ISSN 0001-4346, Mathematical Notes, 2012, Vol. 92, No. 1, pp. 59–63. © Pleiades Publishing, Ltd., 2012.

On Unbounded Probability Theory*

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Received , 2012

Abstract—The value of the empirical expectation coincides with that of the mean energy of an ideal Bose gas for one particle. The exact mathematical identity for these quantities makes it possible to carry over the concept of temperature corresponding to the mean energy to an unboundedly increasing sequence of random values for a new unbounded probability theory and for a generalization of Kolmogorov complexity theory. The notion of spectral gap, which was introduced in superconductivity theory, is carried over to unbounded probability theory.

DOI: `10.1134/S0001434612070073`

Keywords: *Kolmogorov complexity theory, Bose–Einstein entropy, logarithmic scaling, incompressible liquid, gap between the Bose condensate point (zero energy) and energy greater than zero, phase transition of the new ideal gas to the new ideal liquid.*

The present paper is a continuation and complement of the paper [1]. It also contains answers to certain questions from readers specializing in probability theory.

First of all, I wish to recall again the history of unbounded probability theory and present, in greater detail than in my previous papers, the relevant ideas of the great mathematician H. Poincaré, who was head of the department of probability theory at the Sorbonne. In his work "Science and Hypothesis" (Chap. XI "Calculus and Probability"), Poincaré writes[1]:

> "Has the notion of probability ever been defined? And can it really be defined? If not, how can we discuss it? The definition looks very simple: the probability of an event is the ratio of the number of cases favorable to this event to the total number of possible cases.
>
> A simple example shows that this definition is incomplete. Suppose two dice are thrown. What is the probability that at least one of them has 6 on the upper face? Each die can fall in six different ways: the number of possible cases is $6 \times 6 = 36$; the number of favorable cases is 11, and hence the probability is $11/36$.
>
> This is a correct solution. But, on the other hand, I could state that the number of points from a throw of two dice can form $(6 \times 7)/2 = 21$ different combinations; among these combinations, only 6 are favorable, and hence the probability is $6/21$.
>
> Why the first method of calculating the number of possible cases is more correct than the second one? In any case, our definition does not give an answer to this question.
>
> Thus, one must complement this definition by stating "…to the total number of possible cases under the condition that these cases are equiprobable". And thus we see that it is necessary to define the concept of probability in terms of probability itself.
>
> How can we know that any two possible cases are equiprobable? Is this not a result of some sort of agreement? If, at the beginning of each problem, there is an explicit agreement, then everything will take its proper course, and we will only need to apply the rules of arithmetic and algebra to obtain a final indisputable result. But if we wish to apply our result to any problem, even not a very important one, then it is necessary to prove that

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*The article was submitted by the author for the English version of the journal.

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[1] *Translator's note.* The following quotation from Poincaré's work is the translation from the Russian text of his work [3, pp. 115–116 (Russian transl.)].

> our agreement is correct, and we come up against the same difficulty that we have tried to avoid.
>
> Some people may say that the proper foundations can be laid simply by common sense. But here is a simple problem studied by Bertrand: "What is the probability that, in a circle, the chord is greater than the side of the inscribed equilateral triangle?" The famous geometer gave two different agreements, apparently consistent with common sense and found two different answers, 1/2 and 1/3.
>
> The conclusion to be reached from the foregoing examples is that, apparently, probability theory is a useless science and that we should not rely on the vague instinct called common sense which we use to make the needed agreements."

This problem of the dependence of probability theory on common sense was discussed earlier by Laplace; it was solved by Kolmogorov in his complexity theory. We throw a coin without assuming anything about the probabilities of heads or tails and make a sufficiently large number of trials; after that, we note that the majority of possible variants accumulates in the interval $1/2 \pm \sqrt{n}$, where n is the number of trials. In his theory, Kolmogorov defines the minimal code of an isolated trial, which is close to the maximal entropy.

If the value of the random variable does not increase, this remarkable achievement gives a complete answer to the question posed by Poincaré in his philosophical work cited above.

We carry over this Kolmogorov principle to the case of infinitely increasing sequence of random variables. We also use results due to Erdős and Lehner dealing with number theory [4], [5]. In contrast to Kolmogorov's approach based on the notion of Shannon entropy, we studied the case of the Bose–Einstein entropy. In this case, as follows from [1, Remark 1], there is a gap between the Bose condensate point (zero energy) and energy greater than zero (see [7]. In the present paper, we show that the presence of this gap affects the value of the expectation in the case where the growth of the random variable is linear or greater and in the case where it is quadratic or less.

