ARTICLE IN PRESS

Journal of Geometry and Physics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Explicit metrics for a class of two-dimensional cubically superintegrable systems

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ARTICLE INFO

Article history:
Received 4 March 2014
Received in revised form 31 July 2014
Accepted 4 August 2014
Available online xxxx

Keywords:
Superintegrable Hamiltonian systems
Polynomially integrable Riemannian metrics on surfaces
Globally integrable Riemannian metrics on surfaces
Completely integrable systems on the 2-sphere

ABSTRACT

We obtain, in local coordinates, the explicit form of the two-dimensional, superintegrable systems of Matveev and Shevchishin involving linear and cubic integrals. This enables us to determine for which values of the parameters these systems are indeed globally defined on $\mathbb{S}^2$.

1. Introduction

The study of superintegrable dynamical systems has received many important developments reviewed recently in [1]. While integrable systems on the cotangent bundle T^*M of a n-dimensional manifold, M, require a set of functionally independent observables $(H, Q_1, \ldots, Q_{n-1})$ which are all in involution for the Poisson bracket $\{\cdot\,,\,\cdot\}$, a superintegrable system is made out of $\nu \geq n$ functionally independent observables

$$H, \qquad Q_1, \qquad Q_2, \qquad \cdots \quad Q_{\nu-1},$$

with the constraints

$$\{H, Q_i\} = 0, \quad \text{for all } i = 1, 2, \ldots, \nu - 1. \tag{1}$$

The maximal value of ν is $2n - 1$ since the system (1) reads $dH(X_{Q_i}) = 0$, implying that the span of the Hamiltonian vector fields, X_{Q_i}, is, at each point of T^*M, a subspace of the annihilator of the 1-form dH, the latter being of dimension $2n - 1$. Let us observe that for two-dimensional manifolds, a superintegrable system is necessarily maximal since $\nu = 3$.

As is apparent from [1], the large amount of results for superintegrable models is restricted to *quadratically* superintegrable ones, which means that the integrals Q_i are either linear or quadratic in the momenta, and the metrics on which these systems are defined are either flat or of constant curvature. For manifolds of non constant curvature, Koenigs [2] gave examples of quadratically superintegrable models. For some special values of the parameters the metrics happen to be defined on a manifold, M, which is never closed (compact without boundary).

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http://dx.doi.org/10.1016/j.geomphys.2014.08.004
0393-0440/

Please cite this article in press as: G. Valent, et al., Explicit metrics for a class of two-dimensional cubically superintegrable systems, Journal of Geometry and Physics (2014), http://dx.doi.org/10.1016/j.geomphys.2014.08.004

In their quest for superintegrable systems defined on closed manifolds, Matveev and Shevchishin [3] have given a complete classification of all (local) Riemannian metrics on surfaces of revolution, namely

$$G = \frac{dx^2 + dy^2}{h_x^2}, \qquad h = h(x), \qquad h_x = \frac{dh}{dx}, \tag{2}$$

which have a superintegrable geodesic flow (whose Hamiltonian will henceforth be denoted by H), with integrals $L = P_y$ and S respectively linear and cubic in momenta, opening the way to the new field of *cubically* superintegrable models. Let us first recall their main results.

They proved that if the metric G is not of constant curvature, then $\mathcal{I}^3(G)$, the linear span of the cubic integrals, has dimension 4 with a natural basis L^3, LH, S_1, S_2, and with the following structure. The map $\mathcal{L} : S \to \{L, S\}$ defines a linear endomorphism of $\mathcal{I}^3(g)$ and one of the following possibilities hold:

(i) $\mathcal{L}$ has purely real eigenvalues $\pm\mu$ for some real $\mu > 0$, then S_1, S_2 are the corresponding eigenvectors.
(ii) $\mathcal{L}$ has purely imaginary eigenvalues $\pm i\mu$ for some real $\mu > 0$, then $S_1 \pm iS_2$ are the corresponding eigenvectors.
(iii) $\mathcal{L}$ has the eigenvalue $\mu = 0$ with one Jordan block of size 3, in this case

$$\{L, S_1\} = \frac{A_3}{2} L^3 + A_1 LH, \qquad \{L, S_2\} = S_1,$$

for some real constants A_1 and A_3. Superintegrability is then achieved provided the function h be a solution of following non-linear first-order differential equations, namely

$$\begin{aligned}
&\text{(i) } h_x(A_0 h_x^2 + \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x)\\
&\text{(ii) } h_x(A_0 h_x^2 - \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sinh(\mu x)}{\mu} + A_4 \cosh(\mu x)\\
&\text{(iii) } h_x(A_0 h_x^2 - A_1 h + A_2) = A_3 x + A_4
\end{aligned} \tag{3}$$

and the explicit form of the cubic integrals was given in all three cases. For instance, when $\mu = 1$ or $\mu = i$, their structure is

$$S_{1,2} = e^{\pm\mu y}\left(a_0(x) P_x^3 + a_1(x) P_x^2 P_y + a_2(x) P_x P_y^2 + a_3(x) P_y^3\right), \tag{4}$$

where the $a_i(x)$ are explicitly expressed in terms of h and its derivatives; see [3].

For $A_0 = 0$ these equations are easily integrated and one obtains the Koenigs metrics [2], while the cubic integrals have the reducible structure $S_{1,2} = P_y Q_{1,2}$ where the quadratic integrals $Q_{1,2}$ are precisely those obtained by Koenigs.

Furthermore it was proved that in the case (ii), under the conditions

$$\mu > 0, \qquad A_0 > 0, \qquad \mu A_4 > |A_3|, \tag{5}$$

the metric and the cubic integrals are real-analytic and globally defined on $\mathbb{S}^2$.

The aim of this article is on the one hand to integrate explicitly the three differential equations in (3) and, on the other hand, to determine, by a systematic case study, all special cases which lead to superintegrable models *globally* defined on simply-connected, *closed*, Riemann surfaces.

In Section 2 we analyze the trigonometric case (real eigenvalues), integrating explicitly the differential equation (3)(i) to get an explicit local form for the metric and the cubic integrals. The global questions are then discussed, and we show that there is no closed manifold, M, on which the superintegrable model under consideration can be defined.

In Section 3 we investigate the hyperbolic case (purely imaginary eigenvalues). Here too, the integration of the differential equation (3)(ii) provides an explicit form for both the metric and the cubic integrals.

The previous results allows the determination of all superintegrable systems globally defined on $\mathbb{S}^2$, and these are proved in Theorems 1 and 2, namely

Theorem 1. *The metric*

$$G = \rho^2 \frac{dv^2}{D} + \frac{4D}{P} d\phi^2, \quad v \in (a, 1), \ \phi \in \mathbb{S}^1,$$

with

$$D = (v - a)(1 - v^2), \qquad P = (v^2 - 2av + 1)^2, \qquad -\rho = 1 + 4\frac{(v - a)D}{P}, \tag{6}$$

is globally defined on $\mathbb{S}^2$, as well as the Hamiltonian

$$H = \frac{1}{2} G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + \frac{P}{4D} P_\phi^2\right), \quad \Pi = \frac{\sqrt{D}}{\rho} P_v,$$

iff $a \in (-1, +1)$. The two cubic integrals S_1 and S_2, also globally defined on $\mathbb{S}^2$, read

$$S_1 = \cos\phi\, \mathcal{A} + \sin\phi\, \mathcal{B}, \qquad S_2 = -\sin\phi\, \mathcal{A} + \cos\phi\, \mathcal{B}, \tag{7}$$

where

$$\mathcal{A}=\Pi^3-f\,f''\,\Pi\,P_\phi^2,\qquad \mathcal{B}=f'\,\Pi^2\,P_\phi-f\,(1+f'\,f'')\,P_\phi^3,\quad f=\sqrt{D}. \tag{8}$$

Theorem 2. *The metric*

$$G=\rho^2\,\frac{dx^2}{D}+\frac{4D}{P}\,d\phi^2,\quad \rho=\frac{Q}{P},\ x\in(-1,+1),\ \phi\in\mathbb{S}^1, \tag{9}$$

with

$$\begin{cases} D=(x+m)(1-x^2),\\ P=\Big(L_+\,(1-x^2)+2(m+x)\Big)\Big(L_-\,(1-x^2)+2(m+x)\Big),\quad L_\pm=l\pm\sqrt{l^2-1},\\ Q=3x^4+4mx^3-6x^2-12mx-4m^2-1,\end{cases} \tag{10}$$

is globally defined on $\mathbb{S}^2$*, as well as the Hamiltonian*

$$H=\frac{1}{2}\,G^{ij}P_iP_j=\frac{1}{2}\left(\Pi^2+\frac{P}{4D}P_\phi^2\right),\quad \Pi=\frac{\sqrt{D}}{\rho}\,P_x,$$

iff $m>1$*, and* $l>-1$*. The two cubic integrals* S_1 *and* S_2*, still given by the formulas (7) and (8), are also globally defined on* $\mathbb{S}^2$*.*

In Section 3.6 we compare of our results with those of Matveev and Shevchishin [3]. In particular, for a convenience of the reader, we provide the transition formulas between the coordinates and functions used in [3] and the coordinates and function used in the present paper.

In Section 4 we analyze the affine case (zero eigenvalue). As in the trigonometric case, the system is never defined on closed manifolds but we determine in which cases it is globally defined either on $\mathbb{R}^2$ or on $\mathbb{H}^2$.

In Section 5 we draw some conclusions and present some possibly interesting strategy for future developments.

