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HIGHER SCHOOL OF ECONOMICS

Nikolay A. Andreev

WORST-CASE APPROACH TO STRATEGIC OPTIMAL PORTFOLIO SELECTION UNDER TRANSACTION COSTS AND TRADING LIMITS

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Nikolay A. Andreev[1]

WORST-CASE APPROACH TO STRATEGIC OPTIMAL PORTFOLIO SELECTION UNDER TRANSACTION COSTS AND TRADING LIMITS

We study a worst-case scenario approach to the problem of strategic portfolio selection in presence of transaction costs and trading limits under uncertain stochastic process of market parameters. Unlike classic stochastic programming, the approach is model-free, solution of the arising Bellman-Isaacs equation can be easily found numerically under some general assumptions. All results hold for a general class of utility functions and several risky assets. For a special case of proportional transaction costs and CRRA utility, we present a numerical scheme which allows to reduce the dimension of the Bellman-Isaacs equation by a number of risky assets.

JEL Classification: C61, C63, G11.

Key words: portfolio selection, bellman equation, stochastic dynamic programming, transaction costs, worst-case scenario.

[1] National Research University Higher School of Economics. Laboratory for Financial Engineering and Risk Management. Researcher; E-mail: nandreev@hse.ru

Introduction

Stochastic programming approach to optimal portfolio selection is widely used in academic literature since the pioneering works of Merton [Merton, 1969] and Samuelson [Samuelson, 1969] who studied the problem for discrete and continuous time in its simplest form (multi-asset portfolio in costs-free market without price impact). Continuous time strategy modeling usually attracts more interest due to the possibility of a closed-form analytic solution of a Hamilton-Jacobi-Bellman equation (or quasi-variational inequality). The model assumes that m-dimensional price process X is given by geometric Brownian motion with SDE

$$dX_t^i = \mu_t^i X_t^i dt + X_t^i \sum_{j=1}^{m} \sigma_t^{ij} dw_t^j, \quad i = \overline{1, m}, \tag{1}$$

where w_t is a Wiener process. Dynamics of the risk-free asset Y is described by SDE

$$dY_t = rY_t dt \tag{2}$$

where r is a risk-free rate. Problem is solved for isoelasic (CRRA) utility, the solution is to keep a constant part of total portfolio wealth in each risky asset (the so called "Merton line" for single asset).

The approach has been extended in various studies. Richard [Richard, 1979] generalizes results to multi-dimensional Markovian price process. Karatzas et al. [Karatzas et al., 1986] solve the problem for HARA utility. Soon after the works of Merton and Samuelson, Bismut presents an alternative approach to solving the problem using the dual problem [Bismut, 1973], [Bismut, 1975]. It allows Pliska [Pliska, 1986] to find a solution for non-constant market parameters, a terminal phase constraint and general-shaped utility function. Shreve & Xu [Xu and Shreve, 1992] use the dual approach to solve the problem with phase constraints (no short-selling).

Besides the standard framework, there is a number of studies for problem with constrains on probability of default which are popular in actuarial mathematics where investor represents a pension fund. For other optimal criteria, see also [Cvitanić and Karatzas, 1993], [Basak, 1995], [Melnikov and Smirnov, 2012], [Kraft and Steffensen, 2013], [Andreev and Druzhinina, 2013].

Extensive research has been conducted recently for market with transaction costs and price impact. One of the first works in this field is [Magill and Constantinides, 1976] for canonical Merton's framework with infinite horizon and proportional costs function. It shows that optimal strategy allows a constant range of optimal risky position values relative to portfolio wealth. Constantinides' studies were extended by Davis & Norman [Davis and Norman, 1990] and [Dumas and Luciano, 1991] for continuous time. Based on Davis & Norman's work, Shreve & Soner [Shreve and Soner, 1994] study the problem for milder value function assumptions using viscosity solutions.

Above-mentioned continuous control framework does not allow for fixed fee per deal. Zakamouline in [Zakamouline, 2002], [Zakamouline, 2005] considers both fixed and proportional costs while maximizing portfolio terminal value over impulse control strategies. Numerical procedure for finding the solution is also presented for CARA-utility. [Vath et al., 2007] presents characteristics of the solution for price-dependent costs function and permanent price impact. Impulse solution for general-shaped concave costs function is studied in [Ma et al., 2013]. It is noteworthy that there are very few studies considering both transaction costs and trading limits (phase constraints).

The work by Bertsimas & Lo [Bertsimas and Lo, 1998] drew attention to the problem of optimal liquidation, i. e. optimal selection problem with a boundary condition. It has been researched in a series of works by Almgren & Chriss [Almgren and Chriss, 1999], [Almgren and Chriss, 2001], [Almgren, 2003], [Lorenz and Almgren, 2011], [Almgren, 2012] for discrete time, which consider various models of price impact and Markowitz approach to defining optimal criterion using risk-aversion of the portfolio manager. Further extension of the framework can be found in [Andreev et al., 2011], with cubic polinomial costs function with stochastic coefficients.

[Obizhaeva and Wang, 2013][2] became a foundation for further optimal liquidation studies in order-driven market. In this case, costs function can be defined via observable distribution of volumes in limit order book. Based on Almgren & Chriss approach, the work considers the same class of strategies (one-directional trades at given moments) but allows price value and costs dependency on previous actions of investor through resiliency. Alternative definitions of price resiliency were introduced, for example, in [Schied and Schöneborn, 2009], [Schöneborn, 2008], [Schöneborn, 2011] which maximize terminal utility, or [Alfonsi et al., 2008], [Alfonsi et al., 2010], [Predoiu et al., 2011], [Fruth et al., 2013] which minimize total costs of the liquidation.

We present a worst-case approach to optimal selection problem, based on stochastic

[2]The work first appeared in 2005 as a working paper.

dynamic programming principle. The key difference is that specification of the market parameters process is not required. Instead, basic properties of the process must be specified, such as expectation and credible intervals of parameters for subsequent periods (both can be estimated statistically or by an expert). Optimality is defined via maximization of worst-case expected value of general-shaped terminal utility. The approach allows transaction costs and phase constraints (including no short-selling for single risky asset) while being oriented for practical use as a decision support system (DSS) during investment management process. Similar approach, in game-theory terms, was studied in [Deng et al., 2005] for one-period problem and Markowitz optimal criterion without transaction costs.

First chapter considers general framework applied to market without costs. We obtain the sufficient conditions to simplify the arising Bellman-Isaacs equation and study properties of the value function. Second chapter generalizes the results for non-zero transaction costs. Third chapter describes numerical procedure for finding the solution for the particular case of linear costs, which allows reducing dimension of the problem. Fourth chapter presents results for model data, fifth chapter concludes.

1 Market with no transaction costs

Consider an indexed set of filtered probability spaces

$$(\Omega, \mathcal{F}, \mathbb{F}, P_\omega^{(i)}), \quad i \in \mathcal{I},$$

satisfying usual conditions, compact sets $\mathcal{K}_t \subset \mathbb{R}^l$ and vectors $E_t \in \mathcal{K}_t$ for $t \in T$. Let $\Theta\colon \Omega \times T \to \mathbb{R}^l$ be an $\mathcal{F}_t$-adaptive random function such that

$$\begin{aligned} &\Theta_t \in \mathcal{K}_t, \quad t \in T, \\ &\mathbb{E}_{P_\omega^{(i)}}^{\mathcal{F}_{t-}} \Theta_t = E_t, \quad i \in \mathcal{I}, \quad t > \inf T. \end{aligned} \tag{3}$$

We consider a standard financial market of one risk-free and m risky assets (analogous to (B, S)-market in [Shiryaev, 1998]) for discrete time $T = \{t_0, \ldots, t_N\}$). Θ_{t_n} describes stochastic parameters of the market at moment t_n. Distribution of Θ_t is unknown but (3) states that it belongs to a class of distributions with compact support and known expectation, henceforth denoted as $\mathbb{P}$. Let $Q_n^{(i)}(A) \mid \mathcal{F}_{n-1} = P_\omega^{(i)}\{\Theta_{t_n} \in A \mid \mathcal{F}_{n-1}\}$ stand for mutual distribution of parameters at time t_n (dependence on $\mathcal{F}_{n-1}$ will be suppressed in later notation), a set of all such measures is denoted $\mathbb{Q}_n$ for every $n = \overline{1, N}$.

Assume that $H_n^X \in \mathbb{R}^m$ is the volume in risky assets at t_n, and $H_n^Y \in \mathbb{R}$ is the volume in risk-free asset at t_n, while prices are $X_n \in \mathbb{R}^m$ and $Y_n \in \mathbb{R}$ correspondingly.

Definition 1. *Portfolio at time $t_n, n = \overline{0, N}$, is a vector $\mathbf{H}_n = (H_n^X, H_n^Y)$.*

Definition 2. *Market value of portfolio $\mathbf{H}$ at t_n is*

$$W_n = H_n^{X^T} X_n + H_n^Y Y_n. \tag{4}$$

$W_n^X = H_n^{X^T} X_n$, $W_n^Y = H_n^Y Y_n$ are market values of risky and risk-free positions.

In the presence of transaction costs, difference arises between market and liquidation value, i.e. real value of portfolio when liquidated on the market. Henceforth, by "portfolio value" we mean market value.

Definition 3. *Trading strategy is a sequence*

$$\mathbf{H} = \{\mathbf{H}_n\}_{n=1}^N, \quad \mathbf{H}_n \in m(\mathcal{F}_{n-1}).$$

$\mathbf{H}_0$ is a known initial condition. Denote

$$\mathbf{H}_{\leq k} = \{\mathbf{H}_n\}_{n=1}^k, \quad \mathbf{H}_{\geq k} = \{\mathbf{H}_n\}_{n=k}^N.$$

(Throughout the paper inequalities in indices are interpreted analogously.) Outer capital movements are not considered so the following budget equation must hold at every n:

$$\Delta H_n^{X^T} X_{n-1} + \Delta H_n^Y Y_{n-1} = 0 \tag{5}$$

$$\Leftrightarrow$$

$$H_n^Y = Y_{n-1}^{-1}(W_{n-1} - H_n^{X^T} X_{n-1}). \tag{6}$$

A set of $\mathbf{H}_n$ that satisfy (6) is denoted SF_n.

By trading limits at time t_n we mean phase constraints of the form $\mathbf{H}_n \in D_n \subseteq \mathbb{R}^m$, $n = \overline{1, N}$. D_n can be dependent on $\mathbf{H}_{n-1}$, X_{n-1} and W_{n-1}^Y.

Definition 4. *Admissible strategy is a trading strategy $\mathbf{H}$ such that $\mathbf{H}_n \in SF_n \cap D_n = \mathcal{A}_n$ for every $n = \overline{1, N}$. A class of all admissible strategies is denoted $\mathcal{A} = \{\mathbf{H} \colon \mathbf{H}_n \in \mathcal{A}_n,\ n = \overline{1, N}\}$.*

Budget equation demonstrates that H_n^Y can always be expressed through H_n^X, so we will consider H^X as a strategy and define phase constraints in terms of H^X. We also assume that E_n и $\mathcal{K}_n$ are constants that do not depend on market or portfolio state.

Definition 5. *Optimal strategy is an admissible strategy H^{X^*} such that*

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_0} J(X_N, H_N^{X^*}, W_N^{Y^*}) = \sup_{H^X\in\mathcal{A}} \inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_0} J(X_N, H_N^X, W_N^Y). \tag{7}$$

We define value function as

$$V_n(X, H^X, W^Y) = \sup_{H^X_{\geq n+1}\in\mathcal{A}_{\geq n+1}} \inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n} J(X_N, H_N^X, W_N^Y), \quad n = \overline{0, N-1}, \tag{8}$$

where X is vector of risky assets' prices, H^X is vector of volumes of risky assets, and W^Y is market value of risk-free position, all at time t_n. Bellman-Isaacs equation can be written as

$$\begin{aligned} &V_n(X, H^X, W^Y) = \sup_{Z\in D_{n+1}} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z, W_{n+1}^Y), \\ &V_N(X, H^X, W^Y) = J(X, H^X, W^Y). \end{aligned} \tag{9}$$

Theorem 1. *Assume the following assumptions hold*

1. *For any $n = \overline{1, N-1}$, Θ_{n+1} is independent of $\Theta_1, \ldots, \Theta_n$.*

2. *Function $V_n(X, H^X, W^Y)$ is defined according to Bellman-Isaacs equation* (9).

