casimir element

$$\hat{K}_n = \hat{X}^2 + \hat{Y}^2 - \hat{A}^3 - (d_n - 2h)\hat{A}^2 - (d_n+4)h\hat{A} - (d_n+2h)h^2.$$

The Hamiltonian (3.10) is taken in the irreducible representation in which

$$\hat{K}_n = 0. \tag{3.12}$$

The system (3.8)–(3.10) has one degree of freedom. The spectral gaps of the reduced Hamiltonian

$$\hat{E}_n = \alpha\hat{X} - \beta\hat{A}^2 - \gamma_n\hat{A} + \delta_n \tag{3.13}$$

have the other $O(h)$, and thus the spectral gaps of the whole Hamiltonian are estimated as

$$\Delta E \sim \mu B\varepsilon^2 \sim 0.6\cdot 10^{-5}\,\text{eV}. \tag{3.14}$$

The operator $\hat{X}$ in (3.13) is the sum of the creation and annihilation operators in the algebra with relations (3.11) (and $\hat{A}$ is the operator from its Cartan subalgebra).

We have proved the following result.

Theorem 3.1. *Let one break the axial symmetry by deviating the magnetic field by the angle $s\cdot\varepsilon$, and let the bi-resonance condition* (3.9) *hold for geometric and electromagnetic parameters of the Penning planar micro-trap. Then the gaps between energy levels of the electron confined by the trap are estimated by* (3.14)*, i.e., become "visible". These gaps are* 10^3 *times bigger than the gaps* (3.3) *out of resonance, and* 10^2 *times bigger than the resonance gaps* (3.6) *without breaking the axial symmetry. Thus, the symmetry breaking plus tuning at biresonance convert the classical micro-trap to the quantum one.*

Now let us note that if one varies the voltage W on the band electrode of the trap by adding $W + \delta W$, where

$$\delta W/W \approx \varepsilon^2\cdot\frac{3r}{2s}, \tag{3.15}$$

then the constant γ_n in the Hamiltonian (3.13) is replaced by $\gamma_n + r$ (as well as an insignificant constant is added to δ_n). Thus, varying the voltage W in the order ε^2, we can strongly vary the parameter γ_n; in particular, we can change its sign.

By making this sign negative and making $|\gamma_n|$ big enough, one can obtain the separatrix energy level of the Hamiltonian E_n (3.13) on the symplectic leaf $\Omega_n = \{X^2 + Y^2 = A^3 + d_nA^2\}$ corresponding to the irreducible representation (3.12).

Below the separatrix, each energy level $\{E_n = \lambda\}$ consists of two closed curves (trajectories) on the surface $\Omega_n \subset \mathbb{R}^3$. They are candidates for the quantum tunneling and bilocalization of quantum states. Since E_n depends on the free parameter r from (3.15), the "avoided crossing" mechanism is working and the quantum tunneling transfer indeed happens at the same resonance value $r = r_n^*$.

The corresponding classical energy level λ (obeying the Planck–Bohr–Sommerfeld quantization condition) is split into two eigenvalues of the operator $\hat{E}_n$: $\lambda - \Delta/2$ and $\lambda + \Delta/2$ with

$$\Delta \sim \exp\{-S/h\}, \tag{3.16}$$

where S is the instanton under barrier action (see in [10]). Although the parameter $h \sim 10^{-2}$ is very small, but if one chooses the energy λ to be $\sqrt{h}$-close to the separatrix level, then the action

S becomes small as well, and the value of splitting (3.16) can be of order $\Delta \sim h^3$. In physical units this value reads $\mu B\varepsilon^2\Delta/h \sim \mu B\varepsilon^2 h^2$ (see in (3.10)). Thus the tunneling time is estimated as

$$T \sim \pi\hbar/\mu B\varepsilon^2 h^2 \sim 10^{-6} sec. \qquad (3.17)$$

Note that our tunneling-related pairs of closed trajectories of E_n are disposed in $\sqrt{h}$-neighborhood of the separatrix which is a figure eight curve (its center is the unstable stationary point of E_n on Ω_n). In the classical approximation one can say that the tunneling effect actually happens between two loops of this "eight". The classical time to transfer from one loop of the separatrix to another one equals to infinity. But the real tunneling time is given by(3.17).

Also note that the surface Ω_n and the tunneling-related energy curves of E_n belong to the symmetry algebra of the Penning trap Hamiltonian. The blow ups of these curves in the whole $6D$-phase space are presented by the Liouville $3D$-tori near which the real phase trajectories of the electron are placed. In the classical limit these tori join to an eight-shape $3D$-submanifold. Two tunneling-related parts of this "$3D$-separatrix" are mutually intersected by the $2D$-torus corresponding to the separatrix center.

Corollary 3.1. *By an additional tuning of the electric voltage W of the Penning trap at order ε^2, as in* (3.15)*: $r = r_n^*$, the quantum tunneling transfer between closed energy curves of E_n and the bilocalization of states for the Hamiltonian $\hat{E}_n$ appear.*

Therefore near the separatrix energy, after the double tuning (3.9) *plus* (3.15)*(at $r = r_n^*$), the electron trajectories start the quantum tunneling transfers from near one part of the $3D$-separatrix to near another part of it in the $6D$-phase space of the Penning trap. In the case of biresonance microtrap described above the tunneling time is at scale of microseconds* (3.17)*.*

Thus, the strong quantum behavior of a classical system can arise just by breaking its commutative symmetry and tuning its classical parameters to a resonance.

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