2. The trigonometric case

2.1. The explicit form of the metric

The ode (3)(i) obtained in [3] is:

$$h_x\Big(A_0\,h_x^2+\mu^2\,A_0\,h^2-A_1\,h+A_2\Big)=A_3\,\frac{\sin(\mu\,x)}{\mu}+A_4\,\cos(\mu\,x).$$

For the Koenigs metrics $A_0=0$; we thus must consider here a non-vanishing A_0 which can be scaled to 1. By a scaling of x we can also set $\mu=1$. By a translation of x and a scaling of h the right-hand side becomes $\lambda\,\sin x$, with λ a free real parameter. By a translation of h, we can set $A_1=0$ and $A_2=a$. We hence have to solve

$$h_x(h_x^2+h^2+a)=\lambda\,\sin x,\quad a\in\mathbb{R},\ \lambda\in\mathbb{R}\setminus\{0\}. \tag{11}$$

Let us regard now $u=h_x$ as a function of the variable h and define

$$U=u(u^2+h^2+a)\quad \text{with}\ \frac{d^2U}{dx^2}+U=0. \tag{12}$$

This last relation, when expressed in terms of the variable h becomes then

$$\frac{d}{dh}\left(u\,\frac{dU}{dh}\right)+u^2+h^2+a=0,\quad a\in\mathbb{R}, \tag{13}$$

and can be integrated, yielding

$$4hu\,\frac{dU}{dh}=c+(u^2+h^2+a)(3u^2-h^2-a). \tag{14}$$

Since $U=\lambda\,\sin x$ we have also a first order equation

$$U'^2=\lambda^2-U^2\ \Rightarrow\ \left(4hu\,\frac{dU}{dh}\right)^2=16h^2\,(\lambda^2-U^2), \tag{15}$$

and upon using (14) we obtain a quartic equation for u:

$$\Big[c+(u^2+h^2+a)(3\,u^2-h^2-a)\Big]^2=16\,h^2\Big[\lambda^2-u^2(u^2+h^2+a)^2\Big]. \tag{16}$$

If we define $v = u^2 + h^2$, this equation remains a quartic in v but happens to be linear in h^2. Solving for h^2 in terms of the variable v, we find

$$v = u^2 + h^2, \quad h^2 = \frac{D'^2}{8D}, \qquad D(v) = (v+a)(v^2 - a^2 + c) + 2\lambda^2. \tag{17}$$

At this stage, it turns out to be convenient to define

$$f = \sqrt{D} = \sqrt{(v+a)(v^2 - a^2 + c) + 2\lambda^2} \quad \text{and} \quad g = 2v - f'^2 \tag{18}$$

where $f' = df/dv$. This allows, once the old coordinates (x, y) have been expressed in terms of the new ones, (v, y), to get eventually the explicit form of the metric

$$\frac{1}{2}G = \frac{1}{2h_x^2}(dx^2 + dy^2) = \left(\frac{f''}{g}\right)^2 dv^2 + \frac{dy^2}{g} \tag{19}$$

which gives the Hamiltonian

$$H \equiv G^{ij}P_iP_j = \frac{1}{2}\left(\Pi^2 + g\,P_y^2\right), \quad \Pi = \frac{g}{f''}\,P_v. \tag{20}$$

2.2. The cubic integrals

They were given in (4), as borrowed from [3], and become in our new coordinates with a slight change of notation

$$S_\pm = e^{\pm y}\Big(\Pi^3 \mp f'\,\Pi^2\,P_y + f\,f''\,\Pi\,P_y^2 \pm f(1 - f'f'')P_y^3\Big). \tag{21}$$

However due to the relation $dH \wedge dP_y \wedge dS_+ \wedge dS_- = 0$, the four observables involved are not functionally independent. Indeed, we have

$$S_+\,S_- = 8H^3 + 8a\,H^2\,P_y^2 + 2c\,H\,P_y^4 - 2\lambda^2\,P_y^6, \tag{22}$$

so that we may consider two different superintegrable systems

$$\mathcal{I}_+ = (H,\ P_y,\ S_+) \quad \text{and} \quad \mathcal{I}_- = (H,\ P_y,\ S_-). \tag{23}$$

Proposition 1. *The observables S_+ and S_- are integrals and the set $(H,\ P_y,\ S_+,\ S_-)$ generates a Poisson algebra.*

Proof. The Poisson brackets are given by

$$\{H, S_\pm\} = e^{\pm y}\,\frac{g}{f''}\,\Pi\,P_y^2(\Pi \mp f'\,P_y)\Big(f\,f''' - 3(1 - f'\,f'')\Big). \tag{24}$$

Quite remarkably, the ode

$$f\,f''' - 3(1 - f'\,f'') = 0 \tag{25}$$

does linearize upon the substitution $f = \sqrt{D}$ since we have

$$2\Big(f\,f''' - 3(1 - f'\,f'')\Big) = D''' - 6 = 0, \tag{26}$$

which gives for D the most general monic polynomial of third degree

$$D(v) = v^3 - s_1\,v^2 + s_2\,v - s_3, \tag{27}$$

whose coefficients are expressed in terms of the symmetric functions of the roots. As a matter of fact, the function D already obtained in (17) displays exactly 3 parameters a, c, λ. Eqs. (24) and (25) insure then conservation of both cubic integrals S_+ and S_-.

The Poisson algebra structure follows from the following relations, viz.,

$$\begin{aligned} &\{S_+, S_-\} = -16a\,H^2\,P_y - 8c\,H\,P_y^3 + 12\lambda^2\,P_y^5, \\ &S_+\,S_- = 8H^3 + 8a\,H^2\,P_y^2 + 2c\,H\,P_y^4 - 2\lambda^2\,P_y^6; \end{aligned} \tag{28}$$

it is generated by 4 observables in this case. □

2.3. Transformation of the metric and its curvature

Taking for D the expression (27), let us define the following quartic polynomials P and Q, namely

$$P = 8v\,D - D'^2, \qquad Q = 2\,D\,D'' - D'^2 = P + 4(v - s_1)D, \qquad Q' = 12\,D, \tag{29}$$

enabling us to write the metric (19) in the form

$$\frac{1}{2}G = \rho^2\,\frac{dv^2}{D} + \frac{4D}{P}\,dy^2, \quad \rho \equiv \frac{Q}{P} = 1 + (v - s_1)\frac{4D}{P}, \tag{30}$$

the scalar curvature being given by

$$R_G = \frac{1}{4Q^3}\Big(2PQ\,W' - (QP' + 2PQ')\,W\Big), \quad W \equiv DP' - PD' = 8D^2 - QD'. \tag{31}$$

One should bear in mind the following restrictions:

1. The relation $v = u^2 + h^2$ requires $v > 0$.
2. For h to be real we must have $D > 0$.
3. For the metric G to be Riemannian we need $P > 0$.

2.4. Global properties

To study the global geometry of these superintegrable models, we will be using techniques which have proved quite successful in [4,5] for integrable models with either a cubic or a quartic integral.

As emphasized in the Introduction, we will from now on confine considerations to the case of simply connected Riemann surfaces, which, by the Riemann uniformization theorem [6], are conformally related to spaces of constant curvature $\mathbb{S}^2, \mathbb{R}^2, \mathbb{H}^2$.

One has first to determine, from the above positivity conditions, the open interval $I \subset \mathbb{R}$ admissible for the variable v. The end-points are singular points for the metric and the possibility of a manifold structure is related to the behavior of the metric at these end-points. Either they are true singularities (for instance if the scalar curvature is divergent at these points) or they are apparent singularities (also called coordinate singularities) due to a bad choice of the coordinates as, for instance,

$$G = dr^2 + r^2\,d\phi^2, \quad r \in (0, +\infty),\ \phi \in \mathbb{S}^1, \tag{32}$$

for which $r = 0$ is an apparent singularity which can be wiped out, using back Cartesian coordinates.

We will detect true singularities from the scalar curvature:

Lemma 1. *Let us consider the interval $I = (a, b)$, allowed for v, i.e., such that $D(v) > 0$ and $P(v) > 0$ for all $v \in I$. Suppose that Q has a simple real zero $v_* \in I$; then $v = v_*$ is a curvature singularity precluding any manifold structure associated with the metric.*

Proof. The relation (31) entails that

$$\lim_{v\to v_*} Q^3(v)\,R_G(v) = -4\,P(v_*)\,D^2(v_*)\,Q'(v_*) \tag{33}$$

and the right-hand side of this equation does not vanish. The existence of such a curvature singularity for $v_* \in I$ rules out the possibility of a manifold structure. □

We will detect non-closedness by

Lemma 2. *If the variable v takes its values in some interval $I = (a, b)$ and if one of the end-points is a zero of P (and not of Q), then the manifold having infinite measure, it cannot be closed.*

Proof. Let the allowed interval for v be $I = (a, b)$. The measure of the manifold is

$$\mu_G = 4\int_a^b \frac{Q(v)}{P^{3/2}(v)}\,dv \int dy. \tag{34}$$

Now, if P has a zero at one end-point where Q does not vanish, then this integral will be divergent. □

Let us turn ourselves to the analysis of this first case (i). Given any polynomial P we will use the notation $\Delta(P)$ for its discriminant. The discussion will be organized according to the sign of $\Delta(D)$. Let us begin with:

Proposition 2. *If $\Delta(D) = 0$ the superintegrable systems $\mathcal{I}_+$ and $\mathcal{I}_-$ given by (23) are either trivial or are not defined on a closed manifold.*

Proof. If $\Delta(D) = 0$, we may have first $D = (v - v_0)^3$. The scalar curvature, easily computed using (31), is a constant. The following theorem, due to Thompson [7], states that *for Riemannian spaces of constant curvature, namely $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ with $n \geq 2$, every (symmetric) Killing–Stäckel tensor of any degree is fully reducible to symmetrized tensor products of the Killing vectors.* This implies that the cubic integrals are reducible, leaving us with the trivial integrable system (H, P_y).

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For $\Delta(D)=0$ we may also have $D=(v-v_0)(v-v_1)^2$ with $v_0\neq v_1$, which yields

$$\begin{cases} P(v)=-(v-v_1)^2\,p(v), \quad p(v)=v^2-2(2v_0+3v_1)v+(2v_0+v_1)^2,\\ Q=3(v-v_1)^3(v-v_*), \quad v_*=v_0+\dfrac{v_0-v_1}{3}. \end{cases} \tag{35}$$

Let us first observe that for the metric

$$\frac{1}{2}G=\frac{9(v-v_*)^2}{p(v)^2}\,\frac{dv^2}{(v-v_0)}+\frac{4(v-v_0)}{(-p(v))}\,dy^2 \tag{36}$$

to be Riemannian we must have $v>v_0$ and $p(v)<0$. If the roots $w_\pm$ of p are ordered as $w_-<w_+$, positivity of the metric is achieved iff $v\in I=(v_0,+\infty)\cap(w_-,w_+)$, the upper bound of I being w_+. Since $P(w_+)=0$ and $Q(w_+)\neq 0$, the expected manifold cannot be closed by Lemma 2. □

Proposition 3. *If $\Delta(D)<0$ the superintegrable systems $\mathcal{I}_+$ and $\mathcal{I}_-$ given by (23) are never globally defined on a closed manifold.*

Proof. If $\Delta(D)<0$ the polynomial D has only a simple real zero. Using new parameters $(a,\ b)$ we can write

$$D=(v-v_0)\Big((v-a)^2+b^2\Big), \quad v\in\ (v_0,+\infty),\ a\in\mathbb{R},\ b\in\mathbb{R}\setminus\{0\},$$

with

$$\Delta(D)=-4b^2\Big((v_0-a)^2+b^2\Big)^2$$

and, for P and Q,

$$\Delta(P)=16\,384\,a^2\Big((v_0+a)^2+b^2\Big)^2\,\Delta(D), \qquad \Delta(Q)=27\,648\,b^2\Big((v_0-a)^2+b^2\Big)^2\,\Delta(D). \tag{37}$$

We must exclude $a=0$ since $P(v)=-(v^2-2v_0v-b^2)^2$ is negative. Hence, the previous discriminants are strictly negative, implying that both polynomials P and Q have two simple real zeros.