Then

1. *$V_n(X, H^X, W^Y)$ satisfies* (8).

2. *If H^{X^*} satisfies*

$$H_{n+1}^{X^*} \in \operatorname*{Arg\,max}_{\substack{Z\in D_{n+1},\\ H_n^X=H^X, X_n=X,\\ W_n^Y=W^Y}} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z, W_{n+1}^Y) \quad n = \overline{0, N-1}, \tag{10}$$

then H^{X^*} is an optimal solution of the control problem

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_0} J(X_N, H_N^X, W_N^Y) \longrightarrow \sup_{H^X\in\mathcal{A}}. \tag{11}$$

First condition of the Theorem states independence of parameters process which is rather constraining. However it is only required during the proof of the verification theorem, the rest of the results below holds without it as long as optimal solution is a solution of the Bellman-Isaacs equation. Thus strategy can be required to be a solution of the Bellman-Isaacs equation. Besides, dependence of Θ_n values (for example, clustering of price volatility) can be considered while estimating the input parameters of $\mathcal{K}_n$ and E_n. Hereinafter, we assume that optimal solution of the investment problem satisfies (10).

Since Θ describes market parameters, $X_{n+1} = X_{n+1}(\Theta_{n+1}, \cdot)$ and $W^Y_{n+1} = W^Y_{n+1}(\Theta_{n+1}, \cdot)$, while H^X_{n+1} is predictable. Thus, the value function can be viewed as a function of Θ: $V_{n+1}(\Theta_{n+1}, \cdot) = V_{n+1}(X_{n+1}(\Theta_{n+1}, \cdot), H^X_{n+1}, W^Y_{n+1}(\Theta_{n+1}, \cdot))$. The main difficulty in finding optimal strategy is to find the extreme measure in (9). However, when concavity of the value function V_{n+1} in Θ_{n+1} holds, the measure can be found by using the following statement:

Theorem 2. *Let $\mathbb{Q}$ be a class of all measures with compact support $\mathcal{K}$ being a cartesian product of l intervals; consider $f(x)\colon \mathbb{R}^l \to \mathbb{R}$ — a concave function on $\mathcal{K}$, and value $E \in \mathcal{K}$. Then the optimization problem*

$$\begin{cases} \int\limits_K f(x) dQ(x) \longrightarrow \inf\limits_{Q\in\mathbb{Q}}, \\ \int\limits_K x_i dQ(x) = E_i, \ i = \overline{1,l}, \\ \int\limits_K dQ(x) = 1. \end{cases} \tag{12}$$

has an $(l+1)$-atomic solution with mass concentrated in corners of $\mathcal{K}$.

Remark: This result can be obtained from theory of generalized Tchebycheff inequalities, see [Karlin and Studden, 1966, chapter XII]. Similar problem of finding extreme measure is studied in [Goovaerts et al., 2011] for a number of measure classes. The alternative proof presented in Appendix 2 is constructive and provides analytic formula for masses in atoms which is crucial for numerical solution of the Bellman-Isaacs equation.

Hereinafter we assume that for every n, class $\mathbb{Q}_n$ does not depend on $\mathcal{F}_{n-1}$ and contains all distributions with compact support $\mathcal{K}_n$ and expectation E_n.

The rest of the research concentrates on a particular case of Θ being parameters of a general price process. Consider multiplicative price dynamics based on discrete version of geometric Brownian motion (1). Δ means backward difference operator: $\Delta\xi_n = \xi_n - \xi_{n-1}$.

Dynamics of risky price $X \in \mathbb{R}^m$ is given by

$$\Delta X_n = \mu_n X_{n-1} \Delta t_n + \sigma_n X_{n-1} \sqrt{\Delta t_n},\ n = \overline{1, N}, \tag{13}$$

where $\sigma_n \in \mathbb{R}^{m \times m}$ are diagonal matrices with random elements $\sigma_n^1, \ldots, \sigma_n^m$ on the main diagonal, $\mu_n \in \mathbb{R}^{m \times m}$ are matrices with non-negative non-diagonal elements as in [Yaozhong, 2000]. We assume that μ_n is known for all n. Construction of (13) resembles GBM model but does not assume normal, or even symmetrical, distribution of returns, thus avoiding the most criticized assumption of the model (see [Cont, 2001]).

Risk-free dynamics is given by

$$\Delta Y_n = r_n Y_{n-1} \Delta t_n,\ n = \overline{1, N}, \tag{14}$$

where $r_n \geq 0$ is a risk-free rate known for every n.

Theorem 2 can be used to simplify the Bellman-Isaacs equation for specified price dynamics and $\Theta_n = (\sigma_n^1, \ldots, \sigma_n^m)^T$. Assume that at time t_n range of each random variable σ_n^i is $[\underline{\sigma}_n^i, \overline{\sigma}_n^i]$ while expectation is E_n^i. If the value function V_{n+1}, as a function of Θ_{n+1}, is concave in Θ_{n+1}, then solution of the minimization problem in (9) is an atomic measure concentrated on one of $C_{2^m}^{m+1}$ combinations of $m+1$ corners of the support, with particular choice of the combination depending on X, H^X and W^Y if $m > 1$. (Note that for $m = 1$ atomic measure is defined solely by parameters of expectation and support.) A set of corner combinations we denote as $\mathcal{G}$. Then (9) can be simplified to the form

$$\begin{aligned} &V_n(X, H^X, W^Y) = \\ &\quad \sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}(X, H^X, W^Y)} \sum_{i=1}^{m} p_i(G) V_{n+1}\Big((1 + \mu_{n+1} \Delta t_{n+1} + \operatorname{diag}(G_i) \sqrt{\Delta t_{n+1}}) X, Z, \\ &\qquad\qquad (W^Y - (Z - H^X)^T X)(1 + r_{n+1} \Delta t_{n+1}) \Big),\ n < N, \end{aligned} \tag{15}$$

$$V_N(X, H^X, W^Y) = J(X, H^X, W^Y),$$

where $\operatorname{diag}(G_i)$ is a diagonal matrix with elements of vector G_i on the main diagonal. Sufficient conditions for concavity of the value function in this particular framework are presented below.

1.1 Properties of the value function and the optimal strategy

This section presents sufficient conditions to simplify the Bellman-Isaacs equation (9) and transform it to the form (15). We also show that direct specification of expectation E for σ cannot be avoided by solving extreme measure problem on a class with given compact support only. Thus, expectation and support can be thought of as minimal information basis about the unknown distribution required to solve the problem. It is known among practitioners that expectation E is difficult to estimate statistically which is consistent with the Efficient-market hypothesis. Besides, it cannot be separated from μ in this case. For practical purposes, $E_n = 0$ is reasonable assumption since expected value of returns will be defined exclusively by μ. For other choices of Θ (for example, when it includes risk-free rate r) consequences of avoiding E require further research. Hereinafter, for the sake of convenience, we assume that the optimal control problem (11) has finite solution and value function is finite.

Consider set-valued function

$$D(X, H^X, W^Y) : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to 2^{\mathbb{R}}.$$

Assumption 1. *For every X, H^X, W^Y,*

1.

$$Z \in D(X, H^X, W^Y) \quad \Longleftrightarrow \quad A^T Z \in D(A^{-1}X, A^T H^X, W^Y) \tag{16}$$

$$\textit{for all invertible matrices } \; A;$$

2.

$$Z \in D(X, H^X, W^Y) \quad \Longleftrightarrow \quad Z \in D(X, 0, W^Y + {H^X}^T X); \tag{17}$$

3. For every $\alpha \in [0, 1]$,

$$Z_1 \in D(X, H_1^X, W_1^Y),\ Z_2 \in D(X, H_2^X, W_2^Y) \quad \Rightarrow$$

$$\Rightarrow \; \alpha Z_1 + (1-\alpha)Z_2 \; \in \; D(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y). \tag{18}$$

Assumption $1'$**.** *For every* X, H^X, W^Y*,* (17)*,* (18) *hold and*

$$Z \in D(X, H^X, W^Y) \quad \Longleftrightarrow \quad AZ \in D(A^{-1}X, AH^X, W^Y) \tag{19}$$

$$\forall A = \mathrm{diag}(a^1, \ldots, a^m) > 0.$$

As an example of constraint set which satisfies the Assumptions, the following statement can be readily proved:

Statement 1. *Set-valued function*

$$D(X, H^X, W^Y) = \left\{Z \in \mathbb{R}^m \colon -\beta^X W \leq Z^T X \leq (1 + \beta^Y) W\right\}, \tag{20}$$

$$W = W^Y + {H^X}^T X,$$

satisfies Assumption 1.

Proof. Obtained directly by verifying (16)-(18). $\square$

Now, consider function $V(X, H^X, W^Y) : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$.

Assumption 2. *For every* X, H^X, W^Y*,*

$$V(AX, H^X, W^Y) = V(X, A^T H^X, W^Y) \quad \forall A. \tag{21}$$

Assumption $2'$**.** *For every* X, H^X, W^Y*,*

$$V(AX, H^X, W^Y) = V(X, AH^X, W^Y) \quad \forall A = \mathrm{diag}(a^1, \ldots, a^m) > 0. \tag{22}$$

Assumption 3. *For every* X, H^X, W^Y*,*

$$V(X, H^X, W^Y) = V(X, 0, W^Y + {H^X}^T X). \tag{23}$$

The following notations will be used below:

$$
\begin{aligned}
s_{n+1} &= I + \mu_{n+1}\Delta t_{n+1} + \sigma_{n+1}\sqrt{\Delta t_{n+1}}, \\
s_{n+1}(G_i) &= I + \mu_{n+1}\Delta t_{n+1} + \mathrm{diag}(G_i)\sqrt{\Delta t_{n+1}}, \\
\underline{s}_{n+1} &= I + \mu_{n+1}\Delta t_{n+1} + \underline{\sigma}_{n+1}\sqrt{\Delta t_{n+1}}, \\
\overline{s}_{n+1} &= I + \mu_{n+1}\Delta t_{n+1} + \overline{\sigma}_{n+1}\sqrt{\Delta t_{n+1}}, \\
\tilde{r}_{n+1} &= 1 + r_{n+1}\Delta t_{n+1}.
\end{aligned}
\tag{24}
$$

Note that if X_n follows (13), then $X_{n+1} = s_{n+1}X_n$. Thus, if μ_{n+1} is diagonal, then $\mathcal{K}_{n+1}$ must be chosen in such a way that $s_{n+1} > 0$ holds. Otherwise, at time t_{n+1} price of some assets is assumed to become non-positive with non-zero probability which is not considered in the current framework (all issuers are default-free).

Assuming that utility J and constraints D_n satisfy above-mentioned assumptions, we prove that some properties of J is inherited by value functions V_n across all n, along with concavity in W^Y. Then it is easy to obtain sufficient conditions to simplify (9):

Theorem 3. *Let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *satisfy Assumptions 2 and 3.*
2. $J(X, H^X, W^Y)$ *is concave in* W^Y.
3. *For every* $n = \overline{1, N}$, $D_n(X, H^X, W^Y)$ *satisfies Assumption 1.*
4. *Prices* X_n *and* Y_n *follow* (13) *and* (14) *correspondingly.*

Then the Bellman-Isaacs equation (9) *is equivalent to the simplified equation* (15).

Theorem 4. *Let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *satisfy Assumptions* $2'$ *and 3.*
2. $J(X, H^X, W^Y)$ *is concave in* W^Y.
3. *For every* $n = \overline{1, N}$, $D_n(X, H^X, W^Y)$ *satisfies Assumption* $1'$.
4. *Prices* X_n *and* Y_n *follow* (13) *and* (14) *correspondingly.*
5. *For every* $n = \overline{1, N}$, μ_n *are diagonal.*

Then the Bellman-Isaacs equation (9) *is equivalent to the simplified equation* (15).