1. LOGARITHMIC SCALING

In his work on probability theory, Poincaré states that one must not renounce probability theory, because it is connected with the kinetic theory of gases and thermodynamics. This thought of Poincaré is further developed in the present paper on the basis of the new concepts of ideal gas and of new ideal liquid constructed by the author (see [1]).

In order statistics (variational series), logarithmic scaling specifies the one-to-one correspondence of the numbers $A \geq 0$ and $\ln A = a$, $B \geq 0 \to b = \ln B$. Under logarithmic scaling, to the product AB corresponds the sum of the logarithms $\ln A + \ln B$.

Addition under logarithmic scaling. To the sum corresponds

$$A \oplus B = \frac{1}{E_c} \ln(e^{-aE_c} + e^{-bE_c}) = \frac{1}{E_c} \ln(A^{E_c} + B^{E_c}). \tag{1}$$

Using this one-to-one correspondence, we can encode any numbers A and B as well as the semifield AB and obtain the correspondence $A + B \to A \oplus B$.

Example. In the case $\gamma = 0$, i.e., in the two-dimensional case, using the Erdős theorem, we find that the value of E_c equal to

$$E_c = \frac{n_c}{\ln^2 n_c} \frac{\pi^2}{6}, \tag{2}$$

increases with the number of trials.

The addition (1) as $n \to \infty$ becomes the phase transition from the gas state to the liquid state and from the open state to the closed one.

2. THE GAP FOR THE QUANTITIES M AND E

The normalization by Λ and formulas (12)–(16) given in [1] are only required to obtain the number of zeros in the condensate by using a parastatistical correction (see the remark in [1]). This implies that there is a gap between the values of $\varepsilon_0 = 0$ and $\varepsilon_1 > 0$.

By the *energy gap* we mean the difference between the solution of problem (3)

$$\sum_{i=1}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=1}^{\infty} q_i(D) N_i = E, \qquad \varepsilon = \frac{h^2}{2mV^{2/D}}, \qquad q_i(D) = \frac{\Gamma(i+D-1)}{\Gamma(i)\Gamma(D)}, \tag{3}$$

where Γ is the gamma function, and that of problem (4)

$$\sum_{i=1}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=1}^{\infty} q_i(D) N_i < E, \qquad \varepsilon = \frac{h^2}{2mV^{2/D}}, \tag{4}$$

while by the *density gap* we mean the difference between the solution of problem (3) and that of problem (5)

$$\sum_{i=1}^{\infty} N_i < N, \qquad \varepsilon \sum_{i=1}^{\infty} q_i(D) N_i = E, \qquad \varepsilon = \frac{h^2}{2mV^{2/D}}. \tag{5}$$

In other situations, we can pass to the Van der Waals normalization $T_r = T/T_c$. In this case, relation (21) of [1] can be dropped and we can study only the gap that arises for M. In thermodynamics, this corresponds to the gap for the pressure P.

Both in the theory based on the scaling conjecture and in the alternative theory of the author, there is a more rapid growth of the pressure P near the critical point $\mu = 0$ than that predicted by the Van der Waals theory. Therefore, it is natural to expect that the gap for P is greater than the density gap; however, its quantity is independent of the value of Z_c. In our case, the gap depends on Z_c and increases as $Z_c \to 0$.

By [8], we have

$$n_c = b^{-1}\zeta(1-\gamma) + O(b^{-1-\gamma}), \quad \gamma < 0, \qquad b = \frac{1}{T_{\mathrm{r}}},$$

where ζ is the Riemann zeta function.

Thus, for $\gamma < 0$ (i.e., for negative pressures), the value of n_c is proportional to the value of T_c.

Hence, for $\gamma < 0$, the value of $\gamma = \gamma(T)$ is determined on the spinodal.