The relation $Q'=12\,D$ shows that Q is strictly increasing from $Q(v_0)=-[(v_0-a)^2+b^2]$ to $Q(+\infty)=+\infty$, hence there exists a simple zero v_* of Q such that $v_*>v_0$ while the other one lies to the left of v_0 because $Q(-\infty)=+\infty$.

The polynomial P retains the form

$$P(v)=-\Big(v^2-2(v_0+2a)v-a^2-b^2-2av_0\Big)^2+16a\Big((v_0+a)^2+b^2\Big)v$$

showing that for $a<0$ it is never positive as it should; so, we are left with the case $a>0$. From the relations

$$P(v)=Q(v)+4(v_0+2a-v)\,D(v), \qquad P'(v)=8\,D(v)+4(2a+v_0-v)\,D'(v),$$

we see that $P(v_0)$ is strictly negative and that $P'(v)$ is positive from $v=v_0$ to $v=v_0+2a$. Thus P increases to its first zero $v=w_-<v_*$ (since $P(v_*)=4(2a+v_0-v_*)\,D(v_*)>0$), is equal to Q for $v=v_0+2a>v_*$, then vanishes at its second zero w_+ such that $w_+>v_0+2a$ and, at last, decreases to $-\infty$. Therefore, we end up with the ordering

$$v_0<w_-<v_*<v_0+2a<w_+.$$

So, $D>0$ and $P>0$ iff $v\in(w_-,w_+)$, and within this interval Q has a simple zero for $v=v_*$; hence, by Lemma 1, there is no underlying manifold structure. □

Let us conclude this section with

Proposition 4. *If $\Delta(D)>0$ the superintegrable systems $\mathcal{I}_+$ and $\mathcal{I}_-$ given by (23) are never globally defined on a closed manifold.*

Proof. Let us order the roots of D according to $0\le v_0<v_1<v_2$, so that

$$D(v)=(v-v_0)(v-v_1)(v-v_2)=v^3-s_1\,v^2+s_2\,v-s_3,$$

and $D>0$ for $v\in(v_0,v_1)\cup(v_2,+\infty)$. We need to determine now the positivity interval for P. Since

$$\Delta(P)=4096\,\sigma^2\,\Delta(D)>0, \quad \sigma=(v_0+v_1)(v_1+v_2)(v_2+v_0)>0,$$

there will be either four real simple roots or no real root for P. The latter is excluded since $P=8vD-(D')^2$ is negative at the zeros of D, and positive at those of D'. Also, notice that $\Delta(Q)=-6912\,\Delta^2(D)<0$ implies that Q has two simple real roots and one of them is $v_*>v_2$. This is so because $Q(v)=P(v)+4(v-s_1)D(v)$, which shows that $Q(v_2)=P(v_2)=-(v_0-v_2)^2(v_1-v_2)^2<0$; but $Q'=12D$ entails that, for positive D, the function Q is increasing with $Q(+\infty)=+\infty$. Hence $v=v_*$ is a simple zero of Q, forbidding any manifold structure by Lemma 1.

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The zeros of P may appear only when $D > 0$. Let us consider $v \in (v_0, v_1)$. Observing that $P(v_0) = -(v_0 - v_1)^2(v_0 - v_2)^2$ and $P(v_1) = -(v_1 - v_0)^2(v_1 - v_2)^2$ are negative and that there does exist $v = v_- \in (v_0, v_1)$ for which $D'(v_-) = 0$, we get $P(v_-) > 0$ which implies $v_0 < w_0 < v_- < w_1 < v_1$, where (w_0, w_1) is the first pair of simple zeros of P. Positivity of both D and P is therefore obtained for $v \in (w_0, w_1)$. The function Q remains strictly negative for $v \in [v_0, v_1]$, and Lemma 2 help us conclude that the supposed manifold cannot be closed.

The remaining two zeros of P denoted by $w_2 < w_3$ must lie in $(v_2, +\infty)$. Since $Q(v_2) = -(v_2 - v_0)^2(v_2 - v_1)^2 < 0$ and then it increases to $Q(+\infty) = +\infty$ it will have a simple zero $v = v_* > v_2$, and at this point $P(v_*) = 4(s_1 - v_*)D(v_*)$. Let us discuss:

1. If $v_* < s_1$, we have $P(v_*) > 0$, and since $P(+\infty) = -\infty$ we get $v_2 < w_2 < v_* < w_3$. The positivity of D and P requires $v \in (w_2, w_3)$, and there is no manifold structure since the curvature R_G is singular at $v = v_*$.
2. If $v_* \geq s_1$, we have $P(v_*) < 0$ hence $v_2 < v_* < w_2 < w_3$, and the positivity of D and P requires $v \in (w_2, w_3)$. Since $Q(w_3) > 0$ the supposed manifold cannot be closed by Lemma 2. □

We conclude this section by observing that the trigonometric case never leads to superintegrable systems defined on a closed manifold.

3. The hyperbolic case

3.1. The explicit form of the metric

The ode (3)(ii) obtained in [3] is

$$h_x(A_0\, h_x^2 - \mu^2 A_0\, h^2 - A_1\, h + A_2) = A_3 \frac{\sinh(\mu\, x)}{\mu} + A_4\, \cosh(\mu\, x). \tag{38}$$

Again, we may put $A_0 = 1, \mu = 1, A_1 = 0, A_2 = -a$, but, this time, the right-hand side of the previous equation leads to three different cases we will describe according to

$$h_x(h_x^2 - h^2 - a) = \frac{\lambda}{2}\,(e^x + \epsilon\, e^{-x}), \quad \epsilon = 0, \pm 1 \tag{39}$$

where λ is a free parameter.

Let us point out that for $\epsilon = 0$ the changes $x \to -x$ and $\lambda \to -\lambda$ show that there is no need to consider e^{-x} in the right-hand side of (39).

With the definitions

$$u = h_x, \qquad U = u(u^2 - h^2 - a), \quad a \in \mathbb{R},$$

we get similarly

$$U'' - U = 0 \;\Rightarrow\; \frac{d}{dh}\left(u\,\frac{dU}{dh}\right) - (u^2 - h^2 - a) = 0,$$

which can be integrated to yield

$$4hu\,\frac{dU}{dh} = c + (u^2 - h^2 - a)(3u^2 + h^2 + a), \quad c \in \mathbb{R}. \tag{40}$$

Since $U = \frac{\lambda}{2}\,(e^x + \epsilon\, e^{-x})$ we also have the first order ode:

$$U'^2 = U^2 - \epsilon\,\lambda^2 \;\Rightarrow\; \left(4hu\,\frac{dU}{dh}\right)^2 = 16\, h^2\,(U^2 - \epsilon\lambda^2), \tag{41}$$

which, upon use of (40), leaves us with a quartic equation in the variable u. Positing $v = h^2 - u^2$, we still have a quartic in v but the h^2 dependence is merely linear and we can solve for h^2 in terms of the variable v, namely

$$v = u^2 - h^2, \qquad h^2 = \frac{D'^2}{8D}, \qquad D(v) = (a - v)(v^2 - a^2 + c) - 2\epsilon\lambda^2, \tag{42}$$

giving a result surprisingly similar to the case (i), except that v needs not be positive. Upon defining

$$f = \sqrt{D} = \sqrt{(a - v)(v^2 - a^2 + c) - 2\epsilon\lambda^2} \quad \text{and} \quad g = f'^2 + 2v, \tag{43}$$

we obtain the metric in the new coordinates (v, y) in the form

$$\frac{1}{2}G = \frac{1}{2h_x^2}(dx^2 + dy^2) = \left(\frac{f''}{g}\right)^2\, dv^2 + \frac{dy^2}{g}, \tag{44}$$

together with the Hamiltonian

$$H \equiv G^{ij}\, P_i\, P_j = \frac{1}{2}\left(\Pi^2 + g\, P_y^2\right), \quad \Pi = \frac{g}{f''}\, P_v. \tag{45}$$

3.2. The cubic integrals

They were given in (4) and read in our coordinates

$$S_1 = \cos y\,\mathcal{A} + \sin y\,\mathcal{B}, \qquad S_2 = -\sin y\,\mathcal{A} + \cos y\,\mathcal{B}, \tag{46}$$

where

$$\mathcal{A} = \Pi^3 - f f''\,\Pi\,P_y^2, \qquad \mathcal{B} = f'\,\Pi^2\,P_y - f\,(1 + f' f'')\,P_y^3. \tag{47}$$

Proposition 5. *The observables S_1 and S_2 are integrals of the geodesic flow.*

Proof. Let us define the complex object

$$\mathcal{S} = S_1 + iS_2 = e^{-iy}(\mathcal{A} + i\mathcal{B}). \tag{48}$$

The Poisson bracket with the Hamiltonian reads

$$\{H, \mathcal{S}\} = -e^{-iy}\frac{g}{f''}\Pi\,P_y^2(\Pi + if'\,P_y)\Big(f f''' + 3(1 + f' f'')\Big). \tag{49}$$

Again, the transformation $f = \sqrt{D}$ leads to the following linearization:

$$2\Big(f f''' + 3(1 + f' f'')\Big) = D''' + 6 = 0 \implies D = -(v^3 - s_1\,v^2 + s_2\,v - s_3). \tag{50}$$

We conclude via (43) and (49) that $\mathcal{S}$ is an integral. □

As in case (i) we have $dH \wedge dP_y \wedge dS_1 \wedge dS_2 = 0$, which shows that these four observables are not functionally independent. Indeed, we readily find

$$S_1^2 + S_2^2 = \mathcal{A}^2 + \mathcal{B}^2 = 8\,H^3 + 8a\,H^2\,P_y^2 + 2c\,H\,P_y^4 - 2\epsilon\lambda^2\,P_y^6, \tag{51}$$

leading us to consider two different superintegrable systems, namely

$$\mathcal{I}_1 = (H,\ P_y,\ S_1), \qquad \mathcal{I}_2 = (H,\ P_y,\ S_2). \tag{52}$$

The Poisson bracket of the two cubic integrals still reduces to a polynomial in the observables H and P_y, viz.,

$$\{S_1, S_2\} = -8a\,H^2\,P_y - 4c\,H\,P_y^3 + 6\epsilon\,\lambda^2\,P_y^5, \tag{53}$$

as in (51) for $S_1^2 + S_2^2$, but this is no longer true for the product

$$S_1\,S_2 = \cos(2y)\,\mathcal{A}\,\mathcal{B} + \sin(2y)\,\frac{\mathcal{B}^2 - \mathcal{A}^2}{2} \tag{54}$$

which is a new, independent, observable. This time, the set (H, P_y, S_1, S_2) of first integrals of the geodesic flow does not generate a Poisson algebra.