Concluding the section, we return to the question of avoiding E as a required parameter while finding optimal worst-case strategy. For the problem with unspecified E we provide sufficient conditions under which risk-free strategy is always optimal, thus investment process is degenerate.

Theorem 5. *Consider the Bellman-Isaacs equation* (9) *and let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *satisfies Assumptions 2 and 3.*
2. $J(X, H^X, W^Y)$ *is concave in* W^Y*.*
3. *For every* $n = \overline{1, N}$, $D_n(X, H^X, W^Y)$ *satisfies Assumption* $1'$ *and* $0 \in D_n(X, H^X, W^Y)$.
4. *Prices* X_n *and* Y_n *follow* (13) *and* (14) *correspondingly.*
5. *For every* $n = \overline{1, N}$, μ_n *are diagonal.*
6. $(r_{n+1} I - \mu_{n+1})\sqrt{\Delta t_{n+1}} \in \mathcal{K}_{n+1}$.

Then $V_n(X, H^X, W^Y)$ *as a function of* E_{n+1} *at fixed* $\mathcal{K}_{\geq n+1}$, $E_{\geq n+2}$, $\mu_{\geq n+1}$ *and* $r_{\geq n+1}$, *attains minimal value over* $\mathcal{K}_{n+1}$ *at* $E^*_{n+1} = (r_{n+1} I - \mu_{n+1})\sqrt{\Delta t_{n+1}}$, *where* $I \in \mathbb{R}^{m \times m}$ *is identity matrix. Moreover, in this case* $H^{X*}_{n+1} = 0$ *is an optimal strategy.*

2 Market with transaction costs

Denote by $C_n(H, X_n) = C(H, X_n)$ the value of transaction costs from deal of volume H at moment t_n. Presence of costs slightly changes formalization of the problem and obtained results. Budget equation SF_n has the form

$$\Delta H_n^{X^T} X_{n-1} + \Delta H_n^Y Y_{n-1} = -C_{n-1}(\Delta H_n^X, X_{n-1}) \tag{25}$$

$$\Leftrightarrow$$

$$H_n^Y = Y_{n-1}^{-1}(W_{n-1} - H_n^{X^T} X_{n-1} - C_{n-1}(\Delta H_n^X, X_{n-1})) \tag{26}$$

and coincides with (6) when $C_{n-1} \equiv 0$. By liquidation value of the portfolio, we assume $W_n - C_n(H_n^X, X_n)$, i. e. value obtained by liquidating all positions at the market.

Bellman-Isaacs equation (9) remains almost unchanged and the verification theorem 1 still holds. Considering price dynamics (13)-(14), the equation transforms into

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}} V_{n+1}\Big(s_{n+1}X, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} - C_n(Z - H, X)\tilde{r}_{n+1}\Big),\ n < N, \quad (27)$$

$$V_N(X, H^X, W^Y) = J(X, H^X, W^Y).$$

In previous chapter, we presented sufficient conditions in costs-free market to simplify the Bellman-Isaacs equation. Here we provide sufficient conditions under non-zero costs. First, consider some assumptions made for C_n.

Assumption 4. *For every X, H and every $A = \mathrm{diag}(a^1, \ldots, a^m) > 0$,*

1. *$C(H, X)$ is non-negative, non-decreasing in $|H|$ and convex in H;*
2. $$C(AH, X) = C(H, AX); \quad (28)$$

As in costs-free case, one can readily prove analog to Theorem 4:

Theorem 6. *Let the following assumptions hold:*

1. *$J(X, H^X, W^Y)$ is non-decreasing in W^Y.*
2. *$J(X, H^X, W^Y)$ satisfies Assumption $2'$.*
3. *$J(X, H^X, W^Y)$ is jointly concave in H^X, W^Y.*
4. *For every $n = \overline{1, N}$, $D_n(X, H^X, W^Y)$ satisfies Assumption $1'$.*
5. *For every $n = \overline{0, N-1}$, $C_n(H, X)$ satisfies Assumption 4.*
6. *Prices X_n and Y_n follow (13) and (14) correspondingly.*
7. *For every $n = \overline{1, N}$, μ_n are diagonal.*

Then the Bellman-Isaacs equation (27) *is equivalent to the simplified equation*

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}\Big(s_{n+1}(G_i)X,$$
$$Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} - C_n(Z - H^X, X)\tilde{r}_{n+1}\Big),\ n < N. \quad (29)$$

The key difference between Theorems 4 and 6 are the assumption of joint concavity and non-decreasing in W^Y. However, these properties are inherent to a wide range of classic utility functions of the form

$$J(X, H^X, W^Y) = J(W^Y + {H^X}^T X - C_N(H^X, X)), \quad (30)$$

i.e. non-decreasing concave functions of terminal liquidation value of the portfolio.

3 Optimized numeric algorithm for solving the Bellman-Isaacs equation with linear costs

When costs function is linear in volume, it is possible to reformulate the problem in terms of $W^X = {H^X}^T X$ instead of X and H^X separately. This leads to Bellman-Isaacs equation where V_n depends on m less variables compared to the general case, which is useful for numerical purposes. At time t_n, let $C_n(\Delta H, X) = \lambda_n |\Delta H| X$. In the formula, we assume that limit book is symmetrical which is necessary when bid-ask spread is zero and market is arbitrage-free (see [Gatheral, 2010]). Such symmetry is used for the sake of convenience while not required for numerical scheme. Generalization for asymmetrical costs will be presented below.

Consider isoelastic utility $J(X, H^X, W^Y) = (W^Y + {H^X}^T X - \lambda_N |{H^X}^T| X)^\gamma / \gamma$ and $m = 1$, multidimensional case can be readily written out analogously. Then, denoting $\pi^X = \frac{W^X}{W_0}$, $\pi^Y = \frac{W^Y}{W_0}$, we can work in terms of dimensionless variables and obtain the Bellman-Isaacs equation as

$$V_n(\pi^X, \pi^Y) = \sup_{h \in D_{n+1}(W^X, W^Y)} \Big[p_{n+1} V_{n+1}\big(\overline{s}_{n+1} h, \pi^Y \tilde{r}_{n+1} - (h - \pi^X)\tilde{r}_{n+1} - \lambda_n \big| h - \pi^X \big| \tilde{r}_{n+1}\big) +$$

$$+ (1 - p_{n+1})V_{n+1}\big(\underline{s}_{n+1}h, \pi^Y\tilde{r}_{n+1} - (h - \pi^X)\tilde{r}_{n+1} - \lambda_n\big|h - \pi^X\big|\tilde{r}_{n+1})\big)\Big], \quad (31)$$

$$V_N(\pi^X, \pi^Y) = (\pi^Y + \pi^X - \lambda_N|\pi^X|)^\gamma/\gamma, \quad (32)$$

where

$$D_{n+1} = \big\{h : -\beta^X_{n+1}(\pi^X + \pi^Y) \leq h \leq (1 + \beta^Y_{n+1})(\pi^Y + \pi^X)\big\}. \quad (33)$$

Dimension can be reduced up to $m + 1$ for even more general case: when costs function $C_n(\Delta H, X)$ can be represented as a function of $\Delta H^T X$. For general case of asymmetric C_n, in equations (31),(32) all expressions of the form "$\lambda_n|h|$" should be replaced by "$\lambda(h)|h|$" where $\lambda_n(h) \equiv \lambda^+_n = \text{const}$ for $h \geq 0$ and $\lambda_n(h) \equiv \lambda^-_n = \text{const}$ for $h < 0$. (λ^+_n and λ^-_n are proportionality coefficients for transaction costs function for buy and sell deals correspondingly.) However, in this case the value function will depend on (W^X, W^Y) and not on dimensionless (π^X, π^Y).

The described method was implemented for MatLab R2012a. Generally speaking, the framework can be decomposed into several blocks. Actual implementation depends on specific formalization of the problem. The block are:

1. Defining key aspects of the problem which is conducted by both investor and portfolio manager. At this stage, one defines investment horizon, control moments $t_1, \dots, t_N$, initial market and portfolio state, optimal criteria, admissible set of assets for future investments (stock selection).

2. Preliminary analysis of market data: estimation of initial market parameters and a priori distributions for Bayes method.

3. Update procedure for statistically estimated parameters and updates of expert forecasts.

4. Numerical solution of the Bellman-Isaacs equation based on current estimates.

5. Analysis of strategy performance and control characteristics. Strategy can be stopped prematurely due to reset or assumed "bankruptcy".

We consider the case when the value function is concave in parameters Θ, hence the Bellman-Isaacs equation can always be reduced to simplified form. If phase constraints are compact, maximum is always achieved. For example, in one-dimensional case, constraint set (20) is an interval, while in multidimensional case it can be modified as

$$D(X, H^X, W^Y) = \left\{ Z \in \mathbb{R}^m : \begin{array}{rcccl} -\beta^X W & \leq & Z^T X & \leq & (1+\beta^Y)W, \\ & & |Z|^T X & \leq & (1+\tilde{\beta}^Y)W \end{array} \right\}, \tag{34}$$

where $W = W^Y + {H^X}^T X$, so that D is compact and still satisfies Assumption 1 which can be readily verified. (34) can be interpreted as limits for total size of short positions and limit for the amount invested in risky assets. Without the second constraint, one could infinitely short one risky asset and invest in another without violating the limit.

The value function can be calculated recursively according to (31), however this method becomes too slow with the increase of the number of steps N. Hence, we propose a step-by-step reconstruction of the value function on (π^X, π^Y) grid: first, for t_N, then for t_{N-1} by using known values at t_N, and so on up to t_0. As a byproduct, we obtain reconstructed value function for the whole grid which can be used for future analysis and strategy modeling if market parameters and forecasts are assumed unchanged.

At time t_N value function is known from analytic formula of J. Suppose that V_{n+1} is reconstructed for the grid. To find V_n according to (31), we might need V_{n+1} values in points inside and outside the grid. Thus, either interpolation and extrapolation methods or appropriate parametric form $\hat{V}_{n+1}$ of the function is required. The latter approach was used during modeling, parametric form was chosen so that it is concave for any value of calibration coefficients. We find that for isoelastic J, all V_{n+1}, $n < N-1$, can be approximated (even in the presence of costs and constraints) by isoelastic function of the form

$$\hat{V}_{n+1}(\pi^X, \pi^Y) = \left({b^X_{n+1}}^T \pi^X + b^Y_{n+1} \pi^Y + c_{n+1} \right)^\gamma / \gamma, \tag{35}$$

with fitting reduced to simple linear regression. Since V_n depends solely on V_{n+1}, it is possible to calculate values of V_n on the grid in parallel mode.

4 Modeling results

This chapter presents results of implementing the proposed framework to modeled market. We consider one risky asset and stationary parameters μ_n, E_n, $\mathcal{K}_n$. Economic interpretation allows to divide them into two main groups characterizing price forecast and deviation from it. Hence, during investment process, the groups can be estimated by different departments (analyst/trader and risk-manager). We assume that μ is given by an expert analyst since poorly estimated based on market information. $\mathcal{K}$ is estimated and updated via a Bayesian method based on observable data: we assume that the data follows a known stochastic process with unknown parameters (GBM was used since it was the basis for the multiplicative price model). The parameters are estimated and $\mathcal{K}$ is found as credible interval of detrended returns. E is assumed zero to keep all the information about price forecast within μ.

Analyst's forecasts of μ are characterized by the forecasting power. Denote the price change over interval $[t_n, t_{n+1}]$ as ΔX_{n+1}. At time t_n, value of the forecast is modeled as a random variable $\tilde{\mu}_{n+1}$ with normal distribution such that

$$\tilde{\mu}_{n+1} - \frac{\Delta X_{n+1}}{X_n} = \frac{\Delta X_{n+1}}{X_n}\xi, \quad \xi \sim \mathcal{N}(0, \varepsilon^2). \tag{36}$$

ε^{-1} is a measure of forecast's precision (hence, analyst's forecasting power). This method of forecast modeling is extremely rough but allows to define dimensionless measure of forecasting power. Estimates for $\mathcal{K}$ are characterized by credible interval for specified level α.