Using the condition

$$n_r = T_r^{\gamma_0+1}\zeta(\gamma_0+1)$$

on the closed interval $Z = Z_c$ in the plane $Z,\ P$, we obtain the dependence of γ on the temperature for $\tilde\mu = 0$ (for the spinodal) in the region of negative pressures for the isotherm–isochore

$$T_r^{\gamma_0+1}(\zeta(\gamma_0+1)) = \zeta(1+|\gamma(T_r)|)T_r.$$

In particular, for $T_r = 1$ (i.e., $T = T_c$), the value $\gamma = -\gamma_0$ determines the initial reference point for the chemical potential of the liquid phase $\tilde\mu = 0$ from the condition

$$M = \frac{\mathrm{Li}_{2-\gamma_0}(e^{-\tilde\mu/T_c})}{\zeta(2+\gamma_0)} = 1.$$

Thus, we obtain the initial reference point $\tilde\mu$ from the condition on T_r,

$$\mathrm{Li}_{2+\gamma_0}(1) = \mathrm{Li}_{2-\gamma_0}(e^{-\tilde\mu_0/T_c}).$$

We find $\mu = \tilde\mu$ as a function of T_c from the condition

$$T_r^{\gamma_0}\,\mathrm{Li}_{2+\gamma_0}(e^{-\mu/T_r}) = T_r^{-|\gamma(T_r)|}\,\mathrm{Li}_{2-|\gamma(T_t)|}(e^{-\tilde\mu/T_r-\tilde\mu_0/T_c}).$$

Hence, for thermodynamics, we obtain the point μ on the gas branch in which the phase transition of the new ideal gas to the new ideal liquid occurs. The two-phase zone is possible only on a small interval from $T_r = 0.835$ to $T_r = 0.839$.

For smaller temperature T_r, we have $\mu = -\infty$, i.e., the ideal gas becomes an ideal liquid completely. Only by taking into account the interaction and, therefore, the presence of the Zeno line, do we obtain a good agreement with the experimental data. Obviously, without attraction, the transition to a liquid and the dilatation of the liquid is impossible. So we see that a device measuring the density inside a vessel changes our conception of the real (exterior, objective) world and that the "exchange" attraction effect, which is commonplace in quantum physics, remains also valid in classical physics for classical distinguishable particles, but only when their number is sufficiently great (the "heap" paradox).

Note that, although, as $h \to 0$, quantum indistinguishable bosons preserve Bose statistical symmetry, the overlapping region corresponding to exchange interaction contracts to zero. However, as $h \to 0$ jointly with $N \to \infty$, the attraction effect may be preserved.

For $\gamma = 1/2$, we obtain, in particular, a quantum three-dimensional ideal Bose gas. In this case, the new aspect is the transition to the quantum ideal liquid for $T \cong 0.835T_d$ (T_d is the degeneracy temperature). For $0.835T_d < T < T_d$, the two-phase region appears (up to the equality of chemical potentials).

This is the phase transition to another entropy, namely, the entropy of an incompressible Bose liquid. According to Kolmogorov complexity theory, the entropy is directly related to the minimal length code. Thus, we pass to a new encoding. It is impossible to decrease the number of trials, because the number of trials is irreversible. In our case, an increase in the number of trials has no effect, because the superfluous trials contribute to the "Bose condensate" of the Bose liquid. This means that, as the number of trials increases, we inevitably make a contribution to the group with zero income if the general population is the population of a country and the order statistics are determined by the values of the corresponding random variable (the income of each citizen).

T. V. Maslova drew attention to this fact in her diploma paper (Master's degree thesis) presented at the University of the West of England (UWE). In her paper, she considered a program testing procedure for an audio guide of the type used in a picture gallery. The self-learning device under consideration obviously has two translational and one rotational degrees of freedom, i.e., the number γ is 1/2, which corresponds to an ideal Bose gas. This is the case of the values of a random variable with unbounded quadratic growth. This case iss described in [2] as well as in the present paper[2].

However, an incompressible Bose liquid is not superfluid. It follows from the above that, for this liquid to be superfluid, it must be "weakly nonideal" (this fact was noted earlier by N. N. Bogolyubov). The liquid spinodal in the negative energy region terminates at the point $T = 0.39T_d$. Below this temperature, the liquid becomes a dispersive system ("foam").

For $T_r < 1$, for argon, the transition to an ideal liquid is possible only for

$$P = 0.93P_c.$$

Accordingly, the transition to the closed state occurs for $T_r < 1$ only for

$$E < E_c + \delta(\gamma),$$

where $\delta(\gamma)$ is the gap in the expectation values.

For the number of degrees of freedom greater than 4, the chemical potential is identically zero ($\mu \equiv 0$). For $\gamma \geq 0.4$, the pressure gap is independent of Z_c.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 11-01-12058_ofi_m).

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[2] This important result contained in T. V. Maslova's paper was not fully appreciated at the presentation and, as a result, she did not get her diploma with honors.

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