3.3. Transformation of the metric and curvature

Returning to the expression (50) of D, let us define the polynomials

$$P = 8vD + D'^2, \qquad Q = 2DD'' - D'^2 = -P - 4(v - s_1)D, \qquad Q' = -12\,D, \tag{55}$$

which readily yield the metric

$$\frac{1}{2}G = \rho^2\,\frac{dv^2}{D} + \frac{4D}{P}\,dy^2, \quad -\rho \equiv -\frac{Q}{P} = 1 + (v - s_1)\,\frac{4D}{P}, \tag{56}$$

with the restrictions $D > 0$ and $P > 0$ that ensure its Riemannian signature. We notice that the scalar curvature is still given by

$$R_G = \frac{1}{4Q^3}\Big(2PQ\,W' - (QP' + 2PQ')\,W\Big), \quad W \equiv DP' - PD' = 8D^2 + QD', \tag{57}$$

showing that Lemma 1 remains valid.

Lemma 3. *Let $I = (-\infty, v_0)$ be the allowed interval for v where v_0 is a simple zero of D. If for all $v \in I$ one has $P(v) > 0$ and $Q(v) > 0$, then the metric exhibits a conical singularity which precludes any manifold structure.*

Proof. Using the relations given in (55), when $v \to v_0+$ the metric approximates as

$$\frac{1}{2}G \approx \frac{4}{D'(v_0)}(dr^2 + r^2\,dy^2), \quad r = \sqrt{v - v_0} \to 0+ \tag{58}$$

and hence, for this singularity to be apparent, we need to assume $y = \phi \in \mathbb{S}^1$.

For $v \to -\infty$ we get

$$\frac{1}{2}G \approx dr^2 + r^2\left(\frac{d\phi}{3}\right)^2, \quad r = \frac{1}{\sqrt{-v}} \to 0+ \tag{59}$$

and we cannot have $\phi/3 \in \mathbb{S}^1$ as well. This kind of singularity, called *conical*, rules out a manifold structure. □

For further use we will also prove the general result:

Lemma 4. *Assume that the metric*

$$G = A(v)\,dv^2 + B(v)\,d\phi^2, \quad v \in I = [a, b],\ \phi \in \mathbb{S}^1, \tag{60}$$

be globally defined on a closed manifold M. Then its Euler characteristic is given by

$$\chi(M) = \gamma(b) - \gamma(a), \quad \gamma = -\frac{B'}{2\sqrt{AB}}. \tag{61}$$

Proof. Using the orthonormal frame

$$e_1 = \sqrt{A}\,dv, \qquad e_2 = \sqrt{B}\,d\phi,$$

we find that the connection 1-form reads $\omega_{12} = \frac{\gamma}{\sqrt{B}}\,e_2$, where γ is as in (61). The curvature 2-form is then given by

$$R_{12} = d\omega_{12} = \frac{\gamma'}{\sqrt{AB}}\,e_1 \wedge e_2,$$

from which we get

$$\chi(M) = \frac{1}{2\pi}\int_M R_{12} = \int_I \gamma'(v)dv = \gamma(b) - \gamma(a),$$

which was to be proved. □

Let us consider now the global properties of these metrics.

3.4. *The global structure for $\epsilon = 0$*

In this section we will keep the notation

$$D(v) = (a - v)(v^2 - a^2 + c), \qquad \Delta(D) = 4c^2(a^2 - c),$$

and organize the discussion according to the values of the discriminant $\Delta(D)$ of D. We will exclude the single case $a = c = 0$ since then the scalar curvature vanishes, implying that we loose superintegrability as explained in the proof of Proposition 2.

3.4.1. *First case: $\Delta(D) = 0$*

We will begin with

Proposition 6. *There exists no closed manifold for $c = 0$ and $a \neq 0$.*

Proof. We have, in this case,

$$D(v) = (a - v)(v^2 - a^2), \qquad P(v) = (v - a)^4, \qquad Q(v) = 3(v - a)^3(v - v_*), \quad v_* = -\frac{5}{3}\,a, \tag{62}$$

and the metric writes

$$\frac{1}{2}G = 9\frac{(v - v_*)^2}{(a - v)^4}\,\frac{dv^2}{-a - v} + \frac{4}{3}\,\frac{a - v}{(v - v_*)}\,dy^2. \tag{63}$$

For $a > 0$ we have $D > 0$ and $P > 0$ iff $v \in I = (-\infty, -a)$; but since $v_* \in I$ we get no manifold structure by Lemma 2.

For $a < 0$ the positivity of G is satisfied for $v \in (-\infty, a) \cap (a, -a)$. In both cases, a is a zero of P but we cannot use Lemma 2 because $Q(a) = 0$. In fact, the measure of the sought manifold

$$\mu_G = 12\int \frac{(v - v_*)}{(v - a)^3}\,dv \int dy$$

is divergent (since the integrand blows up at $v = a$), prohibiting a closed manifold. □

Please cite this article in press as: G. Valent, et al., Explicit metrics for a class of two-dimensional cubically superintegrable systems, Journal of Geometry and Physics (2014), http://dx.doi.org/10.1016/j.geomphys.2014.08.004

Proposition 7. *There exists no closed manifold for $c = a^2 > 0$.*

Proof. We have now

$$D(v) = v^2(a - v), \qquad P(v) = v^2(v - 2a)^2, \qquad Q(v) = 3v^3(v - v_*), \quad v_* = \frac{4}{3}a. \tag{64}$$

For $a < 0$ we have $D > 0$ and $P > 0$ iff either $v \in I_1 = (2a, a)$ or $v \in I_2 = (-\infty, 2a)$. In the first interval Q has a simple zero $v = v_*$, and $P(v_*)$ and $D(v_*)$ do not vanish; in view of Lemma 1 we get a curvature singularity. As for the second interval, the end-point $v = 2a$ is a zero of P where $Q(2a) \neq 0$; hence by Lemma 2, the sought manifold is not closed.

For $a > 0$ we have $I = (-\infty, a)$. There will be no curvature singularity since Q never vanishes for $v \in I$. Since $v = a$ is a simple zero of D such that $P(a)$ and $Q(a)$ are non-zero; we conclude by Lemma 3. □

3.4.2. Second case: $\Delta(D) < 0$

Here, we have

$$D = (a - v)\left(v^2 + c - a^2\right), \quad c > a^2, \qquad P = (v - w_-)^2(v - w_+)^2, \qquad w_\pm = a \pm \sqrt{c}, \tag{65}$$

and

$$Q = -P + 4(a - v)D, \qquad Q' = -12\,D. \tag{66}$$

Proposition 8. *There exists no closed manifold for $\Delta(D) < 0$.*

Proof. The positivity of D and P holds for any $v \in (-\infty, w_-) \cup (w_-, a)$. The second interval is excluded since Q is strictly decreasing and the relations

$$Q(w_-) = 8\,c^{3/2}\,(\sqrt{c} - a) > 0, \qquad Q(a) = -c^2 < 0, \tag{67}$$

imply that Q has a simple zero inside the interval (w_-, a), inducing a curvature singularity as already explained. This never happens for $v \in (-\infty, w_-)$ since then $Q(v) > 0$. But w_- is a zero of P and $Q(w_-) > 0$; we conclude by Lemma 2. □

3.4.3. The case $\Delta(D) > 0$

This time, $c < a^2$ and we find

$$\begin{aligned} &D(v) = (a - v)(v^2 - v_0^2), \quad v_0 = \sqrt{a^2 - c}, \\ &P(v) = \left((v - a)^2 - c\right)^2, \qquad Q(v) = -P(v) + 4(a - v)D(v). \end{aligned} \tag{68}$$

The parameter c can take its values in the set

$$(-\infty, 0) \cup \{0\} \cup (0, a^2).$$

Let us consider first negative values of c.

Theorem 1. *If $c \in (-\infty, 0)$ the superintegrable systems $\mathcal{I}_1$ and $\mathcal{I}_2$ given in (52) are globally defined on $\mathbb{S}^2$.*

Proof. First of all, we have $P > 0$. The ordering of the zeros of D is $-v_0 < a < v_0$. This implies two possible intervals ensuring its positivity: either $v \in (-\infty, -v_0)$ or $v \in (a, v_0)$.

The first case is easily ruled out since Q decreases from $Q(-\infty) = +\infty$ to $Q(-v_0) = -P(-v_0) < 0$; it thus vanishes in the interval and leads to a curvature singularity.

So let us consider $v \in (a, v_0)$. Then $Q(a) = -P(a) = -c^2$ is negative, and since Q is decreasing it will remain strictly negative everywhere on the interval. Putting $v_0 = 1$ and performing the transformation $G \to 2\,G$ for convenience, we end up with the explicit form of the metric, namely

$$G = \rho^2\,\frac{dv^2}{(v - a)(1 - v^2)} + 4\frac{(v - a)(1 - v^2)}{(v^2 - 2av + 1)^2}d\phi^2, \quad v \in (a, 1),\ \phi \in \mathbb{S}^1, \tag{69}$$

where

$$a \in (-1, 1), \quad -\rho = 1 + 4\frac{(v - a)^2(1 - v^2)}{(v^2 - 2av + 1)^2}. \tag{70}$$

Both end-points are apparent singularities because

$$G(v \to 1-) \sim \frac{2}{1 - a}(dr^2 + r^2\,d\phi^2), \quad r = \sqrt{1 - v}, \tag{71}$$

and

$$G(v \to a+) \sim \frac{4}{1 - a^2}(dr^2 + r^2\,d\phi^2), \quad r = \sqrt{v - a}. \tag{72}$$

Let us compute the Euler characteristic. Resorting to Lemma 4, we find

$$\gamma(v)=\frac{(1-v^2)^2-4(v-a)^2}{Q(v)}\implies \chi(M)=\gamma(1)-\gamma(a)=2, \tag{73}$$

which proves that the manifold is diffeomorphic to $\mathbb{S}^2$. The measure of this surface is

$$\mu_G(\mathbb{S}^2)=\frac{4\pi}{1+a}. \tag{74}$$

Let us investigate now the global status of the integrals H, P_y, S_1, S_2. Using (68), and referring to the Riemann uniformization theorem, we can write

$$H=\frac{1}{2}\left(\Pi^2+P\,\frac{P_\phi^2}{4D}\right)=\frac{1}{2\Omega^2}\left(P_\theta^2+\frac{P_\phi^2}{\sin^2\theta}\right) \tag{75}$$

with

$$t\equiv\tan\frac{\theta}{2}=\sqrt{\frac{(v-a)P}{(1-v^2)}},\qquad \Omega=\frac{1-v^2}{P}+v-a, \tag{76}$$

and the conformal factor is indeed C^∞ for all $v\in[a,1]$.