Below we demonstrate the worst-case strategy and discuss the results for one realized scenario of market price. Parameters are the following: $m = 1$; $N = 5$; $C_n(\Delta H, X) = \lambda|\Delta H|X$; price follows GBM with drift 0.03 and volatility 0.005; risk-free rate equals 0.02; initial prices are 1; initial capital is 10; $\Delta t_n = 1$. The utility function is isoelastic with $\gamma = 0.6$. Strategy is constrained by $D(X, H^X, W^Y)$, defined in (20), with stationary coefficients $\beta_n^X = \beta_n^Y = 1$. Between neighboring t_n 500 price observations are assumed available for Bayesian updates. Figures 1 - 3 demonstrate realized price trajectory and the worst-case strategy results for high forecasting power $\varepsilon = 1$ when $\lambda = 0$ and $\lambda = 0.05$ (loss of 5% of each deal's value).

Since the forecasting power is big enough, all decisions made by DSS were correct in terms of long/short position, hence portfolio value increased at every step. In the presence of costs, transacted volumes are smaller and the total profit becomes less. Further increase in λ shows that at some point costs are so large that risky investments are not worth investing into, even if all the decisions are correct. Figure 4 demonstrates results for the same price scenario and $\lambda = 0.12$.

For the same values of parameters, we simulated market dynamics and compared the expectation of liquidation value. Based on 100 iterations, we obtained that, for $\lambda = 0.05$,

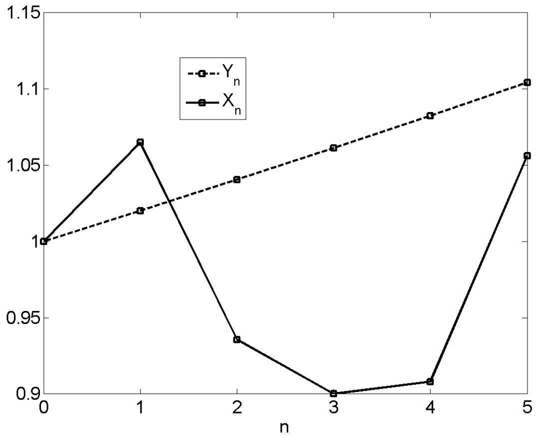

Figure 1: Realized price trajectory.

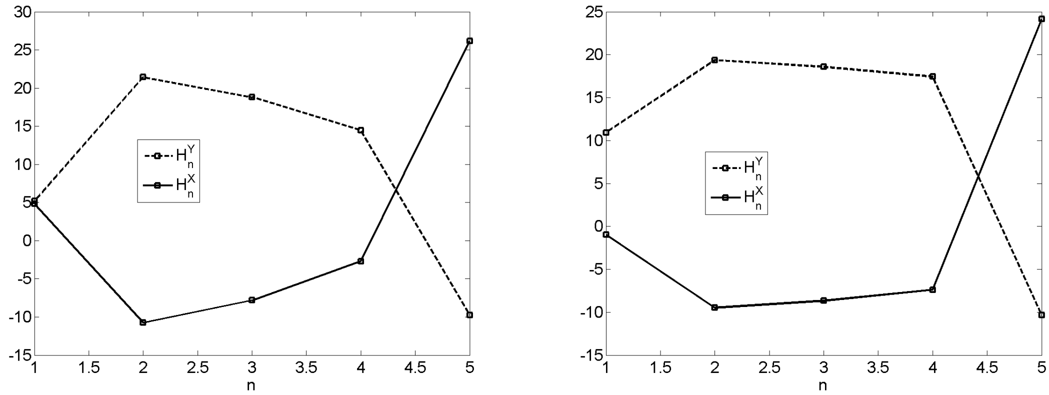

Figure 2: The worst-case optimal strategy: to the left — at costs-free market; to the right — at $\lambda = 0.05$. $\varepsilon = 1$.

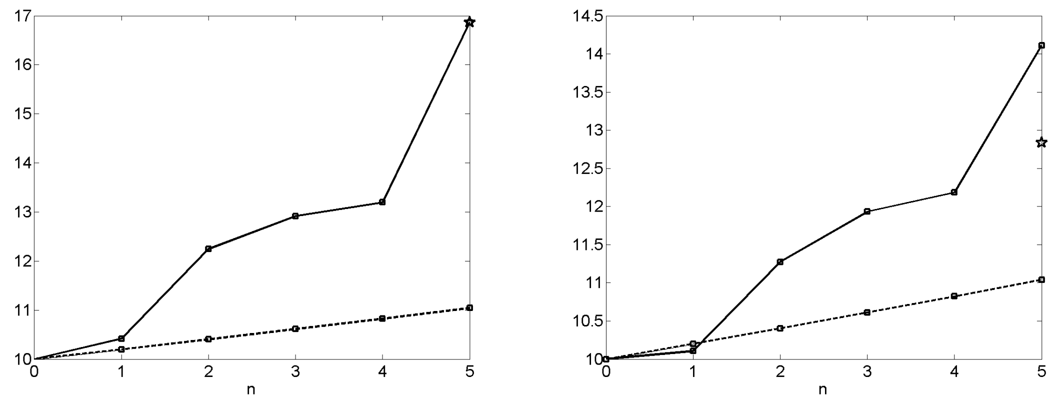

Figure 3: Portfolio market value W_n when using the worst-case optimal strategy: to the left — at costs-free market; to the right — at $\lambda = 0.05$. Dashed line is market value according to risk-free strategy $H^X \equiv 0$. Pentagonal star denotes liquidation value of the portfolio at the end of the strategy. $\varepsilon = 1$.

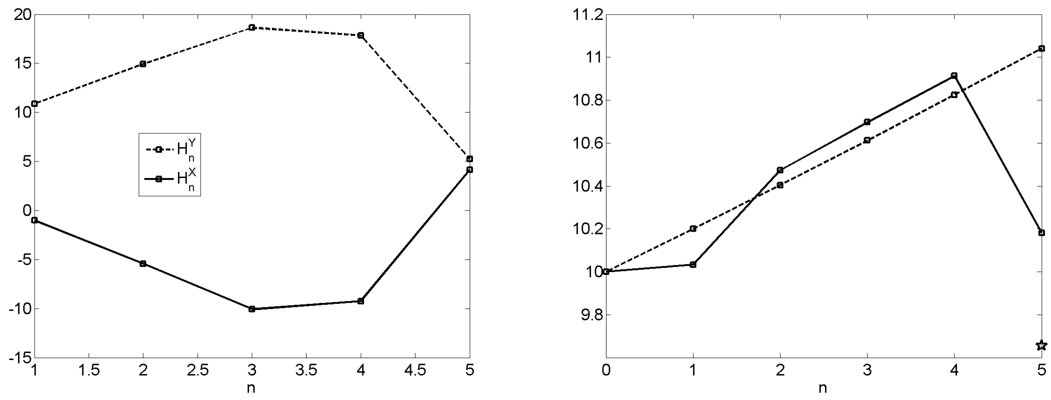

Figure 4: To the left is optimal strategy, to the right is market value of the portfolio for $\lambda = 0.12$. $\varepsilon = 1$. Markup repeats Fig. 2 and 3

expected optimal portfolio value outperforms risk-free value by 3.05%, and by 28.32% for $\lambda = 0$. This shows that even for $\lambda = 0.05$ the worst-case strategy produces better results than risk-free investment.

5 Conclusion

We present a worst-case approach to strategic portfolio selection problem for discrete time when stochastic process of market parameters is not specified. Discrete framework is chosen over continuous because the intended purpose of the work is implementing the approach as a DSS during investment process and not as an automatic trading system. Besides, even high-frequency trading cannot be continuous due to latency of the trading system which, in couple with possibility of fixed transaction costs, makes discrete management model more viable.

The selection problem can be solved if only statistical properties of the process are known such as expected value and range of its values for future time periods. Various choices of parameters produce different frameworks for the problem, hence the paper studies only the case of unknown price process. The key aspect of the worst-case framework is indepedence of price model and the implied assumptions. For example, the canonical model of geometric Brownian motion assumes normal distribution of returns which has been criticized lately [Cont, 2001], while the proposed approach assumes general multiplicative model without the assumptions of GBM. An expert, while working with the DSS, can change forecasts and ranges of possible price movements according to state of the market, which is crucial during crisis and eventual shocks. One of the assumption of the framework is compactness of range which can be considered mild restriction if range is chosen big enough.

The paper presents research of the proposed framework and gives sufficient conditions to reducing the arising Bellman-Isaacs equation to a simpler form. Simplified equation does not require finding extreme element in a class of measures with given expectation and support, thus can be easily solved numerically. The main results hold for several risky assets in presence of convex transaction costs and trading limits. For proportional costs, we present a simpler form of the Bellman-Isaacs equation for numerical solution, which has reduced dimension of the value function's phase space.

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Appendix 1. Verification theorem

To prove Theorem 1, we make use of the following lemma:

Lemma 1. *Let Q be a measure on $X \subseteq \mathbb{R}^l$ and $f\colon X \times Y \to \mathbb{R}$ is such that for every $y \in Y$ the integral*

$$\int\limits_X f(x, y) dQ(x) < \infty.$$

In addition, assume that

$$\exists y^* \in Y\colon \quad \inf_Y \int\limits_X f(x, y) dQ(x) = \int\limits_X f(x, y^*) dQ(x) > -\infty;$$

$$\exists y^{**} \in Y\colon \quad \text{for a.e. } x \in X \quad \inf_Y f(x, y) = f(x, y^{**}) > -\infty.$$

Then

$$\inf_Y \int\limits_X f(x, y) dQ(x) = \int\limits_X \inf_Y f(x, y) dQ(x).$$

Proof. Denote $f(x, y^*) = f^*(x)$, $f(x, y^{**}) = f^{**}(x)$. By the assumptions,

$$\inf_Y \int\limits_X f(x, y) dQ(x) = \int\limits_X f^*(x) dQ(x), \qquad \int\limits_X \inf_Y f(x, y) dQ(x) = \int\limits_X f^{**}(x) dQ(x).$$

When $\int\limits_X f^*(x) dQ(x) > \int\limits_X f^{**}(x) dQ(x)$, then y^* cannot be minimum, thus contradicting the assumptions of the lemma. Hence,

$$\int\limits_X f^*(x) dQ(x) \leq \int\limits_X f^{**}(x) dQ(x).$$

On the other hand,

$$\forall y' \in Y \quad \int\limits_X \inf_Y f(x, y) dQ(x) \leq \int\limits_X f(x, y') dQ(x) \quad \Longrightarrow$$

$$\Longrightarrow \quad \forall y' \in Y \quad \int\limits_X f^{**}(x) dQ(x) \leq \int\limits_X f(x, y') dQ(x),$$

so that $\int\limits_X f^{**}(x)dQ(x) \leq \int\limits_X f^*(x)dQ(x)$. The obtained inequalities imply

$$\inf_Y \int\limits_X f(x,y)dQ(x) = \int\limits_X f^*(x)dQ(x) = \int\limits_X f^{**}(x)dQ(x) = \int\limits_X \inf_Y f(x,y)dQ(x).$$

□

Proof of Theorem 1. 1) For ease of notation, denote $J_N = J(X_N, H_N^X, W_N^Y)$. For $n < N$, we have

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n} J_N = \inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n}\mathbb{E}_P^{\mathcal{F}_{n+1}} J_N = \inf_{P\in\mathbb{P}} \int \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N \; dP(\Theta_{\geq n+1} \mid \mathcal{F}_n) =$$

$$= \inf_{P\in\mathbb{P}} \int \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N \; dP(\Theta_{\geq n+2} \mid \Theta_{n+1}, \mathcal{F}_n) dP(\Theta_{n+1} \mid \mathcal{F}_n) =$$

$$= \inf_{P\in\mathbb{P}} \int \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N \; \underbrace{dP(\Theta_{n+1} \mid \mathcal{F}_n)}_{dQ_{n+1}} \underbrace{\int dP(\Theta_{\geq n+2} \mid \Theta_{n+1}, \mathcal{F}_n)}_{=1} =$$

$$= \inf_{P\in\mathbb{P}} \int \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N \; dQ_{n+1} \overset{\text{Lemma 1}}{=} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \int \inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N \; dQ_{n+1} =$$

$$= \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_{n+1}} J_N. \tag{37}$$

Applying (37) successively for $k \geq n$ leads to

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n} J_N = \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \inf_{Q_{n+2}\in\mathbb{Q}_{n+2}} \mathbb{E}_{Q_{n+2}}^{\mathcal{F}_{n+1}} \ldots \inf_{Q_N\in\mathbb{Q}_N} \mathbb{E}_{Q_N}^{\mathcal{F}_{N-1}} J_N. \tag{38}$$

2) Let H^{X^*} satisfy (10), $\bar{H}^X \in \mathcal{A}$ is an admissible strategy. Then, by using (9), we obtain

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n} J(X_N, \bar{H}_N^X, \bar{W}_N^Y) \overset{(38)}{=} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \ldots \inf_{Q_N\in\mathbb{Q}_N} \mathbb{E}_{Q_N}^{\mathcal{F}_{N-1}} J(X_N, \bar{H}_N^X, \bar{W}_N^Y) =$$

$$= \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \ldots \inf_{Q_N\in\mathbb{Q}_N} \mathbb{E}_{Q_N}^{\mathcal{F}_{N-1}} V_N(X_N, \bar{H}_N^X, \bar{W}_N^Y) \leq$$

$$\leq \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \ldots \inf_{Q_N\in\mathbb{Q}_N} \mathbb{E}_{Q_{N-1}}^{\mathcal{F}_{N-2}} V_{N-1}(X_{N-1}, \bar{H}_{N-1}^X, \bar{W}_{N-1}^Y) \leq \ldots \leq V_n(X_n, H_n, W_n^Y).$$

This proves the first statement of the Theorem. Further transformations give

$$\inf_{P\in\mathbb{P}} \mathbb{E}_P^{\mathcal{F}_n} J(X_N, \bar{H}_N^X, \bar{W}_N^Y) \leq V_n(X_n, H_n, W_n^Y) \overset{(9)}{=} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, H_{n+1}^{X^*}, W_{n+1}^{Y^*}) \overset{(9)}{=} \ldots$$

$$\ldots \overset{(9)}{=} \inf_{Q_{n+1}\in\mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} \ldots \inf_{Q_N\in\mathbb{Q}_N} \mathbb{E}_{Q_N}^{\mathcal{F}_{N-1}} V_N(X_N, H_N^{X^*}, W_N^{Y^*}) \overset{(9)}{=}$$

$$\overset{(9)}{=} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} \cdots \inf_{Q_N \in \mathbb{Q}_N} \mathbb{E}^{\mathcal{F}_{N-1}}_{Q_N} J(X_N, H_N^{X^*}, W_N^{Y^*}) \overset{(38)}{=} \inf_{P \in \mathbb{P}} \mathbb{E}^{\mathcal{F}_n}_{P} J(X_N, H_N^{X^*}, W_N^{Y^*}).$$

Substitution for $n = 0$ verifies the second statement and concludes the proof.

□

Appendix 2. Solution of the extreme measure problem

Lemma 2. *Assume that function* $f(\mathbf{x})$ *is concave on* $K = [\underline{x}^1; \overline{x}^1] \times \ldots \times [\underline{x}^n; \overline{x}^n] \subset \mathbb{R}^n$. *In addition, consider* $\mathbf{A}_1, \ldots, \mathbf{A}_N$ *as corners of* K, *and* $\mathbf{E} \in K$. *Then there exist corners* $\mathbf{A}_{k_1}, \ldots, \mathbf{A}_{k_{n+1}}$ *and affine* $l(\mathbf{x})$, *such that*

1. $l(\mathbf{A}_{k_i}) = f(\mathbf{A}_{k_i})$, $i = \overline{1, n+1}$;
2. $l(\mathbf{x}) \leq f(\mathbf{x})$, $\forall x \in K$.

Proof. 1) One can readily show that concave $f(\mathbf{x})$ attains minimum on one of the corners of K. Consider closed convex hull $M \subset \mathbb{R}_{n+1}$ of points $\mathbf{B}_k = (\mathbf{A}_k, f(A_k)) \in \mathbb{R}^{n+1}$. According to [Artamonov and Latyshev, 2004, p. 45], M is a polyhedron, thus can be defined in terms of a non-singular system of linear equations

$$g_k(\mathbf{y}) = \sum_{i=1}^{n+1} a_i^k y_i + a_0^k, \quad k = \overline{1, r},$$

so that $M = \left\{\mathbf{y} \colon g_k(\mathbf{y}) \geq 0,\ k = \overline{1, r}\right\}$. Since f is finite, M is obviously a bounded set, thus, $a_{n+1}^1 = \ldots = a_{n+1}^r$ cannot hold.

2) Consider a set of facets of M[3]. Our goal is to prove that a bounded polyhedron has a facet which is dominated in the $(n+1)$-th coordinate in the following sense: M possesses hyper plane Π_{k^*}, defined by g_{k^*}, such that $a_{n+1}^{k^*} \neq 0$ and

$$\forall y \in M \;\; y_{n+1} \geq -\sum_{i=1}^{n} \frac{a_i^{k^*}}{a_{n+1}^{k^*}} y_i - \frac{a_0^{k^*}}{a_{n+1}^{k^*}}.$$

By contradiction, assume that for any k, such that $a_{n+1}^k \neq 0$, there is point $\mathbf{y}^{(\mathbf{k})} \in M$ which satisfies

$$y_{n+1}^{(k)} < -\sum_{i=1}^{n} \frac{a_i^k}{a_{n+1}^k} y_i^{(k)} - \frac{a_0^k}{a_{n+1}^k} \quad \Longleftrightarrow \quad a_{n+1}^{k^*} g_{k^*}(\mathbf{y}^{(k)}) < 0.$$

If $a_{n+1}^{k^*} > 0$ then $g_{k^*}(\mathbf{y}^{(k)}) < 0$, thus, $\mathbf{y}^{(k)} \notin M$, which contradicts the assumption. Due to frivolous choice of k, we have to conclude that $a_{n+1}^k \leq 0$ for every $k = \overline{1, r}$. Now, consider any point $\mathbf{z} = (z_1, \ldots, z_{n+1}) \in M$. Then

$$g_k(z_1, \ldots, z_{n+1}) \geq 0 \;\Rightarrow\; g_k(z_1, \ldots, z_{n+1} - \Delta) \geq 0 \quad \forall k = \overline{1, r} \;\Rightarrow$$

[3] According to [Artamonov and Latyshev, 2004, Proposition 2.41], the set is non-empty.

$$\Rightarrow (z_1, \dots, z_{n+1} - \Delta) \in M \quad \forall \Delta > 0,$$

which contradicts boundedness of M, thus proving the statement.

3) Assume that Π_{k^*} has corners $\mathbf{B}_{k_1}, \dots, \mathbf{B}_{k_{n+1}}$. Consider affine function

$$l(\mathbf{x}) = -\sum_{i=1}^{n} \frac{a_i^{k^*}}{a_{n+1}^{k^*}} x_i - \frac{a_0^{k^*}}{a_{n+1}^{k^*}}.$$

By definition of Π_{k^*}, $l(\mathbf{A}_{k_i}) = f(\mathbf{A}_{k_i})\ \forall i = \overline{1, n+1}$. Properties of Π_{k^*} also imply that

$$\forall \mathbf{x} \in K \quad f(\mathbf{x}) \geq -\sum_{i=1}^{n} \frac{a_i^{k^*}}{a_{n+1}^{k^*}} x_i - \frac{a_0^{k^*}}{a_{n+1}^{k^*}} = l(\mathbf{x}).$$

□

Lemma 3. *Under the assumptions of Lemma 2, if $f(\mathbf{x}) \geq c$ on K and $\mathbf{E} \in K$ then there are corners $\mathbf{A}_{k_1}, \dots, \mathbf{A}_{k_{n+1}}$ of K and affine function $l(\mathbf{x}) = -\sum_{i=1}^{n} \frac{a_i^k}{a_{n+1}^k} x_i - \frac{a_0^k}{a_{n+1}^k}$, which has properties 1) and 2) of Lemma 2, such that $l(\mathbf{E}) \geq c$.*

Proof. In terms of proof of Lemma 2, we proved earlier that M has dominated facets $\Pi_1, \dots, \Pi_m$. By contradiction, suppose there is a point $\mathbf{E} \in K$ for which

$$-\sum_{i=1}^{n} \frac{a_i^k}{a_{n+1}^k} E_i - \frac{a_0^k}{a_{n+1}^k} < c \quad \forall i = \overline{1, m}. \tag{39}$$

Denote $\Pi_E = \{y \in M \colon y_1 = E_1, \dots y_n = E_n\}$. (39) implies that

$$\left(\bigcup_{k=1}^{m} \Pi_k\right) \bigcap (\Pi_E \cap M) = \emptyset \Longrightarrow \left(\bigcup_{k=1}^{m} \Pi_k\right) \bigcap (\Pi_E \cap \Gamma) = \emptyset, \tag{40}$$

where Γ denotes boundary of M. Consider non-dominated facets $\Gamma_1, \dots, \Gamma_r$ of M, which have non-empty intersection with Π_E and satisfy equalities $g_1(\mathbf{y}), \dots, g_r(\mathbf{y})$ correspondingly. (40) implies that if $\mathbf{E}' \in \Pi_E \cap \Gamma_k$, then either $a_{n+1}^k = 0$ or $\exists \mathbf{y} \in M \colon y_{n+1} < -\sum_{i=1}^{n} \frac{a_i^k}{a_{n+1}^k} y_i - \frac{a_0^k}{a_{n+1}^k}$. If $a_{n+1}^k > 0$, definition of M implies $g(\mathbf{y}) \geq 0 \Leftrightarrow y_{n+1} \geq -\sum_{i=1}^{n} \frac{a_i^k}{a_{n+1}^k} y_i - \frac{a_0^k}{a_{n+1}^k}$, resulting in contradiction. Hence, if $\mathbf{E}' \in \Pi_E \cap \Gamma_k$ then $a_{n+1}^k \leq 0$.

If $\mathbf{E}' \in \Pi_E \cap \Gamma$ and $a_{n+1}^k \leq 0$, then

$$\begin{cases} g_k(E_1', \dots, E_n', E_{n+1}') \geq 0 \quad \forall k = \overline{1, r}, \\ (E_1', \dots, E_n', E_{n+1}') \notin \bigcup\limits_{k=1}^{m} \Pi_k \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} g_k(E'_1,\dots,E'_n,E'_{n+1}-\Delta)\geq 0 \quad \forall k=\overline{1,r}, \\ (E'_1,\dots,E'_n,E'_{n+1}-\Delta)\notin \bigcup\limits_{k=1}^{m}\Pi_k \end{cases} \qquad \forall\Delta>0 \Longrightarrow$$

$$\Longrightarrow (E'_1,\dots,E'_n,E'_{n+1}-\Delta)\in M \quad \forall\Delta>0,$$

which contradicts boundedness of M.

□

Lemma 4. *Consider affine functions $l, l'\colon \mathbb{R}^n \to \mathbb{R}$ which attain values $l_1,\dots,l_{n+1}$ and $l'_1,\dots,l'_{n+1}$ at corners $\mathbf{A}_1,\dots,\mathbf{A}_{n+1}$ correspondingly, $\mathbf{E}\in \operatorname{conv}\{\mathbf{A}_1,\dots,\mathbf{A}_{n+1}\}$. If $l(\mathbf{A}_i)\geq l'(\mathbf{A}_i)$ $\forall i=\overline{1,n+1}$ then $l(\mathbf{E})\geq l'(\mathbf{E})$.*

Proof. Since $E\in \operatorname{conv}\{\mathbf{A}_1,\dots,\mathbf{A}_{n+1}\}$, $\mathbf{E}=\sum\limits_{i=1}^{n+1}\lambda_i\mathbf{A}_i$ where $\sum\limits_{i=1}^{n+1}\lambda_i=1$ and $\lambda_i\geq 0$. This, together with affinity of l,l', proves the statement.