To ascertain that the previous integrals are globally defined, we will express them in terms of globally defined quantities, e.g., the SO(3) generators on $T^*\mathbb{S}^2$, namely

$$L_1=-\sin\phi\,P_\theta-\frac{\cos\phi}{\tan\theta}\,P_\phi,\qquad L_2=\cos\phi\,P_\theta-\frac{\sin\phi}{\tan\theta}\,P_\phi,\qquad L_3=P_\phi, \tag{77}$$

and the constrained coordinates

$$x^1=\sin\theta\,\cos\phi,\qquad x^2=\sin\theta\sin\phi,\qquad x^3=\cos\theta.$$

The relation $\Pi=-P_\theta/\Omega$ and formulas (46) and (47) yield

$$S_1=-\frac{L_2}{\Omega}\left(\Pi^2-Q\,\frac{P_\phi^2}{4D}\right)+x^2\,L_3\left(A\,\Pi^2-B\,\frac{P_\phi^2}{4D}\right), \tag{78}$$

where the functions A, B of θ retain the form

$$A=\frac{D'-\sqrt{P}\,\cos\theta}{2\sin\theta\,\sqrt{D}},\qquad B=\frac{W-Q\sqrt{P}\,\cos\theta}{2\sin\theta\,\sqrt{D}}. \tag{79}$$

The polynomials P, Q and W are clearly globally defined, as well as the quantities Π^2 and $P_\phi^2/(4D)$ in the Hamiltonian. So, it is sufficient to check that the functions A and B are well-behaved near the poles.

Let us begin with the north-pole ($v\to a+$ or $\theta\to 0+$) for which we get

$$\begin{cases} A=\dfrac{\phi(a)}{2(1-a^2)}-\dfrac{\sin^2\theta}{4(1-a^2)^2}+O(\sin^4\theta),\\ B=-\dfrac{(1-a^2)}{2}\,\phi(a)+\dfrac{3}{4}\sin^2\theta+O(\sin^4\theta),\end{cases}\qquad \phi(a)=a^4-2a^2-2a+1, \tag{80}$$

while for the south pole ($v\to 1-$ or $\theta\to\pi-$) we obtain

$$\begin{cases} A=\dfrac{\psi(a)}{2(1-a)}-\dfrac{(1-a)^4}{2}\,\sin^2\theta+O(\sin^4\theta),\\ B=-2(1-a)\,\psi(a)+6(1-a)^6\,\sin^2\theta+O(\sin^4\theta),\end{cases}\qquad \psi(a)=2a^2-4a+1. \tag{81}$$

We observe that either $\phi(a)$ or $\psi(a)$ may vanish for some $a\in(0,1)$, but this does not jeopardize the conclusion.

For the other integral, due to the relation

$$S_2=\{P_\phi,S_1\}=\frac{L_1}{\Omega}\left(\Pi^2-Q\,\frac{P_\phi^2}{4D}\right)+x^1\,L_3\left(A\,\Pi^2-B\,\frac{P_\phi^2}{4D}\right), \tag{82}$$

there is nothing more to check. □

Let us consider now the second case where c vanishes.

Proposition 9. *For $c = 0$ there exists no closed manifold.*

Proof. The above functions simplify and read

$$D = -(v+a)(v-a)^2, \qquad P = (v-a)^4, \qquad Q = (3v+5a)(v-a)^3, \quad a \neq 0. \tag{83}$$

For $a > 0$ the positivity of D requires $v \in I = (-\infty, -a)$, but since Q has a simple zero $v = -\frac{5}{3}a \in I$, in view of Lemma 1 there is no manifold structure.

For $a < 0$ either $v \in (-\infty, a)$ or $v \in (a, -a)$ ensure the positivity of D. But in both cases P vanishes for $v = a$, and the measure of the would-be manifold

$$\mu_G = 12 \int \frac{(v-v_*)}{(v-a)^3}\, dv \int dy$$

is divergent, excluding a closed manifold. □

The remaining case is $c \in (0, a^2)$. The discussion depends strongly on the sign of a. Beginning with $a > 0$ we have:

Proposition 10. *For $c \in (0,\ a^2)$ and $a < 0$ there exists no closed manifold.*

Proof. The two functions (D, P) are now

$$D(v) = (a-v)(v^2 - v_0^2), \quad v_0 = \sqrt{a^2 - c}, \qquad P = (v-w_-)^2(v-w_+)^2, \qquad w_\pm = a \pm \sqrt{c}, \tag{84}$$

with the ordering $w_- < a < w_+ < -v_0$.

The positivity requirements give three possible intervals:

$$I_1 = (-\infty, w_-), \qquad I_2 = (w_-, a), \qquad I_3 = (-v_0, v_0).$$

- For $v \in I_1$ we notice that w_- is a zero of P for which $Q(w_-) = 4(a - w_-)D(w_-) > 0$, and we conclude by Lemma 2.
- For $v \in I_2$ since $Q(w_-) > 0$ and $Q(a) = -P(a) < 0$, there is a simple zero v_* of Q inside I_2; hence, by Lemma 1, there is no manifold structure.
- For $v \in I_3$ we have $Q(-v_0) = -P(-v_0) < 0$ and then Q decreases to $Q(v_0)$; it thus never vanishes and $P > 0$ in I_3, opening the possibility of a manifold structure.

Putting $v_0 = 1$ and computing the metric brings us back to (69). □

For $a > 0$ we have:

Proposition 11. *For $c \in (0,\ a^2)$ and $a > 0$ there exists no closed manifold.*

Proof. The zeros of D and P interlace as follows $w_- < -|a| < -v_0 < w_+ < 0 < v_0$ giving four possible intervals

$$I_1 = (-\infty, w_-), \qquad I_2 = (w_-, -|a|), \qquad I_3 = (-v_0, w_+), \qquad I_4 = (w_+, v_0).$$

- For $v \in I_1 = (-\infty, w_-)$, and since w_- is a zero of P, we use Lemma 2.
- If $v \in I_2 = (w_-, -|a|)$, then Q is strictly decreasing with

  $$Q(w_-) = 4(-w_- + a)\,D(w_-) > 0 \quad \text{and} \quad Q(-|a|) = -P(-|a|) < 0,$$

  so that Q has a simple zero in I_2; thanks to Lemma 1, there is no manifold structure.
- For $v \in I_3 = (-v_0, w_+)$ or $v \in I_4 = (w_+, v_0)$, since w_+ is a zero of P we invoke again Lemma 2. □

3.5. The global structure for $\epsilon \neq 0$

Let us begin with

Proposition 12. *If $\Delta(D) = 0$ the superintegrable system is never globally defined on a closed manifold.*

Proof. We may have either $D(v) = (v_0 - v)^3$ or $D(v) = (v_0 - v)(v - v_1)^2$ with $v_0 \neq v_1$.

The first case is ruled out as in Proposition 2 since the metric is of constant curvature.

In the second case we have

$$\begin{cases} P(v) = (v-v_1)^2\, p(v), \quad p(v) = v^2 - 2(2v_0 + 3v_1)v + (2v_0 + v_1)^2, \\ Q(v) = 3(v-v_1)^3(v - v_*), \quad v_* = v_0 + \dfrac{v_0 - v_1}{3}. \end{cases} \tag{85}$$

Let us first consider the case $v_0 < v_1$. Then D is positive iff $v \in I = (-\infty, v_0)$. If $\Delta(p) < 0$ then $P > 0$ for all $v \in I$. But, since $v_* < v_0$, there will be a curvature singularity inside I. If $\Delta(p)$ vanishes, we get $p(v) = (v - w_0)^2$ and either $w_0 = v_0 < 0$ or $w_0 = 2v_0 < 0$. In the first case there will be a curvature singularity at $v_* = \frac{4}{3}v_0 \in I$ while, in the second case, the positivity interval becomes $(-\infty, w_0)$; since $v = w_0$ is a zero of P we use Lemma 2. If $\Delta(p) < 0$ we have two real zeros and $p(v) = (v - w_-)(v - w_+)$. The interval of positivity becomes $I = (-\infty, v_0) \cap (w_-, w_+)$ and since at least one of its end-points will correspond to a zero of P we conclude by Lemma 2.

Let us then consider the other case $v_0 > v_1$. Then D is positive iff $v \in I = (-\infty, v_0)$. If $\Delta(p) < 0$ then $P > 0$ for all $v \in I$, and we conclude by Lemma 3. If $\Delta(p) = 0$ we get $p(v) = (v - w_0)^2$, and either $w_0 = v_0 > 0$ or $w_0 = 2v_0 > 0$. In the first case we remain with $v \in (-\infty, v_0)$ and end up with a conical singularity for $v \to -\infty$; in the second case $v \in (w_0, v_0)$ where w_0 is a zero of P, which excludes closedness by Lemma 2. If $\Delta(p) > 0$ we have two real zeros and $p(v) = (v - w_-)(v - w_+)$. The interval of positivity becomes $I = (-\infty, v_0) \cap (w_-, w_+)$ and at least one of its end-points will correspond to a zero of P; we conclude by Lemma 2. □

Let us proceed to

Proposition 13. *If $\Delta(D) < 0$ the superintegrable systems are never globally defined on a closed manifold.*

Proof. In this case, we can write

$$D(v) = (v_0 - v)[(v - a)^2 + b^2], \quad b \neq 0, \qquad Q(v) = -P(v) + 4(v_0 + 2a - v)D(v), \tag{86}$$

and

$$P(v) = p(v)^2 - 16a[(v_0 + a)^2 + b^2]v, \quad p(v) = v^2 - 2(v_0 + 2a)v - a^2 - b^2 - 2av_0. \tag{87}$$

We have $D > 0$ iff $v \in I = (-\infty, v_0)$. Let us also notice that $\Delta(P)$ and $\Delta(Q)$ being negative, P and Q will have two simple real zeros. Since $Q(v_0) < 0$, then Q will have a simple zero $v_* < v_0$.

If $a = 0$ we have $p(v) = (v - w_-)(v - w_+)$, with the ordering $w_- < w_+$; hence P is always positive, but its zeros may change the interval for v: if $w_- < v_*$ the interval for v becomes (w_-, v_0) and then v_* is a curvature singularity inside this interval; if $w_- > v_*$ the interval for v becomes (w_-, v_0) for which Lemma 2 applies.

If $a > 0$, the relation (87) tells us that both roots of P must be positive and, since $P(v_0) = \left((v_0 - a)^2 + b^2\right)^2 > 0$, they must lie to the right of v_0. The interval for v remains $(-\infty, v_0)$ and we conclude by Lemma 3.

If $a < 0$ both roots of P ordered as $w_- < w_+$ must be negative and to the left of v_0. The positivity of P will reduce the interval of v either to $(-\infty, w_-)$ or to (w_+, v_0) and in both cases Lemma 2 allows us to conclude. □

Let us end up this section with:

Theorem 2. *If $\Delta(D) > 0$ one can put $D(v) = -(v - v_0)(v - v_1)(v - v_2)$ with $v_0 < v_1 < v_2$; the superintegrable systems $\mathcal{I}_1$ and $\mathcal{I}_2$ given by (52) are indeed globally defined on $\mathbb{S}^2$ iff $v_0 + v_2 > 0$.*

Proof. Let us define the symmetric polynomials of the roots s_1, s_2, s_3 by

$$D(v) = -(v - v_0)(v - v_1)(v - v_2) = -v^3 + s_1 v^2 - s_2 v + s_3.$$

The function D is positive iff either $v \in (-\infty, v_0)$ or $v \in (v_1, v_2)$. Let us first study the polynomial $Q = 3v^4 - 4s_1 v^3 + \cdots$. Since

$$\Delta(Q) = -6912(v_1 - v_0)^4(v_2 - v_0)^4(v_2 - v_1)^4 < 0$$

we conclude that Q has two simple real zeros. For $v \in (v_0, v_1)$ the relation $Q' = -12\,D$ shows that Q increases from $Q(v_0) = -(v_0 - v_1)^2(v_0 - v_2)^2$ to $Q(v_1) = -(v_1 - v_0)^2(v_1 - v_2)^2$; it then decreases to $Q(v_2) = -(v_2 - v_0)^2(v_2 - v_1)^2$ so that Q is strictly negative for all $v \in (v_0, v_2)$ and, since $Q(\pm\infty) = +\infty$, it will have a simple zero at $v = v_* < v_0$ and at $v = \widetilde{v}_* > v_2$, with the relation $v_* + \widetilde{v}_* = \frac{4}{3}s_1$.

Let us come back to the first positivity interval for D which is $I = (-\infty, v_0)$. As we have already seen, Q has a simple zero $v_* \in I$. Let us prove that $P(v_*) > 0$ which will be sufficient to ascertain, thanks to Lemma 1, that $v = v_*$ is a curvature singularity. To this end we use the relation

$$P(v) = -Q(v) + 4(s_1 - v)D(v) \implies P(v_*) = 4(s_1 - v_*)\,D(v_*). \tag{88}$$

Since $v_* < v_0$ we have $D(v_*) > 0$ and

$$s_1 - v_* = \widetilde{v}_* - \frac{s_1}{3} > v_2 - \frac{s_1}{3} = \frac{2v_2 - v_0 - v_1}{3} > 0.$$

Let us now consider the second positivity interval for D which is $I = (v_1, v_2)$. We find it convenient to define new parameters by

$$d = \frac{v_2 - v_1}{2} > 0, \qquad l = \frac{v_1 + v_2 + 2v_0}{v_2 - v_1} \in \mathbb{R}, \qquad m = \frac{v_1 + v_2 - 2v_0}{v_2 - v_1} > 1, \tag{89}$$

and a new coordinate, x, by

$$v = d\left(x + \frac{l + m}{2}\right), \quad x \in I = [-1, +1]. \tag{90}$$

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Since $d > 0$ we will set $d = 1$. It follows that

$$\begin{cases} D = (x+m)(1-x^2), \\ P = \Big(L_+\,(1-x^2) + 2(m+x)\Big)\Big(L_-\,(1-x^2) + 2(m+x)\Big), \quad L_\pm = l \pm \sqrt{l^2-1}, \\ Q = 3x^4 + 4mx^3 - 6x^2 - 12mx - 4m^2 - 1, \qquad Q' = -12\,D, \end{cases} \tag{91}$$

and the metric (again up to the change $G \to 2\,G$) reads now

$$G = \rho^2\,\frac{dx^2}{D} + \frac{4D}{P}\,d\phi^2, \quad \rho = \frac{Q}{P}. \tag{92}$$

For $x \in I$ the polynomial Q decreases from $Q(-1) = -4(m-1)^2$ to $Q(1) = -4(m+1)^2$ forbidding any curvature singularity. It remains to check the positivity of P. Its factorized expression shows that for $l \in [-1, 1)$ it has no real root. For $l \geq 1$ it has four simple real roots which lie outside I, and for $l < -1$ two of its real roots are still outside I, the remaining two $x_- < x_+$ being contained in I. It follows that I may be reduced to any of the intervals

$$I_1 = (-1, x_-) \quad \text{or} \quad I_2 = (x_-, x_+) \quad \text{or} \quad I_3 = (x_+, 1).$$

Now, at least one end-point is a zero of P, and by Lemma 3, the expected manifold is not closed. So far, we have proved that a manifold can exists iff $l \in (-1, +\infty)$, which translates as $v_0 + v_2 > 0$.

Let us study the behavior of the metric at the end-points of I by setting $x = \cos\vartheta$ with $\vartheta \in (0, \pi)$. We find that

$$G(\vartheta \to 0+) \approx \frac{1}{m+1}(d\vartheta^2 + \sin^2\vartheta\,d\phi^2), \qquad G(\vartheta \to \pi-) \approx \frac{1}{m-1}(d\vartheta^2 + \sin^2\vartheta\,d\phi^2), \tag{93}$$

and $\vartheta = 0, \pi$ are indeed apparent singularities. From Lemma 4 we get

$$\gamma = -\frac{W}{Q\,\sqrt{P}} \quad W = -(x^2 + 2x - 1 + 2m)(x^2 - 2x - 1 - 2m)(x^2 + 2mx + 1) \tag{94}$$

which gives

$$\chi(M) = \gamma(1) - \gamma(-1) = 2,$$

so that the manifold is actually $M \cong \mathbb{S}^2$.

Returning to the integrals, we will define once more

$$H = \frac{1}{2}\left(\Pi^2 + P\,\frac{P_\phi^2}{4D}\right) = \frac{1}{2\Omega^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}\right), \tag{95}$$

which leads to the relations

$$\Omega^2\,\sin^2\theta = \frac{4D}{P}, \qquad \frac{d\theta}{\sin\theta} = \frac{F(x)}{(1-x^2)}\,dx, \quad F(x) = \frac{Q(x)}{2(m+x)\sqrt{P(x)}}, \tag{96}$$

from which we deduce

$$t \equiv \tan\frac{\theta}{2} = \exp\left(\int_0^x \frac{F(u)}{(1-u^2)}\,du\right). \tag{97}$$

We need first to check the behavior of the conformal Ω factor at the north pole for $x \to 1-$. We have

$$t = \sqrt{1-x}\,T_N(x), \quad T_N(x) = \exp(U(x)), \; U(x) = \int_0^x \left(\frac{F(u)}{1+u} - \frac{F(1)}{2}\right)\frac{du}{1-u}, \tag{98}$$

so that T_N is C^∞ in a neighborhood of $x = +1$. This implies that

$$\Omega^2 = \frac{(1+t^2)^2(m+x)(1+x)}{P(x)\,T_N^2(x)} \tag{99}$$

is also C^∞ in a neighborhood of $x = +1$. At the south pole, i.e., for $x \to -1+$ a similar argument works.

The expression of S_1, in view of $\Pi = P_\theta/\Omega$, is now the following:

$$S_1 = \frac{L_2}{\Omega}\left(-\Pi^2 + Q\,\frac{P_\phi^2}{4D}\right) + x^2\,L_3\left(A\,\Pi^2 - B\,\frac{P_\phi^2}{4D}\right) \tag{100}$$

with

$$A=\frac{D'+\sqrt{P}\,\cos\theta}{2\sin\theta\,\sqrt{D}},\qquad B=\frac{W+Q\sqrt{P}\,\cos\theta}{2\sin\theta\,\sqrt{D}},\tag{101}$$

giving at the north pole:

$$\begin{cases}A=\dfrac{1}{\sqrt{2(m+1)}}\,\dfrac{(l+m+2)}{2\,T_N(1)}+O(\sin^2\theta),\\[2ex] B=-\Big(2(m+1)\Big)^{3/2}\,\dfrac{(l+m+2)}{2\,T_N(1)}+O(\sin^2\theta),\end{cases}\tag{102}$$

where the leading coefficients never vanish since $l+m>0$.

To analyze the behavior of S_1 at south pole let us define

$$t=\frac{1}{\sqrt{1+x}\,T_S(x)},\quad T_S(x)=\exp(-V(x)),\ V(x)=\int_0^x\left(\frac{F(u)}{1-u}-\frac{F(-1)}{2}\right)\frac{du}{1+u},\tag{103}$$

from which we deduce

$$\begin{cases}A=-\dfrac{1}{\sqrt{2(m-1)}}\,\dfrac{(l+m-2)}{2\,T_S(-1)}+O(\sin^2\theta),\\[2ex] B=(2(m-1))^{3/2}\,\dfrac{(l+m-2)}{2\,T_S(-1)}+O(\sin^2\theta),\end{cases}\tag{104}$$

which are well-behaved. For $l+m=2$ the power series expansions begin with $\sin^2\theta$, a possibility already observed in the proof of Theorem 1.

As to the integral S_2, the argument given in the proof of Theorem 1 works here just as well. □

3.6. Comparison with the results of Matveev and Shevchishin

In [3] it was stated in Theorem 6.1 that the metric

$$g=\frac{dx^2+dy^2}{h_x^2},\quad h_x=\frac{dh}{dx},$$

where h is a solution of the differential equation (3)(ii) with

$$\mu=1,\qquad A_0=1,\qquad A_1=0,\qquad A_3=A_4=A_e>0\quad\text{and}\quad A_2\in\mathbb{R},\tag{105}$$

is globally defined on S^2. As we will show in what follows, our results are partly in agreement with this Theorem 6.1.

Let us first write again the metric in our (v,ϕ) coordinates:

$$g=\frac{Q^2}{P^2}\,\frac{dv^2}{D}+\frac{4D}{P}\,d\phi^2,\quad \phi\in\mathbb{S}^1,\tag{106}$$

where Q and P are deduced from the knowledge of D by the relations given in (55). So to be able to compare these metrics we have first to notice that

$$h=\frac{D'}{2\sqrt{2D}}\quad D'=\frac{dD}{dv},\qquad h_x=u=\sqrt{h^2+v}=\sqrt{\frac{P}{8D}}>0,$$

and that

$$y=\phi,\qquad \frac{dx}{dv}=\frac{dx}{dh}\frac{dh}{dv}=\frac{1}{h_x}\frac{dh}{dv}=\frac{Q}{2D\sqrt{P}}.$$

Let us notice that to be Riemannian the metric (106) requires $D>0$ and $P>0$ and for the transformation $v\to x$ to be locally bijective we need Q to have a fixed sign.

Under the hypotheses of Theorem 6.1 we have, in our notation,

$$\epsilon=1,\qquad a=-A_2,\qquad \lambda=2A_e,$$

which gives

$$D=-(v+A_2)(v^2-A_2^2+c)-8A_e^2,\tag{107}$$

where c is a constant of integration which does not appear in the proof of Theorem 6.1 and which can be freely chosen.

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The discriminant of D is

$$\Delta(D) = -27\,\xi^2 + 4A_2(8A_2^2 - 9c)\xi + 4c^2(A_2^2 - c), \qquad \xi = 8A_e^2,$$

and the crucial point is that the sign of this discriminant is undefined. If $\Delta(D) > 0$, then Theorem 6.1 of Matveev and Shevchishin agrees with our Theorem 2, and the metric is indeed globally defined on $\mathbb{S}^2$. Nevertheless, if $\Delta(D) \leq 0$ our Propositions 12 and 13 show that either curvature singularities or conical singularities rule out any closed manifold. Let us mention that we have also found a metric globally defined on $\mathbb{S}^2$ for $\epsilon = 0$ (see Theorem 1), a case which has not been studied in [3].