□

Assume $K=[\underline{x}^1;\overline{x}^1]\times\ldots\times[\underline{x}^n;\overline{x}^n]\subset\mathbb{R}^n$. For function $f:K\to\mathbb{R}^+$, $\mathbf{E}\in K$ and a class of measures

$$\mathbb{Q}=\{Q:\mathbb{R}^n\to[0;1],\operatorname{supp}Q\subseteq K\},$$

consider the following optimization problem:

$$\begin{cases} \int\limits_K f(\mathbf{x})dQ(\mathbf{x}) \longrightarrow \inf\limits_{Q\in\mathbb{Q}}, \\ \int\limits_K x_i dQ(\mathbf{x}) = E_i,\ i=\overline{1,n}, \\ \int\limits_K dQ(\mathbf{x})=1. \end{cases} \tag{41}$$

Theorem 7. *If $f(\mathbf{x})$ is concave on K then the optimal solution of (41) is an atomic measure with mass concentrated in $n+1$ or less corners of K.*

Proof. For $f(\mathbf{x})$, consider affine $l(\mathbf{x})=\sum\limits_{i=1}^{n}a_ix_i+a_0$ according to Lemma 3 for corners $\mathbf{A}_0,\dots,\mathbf{A}_n$, where $\mathbf{A}_i=(x_1^i,\dots,x_n^i)$, and given $\mathbf{E}$. We rewrite $l(\mathbf{x})$ as $l(\mathbf{x})=\sum\limits_{i=1}^{n}a_i(x_i-E_i)+a'_0$ and denote $f^i=f(\mathbf{A}_i)$, so that

$$\begin{bmatrix} 1 & x_1^0-E_1 & \dots & x_n^0-E_n \\ \dots & \dots & \dots & \dots \\ 1 & x_1^n-E_1 & \dots & x_n^n-E_n \end{bmatrix}\begin{pmatrix} a'_0 \\ \dots \\ a_n\end{pmatrix}=\begin{pmatrix} f^0 \\ \dots \\ f^n\end{pmatrix} \quad\Longrightarrow\quad a'_0=\frac{\Delta_0}{\Delta},$$

$$\Delta_0 = \begin{vmatrix} f^0 & x_1^0 - E_1 & \dots & x_n^0 - E_n \\ \dots & \dots & \dots & \dots \\ f^n & x_1^n - E_1 & \dots & x_n^n - E_n \end{vmatrix}, \Delta = \begin{vmatrix} 1 & x_1^0 - E_1 & \dots & x_n^0 - E_n \\ \dots & \dots & \dots & \dots \\ 1 & x_1^n - E_1 & \dots & x_n^n - E_n \end{vmatrix}.$$

Denote

$$\delta_i = (-1)^{i+1} \begin{vmatrix} x_1^0 - E_1 & \dots & x_n^0 - E_n \\ \dots & \dots & \dots \\ x_1^{i-1} - E_1 & \dots & x_n^{i-1} - E_n \\ x_1^{i+1} - E_1 & \dots & x_n^{i+1} - E_n \\ \dots & \dots & \dots \\ x_1^n - E_1 & \dots & x_n^n - E_n \end{vmatrix}.$$

Using Laplace expansion along the first column, we derive

$$a_0' = \frac{\sum\limits_{i=1}^{n} \delta_i f^i}{\sum\limits_{i=1}^{n} \delta_i} = \sum_{i=1}^{n} p_i f^i, \quad p_i = \frac{\delta_i}{\sum\limits_{i=1}^{n} \delta_i}.$$

Obviously, $\sum\limits_{i=1}^{n} p_i = 1$. Lemma 4 implies that $a_0' = l(\mathbf{E})$ does not decrease in f^i, thus all $p_i \geq 0$. By definition of $l(\mathbf{x})$, $\forall Q \in \mathbb{Q}$

$$\int\limits_K f(\mathbf{x}) dQ(\mathbf{x}) \geq \int\limits_K l(\mathbf{x}) dQ(\mathbf{x}) = \sum_{i=1}^{n} a_i E_i + a_0 = a_0' = \sum_{i=1}^{n} f(\mathbf{A}_i) p_i = \int\limits_K f(\mathbf{x}) dQ^*(\mathbf{x}),$$

where Q^* is atomic measure concentrated in $\mathbf{A}_0, \dots, \mathbf{A}_n$. $\square$

$\square$

Appendix 3. Properties of the Bellman-Isaacs equation in costs-free market

Lemma 5. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$V_{n+1}(X, H^X, W^Y)$ satisfies Assumptions 2 and 3;*
2. *$V_{n+1}(X, H^X, W^Y)$ is concave in W^Y;*
3. *X_n and Y_n follow* (13) *and* (14) *correspondingly.*

Then (9) *is equivalent to* (15) *for given n.*

Proof. Using Assumptions 2 and 3, (9) can be transformed as:

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, Z, W^Y_{n+1}) =$$

$$= \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(s_{n+1}X, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}) \overset{(23)}{=}$$

$$\overset{(23)}{=} \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(s_{n+1}X, 0, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} + Z^T s_{n+1} X) \overset{(21)}{=}$$

$$\overset{(21)}{=} \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X, 0, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} + Z^T s_{n+1} X).$$

Since V_{n+1} is concave in W^Y, function under the expectation sign is concave in s_{n+1} and Theorem 2 applies. Using (21),(23) for backward transformation, we derive

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}(X, 0,$$
$$W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} + Z^T s_{n+1}(G_i) X) =$$

$$= \sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}(s_{n+1}(G_i)X, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}),$$

which coincides with (15) after substituting formulas for $s_{n+1}(G_i)$ and $\tilde{r}_{n+1}$. □

Similar proof can be readily derived under weaker assumptions when μ_{n+1} (therefore, s_{n+1}) is diagonal:

Corollary 1. *Under the assumptions of Lemma 5, assume that $V_{n+1}(X, H^X, W^Y)$ satisfies Assumptions $2'$ and 3 while μ_{n+1} is diagonal. Then* (9) *is equivalent to* (15) *for given n.*

Assumption 3 can be replaced by joint concavity in H^X and W^Y:

Lemma 6. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$V_{n+1}(X, H^X, W^Y)$ satisfies Assumptions 2;*

2. *$V_{n+1}(X, H^X, W^Y)$ is jointly concave in W^Y and H^X;*

3. *X_n and Y_n follow* (13) *and* (14) *correspondingly.*

Then (9) *is equivalent to* (15) *for given n.*

Proof. Using 2, transform (9):

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, Z, W^Y_{n+1}) =$$

$$= \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(s_{n+1} X, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{Z \in D_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X, s^T_{n+1} Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}).$$

Since V_{n+1} is jointly concave in H^X, W^Y, it is concave in s_{n+1} and Theorem 2 applies. Using (21),(23) for backward transformation, we derive

$$V_n(X, H^X, W^Y) =$$
$$\sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}(X, s^T_{n+1}(G_i) Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{Z \in D_{n+1}} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}(s_{n+1}(G_i) X, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}),$$

which coincides with (15) after substituting formulas for $s_{n+1}(G_i)$ and $\tilde{r}_{n+1}$. □

Corollary 2. *Under the assumptions of Lemma 6, assume that $V_{n+1}(X, H^X, W^Y)$ satisfies Assumptions $2'$ while μ_{n+1} is diagonal. Then* (9) *is equivalent to* (15) *for given n.*

Lemmas 5 and 6 with corollaries provide sufficient conditions to reduce equation (9) to a simpler form where the extreme measure is concentrated in corners of its support. Diagonality of μ_{n+1}, together with diagonality of σ_{n+1}, is quite constraining and seem to lead to independent dynamics of $X_n^1, \ldots, X_n^m$. However, dependence of $\sigma_{n+1}^1, \ldots, \sigma_{n+1}^m$ is still allowed; besides, μ_{n+1} can be estimated according to model which allows dependent dynamics of parameters. Therefore, dependency can be accounted for outside the worst-case framework during practical implementation.

Now we obtain conditions under which properties of V_{n+1} are inherited by V_n.

Statement 2. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$; $D_{n+1}(X, H^X, W^Y)$ satisfies* (17)*. Then $V_n(X, H^X, W^Y)$ satisfies Assumption 3.*

Proof. Direct check together with self-financing condition (6) lead to

$$V_n(X, 0, W^Y + H^{X^T} X) = \sup_{Z \in D_{n+1}(X, 0, W^Y + H^{X^T} X)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z,$$
$$W^Y \tilde{r}_{n+1} + H^{X^T} X \tilde{r}_{n+1} - (Z - 0)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1}) =$$
$$= V_n(X, H^X, W^Y).$$

□

Lemma 7. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$D_{n+1}(X, H^X, W^Y)$ satisfies* (16)*;*
2. *$V_{n+1}(X, H^X, W^Y)$ satisfies Assumption 2;*
3. *X_n and Y_n follow* (13) *and* (14) *correspondingly.*

Then $V_n(X, H^X, W^Y)$ satisfies Assumption 2.

Proof. Letting $Z' = A^T Z$, we obtain

$$V_n(AX, H^X, W^Y) = \sup_{Z \in D_{n+1}(AX, H^X, W^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(AX_{n+1}, Z, W^Y \tilde{r}_{n+1} - (Z - H^X)^T AX \tilde{r}_{n+1}) =$$

$$= \sup_{Z' \in D_{n+1}(X, A^T H^X, W^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(AX_{n+1}, A^{T^{-1}} Z', W^Y \tilde{r}_{n+1} - (A^{T^{-1}} Z' - H^X)^T AX \tilde{r}_{n+1}) =$$

$$= \sup_{Z' \in D_{n+1}(X, A^T H^X, W^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, A^T A^{T^{-1}} Z', W^Y \tilde{r}_{n+1} - (A^T A^{T^{-1}} Z' - A^T H^X)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{Z' \in D_{n+1}(X, A^T H^X, W^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, Z', W^Y \tilde{r}_{n+1} - (Z' - A^T H^X)^T X \tilde{r}_{n+1}) = V_n(X, A^T H^X, W^Y).$$

□

Corollary 3. *Under the assumptions of Lemma 7, assume that $V_{n+1}(X, H^X, W^Y)$ satisfies Assumption 2′, $D_{n+1}(X, H^X, W^Y)$ satisfies* (19) *and μ_{n+1} is diagonal. Then $V_n(X, H^X, W^Y)$ satisfies Assumption 2′.*

Lemma 8. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$D_{n+1}(X, H^X, W^Y)$ satisfies* (18)*;*
2. *X_n and Y_n follow* (13) *and* (14) *correspondingly.*

Then

1. *If $V_{n+1}(X, H^X, W^Y)$ is jointly concave in H^X, W^Y, then $V_n(X, H^X, W^Y)$ is jointly concave in H^X, W^Y.*
2. *If $V_{n+1}(X, H^X, W^Y)$ is concave in W^Y and satisfies Assumption 3, then $V_n(X, H^X, W^Y)$ is jointly concave in H^X, W^Y.*