4. The affine case

In this last case, we will prove that there is no closed manifold for the metric. However, since the analysis is much simpler we will determine the metrics globally defined either on $\mathbb{R}^2$ or on $\mathbb{H}^2$.

4.1. The metric

The differential equation and the metric are

$$h_x\Big(h_x^2 + A_1\,h + A_2\Big) = A_3\,x + A_4, \qquad G = \frac{dx^2 + dy^2}{h_x^2}, \tag{108}$$

see (3)(iii) and (2). Differentiating the equation for h gives

$$\Big(3\,h_x^2 + A_1 h + A_2\Big)h_{xx} + A_1\,h_x^2 = A_3,$$

and regarding again $u = h_x$ as a function of the new variable h, we rewrite the previous equations as

$$u(3u^2 + A_1 h + A_2)\frac{du}{dh} = A_3 - A_1 u^2.$$

Considering the inverse function $h(u)$ we end up with a linear ode, namely

$$(A_3 - A_1 u^2)\frac{dh}{du} - A_1 u\,h = u(3u^2 + A_2). \tag{109}$$

Two cases have to be considered:

1. If $A_1 = 0$ then A_3 cannot vanish; positing $\mu = \frac{3u^2+A_2}{A_3}$, the original variable, x, and the metric, G, are now given by

$$dx = \mu\,du, \quad \mu = \frac{1}{u}\,\frac{dh}{du} \implies G = \frac{1}{u^2}\left(\mu^2\,du^2 + dy^2\right). \tag{110}$$

   Interestingly, the relations

$$h = h_0 + \frac{A_2}{2A_3}u^2 + \frac{3}{4}\frac{u^4}{A_3}, \qquad A_3\,x + A_4 = A_2\,u + u^3$$

   show that we have integrated the ode (108) by expressing the function h and the variable x parametrically in terms of u.
2. If $A_1 \neq 0$ we can set $A_1 = 1$ and, by a shift of h, we may put $A_2 = 0$. To simplify matters, we will perform the following rescalings: $y \to 2y$, and $G \to \frac{1}{4}G$. This time, we will define

$$-2\mu = \frac{1}{u}\,\frac{dh}{du} \implies G = \frac{1}{u^2}\left(\mu^2\,du^2 + dy^2\right),$$

   and we get two possible solutions for μ:

$$\mu = 1 + \frac{C}{(u^2 - A_3)^{3/2}} \quad \text{or} \quad \mu = 1 + \frac{C}{(A_3 - u^2)^{3/2}}, \tag{111}$$

   where C is a real constant of integration.

4.2. Global structure for vanishing A_1

We have just seen that $\mu = \frac{3u^2+A_2}{A_3}$, and must thus discuss two cases separately:

1. First case: $A_2 = 0$, then we can pose $\mu = 2u^2$.
2. Second case: $A_2 \neq 0$, then we can pose $\mu = 1 + au^2$.

4.2.1. The case $A_2 = 0$

The relation (110) and the change $u \to v = u^2$ yield the metric and Hamiltonian, viz.,

$$G = dv^2 + \frac{dy^2}{v} \implies H = \frac{1}{2}(P_v^2 + v\,P_y^2),$$

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while the cubic integrals read now

$$S_1 = \frac{2}{3}\,P_v^3 + P_y^2(v\,P_v + \frac{y}{2}\,P_y) \tag{112}$$

and

$$S_2 = y\,S_1 - \left(\frac{y^2}{4} + \frac{v^3}{9}\right)P_y^3 - \frac{2}{3}\,v^2\,H\,P_y. \tag{113}$$

This last relation shows that S_2 is not algebraically independent, and that the superintegrable system we are considering is just generated by $(H,\ P_y,\ S_1)$. Let us mention, for completeness, the following Poisson brackets, namely

$$\{P_y, S_1\} = \frac{1}{2}\,P_y^3, \qquad \{P_y, S_2\} = S_1, \qquad \{S_1, S_2\} = \frac{3}{2}\,S_2\,P_y^2. \tag{114}$$

Proposition 14. *For $A_2 = 0$ the superintegrable system (H, P_y, S_1) is not globally defined.*

Proof. The Riemannian character of the metric requires $v > 0$ and $y \in \mathbb{R}$. If this metric were defined on a manifold, the scalar curvature would be everywhere defined. An easy computation gives for result $R_G = -\frac{3}{2v^2}$ which is singular for $v \to 0+$. □

4.2.2. The case $A_2 \neq 0$

We have now the Hamiltonian

$$2H = u^2\left(\frac{P_u^2}{\mu^2} + P_y^2\right), \quad u > 0,\ y \in \mathbb{R}, \qquad \mu = 1 + au^2, \quad a \in \mathbb{R}, \tag{115}$$

and the cubic integrals are respectively

$$S_1 = \frac{2a}{3}\left(\frac{u}{\mu}\,P_u\right)^3 + P_y\Big(u\,P_u\,P_y + y\,P_y^2\Big) \tag{116}$$

and

$$S_2 = y\,S_1 - \frac{1}{2}\Big(y^2 + u^2(1 + au^2/3)^2\Big)P_y^3 - \frac{a}{3}u^2(2 + au^2)HP_y. \tag{117}$$

The non-trivial Poisson brackets of the observables are then given by

$$\{P_y, S_1\} = P_y^3, \qquad \{P_y, S_2\} = S_1, \qquad \{S_1, S_2\} = 3\,S_2\,P_y^2\ + 4\,P_y^3\,H + \frac{16}{3}\,a\,P_y\,H^2. \tag{118}$$

Proposition 15. *For $A_2 \neq 0$ the superintegrable system (H, P_y, S_1)*

1. *is not globally defined for $a < 0$,*
2. *is trivial for $a = 0$,*
3. *is globally defined on $M \cong \mathbb{H}^2$ for $a > 0$.*

Proof. The scalar curvature reads now

$$R_G = -\frac{2}{\mu^3}\,(1 + 3au^2), \quad u > 0,\ y \in \mathbb{R}.$$

If $a < 0$ it is singular for $u_0 = |a|^{-1/2}$, and the system cannot be defined on a manifold.

For $a = 0$ the metric reduces to the canonical metric

$$G(\mathbb{H}^2, \text{can}) = \frac{du^2 + dy^2}{u^2}$$

of the hyperbolic plane $\mathbb{H}^2$. As a consequence of Thompson's theorem, which has been recalled above, S_1 and S_2 are reducible. Of course the set $(H,\ P_y)$ still remains an integrable system but it is trivial in the sense that it is no longer superintegrable.

Let us examine the last case for which $a > 0$. The change of coordinates

$$t = u\left(1 + \frac{a}{3}\,u^2\right) \ \longmapsto\ u = \frac{\xi^{1/3}}{a} - \xi^{-1/3}, \qquad \xi(t) = \frac{3}{2}\,a^2\,t + \sqrt{a^3 + \frac{9}{4}\,a^4\,t^2},$$

implies that $u(t)$ is C^∞ for all $t \geq 0$.

In these new coordinates the metric becomes

$$G = \Omega^2\,\frac{dt^2 + dy^2}{t^2} = \Omega^2\,G(\mathbb{H}^2, \text{can}), \qquad \Omega(t) = 1 + \frac{a}{3}\,u^2(t), \quad t > 0,\ y \in \mathbb{R}, \tag{119}$$

and, since Ω never vanishes, it is globally conformally related to the canonical metric of the hyperbolic plane, $M \cong \mathbb{H}^2$.

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Using the generators of sl(2, $\mathbb{R}$) on $T^*\mathbb{H}^2$ (given in the Appendix) allows us to write the Hamiltonian in the new guise

$$H = \frac{t^2}{2\,\Omega^2}\left(P_t^2 + P_y^2\right) = \frac{1}{2\,\Omega^2}\left(M_1^2 + M_2^2 - M_3^2\right). \tag{120}$$

The relations

$$P_y = M_2 + M_3 \quad \text{and} \quad t\,P_t = \frac{M_1 - x^1\,P_1}{1 + (x^1)^2}$$

show that

$$S_1 = \frac{2a}{3}\left(\frac{t\,P_t}{\Omega}\right)^3 + P_y^2\left(\mu\,\frac{t\,P_t}{\Omega} + y\,P_y\right), \qquad \mu(t) = 1 + a\,u^2(t), \quad a > 0, \tag{121}$$

is globally defined on M. The same is true for S_2 (see the relation (117)). $\square$

4.3. Global structure for non-vanishing A_1

In the formula (111) let us change $A_3 \to a$. We have, again, two cases to consider according to $\epsilon = \text{sign}(u^2 - a)$.

4.3.1. First case: $\epsilon = +1$

The metric and the Hamiltonian are given by

$$G = \frac{1}{u^2}\left(\mu^2\,du^2 + dy^2\right), \qquad H = \frac{u^2}{2}\left(\frac{P_u^2}{\mu^2} + P_y^2\right), \quad u^2 - a > 0,\ y \in \mathbb{R}, \tag{122}$$

where

$$\mu = 1 + \frac{C}{(u^2 - a)^{3/2}}.$$

The cubic integrals are then

$$S_1 = \left(\frac{u}{\mu}\,P_u\right)^3 + u(u^2 - a)\,P_u\,P_y^2 - ay\,P_y^3 + 2y\,H\,P_y \tag{123}$$

and

$$S_2 = y\,S_1 + \frac{1}{2}\left(a(u^2 + y^2) - \frac{2Cu^2}{\sqrt{u^2 - a}} + \frac{C^2}{u^2 - a}\right)P_y^3 - \left(u^2 + y^2 - \frac{2C}{\sqrt{u^2 - a}}\right)H\,P_y. \tag{124}$$

The case $C = 0$ corresponds to the canonical metric on $\mathbb{H}^2$, and, as already explained in Proposition 15, the system becomes trivial.

In the following developments, we will discuss the global properties of our superintegrable system according to the sign of $C \neq 0$, rescaling it to ± 1.