3. If $V_{n+1}(X, H^X, W^Y)$ is concave in W^Y, then $V_n(X, H^X, W^Y)$ is concave in W^Y.

Proof. 1) We begin by proofing the first statement. (18) implies, that for any $\alpha \in [0,1]$ the set of Z that can be represented as $\alpha Z_1 + (1-\alpha)Z_2$ with $Z_1 \in D_{n+1}(X, H_1^X, W_1^Y)$ and $Z_2 \in D_{n+1}(X, H_2^X, W_2^Y)$, belongs to the set

$$D'_{n+1} = D_{n+1}(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y).$$

Therefore,

$$V_n(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y) = \sup_{Z \in D'_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}}$$
$$\mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, Z, \alpha W_1^Y \tilde{r}_{n+1} + (1-\alpha)W_2^Y \tilde{r}_{n+1} - \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_{n+1}) \geq$$

$$\geq \sup_{\substack{Z = \alpha Z_1 + (1-\alpha)Z_2 \\ Z_1 \in D_{n+1}(X, H_1^X, W_1^Y) \\ Z_2 \in D_{n+1}(X, H_2^X, W_2^Y)}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, Z,$$
$$\alpha W_1^Y \tilde{r}_{n+1} + (1-\alpha)W_2^Y \tilde{r}_{n+1} - \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_{n+1}) \geq$$

$$\geq \sup_{\substack{Z_1 \in D_{n+1}(X, H_1^X, W_1^Y) \\ Z_2 \in D_{n+1}(X, H_2^X, W_2^Y)}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, \alpha Z_1 + (1-\alpha)Z_2,$$
$$\alpha W_1^Y \tilde{r}_{n+1} + (1-\alpha)W_2^Y \tilde{r}_{n+1} - \big(\alpha Z_1 + (1-\alpha)Z_2 - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{\substack{Z_1 \in D_{n+1}(X, H_1^X, W_1^Y) \\ Z_2 \in D_{n+1}(X, H_2^X, W_2^Y)}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} V_{n+1}(X_{n+1}, \alpha Z_1 + (1-\alpha)Z_2,$$
$$\alpha\big[W_1^Y \tilde{r}_{n+1} - (Z_1 - H_1)^T X \tilde{r}_{n+1}\big] + (1-\alpha)\big[W_2^Y \tilde{r}_{n+1} - (Z_2 - H_2)^T X \tilde{r}_{n+1}\big]) \geq$$

$$\geq \sup_{\substack{Z_1 \in D_{n+1}(X, H_1^X, W_1^Y) \\ Z_2 \in D_{n+1}(X, H_2^X, W_2^Y)}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}} \Big[\alpha V_{n+1}(X_{n+1}, Z_1, W_1^Y \tilde{r}_{n+1} - (Z_1 - H_1)^T X \tilde{r}_{n+1}) +$$
$$+ (1-\alpha) V_{n+1}(X_{n+1}, Z_2, W_2^Y \tilde{r}_{n+1} - (Z_2 - H_2)^T X \tilde{r}_{n+1})\Big]. \quad (42)$$

Since

$$\inf\bigl[\alpha f(x) + (1-\alpha) g(x)\bigr] \geq \alpha \inf f(x) + (1-\alpha) \inf g(x),$$

we obtain

$$V_n(X, \alpha H_1^X + (1-\alpha) H_2^X, \alpha W_1^Y + (1-\alpha) W_2^Y) \geq$$

$$\geq \alpha \sup_{Z_1 \in D_{n+1}(X, H_1^X, W_1^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z_1, W_1^Y \tilde{r}_{n+1} - (Z_1 - H_1)^T X \tilde{r}_{n+1}) +$$

$$+ (1-\alpha) \sup_{Z_1 \in D_{n+1}(X, H_2^X, W_2^Y)} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z_2, W_2^Y \tilde{r}_{n+1} - (Z_2 - H_2)^T X \tilde{r}_{n+1}) =$$

$$= \alpha V_n(X, H_1^X, W_1^Y) + (1-\alpha) V_n(X, H_2^X, W_2^Y).$$

2) The second statement is proven by analog. The key difference is in using Assumption 3 to obtain

$$V_n(X, \alpha H_1^X + (1-\alpha) H_2^X, \alpha W_1^Y + (1-\alpha) W_2^Y) = \sup_{Z \in D'_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}}$$
$$\mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, Z, \alpha W_1^Y \tilde{r}_{n+1} + (1-\alpha) W_2^Y \tilde{r}_{n+1} - \bigl(Z - \alpha H_1^X - (1-\alpha) H_2^X\bigr)^T X \tilde{r}_{n+1}) =$$

$$= \sup_{Z \in D'_{n+1}} \inf_{Q_{n+1} \in \mathbb{Q}_{n+1}} \mathbb{E}_{Q_{n+1}}^{\mathcal{F}_n} V_{n+1}(X_{n+1}, 0, \alpha W_1^Y \tilde{r}_{n+1} + (1-\alpha) W_2^Y \tilde{r}_{n+1} -$$
$$- \bigl(Z - \alpha H_1^X - (1-\alpha) H_2^X\bigr)^T X \tilde{r}_{n+1} + Z^T X_{n+1} \tilde{r}_{n+1}).$$

One can notice that Z is missing in the second argument of V_{n+1} and appears only in the third argument expression which is linear in Z. Therefore we can obtain (42) by using concavity of V_{n+1} only in W^Y. The rest is proved by analog.

3) Proof of the third statement repeats proof of the first when $H_1^X = H_2^X = H^X$. □

Obtained Lemmas lead to Theorems 3 and 4, which provide sufficient conditions for the extreme measure problem in the Bellman-Isaacs equation to have an atomic solution.

Proof of Theorem 3. Properties of D_n and Statement 2 imply that V_n satisfies Assumption 3 for every n. Hence by Lemma 8, concavity in W^Y holds for every n. Lemma 5 concludes the proof. □

Proof of Theorem 4. Proof follows Theorem 3 by using corollaries of the mentioned Lemmas. □

Proof of Theorem 5 is based on analogous theorem for one risky asset:

Theorem 8. *Consider the Bellman-Isaacs equation* (9) *for* $m = 1$. *Assume that the following holds:*

1. $J(X, H^X, W^Y)$ *satisfies Assumptions 2 and 3.*

2. $J(X, H^X, W^Y)$ *is concave in* W^Y.

3. *For every* $n = \overline{1, N}$, $D_n(X, H^X, W^Y)$ *satisfies Assumption 1 and* $0 \in D_n(X, H^X, W^Y)$.

4. X_n *and* Y_n *follow* (13) *and* (14) *correspondingly.*

5. $(r_{n+1} - \mu_{n+1})\sqrt{\Delta t_{n+1}} \in \mathcal{K}_{n+1}$.

Then $V_n(X, H^X, W^Y)$, *as a function of* E_{n+1} *for fixed* $\mathcal{K}_{\geq n+1}$, $E_{\geq n+2}$, $\mu_{\geq n+1}$ *and* $r_{\geq n+1}$, *attains minimum over* $\mathcal{K}_{n+1}$ *at* $E^*_{n+1} = (r_{n+1} - \mu_{n+1})\sqrt{\Delta t_{n+1}}$. *Moreover, in this case* $H^{X*}_{n+1} = 0$ *is an optimal strategy.*

Proof. Lemma 7 implies that $V_n(X, H^X, W^Y)$ satisfies Assumption 2 for every n, while Lemma 8 implies concavity in W^Y for every n. Using Theorem 3, we write

$$V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \Big[p_{n+1} V_{n+1}\Big(\overline{s}_{n+1} X, Z, (W^Y - (Z - H^X)X)\tilde{r}_{n+1}\Big) + \\ + (1 - p_{n+1}) V_{n+1}\Big(\underline{s}_{n+1} X, Z, (W^Y - (Z - H^X)X)\tilde{r}_{n+1}\Big)\Big] =$$

$$= \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \Big[p_{n+1} V_{n+1}\Big(X, 0, (W^Y - (Z - H^X)X)\tilde{r}_{n+1} + \overline{s}_{n+1} Z X\Big) + \\ + (1 - p_{n+1}) V_{n+1}\Big(X, 0, (W^Y - (Z - H^X)X)\tilde{r}_{n+1} + \underline{s}_{n+1} Z X\Big)\Big] =$$

$$= \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \Big[p_{n+1} V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} + (\overline{s}_{n+1} - \tilde{r}_{n+1}) Z X\Big) + \\ + (1 - p_{n+1}) V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} - (\tilde{r}_{n+1} - \underline{s}_{n+1}) Z X\Big)\Big].$$

Since $V_n(X, H^X, W^Y)$ depends on E_{n+1} exclusively through p_{n+1}, it is easier to work in terms of p_{n+1}. We define function

$$F_n(p, X, H^X, W^Y) = \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \Big[p V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} + (\overline{s}_{n+1} - \tilde{r}_{n+1}) Z X\Big) + \\ + (1-p) V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} - (\tilde{r}_{n+1} - \underline{s}_{n+1}) Z X\Big)\Big] \quad (43)$$

and find its minimizer over a set of $p \in [0, 1]$. Maximizer of (43) is denoted $Z^* = Z^*(p, X, H^X, W^Y)$. Since $0 \in D_{n+1}(X, H^X, W^Y)$, we have

$$F_n(p, X, H^X, W^Y) \geq p V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} + (\overline{s}_{n+1} - \tilde{r}_{n+1}) X \cdot 0\Big) + \\ + (1-p) V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} - (\tilde{r}_{n+1} - \underline{s}_{n+1}) X \cdot 0\Big) = \\ = V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1}\Big). \quad (44)$$

On the other hand, $V_{n+1}(X, H^X, W^Y)$ is concave in W^Y, hence

$$F_n(p, X, H^X, W^Y) \leq V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} + \\ + p(\overline{s}_{n+1} - \tilde{r}_{n+1}) Z^* X + (1-p)(\underline{s}_{n+1} - \tilde{r}_{n+1}) Z^* X\Big) = \\ = V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1} + (p\overline{s}_{n+1} + (1-p)\underline{s}_{n+1} - \tilde{r}_{n+1}) Z^* X\Big). \quad (45)$$

(44) and (45) imply that p which satisfies

$$p\overline{s}_{n+1} + (1-p)\underline{s}_{n+1} - \tilde{r}_{n+1} = 0$$

is a minimizer of $F_n(p, X, H^X, W^Y)$ while minimum is $V_{n+1}\Big(X, 0, (W^Y + H^X X)\tilde{r}_{n+1}\Big)$. Therefore,

$$p^*\overline{s}_{n+1} + (1-p^*)\underline{s}_{n+1} - \tilde{r}_{n+1} = 0 \iff \\ \iff p^* = \frac{\tilde{r}_{n+1} - \underline{s}_{n+1}}{\overline{s}_{n+1} - \underline{s}_{n+1}} = \frac{r_{n+1} - \mu_{n+1}\Delta t_{n+1} - \underline{\sigma}_{n+1}\sqrt{\Delta t_{n+1}}}{\overline{\sigma}_{n+1}\sqrt{\Delta t_{n+1}} - \underline{\sigma}_{n+1}\sqrt{\Delta t_{n+1}}} = \\ = \frac{(r_{n+1} - \mu_{n+1})\sqrt{\Delta t_{n+1}} - \underline{\sigma}_{n+1}}{\overline{\sigma}_{n+1} - \underline{\sigma}_{n+1}} = \frac{E^*_{n+1} - \underline{\sigma}_{n+1}}{\overline{\sigma}_{n+1} - \underline{\sigma}_{n+1}} \Rightarrow \\ \Rightarrow E^*_{n+1} = (r_{n+1} - \mu_{n+1})\sqrt{\Delta t_{n+1}}.$$

Moreover,

$$\sup_{Z\in D_{n+1}(X,H^X,W^Y)}\Big[p^* V_{n+1}\Big(X,0,(W^Y+H^X X)\tilde{r}_{n+1}+(\overline{s}_{n+1}-\tilde{r}_{n+1})ZX\Big)+$$
$$+(1-p^*)V_{n+1}\Big(X,0,(W^Y+H^X X)\tilde{r}_{n+1}-(\tilde{r}_{n+1}-\underline{s}_{n+1})ZX\Big)\Big]=$$
$$=F_n(p^*,X,H^X,W^Y)=V_{n+1}\Big(X,0,(W^Y+H^X X)\tilde{r}_{n+1}\Big),\quad (46)$$

where maximum is attained at $Z^*=H^*_{n+1}=0$. □

From Theorem 8, we know that if expected values or risky and risk-free return are the same, then risk-free strategy is optimal. We can also see that in this case optimal expected utility is minimal since all assets are equal in view of portfolio manager, hence any strategy performs as well as risk-free investment. The approach to finding E^*_{n+1} can be extended for $m>1$. The problem of finding extreme measure over a m-dimensional set can be reduced to a sequence of m one-dimensional problems. Diagonality of μ_{n+1} will be required; besides it is crucial that $\mathcal{K}_{n+1}$ is a cartesian product of intervals.

For every measure $P\in\mathbb{P}$, consider marginal measures

$$Q^i_n(A^i)=\mathbb{P}\big(\{\sigma^i_n\in A^i\}\big),\quad A^i\in\mathcal{B}(\mathcal{K}^i_n),\quad i=\overline{1,m}.$$

We denote classes of such measures as $\mathbb{Q}^i_n$. In addition, we denote Lebesgue extension $Q'_n=Q^1_n\otimes\ldots\otimes Q^m_n$ of measure $Q^1_n\times\ldots\times Q^m_n$. For every $P\in\mathbb{P}$, the obtained class of measures is denoted $\mathbb{Q}'_n$. $\mathbb{Q}'_n\subseteq\mathbb{Q}_n$, since $\mathcal{K}_n=\bigotimes_{i=1}^m\mathcal{K}_i$ and $\mathbb{E}^{\mathcal{F}_{n-1}}_{Q'_n}\sigma_n=E_n$.

Proof of Theorem 5. Lemma 7 implies

$$V_n(X,H^X,W^Y)=\sup_{Z\in D_{n+1}}\inf_{Q_{n+1}\in\mathbb{Q}_{n+1}}\mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}}V_{n+1}(X_{n+1},0,W^Y_{n+1}+Z^TX_{n+1})=$$

$$=\inf_{Q_{n+1}\in\mathbb{Q}_{n+1}}\mathbb{E}^{\mathcal{F}_n}_{Q_{n+1}}V_{n+1}(X_{n+1},0,W^Y_{n+1}+Z^{*T}X_{n+1}).$$

Since $\mathbb{Q}'_{n+1}\subseteq\mathbb{Q}_{n+1}$, we obtain

$$V_n(X,H^X,W^Y)\quad\le\quad\inf_{Q'_{n+1}\in\mathbb{Q}'_{n+1}}\mathbb{E}^{\mathcal{F}_n}_{Q'_{n+1}}V_{n+1}(X_{n+1},0,W^Y_{n+1}\quad+\quad Z^{*T}X_{n+1})\quad=$$

$$= \inf_{Q'_{n+1} \in \mathbb{Q}'_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q'_{n+1}} V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} + Z^{*T}(s_{n+1} - \tilde{r}_{n+1} I)X).$$

$V_{n+1}(X, 0, W)$ is concave in W, thus a.s. continuous in W. Therefore by Fubini's theorem, we have

$$V_n(X, H^X, W^Y) \leq \inf_{Q'_{n+1} \in \mathbb{Q}'_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q'_{n+1}} V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} + Z^{*T}(s_{n+1} - \tilde{r}_{n+1} I)X) =$$

$$= \inf_{Q^1_{n+1} \in \mathbb{Q}^1_{n+1}} \dots \inf_{Q^m_{n+1} \in \mathbb{Q}^m_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^m_{n+1}} V_{n+1}(X, 0,$$
$$(W^Y + H^{X^T} X)\tilde{r}_{n+1} + Z^{*T}(s_{n+1} - \tilde{r}_{n+1} I)X) =$$

$$= \inf_{Q^1_{n+1} \in \mathbb{Q}^1_{n+1}} \dots \inf_{Q^m_{n+1} \in \mathbb{Q}^m_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^m_{n+1}} V_{n+1}(X, 0,$$
$$(W^Y + H^{X^T} X)\tilde{r}_{n+1} + \sum_{i=1}^{m} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*}),$$

where the last equality holds due to diagonality of s_{n+1}. Since $\mathbb{Q}^m_{n+1}$ contains all measures with compact support $\mathcal{K}^m_{n+1} \subset \mathbb{R}$ and expectation E^m_{n+1}, we apply Theorem 2 for $l = 1$ to derive that minimizer of the problem

$$\mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^m_{n+1}} V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} + \sum_{i=1}^{m} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*}) \to \inf_{Q^m_{n+1} \in \mathbb{Q}^m_{n+1}}$$

depends exclusively on $\mathcal{K}^m_{n+1}$ и E^m_{n+1} and not on $Q^1_{n+1}, \dots, Q^{m-1}_{n+1}$. Therefore, by applying Lemma 1 for $Y = \mathbb{Q}^m_{n+1}$ and $Q = Q^1_{n+1} \otimes \dots \otimes Q^{m-1}_{n+1}$, we obtain

$$V_n(X, H^X, W^Y) \leq \inf_{Q^1_{n+1} \in \mathbb{Q}^1_{n+1}} \dots \inf_{Q^m_{n+1} \in \mathbb{Q}^m_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^m_{n+1}} V_{n+1}(X, 0,$$
$$(W^Y + H^{X^T} X)\tilde{r}_{n+1} + \sum_{i=1}^{m} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*}) =$$

$$= \inf_{Q^1_{n+1} \in \mathbb{Q}^1_{n+1}} \dots \inf_{Q^{m-1}_{n+1} \in \mathbb{Q}^{m-1}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^{m-1}_{n+1}} \inf_{Q^m_{n+1} \in \mathbb{Q}^m_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^m_{n+1}} V_{n+1}(X, 0,$$
$$(W^Y + H^{X^T} X)\tilde{r}_{n+1} + \sum_{i=1}^{m} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*}) =$$

$$= \inf_{Q^1_{n+1}\in\mathbb{Q}^1_{n+1}} \dots \inf_{Q^{m-1}_{n+1}\in\mathbb{Q}^{m-1}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^{m-1}_{n+1}} \Big[p^m_{n+1} V_{n+1}(X, 0,$$

$$(W^Y + H^{X^T} X)\tilde{r}_{n+1} + \sum_{i=1}^{m-1} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*} + (\overline{s}^m_{n+1} - \tilde{r}_{n+1}) X^m Z^{m*}) +$$

$$+ (1 - p^m_{n+1}) V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} +$$

$$+ \sum_{i=1}^{m-1} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*} + (\underline{s}^m_{n+1} - \tilde{r}_{n+1}) X^m Z^{m*}) \Big] \le$$

$$\le \inf_{Q^1_{n+1}\in\mathbb{Q}^1_{n+1}} \dots \inf_{Q^{m-1}_{n+1}\in\mathbb{Q}^{m-1}_{n+1}} \mathbb{E}^{\mathcal{F}_n}_{Q^1_{n+1}} \dots \mathbb{E}^{\mathcal{F}_n}_{Q^{m-1}_{n+1}} V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} +$$

$$+ \sum_{i=1}^{m-1} (s^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*} + (p^m_{n+1}\overline{s}^m_{n+1} + (1 - p^m_{n+1})\underline{s}^m_{n+1} - \tilde{r}_{n+1}) X^m Z^{m*}).$$

Applying similar technique for $Q^{m-1}_{n+1}, \dots, Q^1_{n+1}$ consequentially, we finally derive:

$$V_n(X, H^X, W^Y) \le V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1} +$$

$$+ \sum_{i=1}^{m} (p^i_{n+1}\overline{s}^i_{n+1} + (1 - p^i_{n+1})\underline{s}^i_{n+1} - \tilde{r}_{n+1}) X^i Z^{i*}).$$

By analogy with the proof of Theorem 8, one can readily obtain that if $0 \in D_{n+1}(X, H^X, W^Y)$, then $V_n(X, H^X, W^Y)$ attains minimum value $V_{n+1}(X, 0, (W^Y + H^{X^T} X)\tilde{r}_{n+1})$ at

$$p^{i^*}_{n+1} = \frac{\tilde{r}_{n+1} - \underline{s}^i_{n+1}}{\overline{s}^i_{n+1} - \underline{s}^i_{n+1}} = \frac{r_{n+1} - \mu^i_{n+1}\Delta t_{n+1} - \underline{\sigma}^i_{n+1}\sqrt{\Delta t_{n+1}}}{\overline{\sigma}^i_{n+1}\sqrt{\Delta t_{n+1}} - \underline{\sigma}^i_{n+1}\sqrt{\Delta t_{n+1}}} =$$

$$= \frac{(r_{n+1} - \mu^i_{n+1})\sqrt{\Delta t_{n+1}} - \underline{\sigma}^i_{n+1}}{\overline{\sigma}^i_{n+1} - \underline{\sigma}^i_{n+1}} = \frac{E^{i^*}_{n+1} - \underline{\sigma}^i_{n+1}}{\overline{\sigma}^i_{n+1} - \underline{\sigma}^i_{n+1}} \Rightarrow$$

$$\Rightarrow E^{i^*}_{n+1} = (r_{n+1} - \mu^i_{n+1})\sqrt{\Delta t_{n+1}} \Rightarrow E^*_{n+1} = (r_{n+1} I - \mu_{n+1})\sqrt{\Delta t_{n+1}}.$$

It can also be shown by analog that $Z^* = H^*_{n+1} = 0$ is an optimal strategy. □

Appendix 4. Properties of the Bellman-Isaacs equation under non-zero transaction costs

Lemma 9. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$D_{n+1}(X, H^X, W^Y)$ satisfies* (19)*;*
2. *$V_{n+1}(X, H^X, W^Y)$ satisfies Assumption* $2'$*;*
3. *$C_n(H^X, X)$ satisfies* (28)*;*
4. *X_n and Y_n follow* (13) *and* (14) *correspondingly;*
5. *μ_{n+1} is diagonal.*

Then $V_n(X, H^X, W^Y)$ satisfies Assumption $2'$

Proof. Proof follows Lemma 7, since transaction costs function satisfies (28). □

Lemma 10. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. *$D_{n+1}(X, H^X, W^Y)$ satisfies* (18)*;*
2. *$V_{n+1}(X, H^X, W^Y)$ is non-decreasing in W^Y;*
3. *$C_n(H, X)$ is convex in H;*
4. *X_n and Y_n follow* (13) *and* (14) *correspondingly;*
5. *μ_{n+1} is diagonal.*

Then

1. *If $V_{n+1}(X, H^X, W^Y)$ is jointly concave in H^X, W^Y, then $V_n(X, H^X, W^Y)$ is jointly concave in H^X, W^Y.*
2. *If $V_{n+1}(X, H^X, W^Y)$ is concave in W^Y, then $V_n(X, H^X, W^Y)$ is concave in W^Y.*

Proof. Proof of the first statement follows Lemma 8 since $-C(H, X)$ is concave in H while V_{n+1} is non-decreasing in W^Y. Proof of the third statement repeats proof of the first when $H_1^X = H_2^X = H^X$. □

Lemma 11. *Assume that $V_n(X, H^X, W^Y)$ is defined by formula of the Bellman-Isaacs equation* (9) *for given $n < N$ and*

1. $V_{n+1}(X, H^X, W^Y)$ *is non-decreasing in W^Y;*

2. $V_{n+1}(X, H^X, W^Y)$ *satisfies Assumption* $2'$;

3. $V_{n+1}(X, H^X, W^Y)$ *is jointly concave in H^X, W^Y;*

4. $C(H, X)$ *satisfies Assumption 4;*

5. X_n *and* Y_n *follow* (13) *and* (14) *correspondingly;*

6. μ_{n+1} *is diagonal.*

Then (9) *is equivalent to* (29) *for given n.*

Proof. Proof follows Lemma 6 since $-C(H, X)$ is concave in H and V_{n+1} is non-decreasing in W^Y. $\square$

Proof of Theorem 6. Analogously to Theorem 4, proof of Theorem 6 is readily derived from corresponding Lemmas. $\square$

Author:

Nikolay Andreev, National Research University Higher School of Economics (Moscow, Russia). Laboratory for Financial Engineering and Risk Management. Researcher;

E-mail: nandreev@hse.ru

Author prefers to publish this text without proofreading.