Proposition 16. *For $C = -1$ the superintegrable system (H, P_y, S_1) is globally defined iff $a < 0$ and $|a| > 1$, in which case the manifold is $M \cong \mathbb{H}^2$.*

Proof. The scalar curvature is

$$R_G = -\frac{2}{\mu^3}\left(1 + \frac{(2u^2 + a)}{(u^2 - a)^{5/2}}\right). \tag{125}$$

For $a \geq 0$ we must have $u > \sqrt{a}$ and R_G will be singular for $u_0 = \sqrt{a + 1}$. For $a < 0$ we must have $u > 0$. Then the curvature is singular for $u_0 = \sqrt{1 - \rho}$ if $\rho = |a| \leq 1$. However for $\rho > 1$ the function μ no longer vanishes and the curvature remains continuous for all $u \geq 0$. The metric then reads

$$G = \frac{1}{u^2}\left(\mu^2\,du^2 + dy^2\right), \quad \mu = 1 - \frac{1}{(\rho + u^2)^{3/2}},\ u > 0,\ y \in \mathbb{R}. \tag{126}$$

Let us define the new variable

$$t = u\left(1 - \frac{1}{\rho\sqrt{\rho + u^2}}\right), \quad u \in [0, +\infty) \longmapsto t \in [0, +\infty).$$

Since $\mu = \frac{dt}{du}$ never vanishes, the inverse function $u(t)$ is $C^\infty([0, +\infty))$ and the metric can be written as

$$G = \Omega^2 \, G(\mathbb{H}^2, \mathrm{can}), \qquad \Omega(t) = 1 - \frac{1}{\rho\sqrt{\rho + u^2(t)}}, \quad \rho > 1, \tag{127}$$

where the conformal factor $\Omega(t)$ is C^∞ and never vanishes: the manifold is again $M \cong \mathbb{H}^2$.

The first cubic integral

$$S_1 = \left(\frac{t\,P_t}{\Omega}\right)^3 + \mu(t)(\rho + u^2(t))\left(\frac{t\,P_t}{\Omega}\right)P_y^2 + \rho\, y\, P_y^3 + 2y\,H\,P_y \tag{128}$$

is therefore globally defined (with same argument as in the proof of Proposition 15), and (124) gives

$$S_2 = y\,S_1 + \frac{1}{2}\left(-\rho(u^2 + y^2) + \frac{2u^2}{\sqrt{\rho + u^2}} + \frac{1}{\rho + u^2}\right)P_y^3 - \left(u^2 + y^2 + \frac{2}{\sqrt{\rho + u^2}}\right)H\,P_y,$$

showing that this is also true for S_2. □

Proposition 17. *For $C = +1$ the superintegrable system (H, P_y, S_1) is globally defined either if $a > 0$ and the manifold is $M \cong \mathbb{R}^2$, or if $a < 0$ and $M \cong \mathbb{H}^2$.*

Proof. The metric reads now

$$G = \frac{1}{u^2}\left(\mu^2\, du^2 + dy^2\right), \qquad \mu = 1 + \frac{1}{(u^2 - a)^{3/2}}. \tag{129}$$

Consider first the case $a > 0$ for which $u > \sqrt{a}$. Let us define the new coordinate

$$t = u\left(1 - \frac{1}{a\sqrt{u^2 - a}}\right), \qquad u \in (\sqrt{a}, +\infty) \longmapsto t \in \mathbb{R}.$$

Since, again, $\mu = \frac{dt}{du}$ does not vanish $u(t)$ is $C^\infty(\mathbb{R})$, and the metric

$$G = \frac{dt^2 + dy^2}{u^2(t)}, \qquad t \in \mathbb{R},\ y \in \mathbb{R}, \tag{130}$$

turns out to be globally conformally related to the flat metric; the manifold is therefore $M \cong \mathbb{R}^2$.

The cubic integral

$$S_1 = (u(t)\,P_t)^3 + \mu(t)(u^2(t) - a)(u(t)\,P_t)\,P_y^2 - ay\,P_y^3 + 2y\,H\,P_y \tag{131}$$

remains hence globally defined, and the same holds true for S_2.

- For $a = 0$ the function $\mu = 1 + \frac{1}{u^3}$ is no longer even, so we must consider that $u \in \mathbb{R}$ and the scalar curvature

  $$R_G = 2u^6\,\frac{(2 - u^3)}{(1 + u^3)^3}$$

  is not defined for $u = -1$; there is thus no obtainable manifold structure.
- For $a < 0$ we set $\rho = |a|$ and we must take $u > 0$; we then define the new coordinate

  $$t = u\left(1 + \frac{1}{\rho\sqrt{\rho + u^2}}\right), \qquad u \in (0, +\infty) \longmapsto t \in (0, +\infty).$$

  Since $\mu = \frac{dt}{du}$ never vanishes, the inverse function $u(t)$ is $C^\infty([0, +\infty))$. The metric

  $$G = \Omega^2\,\frac{dt^2 + dy^2}{t^2}, \qquad \Omega(t) = 1 + \frac{1}{\rho\sqrt{\rho + u^2}}, \quad \rho > 0,\ t > 0,\ y \in \mathbb{R}, \tag{132}$$

  is again globally conformally related to the canonical metric on the manifold $M \cong \mathbb{H}^2$. The proof that the cubic integrals are also globally defined is the same as in Proposition 15. □

4.3.2. Second case: $\epsilon = -1$

The metric and the Hamiltonian are now given by

$$G = \frac{1}{u^2}\left(\mu^2\, du^2 + dy^2\right), \qquad H = \frac{u^2}{2}\left(\frac{P_u^2}{\mu^2} + P_y^2\right), \qquad a - u^2 > 0, \quad y \in \mathbb{R}, \tag{133}$$

where

$$\mu = 1 + \frac{C}{(a-u^2)^{3/2}}.$$

The scalar curvature reads thus

$$R_G = -\frac{2}{\mu^3}\left(1 + C\,\frac{(2u^2+a)}{(a-u^2)^{5/2}}\right). \tag{134}$$

The cubic integral S_1 is the same as in (123) while

$$S_2 = y\,S_1 + \frac{1}{2}\left(a(u^2+y^2) + \frac{2Cu^2}{\sqrt{a-u^2}} + \frac{C^2}{a-u^2}\right)P_y^3 - \left(u^2+y^2+\frac{2C}{\sqrt{a-u^2}}\right)H\,P_y \tag{135}$$

is merely obtained by the substitution $C \to -C$.

Proposition 18. *Either for $C = -1$ and $0 < a < 1$ or for $C = +1$ the superintegrable system (H, P_y, S_1) is globally defined on the manifold $M \cong \mathbb{H}^2$.*

Proof. We must have $a > 0$ to ensure $u \in (0, \sqrt{a})$.

- For $C = -1$ the scalar curvature is singular when μ vanishes. This happens for $u_0 = \sqrt{a-1}$ and $a \geq 1$; in this case there exists no manifold structure. However for $0 < a < 1$ the function μ never vanishes, so we can define

$$t = -u\left(1 - \frac{1}{a\sqrt{a-u^2}}\right), \quad u \in (0, \sqrt{a}) \longmapsto t \in (0, +\infty),$$

and the inverse function $u(t)$ is in $C^\infty([0, +\infty))$; this leads to the metric

$$G = \Omega^2\, G(\mathbb{H}^2, \text{can}), \qquad \Omega(t) = -1 + \frac{1}{a\sqrt{a-u^2(t)}}, \quad 0 < a < 1, \tag{136}$$

where the conformal factor never vanishes; hence, the manifold is again $M \cong \mathbb{H}^2$. The proof that the cubic integrals are also globally defined is the same as in Proposition 15.

- For $C = +1$ the function

$$\mu = 1 + \frac{1}{(a-u^2)^{3/2}}$$

never vanishes, implying that the curvature is defined everywhere for $u \in (0, \sqrt{a})$. If we define

$$t = u\left(1 + \frac{1}{a\sqrt{a-u^2}}\right), \quad u \in (0, \sqrt{a}) \longmapsto t \in (0, +\infty),$$

the metric retains the form

$$G = \Omega^2\, G(\mathbb{H}^2, \text{can}), \quad \Omega = 1 + \frac{1}{a\sqrt{a-u^2(t)}}, \; a > 0, \tag{137}$$

where the conformal factor, Ω, never vanishes; hence, the manifold is again $M \cong \mathbb{H}^2$. At last, the proof that the cubic integrals S_1 and S_2 are also globally defined is the same as in Proposition 15. □

5. Conclusion

We have completed the work initiated by Matveev and Shevchishin in [3] by providing the explicit form of their metrics in local coordinates. This allowed us to determine systematically all the cases in which their superintegrable systems can be hosted by a simply-connected, two-dimensional smooth manifold M. Let us emphasize that we have achieved, via Theorems 1 and 2, the classification of all these metrics on closed, simply-connected, surfaces, namely on $M \cong \mathbb{S}^2$.

As pointed out in [3] superintegrable systems on a closed manifold should lead to Zoll metrics [8], i.e., to metrics whose geodesics are all closed and of the same length. Using the explicit formulas obtained here for the metrics it has been proved by a direct analysis in [9] that all the metrics defined on S^2 that we have obtained here are indeed Zoll metrics. Generalizing this analysis to closed orbifolds gives either Tannery or Zoll metrics.

Another obvious line of research would be the generalization of these results to the case of observables of fourth or even higher degree, as well as the challenging problem of their quantization. An interesting approach could be to use a well and uniquely defined quantization procedure, in our case the conformally-equivariant quantization [10]. The latter, from its very definition and construction, could be perfectly fitted to deal with integrable systems on Riemann surfaces.

Acknowledgments

We wish to warmly thank V. Matveev, and J.-P. Michel for their interest in this work, and for enlightening discussions.

Appendix. The hyperbolic plane

Let us recall that the hyperbolic plane

$$\mathbb{H}^2 = \{(x^1,\ x^2,\ x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 - (x^3)^2 = -1,\ x^3 > 0\} \tag{138}$$

may be embedded in $\mathbb{R}^{2,1}$ as follows

$$x^1 = \frac{y}{t}, \qquad x^2 = \frac{1}{2t}(t^2 + y^2 - 1), \qquad x^3 = \frac{1}{2t}(t^2 + y^2 + 1). \tag{139}$$

This choice of coordinates leads to the induced metric

$$G(\mathbb{H}^2, \text{can}) = \frac{dt^2 + dy^2}{t^2}, \quad t > 0,\ y \in \mathbb{R}. \tag{140}$$

The generators on $T^*(\mathbb{H}^2)$ of the group of isometries of $\mathbb{H}^2$ given by

$$\begin{aligned} M_1 &= x^2\, P_3 + x^3\, P_2 = t\, P_t + y\, P_y, \\ M_2 &= x^3\, P_1 + x^1\, P_3 = -ty\, P_t + \frac{(1 + t^2 - y^2)}{2}\, P_y, \\ M_3 &= x^1\, P_2 - x^2\, P_1 = +ty\, P_t + \frac{(1 - t^2 + y^2)}{2}\, P_y, \end{aligned} \tag{141}$$

are globally defined and generate, with respect to the Poisson bracket, the Lie algebra $\text{sl}(2, \mathbb{R})$, namely

$$\{M_1, M_2\} = -M_3, \qquad \{M_2, M_3\} = M_1, \qquad \{M_3, M_1\} = M_2.$$

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Please cite this article in press as: G. Valent, et al., Explicit metrics for a class of two-dimensional cubically superintegrable systems, Journal of Geometry and Physics (2014), http://dx.doi.org/10.1016/j.geomphys.2014.08.004